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## Hamiltonian Structures and Integrability of Frobenius Algebra-Valued $(n, m)^{\text{th}}$ KdV Hierarchy

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We introduce Frobenius algebra  $\mathcal{F}$ -valued  $(n, m)^{\text{th}}$  KdV hierarchy and construct its bi-Hamiltonian structures by employing  $\mathcal{F}$ -valued pseudo-differential operators. As an illustrative example, the  $(1, 1)^{\text{th}}$   $\mathcal{L}_2$ -valued case is analyzed in detail. Its Hamiltonian structures and recursion operator are derived. Infinitely many symmetries, conservation laws and explicit flow equations are also obtained.

*Keywords:*  $\mathcal{F}$ -valued  $(n, m)^{\text{th}}$  KdV hierarchy; Hamiltonian structure; hereditary recursion operator; flow equation

2000 Mathematics Subject Classification: 37K10, 35Q53

### 1. Introduction

In recent years algebra-valued generalizations of integrable soliton equations have been much explored. For them, field variables take their values in a particular algebra, usually an associative algebra. Some important examples include matrix or operator algebras, Clifford algebras and group algebras. Svinolupov [1, 2] and Olver & Sokolov [3] have started the systematic classification of certain algebra-valued systems.

Recently, Zuo & Strachan [4–6] have considered Frobenius algebra valued integrable systems. In Ref. [4] Zuo constructed the bi-Hamiltonian structures of  $\mathcal{L}_m$ -KP hierarchy by introducing a new trace-type map. In Ref. [5] they derived the Hamiltonian structures and  $\tau$  function for  $\mathcal{F}$ -KP hierarchy. The  $\mathcal{F}$ -valued counterpart of the so-called  $n^{\text{th}}$  constrained KP hierarchy has been studied in Ref. [7].

In the literatures, besides the  $n^{\text{th}}$  constrained hierarchy, classical KP hierarchy admits another class of KdV-type hierarchy, which is also studied extensively [8–13]. It is called  $(n, m)^{\text{th}}$  KdV hierarchy in Refs. [8, 9]. It is natural to study its Frobenius algebra valued generalization. The aim of this paper is to introduce  $\mathcal{F}$ -valued  $(n, m)^{\text{th}}$  KdV hierarchy, and investigate its Hamiltonian structures and some related integrability properties.

This paper is organized as follows. In Section 2, we introduce  $\mathcal{F}$ -valued  $(n, m)^{\text{th}}$  KdV hierarchy and derive its bi-Hamiltonian structures by making use of pseudo-differential operators. As an illustrative example, in Section 3 we analyze the  $(1, 1)^{\text{th}}$   $\mathcal{L}_2$ -KdV hierarchy. The  $t_2$ -flow equation and its explicit Hamiltonian structures are calculated. Employing hereditary recursion operator we derive infinitely many symmetries and conservation laws of the hierarchy, along with the  $t_3$ -flow equation.

**2.  $\mathcal{F}$ -valued  $(n, m)^{\text{th}}$  KdV hierarchy**

Recall that [5, 14] a Frobenius algebra  $\{\mathcal{F}, \circ, \mathbf{1}, \omega\}$  is a vector space over  $\mathbb{R}$  which satisfies the conditions: (i)  $(\mathcal{F}, +, \circ)$  is a commutative, associative algebra with multiplicative unit  $\mathbf{1}$ ; (ii)  $\omega \in \mathcal{F}^*$  defines a non-degenerate inner product  $\langle a, b \rangle = \omega(a \circ b)$ . The linear form  $\omega$  is often called a trace map, denoted as  $\omega = \text{tr}$ .

For example [4, 5], let  $\mathcal{L}_2$  be a 2-dimensional commutative and associative algebra with a basis  $(e_1 = \mathbf{1}, e_2)$  satisfying

$$e_1 \circ e_1 = e_1, \quad e_1 \circ e_2 = e_2, \quad e_2 \circ e_2 = 0. \tag{2.1}$$

The trace map is defined as  $\text{tr}(a) = a_2$ ,  $a = a_1 e_1 + a_2 e_2 \in \mathcal{L}_2$ . This is the Frobenius algebra we mainly use later. It has a matrix representation as follows

$$e_1 \mapsto I_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad e_2 \mapsto \Lambda = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}. \tag{2.2}$$

For the basic facts about ( $\mathcal{F}$ -valued) pseudo-differential operator and its application in the theory of integrable systems, the reader is referred to e.g. [5, 7, 12, 15].

**2.1.  $\mathcal{F}$ -valued  $(n, m)^{\text{th}}$  KdV hierarchy**

Suppose  $A, B$  are two  $\mathcal{F}$ -valued differential operators, of order  $n + m$ ,  $m$ , ( $m \geq 0, n \geq 1$ ) respectively,

$$A = \mathbf{1} \partial^{n+m} + \sum_{j=0}^{n+m-1} A_j \partial^j, \quad B = \mathbf{1} \partial^m + \sum_{j=0}^{m-1} B_j \partial^j, \quad \partial = \frac{d}{dx}$$

with identical second coefficients,  $A_{n+m-1} = B_{m-1}$ . Here  $A_j, B_j$  denote the coefficients of  $\partial^j$  in  $A$  and  $B$ , which are  $\mathcal{F}$ -valued smooth functions of variable  $x$ .

Let  $(e_1, \dots, e_s)$  be a linear basis of  $\mathcal{F}$ , the  $\mathcal{F}$ -valued coefficients decompose in terms of basis as

$$A_j = \sum_{k=1}^s a_{jk} e_k, \quad B_j = \sum_{k=1}^s b_{jk} e_k,$$

where the components  $a_{jk}, b_{jk}$  are scalar-valued (real-valued) smooth functions of  $x$ .

The differential algebra under consideration is denoted by  $\mathcal{A}$ , which consists of polynomials with real coefficients in  $\{a_{jk}, b_{jk}\}$  and their derivatives of arbitrary orders, i.e. differential polynomials in  $\{a_{jk}, b_{jk}\}$ . The flow equations and Hamiltonian structures of our hierarchy will be established in terms of these dynamical coordinates, or evolutionary fields  $a_{jk}, b_{jk}$ .

The operator  $\partial = d/dx$  is a derivation of algebra  $\mathcal{A}$ , acting on it by the rule

$$\partial(fg) = (\partial f)g + f(\partial g).$$

When a real-valued function  $f$  is regarded as a multiplication operator, the composition rule of  $\partial^j$  with  $f$  reads [15]

$$\partial^j \circ f = f \partial^j + \binom{j}{1} f' \partial^{j-1} + \dots, \quad j \in \mathbb{Z}, \tag{2.3}$$

where  $\binom{j}{k}$  is the standard binomial coefficient. Whilst in the algebra-valued setting, the composition rule is defined as below. Let  $A_j = \sum_{k=1}^s a_{jk} e_k, F = \sum_{i=1}^s f_i e_i$  be two  $\mathcal{F}$ -valued functions, the

composition of  $\mathcal{F}$ -valued operators  $A_j \partial^j$  and  $F$  is given by

$$\begin{aligned} A_j \partial^j \circ F &= \sum_{k=1}^s a_{jk} e_k \partial^j \circ \sum_{i=1}^s f_i e_i = \sum_{k=1}^s \sum_{i=1}^s a_{jk} (e_k \circ e_i) \partial^j \circ f_i \\ &= \sum_{k=1}^s \sum_{i=1}^s \sum_{\ell=1}^s a_{jk} \Gamma_{ki}^\ell (\partial^j \circ f_i) e_\ell, \end{aligned}$$

where  $\partial^j \circ f_i$  is given in (2.3),  $\Gamma_{ki}^\ell$  is the structure constant of the algebra  $\mathcal{F}$  defined by  $e_k \circ e_i = \sum_{\ell=1}^s \Gamma_{ki}^\ell e_\ell$ .

The following notation will be used throughout this paper. Let  $P = \sum_{i=-\infty}^m P_i \partial^i$  be an  $\mathcal{F}$ -valued pseudo-differential operator ( $\Psi$ DO briefly),  $P_+$  the pure differential part of the operator  $P$  and  $P_- = P - P_+$ ,  $\text{res}(P) = P_{-1}$ ,  $\text{ord}(P) = m$ . For natural number  $k$  we define (pseudo)-differential operators

$$\mathcal{L} = AB^{-1}, \quad L = \mathcal{L}^{1/n}, \quad P_k = L_+^k.$$

The first property about them is

**Proposition 2.1.** *None of the three operators admits the second leading term.*

**Proof.** We rewrite  $\mathcal{L} = AB^{-1}$  as  $\mathcal{L}B = A$ . The left hand side is the composition of operators  $\mathcal{L}$  and  $B$ . Assume that  $\mathcal{L} = \mathbf{1} \partial^n + \sum_{j=n-1}^{-\infty} V_j \partial^j$ , the second leading term of  $\mathcal{L}B$ , i.e. the term of order  $(n+m-1)$ , is determined by two parts,

$$\mathbf{1} \partial^n \circ B_{m-1} \partial^{m-1} = B_{m-1} \partial^{n+m-1} + nB'_{m-1} \partial^{n+m-2} + \dots$$

and  $V_{n-1} \partial^{n-1} \circ \mathbf{1} \partial^m = V_{n-1} \partial^{n+m-1}$  only. It follows that the coefficient of second leading term of  $A$  is

$$B_{m-1} + V_{n-1} = A_{n+m-1}.$$

Since  $B_{m-1} = A_{n+m-1}$ , it simply implies the vanishing of  $V_{n-1}$ .

If we assume  $L = \mathbf{1} \partial + U_0 + U_{-1} \partial^{-1} + \dots$ , similar calculation leads to  $nU_0 = V_{n-1} = 0$  from the equality  $L^n = \mathcal{L}$ . Similarly the second leading terms of  $L^k$  and  $L_+^k$  also vanish.  $\square$

**Definition 2.1.** The following nonlinear system

$$\partial_k A = A (A^{-1} [P_k, A])_-, \tag{2.4}$$

$$\partial_k B = B (B^{-1} [P_k, B])_-, \tag{2.5}$$

is called  $\mathcal{F}$ -valued  $(n, m)^{\text{th}}$  KdV hierarchy, where  $\partial_k = \frac{\partial}{\partial t_k}$ ,  $t_k$  is the evolution time of  $k$ -th flow.

When the algebra  $\mathcal{F}$  coincides with  $\mathbb{R}$ , the above is just the definition of classical  $(n, m)^{\text{th}}$  KdV hierarchy, which was proposed and studied in [8, 9, 11]. Therefore our defined hierarchy generalizes the classical systems.

From the definition, one can write the explicit flow equations. Each  $t_k$ -flow equation is a  $(1+1)$ -dimensional integrable system. It is not hard to show that  $t_1$ -flow equation is trivial,

$$A_{i,t_1} = A_{i,x}, \quad B_{j,t_1} = B_{j,x}, \quad 0 \leq i \leq n+m-1, \quad 0 \leq j \leq m-1.$$

Similar to [12], in the algebra-valued setting we also have

**Proposition 2.2.**

$$\partial_k(AB^{-1}) = [P_k, AB^{-1}].$$

**Proof.** First the identity  $BB^{-1} = 1$  gives  $\partial_k(BB^{-1}) = 0$ , which implies

$$\partial_k B^{-1} = -B^{-1}(\partial_k B)B^{-1}.$$

The right-hand side is

$$[P_k, AB^{-1}] = P_k AB^{-1} - AB^{-1} P_k = ASB^{-1}$$

where we denote  $S := A^{-1}P_k A - B^{-1}P_k B$ . The left-hand side is

$$\begin{aligned} \partial_k(AB^{-1}) &= (\partial_k A)B^{-1} - AB^{-1}(\partial_k B)B^{-1} \\ &= A(A^{-1}[\partial_k A])_B^{-1} - A(B^{-1}[\partial_k B])_B^{-1} = AS_B^{-1} \end{aligned}$$

It suffices to show that  $S$  is an integral operator, which follows from an analysis of the order of  $[P_k, AB^{-1}]$ . □

By Proposition 2.2 if  $A, B$  satisfy (2.4)–(2.5), then  $\mathcal{L} = AB^{-1}$  satisfies  $\mathcal{F}$ -KP equation  $\partial_k \mathcal{L} = [P_k, \mathcal{L}]$  (see [5]). This implies  $(n, m)$ <sup>th</sup>  $\mathcal{F}$ -KdV hierarchy is a reduction of  $\mathcal{F}$ -KP hierarchy.

**2.2. Hamiltonian structures**

The  $\mathcal{F}$ -KP hierarchy admits infinitely many bi-Hamiltonian structures [5]. More precisely, if it is expressed by

$$\mathcal{L} = \mathbf{1} \partial^n + \sum_{i=-\infty}^{n-1} V_i \partial^i,$$

there exist one pair of bi-Hamiltonian structures, usually called the  $n$ <sup>th</sup> pair [5].

According to [5] we denote the space of functionals by

$$\mathcal{D} = \left\{ \tilde{f} = \int f(v) dx = \int \text{tr} F(V) dx \mid F(V) \text{ is } \mathcal{F}\text{-valued smooth function} \right\}$$

The integral  $\int$  here is an abstract operation. Note that the integrand is still a real-valued function, instead of  $\mathcal{F}$ -valued one. It is the same operation as in Ref. [15]. It has two features, i.e., being linear, and satisfying  $\int f' dx = 0$ . By [5] the  $\mathcal{F}$ -valued variational derivative  $\delta f / \delta V$  for  $V = \sum_{i=1}^s v_s e_s$  is defined through

$$\tilde{f}(v + \delta v) - \tilde{f}(v) = \int \text{tr} \left( \frac{\delta f}{\delta V} \circ \delta V + o(\delta V) \right) dx = \int \sum_{i=1}^s \left( \frac{\delta f}{\delta v_i} \delta v_i + o(\delta v) \right) dx,$$

where  $\delta V = \sum_i e_i \delta v_i$  and

$$\frac{\delta f}{\delta v_i} = \sum_{j=0}^{\infty} (-\partial)^j \frac{\delta f}{\delta v_i^{(j)}}$$

is the ordinary (real-valued) variational derivative [15]. Let us define

$$\frac{\delta f}{\delta \mathcal{L}} = \sum_{i=-\infty}^{n-1} \partial^{-i-1} \circ \frac{\delta f}{\delta V_i},$$

then the second Poisson bracket of  $\mathcal{F}$ -KP hierarchy reads [5]

$$\{\tilde{f}, \tilde{g}\}_{\mathcal{L}}^{(2)} = \int \text{trres} \left[ \left( \frac{\delta f}{\delta \mathcal{L}} \mathcal{L} \right)_- \frac{\delta g}{\delta \mathcal{L}} \mathcal{L} - \left( \mathcal{L} \frac{\delta f}{\delta \mathcal{L}} \right)_- \mathcal{L} \frac{\delta g}{\delta \mathcal{L}} \right] dx. \quad (2.6)$$

Employing the arguments similar to [12], we prove the following variational derivatives identities. Later we will see these functionals serve as the desired Hamiltonians.

**Lemma 2.1.** *Define functional*

$$\tilde{h}_k := \frac{n}{k} \int \text{trres} \mathcal{L}^{k/n} dx = \frac{n}{k} \int \text{trres} L^k dx, \quad (2.7)$$

then its variational derivatives are

$$\frac{\delta h_k}{\delta A} = \left( B^{-1} L^{k-n} \right)_- = \left( A^{-1} L^k \right)_-, \quad (2.8)$$

$$\frac{\delta h_k}{\delta B} = - \left( B^{-1} L^k \right)_-. \quad (2.9)$$

**Proof.** The comparison of variations of  $\tilde{h}_k$ ,

$$\begin{aligned} \delta \tilde{h}_k &= \int \text{trres} \frac{\delta h_k}{\delta \mathcal{L}} \delta(AB^{-1}) dx \\ &= \int \text{trres} \left( \frac{\delta h_k}{\delta \mathcal{L}} (\delta A) B^{-1} - \frac{\delta h_k}{\delta \mathcal{L}} A B^{-1} (\delta B) B^{-1} \right) dx \end{aligned}$$

with

$$\delta \tilde{h}_k = \int \text{trres} \left( \frac{\delta h_k}{\delta A} \delta A + \frac{\delta h_k}{\delta B} \delta B \right) dx$$

implies

$$\frac{\delta h_k}{\delta A} = \left( B^{-1} \frac{\delta h_k}{\delta \mathcal{L}} \right)_- = \left( B^{-1} L^{k-n} \right)_- = \left( A^{-1} L^k \right)_-$$

and similarly for  $\delta h_k / \delta B$ , whereas we have used the variational identity  $\delta h_k / \delta \mathcal{L} = L^{k-n} \text{ mod } O(\partial^{-n-1})$ , which is proved in [5].  $\square$

When constrained to  $(n, m)^{\text{th}}$   $\mathcal{F}$ -KdV hierarchy, the second Poisson bracket (2.6) becomes the counterpart of the  $(n, m)^{\text{th}}$  hierarchy, which reads  $\{\tilde{f}, \tilde{g}\}^{(2)} = \{\tilde{f}, \tilde{g}\}_A^{(2)} - \{\tilde{f}, \tilde{g}\}_B^{(2)}$ , here

$$\{\tilde{f}, \tilde{g}\}_A^{(2)} = \int \text{trres} \left[ \left( \frac{\delta f}{\delta A} A \right)_- \frac{\delta g}{\delta A} A - \left( A \frac{\delta f}{\delta A} \right)_- A \frac{\delta g}{\delta A} \right] dx, \quad (2.10)$$

and similarly for  $\{\tilde{f}, \tilde{g}\}_B^{(2)}$ . Moreover, we have

**Proposition 2.3** (2<sup>nd</sup> Hamiltonian structure).  *$(n, m)^{\text{th}}$   $\mathcal{F}$ -KdV hierarchy is a Hamiltonian system with Poisson bracket  $\{\tilde{f}, \tilde{g}\}^{(2)} = \{\tilde{f}, \tilde{g}\}_A^{(2)} - \{\tilde{f}, \tilde{g}\}_B^{(2)}$  and Hamiltonian  $\tilde{h}_k$  (2.7).*

That is to say,  $(n, m)^{\text{th}}$   $\mathcal{F}$ -KdV hierarchy can be written as

$$\partial_k A = \left( A \frac{\delta h_k}{\delta A} \right)_+ A - A \left( \frac{\delta h_k}{\delta A} A \right)_+ \tag{2.11}$$

$$\partial_k B = B \left( \frac{\delta h_k}{\delta B} B \right)_+ - \left( B \frac{\delta h_k}{\delta B} \right)_+ B. \tag{2.12}$$

**Proof.** By (2.8), we have

$$\begin{aligned} \left( A \frac{\delta h_k}{\delta A} \right)_+ A - A \left( \frac{\delta h_k}{\delta A} A \right)_+ &= \left( A(B^{-1}L^{k-n})_- \right)_+ A - A \left( (B^{-1}L^{k-n})_- A \right)_+ \\ &= (AB^{-1}L^{k-n})_+ A - A(B^{-1}L^{k-n}A)_+ \\ &= P_k A - A(A^{-1}L^k A)_+ = \partial_k A, \end{aligned}$$

Similarly from (2.9) one obtains (2.12). □

To derive the first Poisson bracket, we can replace  $\mathcal{L}$  with  $\tilde{\mathcal{L}} = (\mathcal{L} + \lambda \mathbf{1})$  in calculating (2.6). Thus we obtain the second Poisson bracket  $\{\tilde{f}, \tilde{g}\}_{\tilde{\mathcal{L}}}^{(2)}$  corresponding to  $\tilde{\mathcal{L}}$ . This bracket can be expanded in  $\lambda$  as

$$\{\tilde{f}, \tilde{g}\}_{\tilde{\mathcal{L}}}^{(2)} = \{\tilde{f}, \tilde{g}\}^{(2)} + \lambda \cdot \{\tilde{f}, \tilde{g}\}^{(1)} + \lambda^2 \cdot 0.$$

The coefficient of  $\lambda^2$  vanishes, and that of  $\lambda^0$  is  $\{\tilde{f}, \tilde{g}\}^{(2)}$ . That of  $\lambda$  reads

$$\begin{aligned} \{\tilde{f}, \tilde{g}\}^{(1)} &= \int \text{tr res} \left[ \left( A \frac{\delta f}{\delta A} \right)_+ B \frac{\delta g}{\delta A} + \left( B \frac{\delta f}{\delta A} \right)_+ A \frac{\delta g}{\delta A} - \right. \\ &\quad \left. - \left( \frac{\delta f}{\delta A} A \right)_+ \frac{\delta g}{\delta A} B - \left( \frac{\delta f}{\delta A} B \right)_+ \frac{\delta g}{\delta A} A + \right. \\ &\quad \left. + \left( B \frac{\delta f}{\delta B} \right)_+ B \frac{\delta g}{\delta A} + \left( B \frac{\delta f}{\delta A} \right)_+ B \frac{\delta g}{\delta B} - \right. \\ &\quad \left. - \left( \frac{\delta f}{\delta B} B \right)_+ \frac{\delta g}{\delta A} B - \left( \frac{\delta f}{\delta A} B \right)_+ \frac{\delta g}{\delta B} B \right] dx, \end{aligned} \tag{2.13}$$

which is exactly the desired first Poisson bracket. Moreover, we have

**Proposition 2.4 (1<sup>st</sup> Hamiltonian structure).**  $(n, m)^{\text{th}}$   $\mathcal{F}$ -KdV hierarchy has another Hamiltonian structure, where Poisson bracket is given by (2.13), the Hamiltonian is  $\tilde{h} = \tilde{h}_{k+n}$  defined in (2.7).

That is to say,  $(n, m)^{\text{th}}$  KdV hierarchy (2.4)–(2.5) can be written as

$$\begin{aligned} \partial_k A &= \left( A \frac{\delta h}{\delta A} \right)_+ B + \left( B \frac{\delta h}{\delta A} \right)_+ A - B \left( \frac{\delta h}{\delta A} A \right)_+ - \\ &\quad - A \left( \frac{\delta h}{\delta A} B \right)_+ + \left( B \frac{\delta h}{\delta B} \right)_+ B - B \left( \frac{\delta h}{\delta B} B \right)_+, \end{aligned} \tag{2.14}$$

$$\partial_k B = \left( B \frac{\delta h}{\delta A} \right)_+ B - B \left( \frac{\delta h}{\delta A} B \right)_+. \tag{2.15}$$

**Proof.** Let  $\tilde{h} = \tilde{h}_{k+n}$ , by (2.8), (2.9) we have

$$\frac{\delta h}{\delta A} = (B^{-1}L^k)_- = (A^{-1}L^{k+n})_-, \quad \frac{\delta h}{\delta B} = -(B^{-1}L^{k+n})_-. \quad (2.16)$$

Eq. (2.15) follows from

$$\begin{aligned} \left( B \frac{\delta h}{\delta A} \right)_+ B - B \left( \frac{\delta h}{\delta A} B \right)_+ &= \left( B(B^{-1}L^k)_\ominus \right)_+ B - B \left( (B^{-1}L^k)_\ominus B \right)_+ \\ &= (BB^{-1}L^k)_+ B - B(B^{-1}L^k B)_+ \\ &= P_k B - B(B^{-1}L^k B)_+ = \partial_k B, \end{aligned}$$

where  $\ominus$  denote the  $-$  can be removed, since

$$(B(B^{-1}L^k)_+)_+ B - B((B^{-1}L^k)_+ B)_+ = (B(B^{-1}L^k)_+)B - B((B^{-1}L^k)_+ B) = 0.$$

As for Eq. (2.14), we decompose its right side to three groups

$$\begin{aligned} \left( A \frac{\delta h}{\delta A} \right)_+ B - A \left( \frac{\delta h}{\delta A} B \right)_+ &= (A(B^{-1}L^k))_+ B - A((B^{-1}L^k)B)_+ \\ &= P_{n+k} B - A(B^{-1}L^k B)_+, \end{aligned}$$

$$\begin{aligned} \left( B \frac{\delta h}{\delta A} \right)_+ A - B \left( \frac{\delta h}{\delta A} A \right)_+ &= (B(B^{-1}L^k))_+ A - B((B^{-1}L^k)A)_+ \\ &= P_k A - B(B^{-1}L^k A)_+, \end{aligned}$$

$$\begin{aligned} \left( B \frac{\delta h}{\delta B} \right)_+ B - B \left( \frac{\delta h}{\delta B} B \right)_+ &= -(B(B^{-1}L^{k+n}))_+ B + B((B^{-1}L^{k+n})B)_+ \\ &= -P_{k+n} B + B(B^{-1}L^k A)_+. \end{aligned}$$

In the above subscripts in  $\delta h/\delta A, \delta h/\delta B$  have been removed. The summation of them gives exactly Eq. (2.14).  $\square$

### 3. Example

In order to illustrate the general results of the previous section, in this section we consider the case of  $n = m = 1$  and  $\mathcal{F} = (\mathcal{L}_2, \text{tr})$  for the sake of simplicity. This corresponds to  $(1, 1)^{\text{th}}$   $\mathcal{F}$ -KdV hierarchy. Other cases of larger  $n, m$  can be manipulated in the same way, although the calculations are much more involved.

#### 3.1. $t_1, t_2$ -flow equations

For the  $(1, 1)^{\text{th}}$   $\mathcal{L}_2$ -KdV hierarchy, the differential operators are

$$A = \mathbf{1} \partial^2 + A_1 \partial + A_0, \quad B = \mathbf{1} \partial + B_0, \quad A_1 = B_0. \quad (3.1)$$

First coefficients of Lax operators  $\mathcal{L} = \mathbf{1} \partial + \sum_{i \leq -1} U_i \partial^i$ , are given by

$$U_{-1} = A_0 - B'_0, \quad U_{-2} = -B_0 U_{-1}, \quad U_{-3} = (B_0^2 + B'_0) U_{-1}.$$

Moreover, differential operators  $P_1 = \mathbf{1} \partial, P_2 = \mathbf{1} \partial^2 + 2U_{-1}$ .



Below we compute in detail some flows equations of lower orders. Throughout this paper, we denote by  $X_k$  the vector field corresponding to  $t_k$  flow.

**$t_1$ -flow equation.** This is the trivial equation  $A_{0,t_1} = A'_0, B_{0,t_1} = B'_0$ . When we choose  $\mathcal{F} = \mathcal{L}_2$ , and assume

$$A_0 = \begin{pmatrix} \phi_0 & 0 \\ \phi_1 & \phi_0 \end{pmatrix}, \quad B_0 = \begin{pmatrix} \psi_0 & 0 \\ \psi_1 & \psi_0 \end{pmatrix},$$

the algebra-valued equations become the following trivial system

$$(\phi_0, \phi_1, \psi_0, \psi_1)_{t_1}^T = (\phi'_0, \phi'_1, \psi'_0, \psi'_1)^T = X_1. \tag{3.2}$$

**$t_2$ -flow equation.** One can derive that

$$[P_2, B] = 2B'_0 \partial + 3B''_0 - 2A'_0,$$

and

$$[P_2, A] = 2B'_0 \partial^2 + (5B''_0 - 2A'_0) \partial + 2B'''_0 + 2B_0 B''_0 - A''_0 - 2B_0 A'_0.$$

Let us denote that

$$M = B^{-1}[P_2, B] = \sum_{i \leq 0} M_i \partial^i, \quad N = A^{-1}[P_2, A] = \sum_{i \leq 0} N_i \partial^i,$$

it follows  $M_+ = M_0 = 2B'_0, N_+ = N_0 = 2B'_0$ . Finally  $t_2$ -flow equation reads

$$\begin{cases} A_{0,t_2} = -A''_0 - 2(A_0 B_0)', \\ B_{0,t_2} = B''_0 - 2A'_0 - (B_0^2)'. \end{cases} \tag{3.3}$$

Again we choose  $\mathcal{F} = \mathcal{L}_2$ , then algebra-valued Eqs. (3.3) becomes the following system  $X_2$  with four components

$$\begin{cases} \phi_{0,t_2} = -\phi''_0 - 2(\phi_0 \psi_0)', \\ \phi_{1,t_2} = -\phi''_1 - 2(\phi_0 \psi_1 + \phi_1 \psi_0)', \\ \psi_{0,t_2} = \psi''_0 - 2\phi'_0 - (\psi_0^2)', \\ \psi_{1,t_2} = \psi''_1 - 2\phi'_1 - 2(\psi_0 \psi_1)'. \end{cases} \tag{3.4}$$

### 3.2. Bi-Hamiltonian structures of $t_2$ -flow

The  $(1, 1)^{\text{th}}$   $\mathcal{L}_2$ -KdV hierarchy has *four* independent evolutionary fields:  $(\phi_0, \phi_1, \psi_0, \psi_1)$ . Using propositions of Section 2, we now derive bi-Hamiltonian structures of its second flow equation (3.4). Unlike the conclusions of Section 2, these structures will be explicitly dependent upon the evolution fields, instead of the abstract  $\mathcal{F}$ -valued pseudo-differential operators.

**First Hamiltonian Structure.** By the definition (2.7), we calculate first

$$\begin{aligned} \text{res } L^3 &= W'_{-1} + W_{-2} + U_{-1}W_0 + U_{-3} \\ &= Z' + 3(U_{-3} + U_{-1}^2) = Z' + 3(A_0 + B_0^2)U_{-1} \\ &= Z' + 3(A_0^2 + A_0B_0^2 + A_0B_0'), \end{aligned}$$

$Z'$  denotes an exact derivative w.r.t.  $x$ , hence its corresponding functional vanishes; these  $Z$ 's may be distinct. The Hamiltonian of first Hamiltonian structure is

$$\begin{aligned} \tilde{h}_3 &= \int \text{tr}(A_0^2 + A_0B_0^2 + A_0'B_0) \, dx \\ &= \int (2\phi_0\phi_1 + 2\phi_0\psi_0\psi_1 + \phi_1\psi_0^2 + \phi_0'\psi_1 + \phi_1'\psi_0) \, dx. \end{aligned}$$

By Proposition 2.4, the algebra-valued system (3.3) can be represented as

$$\begin{pmatrix} A_0 \\ B_0 \end{pmatrix}_{t_2} = \begin{pmatrix} 0 & -\mathbf{1}\partial \\ -\mathbf{1}\partial & 0 \end{pmatrix} \begin{pmatrix} \delta\tilde{h}_3/\delta A_0 \\ \delta\tilde{h}_3/\delta B_0 \end{pmatrix}, \tag{3.5}$$

According to [5], the algebra-valued variational derivatives are related with real-valued variational derivatives by

$$\frac{\delta\tilde{h}_3}{\delta A_0} = \frac{\delta\tilde{h}_3}{\delta\phi_1}I_2 + \frac{\delta\tilde{h}_3}{\delta\phi_0}\Lambda, \quad \frac{\delta\tilde{h}_3}{\delta B_0} = \frac{\delta\tilde{h}_3}{\delta\psi_1}I_2 + \frac{\delta\tilde{h}_3}{\delta\psi_0}\Lambda, \tag{3.6}$$

Combing the above together, one has the first Hamiltonian structure of four-component system (3.4)

$$\theta_{t_2} = X_2 = J \frac{\delta\tilde{h}_3}{\delta\theta} \tag{3.7}$$

where we denote the potentials  $\theta = (\phi_0, \phi_1, \psi_0, \psi_1)^T$ , variational derivatives

$$\frac{\delta\tilde{h}_3}{\delta\theta} = \left( \frac{\delta\tilde{h}_3}{\delta\phi_0}, \frac{\delta\tilde{h}_3}{\delta\phi_1}, \frac{\delta\tilde{h}_3}{\delta\psi_0}, \frac{\delta\tilde{h}_3}{\delta\psi_1} \right)^T,$$

the Hamiltonian operator is

$$J = \begin{pmatrix} 0 & 0 & 0 & -\partial \\ 0 & 0 & -\partial & 0 \\ 0 & -\partial & 0 & 0 \\ -\partial & 0 & 0 & 0 \end{pmatrix}. \tag{3.8}$$

**Second Hamiltonian Structure.** The Hamiltonian is

$$\begin{aligned} \tilde{h}_2 &= \frac{1}{2} \int \text{tr}(U_{-1,x} + 2U_{-2}) \, dx = \int \text{tr}U_{-2} \, dx \\ &= \int [\psi_0(\psi_1' - \phi_1) + \psi_1\psi_0' - \phi_0] \, dx \\ &= - \int (\psi_0\phi_1 + \psi_1\phi_0) \, dx. \end{aligned}$$

By Proposition 2.3 algebra-valued system (3.3) can also be represented as

$$\begin{pmatrix} A_0 \\ B_0 \end{pmatrix}_{t_2} = \begin{pmatrix} \partial A_0 + A_0 \partial & \mathbf{1} \partial^2 + B_0 \partial \\ -\mathbf{1} \partial^2 + \partial B_0 & 2 \partial \end{pmatrix} \begin{pmatrix} \delta \tilde{h}_2 / \delta A_0 \\ \delta \tilde{h}_2 / \delta B_0 \end{pmatrix}, \quad (3.9)$$

the above algebra-valued variation derivatives are given similarly as in (3.6).

After expanding (3.9) one eventually obtains the second Hamiltonian structure of system (3.4)

$$\theta_{t_2} = X_2 = M \frac{\delta \tilde{h}_2}{\delta \theta} \quad (3.10)$$

with Hamiltonian operator

$$M = \begin{pmatrix} 0 & \partial \phi_0 + \phi_0 \partial & 0 & \partial^2 + \psi_0 \partial \\ \partial \phi_0 + \phi_0 \partial & \partial \phi_1 + \phi_1 \partial & \partial^2 + \psi_0 \partial & \psi_1 \partial \\ 0 & -\partial^2 + \partial \psi_0 & 0 & 2 \partial \\ -\partial^2 + \partial \psi_0 & \partial \psi_1 & 2 \partial & 0 \end{pmatrix}. \quad (3.11)$$

### 3.3. Hereditary recursion operator

According to Propositions 2.3 and 2.4 of Section 2, it follows that operators  $J$  and  $M$  constitute a Hamiltonian pair, i.e., any linear combination  $N$  of  $J$  and  $M$  satisfies

$$\int \alpha^T N'(u) [N \beta] \gamma dx + \text{cycle}(\alpha, \beta, \gamma) = 0 \quad (3.12)$$

for all one-forms  $\alpha, \beta$ , and  $\gamma$  (see e.g. [16, 17]).

It is obvious that  $J$  is invertible and we have the recursive operator

$$\mathcal{R} = MJ^{-1} = - \begin{pmatrix} \partial + \psi_0 & 0 & \partial \phi_0 \partial^{-1} + \phi_0 & 0 \\ \psi_1 & \partial + \psi_0 & \partial \phi_1 \partial^{-1} + \phi_1 & \partial \phi_0 \partial^{-1} + \phi_0 \\ 2 & 0 & -\partial + \partial \psi_0 \partial^{-1} & 0 \\ 0 & 2 & \partial \psi_1 \partial^{-1} & -\partial + \partial \psi_0 \partial^{-1} \end{pmatrix} \quad (3.13)$$

It follows that the operator  $\mathcal{R}$  is hereditary (see Ref. [18]), i.e., it satisfies for all vector fields  $K$  and  $S$ ,

$$\mathcal{R}'(u) [\mathcal{R}K]S - \mathcal{R} \mathcal{R}'(u) [K]S = \mathcal{R}'(u) [\mathcal{R}S]K - \mathcal{R} \mathcal{R}'(u) [S]K. \quad (3.14)$$

The condition (3.14) for the hereditary operators is equivalent to

$$L_{\mathcal{R}K} \mathcal{R} = \mathcal{R} L_K \mathcal{R}, \quad (3.15)$$

where  $K$  is an arbitrary vector field,  $L_K$  is the Lie derivative along  $K$ . Note that an autonomous operator  $\mathcal{R} = \mathcal{R}(u, u_x, \dots)$  is a recursion operator of a given evolution equation  $u_t = K = K(u)$  if and only if  $\mathcal{R}$  satisfies  $L_K \mathcal{R} = 0$ .

One can verify directly the operator  $\mathcal{R}$  defined by (3.13) satisfies

$$L_{X_1}\mathcal{R} = 0, \quad X_1 = (\phi'_0, \phi'_1, \psi'_0, \psi'_1)^T,$$

and thus for each  $k \geq 1$ ,

$$L_{X_{k+1}}\mathcal{R} = L_{\mathcal{R}X_k}\mathcal{R} = \mathcal{R}L_{X_k}\mathcal{R} = 0,$$

where  $X_k$  is the vector field of  $t_k$  flow. This shows that the operator  $\mathcal{R}$  is a common hereditary recursion operator for each member of the  $(1, 1)^{\text{th}}$   $\mathcal{L}_2$ -KdV hierarchy.

Summarizing above we have the following

**Proposition 3.1.** *The  $(1, 1)^{\text{th}}$   $\mathcal{L}_2$ -KdV hierarchy admits a bi-Hamiltonian representation in explicit form,*

$$\theta_{t_k} = X_k = J \frac{\delta \tilde{h}_{k+1}}{\delta \theta} = M \frac{\delta \tilde{h}_k}{\delta \theta}, \quad k \geq 1, \tag{3.16}$$

where the compatible Hamiltonian operators  $J, M$  are given by (3.8), (3.11) respectively, the Hamiltonian  $\tilde{h}_k$  is given by (2.7).

There exists a natural recursion relation between nearby functional gradients  $G_k := \delta \tilde{h}_k / \delta \theta$ ,

$$G_{k+1} = \mathcal{R}^* G_k, \quad k \geq 1, \tag{3.17}$$

here  $\mathcal{R}^*$  is the operator conjugate to  $\mathcal{R}$ .

It follows that the hierarchy is Liouville integrable, i.e., it possesses infinitely many commuting symmetries and conservation laws. More precisely, we have the following Abelian symmetry algebra,

$$[X_k, X_l] = X'_k(u)[X_l] - X'_l(u)[X_k] = 0, \quad k, l \geq 1, \tag{3.18}$$

and the two Abelian algebras of conserved functionals,

$$\{\tilde{h}_k, \tilde{h}_l\}_J = \{\tilde{h}_k, \tilde{h}_l\}_M = 0, \quad k, l \geq 1, \tag{3.19}$$

To conclude this paper, we employ the recursion relation (3.16) and derive the next nonlinear integrable system in the hierarchy,

$$\theta_{t_3} = X_3 = M \frac{\delta \tilde{h}_3}{\delta \theta}.$$

Its explicit form is given as follows

$$\begin{aligned} \phi_{0,t_3} &= 6\phi_0\phi'_0 + 3\phi'_0\psi'_0 + 3\phi'_0\psi_0^2 + 3\phi_0''\psi_0 + 6\phi_0\psi_0\psi'_0 + \phi_0''', \\ \phi_{1,t_3} &= 6\phi'_0\phi_1 + 6\phi_0\phi'_1 + 3\phi'_1\psi_0^2 + \phi_1'''' + 6\phi'_0\psi_0\psi_1 + 6\phi_0\psi'_0\psi_1 \\ &\quad + 6\phi_0\psi_0\psi'_1 + 6\phi_1\psi_0\psi'_0 + 3\phi_0''\psi_1 + 3\phi'_0\psi'_1 + 3\phi_1''\psi_0 + 3\phi'_1\psi'_0, \\ \psi_{0,t_3} &= -2\phi_1'' - 2\psi_0''\psi_1 - 5\psi'_0\psi'_1 - 3\psi_0\psi_1'' + \psi_1'''' + 2\phi_1\psi'_0 \\ &\quad + 4\psi'_0\psi_0\psi_1 + 2\phi_1'\psi_0 + 2\psi_0^2\psi'_1 + 4\phi'_0\psi_0 + 4\phi_0\psi'_0 + 2\phi_0'', \\ \psi_{1,t_3} &= -3\psi_0''\psi_1 - 6\psi'_0\psi'_1 - 3\psi_0\psi_1'' + \psi_1'''' + 6\phi_1\psi'_0 \\ &\quad + 6\psi'_0\psi_0\psi_1 + 6\phi_1'\psi_0 + 3\psi_0^2\psi'_1 + 6\phi_0\psi'_1 + 6\phi'_0\psi_1. \end{aligned}$$

#### 4. Conclusion

In this paper, we have introduced  $\mathcal{F}$ -valued  $(n, m)^{\text{th}}$  KdV hierarchy and derived its bi-Hamiltonian structures. In particular we analyzed  $(1, 1)^{\text{th}}$   $\mathcal{L}_2$ -KdV hierarchy in detail. We have constructed its recursion operator, Hamiltonian structures, infinitely many symmetries and conservation laws and some explicit flow equations. Other objects of it may also be explored, such as Darboux transformation, soliton solutions [19], additional symmetries [20] etc.

As we remarked earlier, when  $\mathcal{F}$  coincides with  $\mathbb{R}$ , the hierarchy reduces to the well-known classical  $(n, m)^{\text{th}}$  KdV hierarchy. When other algebra  $\mathcal{F}$  is chosen, we can obtain some novel real-valued integrable systems by using a basis of  $\mathcal{F}$ . The example given in Section 3 summaries the major features. For an arbitrary algebra  $\mathcal{F}$ , the corresponding results hold similarly, although the complete expressions are much more complicated than those appeared in Section 3. It is an interesting question to investigate the systematic relations between these new systems and those in the literature.

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