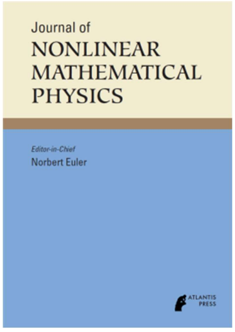

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On Hybrid Ermakov-Painlevé Systems. Integrable Reduction

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Hybrid Ermakov-Painlevé II-IV systems are introduced here in a unified manner. Their admitted Ermakov invariants together with associated canonical Painlevé equations are used to establish integrability properties.

1. Introduction

Prototype hybrid Ermakov-Painlevé II systems were recently introduced via symmetry reduction of an n+1-dimensional Manakov-type system in [36]. Two-point Dirichlet boundary value problems for a particular Ermakov-Painlevé II reduction arising out of a Nernst-Planck three-ion electrodiffusion system have been subsequently treated in [2].

Integrable Ermakov-Painlevé II systems with underlying Hamiltonian structure as recently set down in [37] adopt the form

$$\begin{aligned} \ddot{W}_{\mathrm{I}} + \left[\frac{t}{2} + \varepsilon(W_{\mathrm{I}}^2 + W_{\mathrm{II}}^2)\right] W_{\mathrm{I}} &= \frac{1}{W_{\mathrm{I}}^2 W_{\mathrm{II}}} S'(W_{\mathrm{II}}/W_{\mathrm{I}}) \ , \\ \ddot{W}_{\mathrm{II}} + \left[\frac{t}{2} + \varepsilon(W_{\mathrm{I}}^2 + W_{\mathrm{II}}^2)\right] W_{\mathrm{II}} &= \frac{1}{W_{\mathrm{I}} W_{\mathrm{II}}^2} T(W_{\mathrm{I}}/W_{\mathrm{II}}) \end{aligned} \tag{1.1}$$

with

$$\begin{aligned} S(W_{\mathrm{II}}/W_{\mathrm{I}}) &= 2\frac{W_{\mathrm{II}}}{W_{\mathrm{I}}} J(W_{\mathrm{II}}/W_{\mathrm{I}}) + \frac{W_{\mathrm{II}}^2}{W_{\mathrm{I}}^2} J'(W_{\mathrm{II}}/W_{\mathrm{I}}) \ , \\ T(W_{\mathrm{I}}/W_{\mathrm{II}}) &= -\frac{W_{\mathrm{II}}^2}{W_{\mathrm{I}}^2} J'(W_{\mathrm{II}}/W_{\mathrm{I}}) \ , \end{aligned} \tag{1.2}$$

where in the above, a dot denotes a derivative with respect to the independent variable t and the prime denotes a derivative with respect to the argument $W_{\mathrm{II}}/W_{\mathrm{I}}$. The novel Ermakov-NLS systems introduced in [37] admit symmetry reduction to the hybrid Ermakov-Painlevé II system (1.1) via a wave packet ansatz with genesis in a nonlinear optics context [23].

The nonlinear coupled systems as introduced by Ray and Reid in [34, 35] have roots in work of Ermakov [20] and adopt the form

$$\begin{aligned} \ddot{W}_{\mathrm{I}} + \omega(t) W_{\mathrm{I}} &= \frac{1}{W_{\mathrm{I}}^2 W_{\mathrm{II}}} S(W_{\mathrm{II}}/W_{\mathrm{I}}) \ , \\ \ddot{W}_{\mathrm{II}} + \omega(t) W_{\mathrm{II}} &= \frac{1}{W_{\mathrm{II}}^2 W_{\mathrm{I}}} T(W_{\mathrm{I}}/W_{\mathrm{II}}) \ . \end{aligned} \tag{1.3}$$

They admit a distinctive integral of motion and have diverse physical applications, in such areas as nonlinear optics [16, 24, 25, 27, 39, 40, 57], hydrodynamics [41], spinning gas cloud theory [42], magnetogasdynamics [43] and oceanographic warm core eddy theory [44]. Ermakov-Ray-Reid systems also occur in connection with the contribution of orbital angular momentum to the suppression of the collapse of spiralling elliptic solutions in nonlinear Kerr media [17] as well as in the analysis of cloud evolution in a Bose-Einstein condensate [1].

In [45], a 2+1-dimensional version of (1.3) was introduced, while subsequently, multi-component Ermakov-Ray-Reid systems were derived in a hydrodynamics context via symmetry reduction of a multi-layer fluid model [46]. Therein it was shown that sequences of Ermakov-Ray-Reid systems may be linked by Darboux transformations.

The six classical Painlevé equations $P_{\mathrm{I}} - P_{\mathrm{VI}}$ likewise arise in a wide range of physical applications and play a fundamental role in modern soliton theory (see e.g. Conte [14] and Clarkson [11] together with literature cited therein). Painlevé equations may be shown to possess nonlinear superposition principles associated with the admittance of Bäcklund transformations (see e.g. [26]). Ermakov-Ray-Reid systems likewise admit nonlinear superposition principles albeit of another kind [34, 35]. Painlevé equations characteristically admit Lax representations while Ermakov-Ray-Reid systems also have been shown to admit underlying linear structure [4].

In general, the studies of Painlevé equations and Ermakov-type systems have proceeded independently. Thus, the only known hybrid solitonic-Ermakov system seems to be that obtained in [54, 55] where a 2+1-dimensional Ernst-type system of general relativity as derived in [56], suitably constrained, leads to a novel composition of the integrable 2+1-dimensional sinh-Gordon equation of [30, 31] and of a generalised Ermakov-Ray-Reid system. The work of [2, 36–38] on Ermakov-Painlevé II systems has recently been augmented by the introduction in [47] of prototype Ermakov-Painlevé IV systems via a symmetry reduction of a coupled derivative resonant NLS triad. Dirichlet type two-point boundary value problems for a single hybrid Ermakov-Painlevé IV equation have been investigated with regard to existence and uniqueness properties in [3].

The preceding motivates the present work wherein hybrid Ermakov-Painlevé II, Ermakov-Painlevé III and Ermakov-Painlevé IV systems are derived in a unified manner. In this context, whereas the classical Ermakov-equation is seen to underlie the standard Ermakov-Ray-Reid system (1.3), the classical Painlevé II $-$ Painlevé IV equations underlie their hybrid Ermakov counterparts. The algorithmic solution of the latter is demonstrated in the case of underlying Hamiltonian-type structure by a combined use of the classical Painlevé II $-$ IV components for the amplitudes of the systems and of admitted Ermakov invariants for the phases.

2. Extended Ermakov-Ray-Reid Systems

Here, extended Ermakov-Ray-Reid systems are introduced of the type

$$\begin{gathered}
\ddot{W}_{\mathrm{I}} - \frac{1}{\Phi}\left[\ddot{\Phi} - \frac{\zeta}{\Phi^3}\right] W_{\mathrm{I}} = \frac{1}{W_{\mathrm{I}}^2 W_{\mathrm{II}}} S(W_{\mathrm{II}}/W_{\mathrm{I}}) \ , \\
\ddot{W}_{\mathrm{II}} - \frac{1}{\Phi}\left[\ddot{\Phi} - \frac{\zeta}{\Phi^3}\right] W_{\mathrm{II}} = \frac{1}{W_{\mathrm{I}} W_{\mathrm{II}}^2} T(W_{\mathrm{I}}/W_{\mathrm{II}}) \ , \\
\zeta \varepsilon \mathbb{R} \ .
\end{gathered} \tag{2.1}$$

The standard Ermakov-Ray-Reid system (1.3) is retrieved in the specialisation when Φ is determined by the classical Ermakov equation

$$\ddot{\Phi} + \omega(t)\Phi = \frac{\zeta}{\Phi^3} \; . \tag{2.2}$$

It is recalled that the latter admits general solution via a well-known nonlinear superposition principle as readily established by Lie group methods (see e.g. [49, 50]). Thus,

$$\Phi = \sqrt{ax_1^2 + 2cx_1x_2 + bx_2^2} \tag{2.3}$$

where x_1, x_2 are linearly independent solutions of the canonical linear equation

$$\ddot{W} + \omega(t)W = 0 \tag{2.4}$$

and the constants a, b and c are related by

$$ab - c^2 = \frac{\zeta}{\mathscr{W}(x_1;x_2)} \tag{2.5}$$

where $\mathscr{W}(x_1;x_2) = x_1\dot{x}_2 - x_2\dot{x}_1$ is the Wronskian of x_1 and x_2. It is noted that the Ermakov-Ray-Reid system (1.3) may be rendered autonomous by the introduction of new dependent and independent variables α, β and z according to [4, 46]

$$\begin{gathered} W_{\mathrm{I}} = \alpha(z)U \quad , \quad W_{\mathrm{II}} = \beta(z)U \; , \\ z = V/U \end{gathered} \tag{2.6}$$

where U, V are linearly independent solutions of (2.4) with unit Wronskian. Thus, under (2.6), reduction of (1.3) is obtained to the autonomous system

$$\alpha_{zz} = \frac{1}{\alpha^2\beta}S(\beta/\alpha) \quad , \quad \beta_{zz} = \frac{1}{\alpha\beta^2}T(\alpha/\beta) \; . \tag{2.7}$$

In general, the system (2.1) yields

$$\begin{aligned} \ddot{W}_{\mathrm{I}}\Phi - W_{\mathrm{I}}\ddot{\Phi} + \frac{\zeta W_{\mathrm{I}}}{\Phi^3} &= \frac{\Phi}{W_{\mathrm{I}}^2 W_{\mathrm{II}}} S(W_{\mathrm{II}}/W_{\mathrm{I}}) \; , \\ \ddot{W}_{\mathrm{II}}\Phi - W_{\mathrm{II}}\ddot{\Phi} + \frac{\zeta W_{\mathrm{II}}}{\Phi^3} &= \frac{\Phi}{W_{\mathrm{I}} W_{\mathrm{II}}^2} T(W_{\mathrm{I}}/W_{\mathrm{II}}) \end{aligned} \tag{2.8}$$

whence, on setting

$$\begin{gathered} \Phi_{\mathrm{I}} = W_{\mathrm{I}}/\Phi \quad , \quad \Phi_{\mathrm{II}} = W_{\mathrm{II}}/\Phi \; , \\ dt^* = \Phi^{-2}dt \end{gathered} \tag{2.9}$$

the system (2.1) is reduced to

$$\begin{aligned} \Phi_{\mathrm{I},t^*t^*} + \zeta\Phi_{\mathrm{I}} &= \frac{1}{\Phi_{\mathrm{I}}^2\Phi_{\mathrm{II}}} S(\Phi_{\mathrm{II}}/\Phi_{\mathrm{I}}) \; , \\ \Phi_{\mathrm{II},t^*t^*} + \zeta\Phi_{\mathrm{II}} &= \frac{1}{\Phi_{\mathrm{I}}\Phi_{\mathrm{II}}^2} T(\Phi_{\mathrm{I}}/\Phi_{\mathrm{II}}) \; . \end{aligned} \tag{2.10}$$

with independent variable t^*.

Here, the integrability of the extended Ermakov-Ray-Reid system (2.1) is addressed with $S(W_{\rm II}/W_{\rm I})$ and $T(W_{\rm I}/W_{\rm II})$ parametrised in terms of $J(W_{\rm II}/W_{\rm I})$ as given by relations (1.2) originally derived in [41] to characterise standard systems (1.3) with underlying Hamiltonian structure. Hence, the extended system (2.1) becomes

$$\begin{aligned} \ddot{W}_{\rm I} - \frac{1}{\Phi}\left[\ddot{\Phi} - \frac{\zeta}{\Phi^3}\right] W_{\rm I} &= \frac{2}{W_{\rm I}^3} J(W_{\rm II}/W_{\rm I}) + \frac{W_{\rm II}}{W_{\rm I}^4} J'(W_{\rm II}/W_{\rm I})\ , \\ \ddot{W}_{\rm II} - \frac{1}{\Phi}\left[\ddot{\Phi} - \frac{\zeta}{\Phi^3}\right] W_{\rm II} &= -\frac{1}{W_{\rm I}^3} J'(W_{\rm II}/W_{\rm I}) \end{aligned} \tag{2.11}$$

and admits the Ermakov invariant

$$\mathscr{E} = \frac{1}{2}(W_{\rm I}\dot{W}_{\rm II} - W_{\rm II}\dot{W}_{\rm I})^2 + \left(\frac{W_{\rm I}^2 + W_{\rm II}^2}{W_{\rm I}^2}\right) J(W_{\rm II}/W_{\rm I})\ . \tag{2.12}$$

Accordingly, the identity

$$(W_{\rm I}^2 + W_{\rm II}^2)(\dot{W}_{\rm I}^2 + \dot{W}_{\rm II}^2) - (W_{\rm I}\dot{W}_{\rm II} - W_{\rm II}\dot{W}_{\rm I})^2 \equiv (W_{\rm I}\dot{W}_{\rm I} + W_{\rm II}\dot{W}_{\rm II})^2 \tag{2.13}$$

on use of (2.12) yields

$$(W_{\rm I}^2 + W_{\rm II}^2)(\dot{W}_{\rm I}^2 + \dot{W}_{\rm II}^2) - 2\left[\mathscr{E} - \left(\frac{W_{\rm I}^2 + W_{\rm II}^2}{W_{\rm I}^2}\right) J(W_{\rm II}/W_{\rm I})\right] = \frac{1}{4}\dot{\Sigma}^2 \tag{2.14}$$

where $\Sigma = W_{\rm I}^2 + W_{\rm II}^2$. Thus,

$$\dot{W}_{\rm I}^2 + \dot{W}_{\rm II}^2 - \frac{2\mathscr{E}}{\Sigma} + \frac{2J(W_{\rm II}/W_{\rm I})}{W_{\rm I}^2} = \frac{1}{4}\dot{\Sigma}^2/\Sigma \tag{2.15}$$

whence

$$\dot{W}_{\rm I}\ddot{W}_{\rm I} + \dot{W}_{\rm II}\ddot{W}_{\rm II} + \mathscr{E}\left(\frac{\dot{\Sigma}}{\Sigma^2}\right) + d\left[\frac{J(W_{\rm II}/W_{\rm I})}{W_{\rm I}^2}\right]/dt = \frac{1}{8}\left[2\frac{\dot{\Sigma}\ddot{\Sigma}}{\Sigma} - \frac{\dot{\Sigma}^3}{\Sigma^2}\right]\ . \tag{2.16}$$

But, the system (2.11) shows that

$$\begin{aligned} \dot{W}_{\rm I}\ddot{W}_{\rm I} + \dot{W}_{\rm II}\ddot{W}_{\rm II} - \Delta(W_{\rm I}\dot{W}_{\rm I} + W_{\rm II}\dot{W}_{\rm II}) &= \left(\frac{2}{W_{\rm I}^3}J + \frac{W_{\rm II}}{W_{\rm I}^4}J'\right)\dot{W}_{\rm I} - \frac{1}{W_{\rm I}^3}J'\dot{W}_{\rm II} \\ &= -d\left[\frac{J(W_{\rm II}/W_{\rm I})}{W_{\rm I}^2}\right]/dt \end{aligned} \tag{2.17}$$

where

$$\Delta = \frac{1}{\Phi}\left[\ddot{\Phi} - \frac{\zeta}{\Phi^3}\right]\ . \tag{2.18}$$

Subtraction of (2.16) and (2.17) now yields

$$\frac{\mathscr{E}}{\Sigma^2} + \frac{\Delta}{2} = \frac{1}{4}\frac{\ddot{\Sigma}}{\Sigma} - \frac{1}{8}\frac{\dot{\Sigma}^2}{\Sigma^2} = \frac{1}{2}\frac{(\ddot{\Sigma^{1/2}})}{\Sigma^{1/2}} \tag{2.19}$$

so that, on use of (2.18), a basic Ermakov equation in the ratio $\Sigma^{1/2}/\Phi$ results, namely

$$d^2(\Sigma^{1/2}/\Phi)/dt^{*2} + \zeta(\Sigma^{1/2}/\Phi) = 2\mathscr{E}(\Sigma^{1/2}/\Phi)^{-3} \tag{2.20}$$

where the independent variable t^* is given by integration of $(2.9)_3$. Thus, $\Sigma^{1/2}/\Phi$ may be readily obtained by the nonlinear superposition principle admitted by (2.20).

3. Hybrid Ermakov-Painlevé Systems

3.1. *The Ermakov-Painlevé II System*

Here, it is required that Φ be governed by the prototype integrable Ermakov-Painlevé II equation [36]

$$\ddot{\Phi} + \frac{t}{2}\Phi + \varepsilon\Phi^3 + \frac{(\alpha - \varepsilon/2)^2}{4\Phi^3} = 0 \quad , \quad \varepsilon^2 = 1 \ , \tag{3.1}$$

whence, on setting $\zeta = -(\alpha - \varepsilon/2)^2/4$ in (2.1), the system becomes

$$\begin{aligned} \ddot{W}_{\mathrm{I}} + \left[\frac{t}{2} + \varepsilon\Phi^2\right] W_{\mathrm{I}} &= \frac{1}{W_{\mathrm{I}}^2 W_{\mathrm{II}}} S(W_{\mathrm{II}}/W_{\mathrm{I}}) \ , \\ \ddot{W}_{\mathrm{II}} + \left[\frac{t}{2} + \varepsilon\Phi^2\right] W_{\mathrm{II}} &= \frac{1}{W_{\mathrm{I}} W_{\mathrm{II}}^2} T(W_{\mathrm{I}}/W_{\mathrm{II}}) \ . \end{aligned} \tag{3.2}$$

Importantly, as observed in the three-ion electrodiffusion context of [2], in terms of $w = \Phi^2$, the Ermakov-Painlevé II equation (3.1) in Φ delivers the integrable Painlevé XXXIV equation in w, namely

$$\ddot{w} + \frac{\dot{w}^2}{2w} + tw + 2\varepsilon w^2 + \frac{(\alpha - \varepsilon/2)^2}{2w} = 0 \quad , \quad \varepsilon^2 = 1 \tag{3.3}$$

while $(2.9)_3$ shows that

$$t^* = \int w^{-1} dt \tag{3.4}$$

in (2.20).

Here, the system (2.11) becomes

$$\begin{aligned} \ddot{W}_{\mathrm{I}} + \left[\frac{t}{2} + \varepsilon\Phi^2\right] W_{\mathrm{I}} &= \frac{2}{W_{\mathrm{I}}^3} J(W_{\mathrm{II}}/W_{\mathrm{I}}) + \frac{W_{\mathrm{II}}}{W_{\mathrm{I}}^4} J'(W_{\mathrm{II}}/W_{\mathrm{I}}) \ , \\ \ddot{W}_{\mathrm{II}} + \left[\frac{t}{2} + \varepsilon\Phi^2\right] W_{\mathrm{II}} &= -\frac{1}{W_{\mathrm{I}}^3} J'(W_{\mathrm{II}}/W_{\mathrm{I}}) \ , \end{aligned} \tag{3.5}$$

so that the Ermakov-Painlevé II system (1.1)–(1.2) is retrieved corresponding to the particular solution $\Phi = \Sigma^{1/2}$ of the Ermakov equation (2.20) with the relation

$$\zeta = 2\mathscr{E} = -(\alpha - \varepsilon/2)^2/4 \ , \tag{3.6}$$

linking the Painlevé parameter α and Ermakov invariant $\mathscr{E} < 0$. It is seen that (2.12) implies that $J(W_{\mathrm{II}}/W_{\mathrm{I}}) < 0$. In this specialisation with $\Phi = \Sigma^{1/2}$, (3.1) shows that the amplitude $\Sigma^{1/2} = (W_{\mathrm{I}}^2 + W_{\mathrm{II}}^2)^{1/2}$ is governed by an Ermakov-Painlevé II equation directly related with $w = \Sigma$ to the canonical integrable Painlevé XXXIV equation.

To determine the phase $\Lambda = W_{\mathrm{II}}/W_{\mathrm{I}}$, return is made to the Ermakov invariant relation (2.12) which shows that

$$\frac{1}{2}\left(\Sigma\frac{d}{dt}\tan^{-1}\left(\frac{W_{\mathrm{II}}}{W_{\mathrm{I}}}\right)\right)^2 + \left(1+\left(\frac{W_{\mathrm{II}}}{W_{\mathrm{I}}}\right)^2\right)J\left(\frac{W_{\mathrm{II}}}{W_{\mathrm{I}}}\right) = \mathscr{E} \tag{3.7}$$

whence, on introduction of the new independent variable t^* according to

$$dt^* = \Sigma^{-1}dt\ , \tag{3.8}$$

(3.8) shows that

$$\int \frac{d(\tan^{-1}\Lambda)}{\sqrt{2(\mathscr{E}-(1+\Lambda^2)J(\Lambda))}} = \pm t^*\ . \tag{3.9}$$

Once the amplitude $\Sigma^{1/2}$ has been determined corresponding to a positive solutions $w = \Sigma$ of Painlevé XXXIV, and the phase Λ obtained via (3.9), the original variables W_{I} and W_{II} in the Ermakov-Painlevé II system (1.1)−(1.2) are given by the relations

$$W_{\mathrm{I}} = \pm\sqrt{\frac{\Sigma}{1+\Lambda^2}}\quad ,\quad W_{\mathrm{II}} = \pm\Lambda\sqrt{\frac{\Sigma}{1+\Lambda^2}}\ . \tag{3.10}$$

Thus, in summary, the algorithm for generation of solutions of the Ermakov-Painlevé II system (1.1)−(1.2) decomposes into the isolation of solutions $\Sigma > 0$ of the integrable Painlevé XXXIV equation (3.3) together with evaluation of the quadrature in (3.9) to determine the phase Λ.

The importance of positive solutions of Painlevé XXXIV arises, interestingly, elsewhere in the context of boundary value problems for a Painlevé II reduction of the classical two-ion Nernst-Planck system [6, 10, 15, 48]. Therein, the scaled electric field E is governed by the canonical integrable Painlevé II equation

$$E_{zz} = 2E^3 + zE + \alpha\ , \tag{3.11}$$

while the associated ion concentrations are given by

$$c_{\pm} = \pm E_z + E^2 = \frac{z}{2}\ . \tag{3.12}$$

Thus, on elimination of E in (3.12), the concentrations $c_{\pm}$, which by physical considerations are required to be positive, are seen to be governed by the Painlevé XXXIV equation (3.3) with $\varepsilon = -1$. In [6], sequences of exact solutions of this nonlinear equation in terms of Yablonski-Vorob'ev polynomials or classical Airy functions as generated by the iterated action of a Bäcklund transformation were investigated in detail with regard to this positivity constraint. Moreover, in [37], the celebrated Bäcklund transformation for Painlevé II has been recently interpreted at the level of Painlevé XXXIV and used to demonstrate that the problem of integration of the reciprocal of a solution Σ of Painlevé XXXIV as required in (3.8) is equivalent to the problem of integration of an associated solution of Painlevé II. The latter problem was then shown to be amenable to solution via the iterated action of the Bäcklund transformation for Painlevé II on a seed solution.

3.2. *The Ermakov-Painlevé III System*

Painlevé III, namely

$$w_{zz} = \frac{w_z^2}{w} - \frac{w_z}{z} + \frac{1}{z}(\alpha w^2 + \beta) + \gamma w^3 + \frac{\delta}{w} \tag{3.13}$$

where α, β, γ and δ are arbitrary parameters has been derived in a wide range of physical contexts, notably in general relativity, electromagnetic radiation and statistical mechanics (see e.g. the literature cited in [33]). It has been derived via symmetry reduction of nonlinear equations in soliton theory, such as the Ernst equations of general relativity [58], the Pohlmeyer-Regge-Lund system [28], the stimulated Raman scattering system [21], as well as Yang-Mills and Bianchi-IX systems in [53] and [18] respectively.

On setting

$$w = e^W \quad , \quad z = e^t \tag{3.14}$$

in (3.13), it adopts the novel symmetric form

$$\ddot{W} = \alpha e^{t+W} + \beta e^{t-W} + \gamma e^{2(t+W)} + \delta e^{2(t-W)} \tag{3.15}$$

which encapsulates positive solutions of Painlevé III on regions $z > 0$. In this connection, it is noted that (3.15) is derived under the similarity transformation

$$w = e^W \quad , \quad z = e^t = XT \tag{3.16}$$

of the modulated nonlinear hyperbolic equation

$$W_{XT} = \alpha e^W + \beta e^{-W} + XT(\gamma e^{2W} + \delta e^{-2W}) \tag{3.17}$$

symmetric in X and T. In the case $\beta = \gamma = 0$, (3.17) constitutes a modulated Tzitzeica equation and has been derived in the context of affine geometry in [19]. It is remarked that modulated systems arise naturally in continuum mechanics in such areas as the visco-elastodynamics, elastostatics, and elastodynamics of inhomogeneous media (see e.g. [5,12,29] and literature cited therein). Nonlinear Schrödinger (NLS) equations with modulation also are important, notably in soliton management (Malomed [32]). The structure and application of NLS models with inhomogeneities determined by associated Ermakov-type systems is a subject of current interest and has been recently investigated in [8,9,36,51,52,59].

The above motivates introduction here of hybrid Ermakov-Painlevé III systems with Φ governed by (3.15), so that,

$$\ddot{\Phi} = \alpha e^{t+\Phi} + \beta e^{t-\Phi} + \gamma e^{2(t+\Phi)} + \delta e^{2(t-\Phi)} \tag{3.18}$$

whence, the system (2.1) becomes

$$\begin{aligned} W_{\mathrm{I}} - \frac{1}{\Phi}\left[\alpha e^{t+\Phi} + \beta e^{t-\Phi} + \gamma e^{2(t+\Phi)} + \delta e^{2(t-\Phi)} - \frac{\zeta}{\Phi^3}\right] W_{\mathrm{I}} &= \frac{1}{W_{\mathrm{I}}^2 W_{\mathrm{II}}} S(W_{\mathrm{II}}/W_{\mathrm{I}}) \ , \\ W_{\mathrm{II}} - \frac{1}{\Phi}\left[\alpha e^{t+\Phi} + \beta e^{t-\Phi} + \gamma e^{2(t+\Phi)} + \delta e^{2(t-\Phi)} - \frac{\zeta}{\Phi^3}\right] W_{\mathrm{II}} &= \frac{1}{W_{\mathrm{I}} W_{\mathrm{II}}^2} T(W_{\mathrm{I}}/W_{\mathrm{II}}) \end{aligned} \tag{3.19}$$

If we proceed with the particular solution of the Ermakov equation (2.20) with $\Phi = \Sigma^{1/2}$ and $\zeta = 2\mathscr{E}$ where $\mathscr{E}$ is the Ermakov invariant of the system (3.19) then the amplitude $\Sigma^{1/2}$ is governed by the variant

$$d^2\Sigma^{1/2}/dt^2 = \alpha e^{t+\Sigma} + \beta e^{t-\Sigma} + \gamma e^{2(t+\Sigma)} + \delta e^{2(t-\Sigma)} \tag{3.20}$$

of the canonical integrable Painlevé III equation (3.13). In the case of underlying Hamiltonian-type structure, so that $S(W_{\rm II}/W_{\rm I})$ and $T(W_{\rm I}/W_{\rm II})$ given by (1.2), the Ermakov-Painlevé III system (3.19) with $\Phi = \Sigma^{1/2}$ becomes

$$\begin{aligned}
&\ddot{W}_{\rm I} - \frac{1}{\Sigma^{1/2}}\left[\alpha e^{t+\Sigma^{1/2}} + \beta e^{t-\Sigma^{1/2}} + \gamma e^{2(t+\Sigma^{1/2})} + \delta e^{2(t-\Sigma^{1/2})} - \frac{2\mathscr{E}}{\Sigma^{3/2}}\right] W_{\rm I} \\
&\quad = \frac{2}{W_{\rm I}^3} J(W_{\rm II}/W_{\rm I}) + \frac{W_{\rm II}}{W_{\rm I}^4} J'(W_{\rm II}/W_{\rm I}) \;, \\
&\ddot{W}_{\rm II} - \frac{1}{\Sigma^{1/2}}\left[\alpha e^{t+\Sigma^{1/2}} + \beta e^{t-\Sigma^{1/2}} + \gamma e^{2(t+\Sigma^{1/2})} + \delta e^{2(t-\Sigma^{1/2})} - \frac{2\mathscr{E}}{\Sigma^{3/2}}\right] W_{\rm II} \\
&\quad = -\frac{1}{W_{\rm I}^3} J'(W_{\rm II}/W_{\rm I}) \;.
\end{aligned} \tag{3.21}$$

The Ermakov integral of motion (2.12) generic to the general hybrid system (2.1) again determines the ratio $\Lambda = W_{\rm II}/W_{\rm I}$ via (3.9) but now in the integration of the relation (3.8) to determine t^*, the squared amplitude Σ is determined via (3.20).

It is evident that (3.20) admits the reciprocal invariance

$$R : \Phi_R = -\Phi \quad , \quad \alpha_R = -\beta \quad , \quad \beta_R = -\alpha \quad , \quad \gamma_R = -\delta \quad , \quad \delta_R = -\gamma \tag{3.22}$$

with induced invariance with $w_R = e^{\Phi_R} = w^{-1}$ in Painlevé III. This invariance, augmented by a triad of Bäcklund transformations admitted by Painlevé III allows the iterative generation of sequences of its exact solutions and hence of (3.20) via action on appropriate seed solutions. Bäcklund transformations for Painlevé III have been set down in [33] where rational and classical Bessel function solutions are recorded. In the present context, in view of the relations (3.16) interest is restricted to regions $z > 0$ on which such solutions are positive. Once solutions $\Sigma^{1/2}$ and Λ have been determined, in turn, via the integrable Painlevé III variant (3.20) and the integral relation (3.9) derived by means of the Ermakov integral of motion (2.12), the associated solutions $W_{\rm I}$ and $W_{\rm II}$ in the hybrid Ermakov-Painlevé III system (3.21) are again given by the relations of the type (3.10).

3.3. *The Ermakov-Painlevé IV System*

In this case, it is required that Φ be governed by the prototype Ermakov-Painlevé IV equation

$$\ddot{\Phi} - \left[\frac{3}{4}\Phi^4 + 2t\Phi^2 + t^2 - \alpha\right]\Phi = \frac{\zeta}{\Phi^3} \tag{3.23}$$

as originally derived in [36] via a symmetry reduction of a coupled derivative NLS system. It is seen that with $w = \Phi^2$, the canonical Painlevé IV equation in w is obtained, namely

$$w\ddot{w} = \frac{1}{2}\dot{w}^2 + \frac{3}{2}w^4 + 4tw^3 + 2(t^2 - \alpha)w^2 + \beta \;. \tag{3.24}$$

The latter has applications ‘inter alia’, in nonlinear lattice theory and the propagation of ion sound waves in plasma physics. It was shown in [13] to arise as a similarity reduction of the classical

Boussinesq equation, and also occurs, notably, in connection with a symmetry reduction of the discrete Kac-Moerbeke equation [22].

Use of (3.23) in (2.1) produces the hybrid Ermakov-Painlevé IV system

$$\begin{aligned} \ddot{W}_{\mathrm{I}} - \left[\frac{3}{4}\Phi^4 + 2t\Phi^2 + t^2 - \alpha\right] W_{\mathrm{I}} &= \frac{1}{W_{\mathrm{I}}^2 W_{\mathrm{II}}} S(W_{\mathrm{II}}/W_{\mathrm{I}}) \ , \\ \ddot{W}_{\mathrm{II}} - \left[\frac{3}{4}\Phi^4 + 2t\Phi^2 + t^2 - \alpha\right] W_{\mathrm{II}} &= \frac{1}{W_{\mathrm{I}} W_{\mathrm{II}}^2} T(W_{\mathrm{I}}/W_{\mathrm{II}}) \ . \end{aligned} \tag{3.25}$$

In the case when S, T are given by the expressions (1.2) and with the particular solution $\Phi = \Sigma^{1/2}$, $\zeta = 2\mathscr{E}$ of the Ermakov equation (2.20), the solutions W_{I}, W_{II} are given by the expressions (3.10) where now Σ therein is determined by the Ermakov-Painlevé IV equation

$$d^2\Sigma^{1/2}/dt^2 - \left[\frac{3}{4}\Sigma^2 + 2t\Sigma + t^2 - \alpha\right]\Sigma^{1/2} = \frac{2\mathscr{E}}{\Sigma^{3/2}} \ . \tag{3.26}$$

In the latter connection, it is remarked that a privileged sequence of bound state solutions which are non-negative may be generated by the iterated action of a Bäcklund transformation and corresponds to parameters α an odd integer and $\beta = 0$ (see [7]). The ratio $\Lambda = W_{\mathrm{I}}/W_{\mathrm{II}}$ is again determined via the relations (3.10) but where now Σ is a positive solution of Painlevé IV.

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