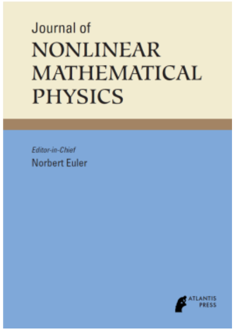

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LETTER TO THE EDITOR

Bäcklund transformations between four Lax-integrable 3D equations

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Recently a classification of contactly-nonequivalent three-dimensional linearly degenerate equations of the second order was presented by E.V. Ferapontov and J. Moss. The equations are Lax-integrable. In our paper we prove that all these equations are connected with each other by appropriate Bäcklund transformations.

Keywords: Lax-integrable equations; Bäcklund transformations

2000 Mathematics Subject Classification: 58H05, 58J70, 35A30

The aim of this note is to construct Bäcklund transformations, [4, 8], between the following four equations

$$u_{yy} = u_{tx} + u_y\, u_{xx} - u_x\, u_{xy}, \tag{1}$$

$$u_{ty} = u_x\, u_{xy} - u_y\, u_{xx}, \tag{2}$$

$$u_{yy} = u_y\, u_{tx} - u_x\, u_{ty}, \tag{3}$$

$$u_{ty} = u_t\, u_{xy} - u_y\, u_{tx}, \tag{4}$$

In different contexts, these equations have appeared before in [1–3, 5, 10–18]. These quasilinear equations have plenty interesting and important properties. For instance, all of them possess infinitely many global solutions (see [6]), because these equations have infinitely many two-dimensional dispersive reductions, which are known as integrable two-dimensional systems (see [19]), whose multi-phase solutions illustrate impossibility of breakdown of smooth initial data.

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Recently in [7] a classification of three-dimensional linearly degenerate[a] integrable equations of the second order (non-equivalent up to contact transformations) was presented. Then this list of equations was recognised as a set of equations which belong to the same linearly-degenerate integrable hierarchy (see equations (56), (57) in [19]). In this paper we prove that all these three-dimensional linearly degenerate integrable equations of second order are connected with each other via appropriate Bäcklund transformations. Thus, any known particular solution of one of these equations can be mapped into corresponding solutions of other equations. At this moment this phenomenon (existence of several three-dimensional equations connected by Bäcklund transformations and belonging to the same integrable hierarchy) is observed just for above linearly degenerate equations (1), (2), (3), (4).

The equations are Lax-integrable, that is, each equation has a differential covering linear in the covering variable (or a Lax pair) with a non-removable spectral parameter. The coverings for (1), (2), (3), (4), are defined, [1,5,11,12,17,18], by the following over-determined systems, respectively:

$$\begin{cases} v_t = (\lambda^2 - \lambda\, u_x - u_y)\, v_x, \\ v_y = (\lambda - u_x)\, v_x, \end{cases} \tag{5}$$

$$\begin{cases} v_t = (u_x - \lambda)\, v_x, \\ v_y = \lambda^{-1}\, u_y\, v_x, \end{cases} \tag{6}$$

$$\begin{cases} v_t = \lambda^{-1}\, u_y^{-1}\, v_y, \\ v_x = (\lambda + u_y\, u_x^{-1})\, v_y, \end{cases} \tag{7}$$

$$\begin{cases} v_t = \lambda\, (\lambda+1)^{-1}\, u_t\, v_x, \\ v_y = \lambda\, u_y\, v_x. \end{cases} \tag{8}$$

The compatibility conditions for (5), (6), (7), and (8) coincide with equations (1), (2), (3), and (4), respectively. Eliminating u from (5), (6), (7), (8) and rescaling the coordinates yields equations

$$v_{yy} = v_{tx} + \frac{v_y - v_t}{v_x}\, v_{xx} + \frac{v_y - v_x}{v_x}\, v_{xy}, \tag{9}$$

$$v_{ty} = \frac{v_t + v_x}{v_x}\, v_{xy} - \frac{v_y}{v_x}\, v_{xx}, \tag{10}$$

$$v_{yy} = \frac{v_y}{v_t}\, v_{tx} + \frac{v_y - v_x}{v_t}\, v_{ty}, \tag{11}$$

$$v_{ty} = \frac{\lambda+1}{\lambda}\, \frac{v_t}{v_x}\, v_{xy} - \frac{1}{\lambda}\, \frac{v_y}{v_x}\, v_{tx}. \tag{12}$$

In other words, systems (5), (6), (7), (8) define Bäcklund transformations between pairs of equations (1) and (9), (2) and (10), (3) and (11), (4) and (12), respectively. Equation (12) was considered in [20].

[a]The concept of linearly degenerate three-dimensional equation is based on an existence of global solutions. See, for instance, [6] and [7].

Theorem. *The following pairs of equations are equivalent via point transformations:*

(i) (9) and (2),
(ii) (10) and (11),
(iii) (11) and (4).

Proof. (i) Write equation (9) as

$$\tilde{v}_{\tilde{y}\tilde{y}} = \tilde{v}_{\tilde{t}\tilde{x}} + \frac{\tilde{v}_{\tilde{y}} - \tilde{v}_{\tilde{t}}}{\tilde{v}_{\tilde{x}}}\,\tilde{v}_{\tilde{x}\tilde{x}} + \frac{\tilde{v}_{\tilde{y}} - \tilde{v}_{\tilde{x}}}{\tilde{v}_{\tilde{x}}}\,\tilde{v}_{\tilde{x}\tilde{y}}. \tag{13}$$

Then the change of variables

$$\tilde{t} = t, \quad \tilde{x} = -u + x, \quad \tilde{y} = x, \quad \tilde{v} = y \tag{14}$$

maps equation (13) to equation (2).

(ii) Write equation (11) as

$$\tilde{v}_{\tilde{y}\tilde{y}} = \frac{\tilde{v}_{\tilde{y}}}{\tilde{v}_{\tilde{t}}}\,\tilde{v}_{\tilde{t}\tilde{x}} + \frac{\tilde{v}_{\tilde{y}} - \tilde{v}_{\tilde{x}}}{\tilde{v}_{\tilde{t}}}\,\tilde{v}_{\tilde{t}\tilde{y}}. \tag{15}$$

Then the change of variables

$$\tilde{t} = y, \quad \tilde{x} = t, \quad \tilde{y} = -x, \quad \tilde{v} = v \tag{16}$$

maps equation (15) to equation (10).

(iii) The change of variables

$$\tilde{t} = t, \quad \tilde{x} = x, \quad \tilde{y} = u, \quad \tilde{v} = y \tag{17}$$

maps equation (15) to equation (4). □

Remark. The observation of the theorem can be hinted by the symmetry algebras of the Bäcklund related equations. For instance, two equations (2) and (9) as well as three equations (10), (4), and (11) have the same contact symmetry algebras. Although the coincidence of the symmetry algebras is only a necessary condition for the equivalence of two equations, in the above cases, we have the equivalences defined by transformations (14), (16), and (17). This observation was already exploited in the paper [9] to find a Bäcklund transformation between the four-dimensional Martínez Alonso–Shabat and Ferapontov–Khusnutdinova equations.

Corollary. *Each pair of equations (1), (2), (3), (4), (9), (10), (11), (12) is related via an appropriate combination of transformations (5), (6), (7), (8), (14), (16), (17).*

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