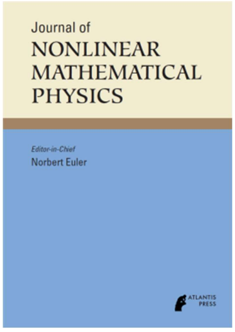

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New Double Wronskian Solutions of the Whitham-Broer-Kaup System: Asymptotic Analysis and Resonant Soliton Interactions

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In this paper, by the Darboux transformation together with the Wronskian technique, we construct new double Wronskian solutions for the Whitham-Broer-Kaup (WBK) system. Some new determinant identities are developed in the verification of the solutions. Based on analyzing the asymptotic behavior of new double Wronskian functions as $t \to \pm\infty$, we make a complete characterization of asymptotic solitons for the non-singular, nontrivial and irreducible soliton solutions. It turns out that the solutions are the linear superposition of two fully-resonant multi-soliton configurations, in each of which the amplitudes, velocities and numbers of asymptotic solitons are in general not equal as $t \to \pm\infty$. To illustrate, we present the figures for several examples of soliton interactions occurring in the WBK system.

Keywords: Soliton interactions; Whitham-Broer-Kaup system; asymptotic analysis; double Wronskian.

2000 Mathematics Subject Classification: 35Q51, 37K40

1. Introduction

It has been an important topic to study the soliton interactions in theory and in experiment [2,25,26, 34]. The soliton interactions in (1+1)-dimensional integrable systems are usually elastic, that is, the interacting solitons retain their energies, amplitudes and velocities upon an interaction except for the phase shift [2]. For many (2+1)-dimensional integrable systems, they also admit the soliton resonant

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interaction in addition to the elastic one [22, 23]. In the past decade, the Kadomtsev-Petviashvilli II (KPII) equation, which is a prototype model of $(2+1)$-dimensional integrable systems [13], has been found to admit a large variety of soliton interactions which are in general not elastic in the sense that the amplitudes, directions and even the number of asymptotic solitons as $y \to -\infty$ are different from those as $y \to \infty$ [4, 5, 7, 8]. Such nontrivial soliton interaction behavior in the KPII equation has aroused much attention, and new progress includes the asymptotic analysis and classification scheme of the soliton solutions [5, 7], the connection between soliton structures and the theory of total positivity for the Grassmannian [15, 16], algebraic method for the construction of exact soliton solutions from the web-patterns of shallow water waves [9], vertex dynamics of the multi-soliton solutions [35], and so on.

In fact, some (1+1)-dimensional nonlinear evolution equations [(1+1)-DNLEEs] may also exhibit abundant soliton interactions except for the standard elastic one. For example, the Sawada-Kotera equation [12], the "good" Boussinesq equation [18, 36] and the Sasa-Satsuma equation [30] admit both the "X"-type elastic interaction and "Y"-type resonant interaction, and various partially/completely inelastic interactions consisting of such two fundamental ones. However, it has been shown that the soliton interactions in the (1+1)-DNLEEs with multiple fields are much richer than the ones in those with a single field. Several examples can be seen as follows: the multi-component nonlinear Schrödinger system possesses the vector soliton solutions which can exhibit the properties of polarization change and energy redistribution among the components under certain parametric conditions [1,31], the Broer-Kaup (BK) system has the resonant solutions describing the fusion of N solitons merging into $N-M$ ones or the fission of $N-M$ solitons into N ones after their interactions [24], the Whitham-Broer-Kaup (WBK) system admits the soliton structures which are the linear superposition of elastic and resonant soliton interactions [19–21, 28].

As a dispersive long wave model in shallow water [6, 14, 29], the WBK system is written in the form

$$u_t + u u_x + v_x + \beta u_{xx} = 0, \tag{1.1a}$$

$$v_t + (uv)_x - \beta v_{xx} + \alpha u_{xxx} = 0, \tag{1.1b}$$

where $u = u(x,t)$ is the field of horizontal velocity, $v = v(x,t)$ is the height deviating from the equilibrium position of liquid, α and β are real constants that represent different diffusion powers, and this system is reduced to the BK system [6, 14] if $\beta = 0$ but $\alpha \neq 0$. System (1.1) is integrable and has many remarkable properties such as the Hamiltonian structure [17], Lax pair [19], Darboux transformation (DT) [19, 28], Painlevé property [37], etc. Recently, based on the asymptotic analysis method developed for the KPII soliton solutions [5], Ref. [32] has studied the soliton solutions in terms of two double Wronskians for the WBK system. The results show that the soliton solutions are linearly combined of two fully-resonant multi-soliton configurations. Moreover, the WBK system permits all types of fully-resonant soliton interactions, which are more general than those in the other (1+1)-dimensional models admitting the soliton resonant phenomena [12, 18, 30, 36]. It is possible that the soliton resonance is a fundamental way for the nonlinear interactions of solitary waves in shallow water. In fact, the multi-soliton fission based on the the Korteweg-de Vries description has just been confirmed in experiment [26]. Since System (1.1) governs the bidirectional wave propagation dynamics and does not exclude the effects of wave interactions and/or wave reflections [27], it is expected that the multi-soliton resonant interactions can be observed in a flat channel or long tank.

In this paper, we will make a further study on the soliton solutions of System (1.1). On one hand, a more general family of soliton solutions will be derived by using the DT method and Wronskian technique. On the other hand, the asymptotic analysis will be performed to make a full characterization of the asymptotic behavior of multi-soliton solutions. The structure of this paper is organized as follows:

(i) In Section 2, we transform the WBK system to the second-order Ablowitz-Kaup-Newell-Segur (AKNS) system. Then, by using the DT method and starting from a nonzero seed, we derive new determinant solutions of the AKNS system in terms of the (N,N)-component double Wronskian. On this basis, we prove that the AKNS system also admits the general (N,M)-component double Wronskian solutions via the Wronskian technique. Hence, we derive a new family of double Wronskian solutions which are more general than those obtained in Refs. [20, 32] because an additional free parameter is included.

(ii) In Section 3, we expand the new double Wronskians by the Laplace expansion technique and Binet-Cauchy theorem, and further analyze their algebraic properties so as to find the parametric condition for the non-singular, non-trivial and irreducible soliton solutions.

(iii) In Section 4, following the asymptotic analysis method [5, 7], we study the asymptotic behavior of the double Wronskian functions as $t \to \pm\infty$, and make a complete characterization of asymptotic solitons for the non-singular, non-trivial and irreducible soliton solutions. It turns out that the solutions are linearly superposed of the fully-resonant $(M+1,N)$- and $(M,N+1)$-soliton configurations, in each of which the amplitudes, velocities and numbers of asymptotic solitons are in general not equal as $t \to \pm\infty$. Also, we present an algebraic procedure on how to determine the asymptotic solitons and dominant exponentials for any given multi-soliton solution.

(iv) In Section 5, we graphically demonstrate some examples of soliton interactions in System (1.1), including the $(2,1)$-, $(1,2)$-, $(3,1)$-, $(1,3)$-, $(2,2)$-, $(3,2)$-, $(2,3)$-soliton resonant interactions as well as some complex soliton structures linearly superposed of two resonant multi-soliton interactions.

(v) In Section 6, we address the conclusions of this work and compare the results with those obtained in Ref. [32].

2. New double Wronskian solutions

In this section, we first transform the WBK system to the second-order AKNS system. Then, by using the DT method and starting from a nonzero seed, we derive new determinant solutions in terms of the (N,N)-component double Wronskian. Furthermore, we guess that the general (N,M)-component double Wronskian solutions also satisfy the AKNS system, and prove our conjecture by the Wronskian technique.

By introducing the transformations [28]

$$u = -2\gamma(\ln p)_x, \tag{2.1a}$$

$$v = -4\gamma^2 pq + 2(\beta\gamma + \gamma^2)(\ln p)_{xx}, \tag{2.1b}$$

with $\gamma = \sqrt{\alpha+\beta^2}$ $(\alpha+\beta^2 > 0)$, we can transform the WBK system into the second-order AKNS system [3]:

$$p_t + 2\gamma p^2 q - \gamma p_{xx} = 0, \tag{2.2a}$$

$$q_t - 2\gamma q^2 p + \gamma q_{xx} = 0. \tag{2.2b}$$

The Lax pair of System (2.2a) and (2.2b) is given as follows:

$$\begin{pmatrix}\phi_1\\ \phi_2\end{pmatrix}_x = \begin{pmatrix}\lambda & p\\ q & -\lambda\end{pmatrix}\begin{pmatrix}\phi_1\\ \phi_2\end{pmatrix}, \tag{2.3a}$$

$$\begin{pmatrix}\phi_1\\ \phi_2\end{pmatrix}_t = \begin{pmatrix}2\gamma\lambda^2 - \gamma pq & 2\gamma\lambda p + p_x\\ 2\gamma\lambda q - \gamma q_x & \gamma pq - 2\gamma\lambda^2\end{pmatrix}\begin{pmatrix}\phi_1\\ \phi_2\end{pmatrix}, \tag{2.3b}$$

where λ is the spectral parameter.

Based on the work in Ref. [33], the N-time iterated DT of System (2.2a) and (2.2b) can be constructed in the following scheme:

$$\begin{pmatrix}\phi_1[N]\\ \phi_2[N]\end{pmatrix} = T[N]\begin{pmatrix}\phi_1\\ \phi_2\end{pmatrix}, \quad T[N] = \begin{pmatrix}\lambda^N - \sum\limits_{j=0}^{N-1} a_j(x,t)\lambda^n & -\sum\limits_{j=0}^{N-1} b_j(x,t)(-\lambda)^n\\ -\sum\limits_{j=0}^{N-1} c_j(x,t)\lambda^n & \lambda^N - \sum\limits_{j=0}^{N-1} d_j(x,t)(-\lambda)^n\end{pmatrix}, \tag{2.4}$$

where $(\phi_1[N], \phi_2[N])^T$ is the N-time iterated eigenfunction. Via Cramer's rule, the functions a_j's, b_j's, c_j's and d_j's ($0 \leqslant j \leqslant N-1$) in $T[N]$ can be determined by solving the following equations:

$$T[N]|_{\lambda=\lambda_n}(f_n, g_n)^T = 0 \quad (1 \leqslant n \leqslant 2N), \tag{2.5}$$

where the vector function $(f_n, g_n)^T$ is the solution of Lax pair (2.3a) and (2.3b) with $\lambda = \lambda_n$. It can be verified that under Transformation (2.4) the N-time iterated eigenfunction $(\phi_1[N], \phi_2[N])^T$ satisfies Lax pair (2.3a) and (2.3b) with p and q instead of $p[N]$ and $q[N]$, which are given by

$$p[N] = p - 2(-1)^N b_{N-1}, \quad q[N] = q - 2c_{N-1}. \tag{2.6}$$

Hence, the eigenfunction transformation (2.4) and potential transformation (2.6) constitute the N-time iterated DT of System (2.2a) and (2.2b).

Starting from $p = p_0 \neq 0$ and $q = 0$, one can obtain the solution for Lax pair (2.3a) and (2.3b) with $\lambda = \lambda_n$ as follows:

$$f_n = \alpha_n \mathrm{e}^{\theta_n} - \frac{p_0\beta_n}{2\lambda_n}\mathrm{e}^{-\theta_n}, \qquad g_n = \beta_n \mathrm{e}^{-\theta_n} \quad (1 \leqslant n \leqslant 2N), \tag{2.7}$$

where $\theta_n := \lambda_n x + 2\gamma\lambda_n^2 t + \delta_n$, α_n, β_n and δ_n are arbitrary real constants. Thus, the N-time iterated solutions $p[N]$ and $q[N]$ can be represented as

$$p[N] = p_0 + 2\frac{|F_{2N\times(N+1)}, G_{2N\times(N-1)}|}{|F_{2N\times N}, G_{2N\times N}|}, \quad q[N] = -2\frac{|F_{2N\times(N-1)}, G_{2N\times(N+1)}|}{|F_{2N\times N}, G_{2N\times N}|}, \tag{2.8}$$

where the block matrices $F_{2N\times M} = [(\lambda_n)^{m-1} f_n]_{\substack{1\leqslant n\leqslant 2N,\\ 1\leqslant m\leqslant M}}$ and $G_{2N\times M} = [(-\lambda_n)^{m-1} g_n]_{\substack{1\leqslant n\leqslant 2N,\\ 1\leqslant m\leqslant M}}$ ($M = N-1, N, N+1$). Looking at $f_n = \alpha_n \mathrm{e}^{\theta_n} - \frac{p_0}{2\lambda_n} g_n$ and using some determinant properties, we can write Solution (2.8) in the compact form:

$$p[N] = 2\frac{\tau_{N+1,N-1}}{\tau_{N,N}}, \qquad q[N] = -2\frac{\tau_{N-1,N+1}}{\tau_{N,N}}, \tag{2.9}$$

with $\tau_{N,N}$ represented as the superposition of two (N,N)-component double Wronskians as follows:

$$\tau_{N,N} = \tau^{\mathrm{I}}_{N,N} + \frac{1}{2}(-1)^{N-1} p_0 \tau^{\mathrm{II}}_{N-1,N+1}, \tag{2.10}$$

$$\tau^{\mathrm{I}}_{N,N} = \left|\Phi_1, \partial_x\Phi_1, \ldots, \partial_x^{N-1}\Phi_1; \Phi_2, \partial_x\Phi_2, \ldots, \partial_x^{N-1}\Phi_2\right|, \tag{2.11}$$

$$\tau^{\mathrm{II}}_{N-1,N+1} = \left|\partial_x\Phi_1, \ldots, \partial_x^{N-1}\Phi_1; \partial_x^{-1}\Phi_2, \Phi_2, \ldots, \partial_x^{N-1}\Phi_2\right|, \tag{2.12}$$

where $\Phi_1 = (\alpha_1 \mathrm{e}^{\theta_1}, \ldots, \alpha_{2N}\mathrm{e}^{\theta_{2N}})^T$ and $\Phi_2 = (\beta_1 \mathrm{e}^{-\theta_1}, \ldots, \beta_{2N}\mathrm{e}^{-\theta_{2N}})^T$.

It should be noted that System (2.2a) and (2.2b) has the (N,M)-component double Wronskian solutions $p = 2\frac{\tau^{\mathrm{I}}_{N+1,M-1}}{\tau^{\mathrm{I}}_{N,M}}$, $q = -2\frac{\tau^{\mathrm{I}}_{N-1,M+1}}{\tau^{\mathrm{I}}_{N,M}}$ (N is in general not equal to M) [32], which include Solutions (2.8) with $p_0 = 0$ as a very special case. That inspires us to guess that this system also admits the following general determinant solutions:

$$p = 2\frac{\tau_{N+1,M-1}}{\tau_{N,M}}, \quad q = -2\frac{\tau_{N-1,M+1}}{\tau_{N,M}}, \tag{2.13}$$

where the determinant $\tau_{N,M}$ is defined by

$$\tau_{N,M} = \tau^{\mathrm{I}}_{N,M} + \frac{1}{2}(-1)^{N-1} p_0 \tau^{\mathrm{II}}_{N-1,M+1}, \tag{2.14a}$$

$$\tau^{\mathrm{I}}_{N,M} = \left|\Phi_1, \partial_x\Phi_1, \ldots, \partial_x^{N-1}\Phi_1; \Phi_2, \partial_x\Phi_2, \ldots, \partial_x^{M-1}\Phi_2\right|, \tag{2.14b}$$

$$\tau^{\mathrm{II}}_{N,M} = \left|\partial_x\Phi_1, \ldots, \partial_x^{N}\Phi_1; \partial_x^{-1}\Phi_2, \Phi_2, \ldots, \partial_x^{M-2}\Phi_2\right|, \tag{2.14c}$$

with $\Phi_1 = (\alpha_1 \mathrm{e}^{\theta_1}, \ldots, \alpha_{N+M}\mathrm{e}^{\theta_{N+M}})^T$ and $\Phi_2 = (\beta_1 \mathrm{e}^{-\theta_1}, \ldots, \beta_{N+M}\mathrm{e}^{-\theta_{N+M}})^T$. For convenience, the determinants $\tau_{N,M}$, $\tau_{N+1,M-1}$ and $\tau_{N-1,M+1}$ are notated as

$$\tau_{N,M} = |\widehat{N-1}; \widehat{M-1}| + \frac{1}{2}(-1)^{N-1} p_0 |\overline{N-1}; \widetilde{M-1}|, \tag{2.15a}$$

$$\tau_{N+1,M-1} = |\widehat{N}; \widehat{M-2}| + \frac{1}{2}(-1)^{N} p_0 |\overline{N}; \widetilde{M-2}|, \tag{2.15b}$$

$$\tau_{N-1,M+1} = |\widehat{N-2}; \widehat{M}| + \frac{1}{2}(-1)^{N-2} p_0 |\overline{N-2}, \widetilde{M}|, \tag{2.15c}$$

where the notations are defined by

$$\widehat{N} = (\Phi_1, \partial_x\Phi_1, \ldots, \partial_x^{N}\Phi_1), \quad \widehat{N-1} = (\Phi_1, \partial_x\Phi_1, \ldots, \partial_x^{N-1}\Phi_1), \tag{2.16a}$$

$$\widehat{N-2} = (\Phi_1, \partial_x\Phi_1, \ldots, \partial_x^{N-2}\Phi_1), \quad \widehat{M} = (\Phi_2, \partial_x\Phi_2, \ldots, \partial_x^{M}\Phi_2), \tag{2.16b}$$

$$\widehat{M-1} = (\Phi_2, \partial_x\Phi_2, \ldots, \partial_x^{M-1}\Phi_2), \quad \widehat{M-2} = (\Phi_2, \partial_x\Phi_2, \ldots, \partial_x^{M-2}\Phi_2), \tag{2.16c}$$

$$\overline{N} = (\partial_x\Phi_1, \ldots, \partial_x^{N}\Phi_1), \quad \overline{N-1} = (\partial_x\Phi_1, \ldots, \partial_x^{N-1}\Phi_1), \tag{2.16d}$$

$$\overline{N-2} = (\partial_x\Phi_1, \ldots, \partial_x^{N-2}\Phi_1), \quad \widetilde{M} = (\partial_x^{-1}\Phi_2, \Phi_2, \ldots, \partial_x^{M}\Phi_2), \tag{2.16e}$$

$$\widetilde{M-1} = (\partial_x^{-1}\Phi_2, \Phi_2, \ldots, \partial_x^{M-1}\Phi_2), \quad \widetilde{M-2} = (\partial_x^{-1}\Phi_2, \Phi_2, \ldots, \partial_x^{M-2}\Phi_2). \tag{2.16f}$$

In order to prove the above conjecture, we let

$$W = \tau_{N,M}, \quad U = 2\,\tau_{N+1,M-1}, \quad V = 2\,\tau_{N-1,M+1}, \tag{2.17}$$

and substitute them into System (2.2a) and (2.2b), yielding the splitting equations:

$$W W_{xx} - W_x^2 - UV = 0, \tag{2.18}$$

$$UW_t - WU_t + \gamma(WU_{xx} + UW_{xx} - 2W_xU_x) = 0, \tag{2.19}$$
$$VW_t - WV_t - \gamma(WV_{xx} + VW_{xx} - 2W_xV_x) = 0. \tag{2.20}$$

Based on the notations in Eqs. (2.15a)–(2.15c), we can also obtain the compact notations of various order derivatives of U, V and W [see Eqs. (A.1)–(A.9) in Appendix A]. Meanwhile, we give three lemmas for some useful double Wronskian identities in Appendix B. Thus, we can verify that U, V and W exactly satisfy Eqs. (2.18)–(2.20), and then arrive at the following theorem:

Theorem 2.1. *System (2.2a) and (2.2b) admits the general (N,M)-component double Wronskian solution (2.13), where $\tau_{N,M}$, $\tau_{N+1,M-1}$ and $\tau_{N-1,M+1}$ are given by Eqs. (2.15a)–(2.15c), and p_0 is a real nonzero constant.*

The proof of Theorem 2.1 is given in Appendix C. Finally, substituting Solution (2.13) into Transformations (2.1a) and (2.1b), we obtain a new family of double Wronskian solutions for the WBK system as follows:

$$u = 2\gamma(\ln\tau_{N,M})_x - 2\gamma(\ln\tau_{N+1,M-1})_x, \tag{2.21a}$$
$$v = 2(\beta\gamma+\gamma^2)(\ln\tau_{N+1,M-1})_{xx} - 2(\beta\gamma-\gamma^2)(\ln\tau_{N,M})_{xx}, \tag{2.21b}$$

where $\tau_{N,M}$ and $\tau_{N+1,M-1}$ are given by Eqs. (2.15a) and (2.15b). It is mentioned that the following identity

$$\tau_{N,M}^2(\ln\tau_{N,M})_{xx} = 4\,\tau_{N-1,M+1}\tau_{N+1,M-1}, \tag{2.22}$$

is used for obtaining Eq. (2.21b), where Eq. (2.22) is exactly equivalent to Eq. (2.18). We point out that Solutions (2.21a) and (2.21b) can display a variety of multi-soliton structures with unequal numbers of asymptotic solitons as $t \to \pm\infty$, and they are more general than the results obtained in Ref. [32] in the sense that the former coincides with the latter when $p_0 = 0$.

3. Properties of the double Wronskians

In this section, we analyze the properties of $\tau_{N,M}$ ($\tau_{N+1,M-1}$ can be analyzed in a similar way) so as to find the parametric condition for Solutions (2.21a) and (2.21b) to be non-singular, non-trivial and irreducible. Noticing that $\lambda_i = \lambda_j$ makes the double Wronskians $\tau^{\mathrm{I}}_{N,M}$ and $\tau^{\mathrm{II}}_{N-1,M+1}$ become zero, we require $\lambda_i \neq \lambda_j$ and assume that the phase parameters $\{\lambda_i\}_{i=1}^K$ ($K := N+M$) are well ordered as $\lambda_1 < \lambda_2 < \cdots < \lambda_K$.

In order to understand the properties of $\tau_{N,M}$ in Eq. (2.15a), we perform our analysis on the double Wronskians $\tau^{\mathrm{I}}_{N,M}$ and $\tau^{\mathrm{II}}_{N-1,M+1}$. First, we express $\tau^{\mathrm{I}}_{N,M}$ and $\tau^{\mathrm{II}}_{N-1,M+1}$ as

$$\tau^{\mathrm{I}}_{N,M} = \left|A\Theta^+\Lambda_1^+, B\Theta^-\Lambda_1^-\right|, \quad \tau^{\mathrm{II}}_{N-1,M+1} = \left|A\Theta^+\Lambda_2^+, B\Theta^-\Lambda_2^-\right|, \tag{3.1}$$

with

$$\begin{aligned}
&A = \mathrm{diag}(\alpha_1,\alpha_2,\cdots,\alpha_K), \quad B = \mathrm{diag}(\beta_1,\beta_2,\cdots,\beta_K),\\
&\Theta^+ = \mathrm{diag}(\mathrm{e}^{\theta_1},\mathrm{e}^{\theta_2},\cdots,\mathrm{e}^{\theta_K}), \quad \Theta^- = \mathrm{diag}(\mathrm{e}^{-\theta_1},\mathrm{e}^{-\theta_2},\cdots,\mathrm{e}^{-\theta_K}),\\
&\Lambda_1^+ = [(\lambda_n)^{m-1}]_{\substack{1\leqslant n\leqslant K,\\ 1\leqslant m\leqslant N}}, \quad \Lambda_1^- = [(-\lambda_n)^{m-1}]_{\substack{1\leqslant n\leqslant K,\\ 1\leqslant m\leqslant M}},\\
&\Lambda_2^+ = [(\lambda_n)^{m}]_{\substack{1\leqslant n\leqslant K,\\ 1\leqslant m\leqslant N-1}}, \quad \Lambda_2^- = [(-\lambda_n)^{m-2}]_{\substack{1\leqslant n\leqslant K,\\ 1\leqslant m\leqslant M+1}}.
\end{aligned}$$

Then, applying the Laplace expansion technique and Binet-Cauchy theorem, we can expand $\tau^{\mathrm{I}}_{N,M}$ and $\tau^{\mathrm{II}}_{N-1,M+1}$ as follows:

$$\begin{aligned}\tau^{\mathrm{I}}_{N,M} &= \sum_{\substack{\mathscr{I}_N\cap\mathscr{J}_M=\varnothing,\\ \mathscr{I}_N\cup\mathscr{J}_M=[K]}} (-1)^{\frac{(N+1)N}{2}+\frac{(M-1)M}{2}+\sum\limits_{n=1}^{N} i_n} \\ &\quad\times\prod_{n=1}^{N}\alpha_{i_n}\prod_{m=1}^{M}\beta_{j_m}V_{\mathscr{I}_N}V_{\mathscr{J}_M}\exp\left(\sum_{n=1}^{N}\theta_{i_n}-\sum_{m=1}^{M}\theta_{j_m}\right), \qquad (3.2a)\\ \tau^{\mathrm{II}}_{N-1,M+1} &= \sum_{\substack{\mathscr{I}_{N-1}\cap\mathscr{J}_{M+1}=\varnothing,\\ \mathscr{I}_{N-1}\cup\mathscr{J}_{M+1}=[K]}} (-1)^{\frac{(N-1)N}{2}+\frac{(M+1)M}{2}+M+1+\sum\limits_{n=1}^{N-1} i_n} \\ &\quad\times\frac{\prod\limits_{n=1}^{N-1}\lambda_{i_n}}{\prod\limits_{m=1}^{M+1}\lambda_{j_m}}\prod_{n=1}^{N-1}\alpha_{i_n}\prod_{m=1}^{M+1}\beta_{j_m}V_{\mathscr{I}_{N-1}}V_{\mathscr{J}_{M+1}}\exp\left(\sum_{n=1}^{N-1}\theta_{i_n}-\sum_{m=1}^{M+1}\theta_{j_m}\right), \qquad (3.2b)\end{aligned}$$

with

$$\begin{aligned}&V_{\mathscr{I}_N}=\prod_{1\leqslant n<l\leqslant N}(\lambda_{i_l}-\lambda_{i_n}),\quad V_{\mathscr{I}_{N-1}}=\prod_{1\leqslant n<l\leqslant N-1}(\lambda_{i_l}-\lambda_{i_n}),\\ &V_{\mathscr{J}_M}=\prod_{1\leqslant h<m\leqslant M}(\lambda_{j_m}-\lambda_{j_h}),\quad V_{\mathscr{J}_{M+1}}=\prod_{1\leqslant h<m\leqslant M+1}(\lambda_{j_m}-\lambda_{j_h}),\\ &\mathscr{I}_N=\{i_1,\ldots,i_N\},\quad \mathscr{J}_M=\{j_1,\ldots,j_M\},\\ &\mathscr{I}_{N-1}=\{i_1,\ldots,i_{N-1}\},\quad \mathscr{J}_{M+1}=\{j_1,\ldots,j_{M+1}\}.\end{aligned}$$

Based on the expansions of $\tau^{\mathrm{I}}_{N,M}$ and $\tau^{\mathrm{II}}_{N-1,M+1}$ in Eqs. (3.2a) and (3.2b), we have the following properties of the function $\tau_{N,M}$.

(i) The expansions of $\tau^{\mathrm{I}}_{N,M}$ and $\tau^{\mathrm{II}}_{N-1,M+1}$ are, respectively, the summations of exponentials $\exp(\sum\limits_{n=1}^{N}\theta_{i_n}-\sum\limits_{m=1}^{M}\theta_{j_m})$ and $\exp(\sum\limits_{n=1}^{N-1}\theta_{i_n}-\sum\limits_{m=1}^{M+1}\theta_{j_m})$. An exponential $\exp(\sum\limits_{n=1}^{N}\theta_{i_n}-\sum\limits_{m=1}^{M}\theta_{j_m})$ is actually present in the expansion of $\tau^{\mathrm{I}}_{N,M}$ if and only if $\alpha_{i_1},\ldots,\alpha_{i_N},\beta_{j_1},\ldots,\beta_{j_M}\neq 0$; while if $\exp(\sum\limits_{n=1}^{N-1}\theta_{i_n}-\sum\limits_{m=1}^{M+1}\theta_{j_m})$ appears in the expansion of $\tau^{\mathrm{II}}_{N-1,M+1}$, one still need require that none of λ_n's ($1\leqslant n\leqslant K$) is zero in addition to $\alpha_{i_1},\ldots,\alpha_{i_{N-1}},\beta_{j_1},\ldots,\beta_{j_{M+1}}\neq 0$. Thus, there are at most $\binom{K+1}{N}$ exponential terms for the expansion of $\tau_{N,M}$ if $\alpha_n,\beta_n,\lambda_n\neq 0$ for all $n\in[K]$.

(ii) A sufficient condition for Solutions (2.21a) and (2.21b) to be nonsingular is that the functions $\tau_{N,M}$ and $\tau_{N+1,M-1}$ are sign-definite for all $(x,t)\in\mathbb{R}^2$. The sign-definiteness of $\tau_{N,M}$ implies that all the exponentials actually appearing in the expansions of $\tau^{\mathrm{I}}_{N,M}$ and $\frac{1}{2}(-1)^{N-1}p_0\tau^{\mathrm{II}}_{N-1,M+1}$ have the same signs. According to the proof in Ref. [32], both the expansions of $\tau^{\mathrm{I}}_{N,M}$ and $\tau^{\mathrm{II}}_{N-1,M+1}$ keep the sign-definiteness under the condition

$$\alpha_n\alpha_{n+1}\beta_n\beta_{n+1}\leqslant 0 \quad (1\leqslant n\leqslant K-1). \qquad (3.3)$$

On the other hand, *any* exponential in $\tau^{\mathrm{I}}_{N,M}$ has the same sign as that of *any* one in $\frac{1}{2}(-1)^{N-1}p_0\tau^{\mathrm{II}}_{N-1,M+1}$ if and only if

$$\sigma p_0\alpha_1\beta_1 \leqslant 0 \text{ with } \sigma = \mathrm{sign}(\prod_{n=1}^{K}\lambda_n). \tag{3.4}$$

Hence, Conditions (3.3) and (3.4) are the sufficient and necessary conditions for $\tau_{N,M}$ to be sign-definite in the *xt* plane. It can be checked that $\tau_{N+1,M-1}$ also keeps the sign-definiteness under such two conditions.

(iii) If $\alpha_k = 0$ for some $k \in [K]$, there are only the exponentials with $k \notin \mathscr{I}_N$ but $k \in \mathscr{J}_M$ appearing in the expansion of $\tau^{\mathrm{I}}_{N,M}$, and those with $k \notin \mathscr{I}_{N-1}$ but $k \in \mathscr{J}_{M+1}$) appearing in the expansion of $\tau^{\mathrm{II}}_{N-1,M+1}$. In such reducible case, the function $\tau_{N,M}$ can be represented as

$$\tau_{N,M} = (-1)^{M-1}\beta_k e^{\theta_k}\tilde{\tau}^{\mathrm{I}}_{N,M-1} \quad (\tilde{\tau}_{N,M-1} = \tilde{\tau}^{\mathrm{I}}_{N,M-1} + \frac{1}{2}(-1)^{N-1}p_0'\tilde{\tau}^{\mathrm{II}}_{N-1,M}), \tag{3.5}$$

where $p_0' = p_0/\lambda_k$, $\tilde{\tau}^{\mathrm{I}}_{N,M-1}$ and $\tilde{\tau}^{\mathrm{II}}_{N-1,M}$ are, respectively, the following double Wronskians by deleting the k-th row and K-th column of $\tau^{\mathrm{I}}_{N,M}$ and $\tau^{\mathrm{II}}_{N-1,M+1}$, which can be expanded as follows:

$$\begin{aligned}\tilde{\tau}^{\mathrm{I}}_{N,M-1} = &\sum_{\substack{\mathscr{I}_N\cap\mathscr{J}_{M-1}=\varnothing,\\ \mathscr{I}_N\cup\mathscr{J}_{M-1}=[K]\setminus\{k\}}} (-1)^{\frac{(N+1)N}{2}+\frac{(M-2)(M-1)}{2}+\sum_{n=1}^{N} i_n}\\ &\times\prod_{n=1}^{N}\alpha_{i_n}\prod_{m=1}^{M-1}\tilde{\beta}_{j_m}V_{\mathscr{I}_N}V_{\mathscr{J}_{M-1}}\exp\left(\sum_{n=1}^{N}\theta_{i_n}-\sum_{m=1}^{M-1}\theta_{j_m}\right),\end{aligned} \tag{3.6a}$$

$$\begin{aligned}\tilde{\tau}^{\mathrm{II}}_{N-1,M} = &\sum_{\substack{\mathscr{I}_{N-1}\cap\mathscr{J}_M=\varnothing,\\ \mathscr{I}_{N-1}\cup\mathscr{J}_M=[K]\setminus\{k\}}} (-1)^{\frac{(N-1)N}{2}+\frac{(M-1)M}{2}+M+\sum_{n=1}^{N-1} i_n}\\ &\times\frac{\prod_{n=1}^{N-1}\lambda_{i_n}}{\prod_{m=1}^{M}\lambda_{j_m}}\prod_{n=1}^{N-1}\alpha_{i_n}\prod_{m=1}^{M}\tilde{\beta}_{j_m}V_{\mathscr{I}_{N-1}}V_{\mathscr{J}_M}\exp\left(\sum_{n=1}^{N-1}\theta_{i_n}-\sum_{m=1}^{M}\theta_{j_m}\right),\end{aligned} \tag{3.6b}$$

with $\tilde{\beta}_{j_m} = \left|\lambda_{j_m}-\lambda_k\right|\beta_{j_m}$. Because the scaling transformation on $\tau_{N,M}$ and $\tau_{N+1,M-1}$ leaves Solutions (2.21a) and (2.21b) invariant, $\tau^{\mathrm{I}}_{N,M}$ and $\tau^{\mathrm{II}}_{N-1,M+1}$ are actually equivalent to the reduced double Wronskians $\tilde{\tau}^{\mathrm{I}}_{N,M-1}$ and $\tilde{\tau}^{\mathrm{II}}_{N-1,M}$, respectively. That is, $\tau_{N,M}$ is effectively equivalent to the function $\widetilde{\tau}_{N,M-1}$. Similarly, as for $\beta_k = 0$ for some $k \in [K]$, one can also obtain that Solutions (2.21a) and (2.21b) are kept invariant. However, if α_k and β_k are equal to 0 for the same k, we have $\tau_{N,M} = 0$.

Note that the above properties also apply to $\tau_{N+1,M-1}$ if changing $N \to N+1$ and $M \to M-1$. Therefore, we give the non-singular, non-trivial and irreducible condition for Solutions (2.21a) and (2.21b) as follows:

$$\alpha_n\alpha_{n+1}\beta_n\beta_{n+1} < 0 \; (1 \leqslant n \leqslant K-1) \text{ and } \sigma p_0\alpha_1\beta_1 < 0, \tag{3.7}$$

where $N \geq 1, M \geq 1, \sigma = \text{sign}(\prod_{n=1}^{K} \lambda_n)$ and $\lambda_n \neq 0$ for all $n \in [K]$. Since Condition (3.7) implies that $\alpha_n \neq 0$ for all $n \in [K]$, we can without loss of generality take $\alpha_n = 1$ $(1 \leqslant n \leqslant K)$. In the next two sections, the asymptotic behavior and soliton interactions in Solutions (2.21a) and (2.21b) will be studied under Condition (3.7).

4. Asymptotic analysis of soliton solutions

The temporal-spatial patterns of Solutions (2.21a) and (2.21b) are determined by the exponentials of $\tau_{N,M}$ and $\tau_{N+1,M-1}$ dominating in different regions of the xt-plane. In this section, following the way in Refs. [5, 7], we first analyze the asymptotic behavior of $\tau_{N,M}$ and $\tau_{N+1,M-1}$ as $t \to \pm\infty$, and then make a characterization of asymptotic solitons in Solutions (2.21a) and (2.21b) under the non-singular, non-trivial and irreducible condition (3.7). In what follows, we focus our analysis on the function $\tau_{N,M}$ since the asymptotic properties of $\tau_{N+1,M-1}$ can be obtained with the change $N \to N+1$ and $M \to M-1$.

To begin with, we note that Solutions (2.21a) and (2.21b) are kept invariant under the transformation $\tau^{\diamond}_{N,M} \to (-1)^{\frac{(N+1)N}{2}+\frac{(M-1)M}{2}} \mathrm{e}^{\sum_{n=1}^{K}\theta_n} \tau_{N,M}$. Let us define λ'_n, β'_n and δ'_n $(1 \leqslant n \leqslant K' := K+1)$ as

$$\lambda'_n = \begin{cases} \lambda_n, & 1 \leqslant n \leqslant k-1, \\ 0, & n = k, \\ \lambda_{n-1}, & k+1 \leqslant n \leqslant K', \end{cases} \tag{4.1a}$$

$$\beta'_n = \begin{cases} \beta_n, & 1 \leqslant n \leqslant k-1, \\ 0, & n = k, \\ \beta_{n-1}, & k+1 \leqslant n \leqslant K', \end{cases} \qquad \delta'_n = \begin{cases} \delta_n, & 1 \leqslant n \leqslant k-1, \\ 0, & n = k, \\ \delta_{n-1}, & k+1 \leqslant n \leqslant K', \end{cases} \tag{4.1b}$$

where k is an integer in $[K']$ such that $\lambda_1 < \cdots < \lambda_{k-1} < 0 < \lambda_k < \cdots < \lambda_K$. Thus, the function $\tau^{\diamond}_{N,M}$ can be expanded as

$$\begin{aligned} \tau^{\diamond}_{N,M} = & \sum_{\substack{\mathscr{I}_N \cap \mathscr{J}_M = \varnothing, \\ \mathscr{I}_N \cup \mathscr{J}_M = [K'] \setminus \{k\}}} (-1)^{\sum_{n=1}^{N} i_n} \prod_{m=1}^{M} \beta'_{j_m} V_{\mathscr{I}_N} V_{\mathscr{J}_M} E(i_1, \ldots, i_N) \\ & + \frac{1}{2} p_0 \sum_{\substack{\mathscr{I}_{N-1} \cap \mathscr{J}_{M+1} = \varnothing, \\ \mathscr{I}_{N-1} \cup \mathscr{J}_{M+1} = [K'] \setminus \{k\}}} (-1)^{\sum_{n=1}^{N-1} i_n} \frac{\prod_{n=1}^{N-1} \lambda'_{i_n}}{\prod_{m=1}^{M+1} \lambda'_{j_m}} \prod_{m=1}^{M+1} \beta'_{j_m} V_{\mathscr{I}_{N-1}} V_{\mathscr{J}_{M+1}} E(i_1, \ldots, i_{N-1}, k), \end{aligned} \tag{4.2}$$

with

$$\begin{aligned} & V_{\mathscr{I}_N} = \prod_{1 \leqslant n < l \leqslant N} (\lambda'_{i_l} - \lambda'_{i_n}), \quad V_{\mathscr{I}_{N-1}} = \prod_{1 \leqslant n < l \leqslant N-1} (\lambda'_{i_l} - \lambda'_{i_n}), \\ & V_{\mathscr{J}_M} = \prod_{1 \leqslant h < m \leqslant M} (\lambda'_{j_m} - \lambda'_{j_h}), \quad V_{\mathscr{J}_{M+1}} = \prod_{1 \leqslant h < m \leqslant M+1} (\lambda'_{j_m} - \lambda'_{j_h}), \\ & E(i_1, \ldots, i_N) = \exp\big(2 \sum_{n=1}^{N} \theta'_{i_n}\big), \quad E(i_1, \ldots, i_{N-1}, k) = \exp\big(2 \sum_{n=1}^{N-1} \theta'_{i_n} + \theta'_k\big), \end{aligned}$$

where $\theta'_n = \lambda'_n x + 2\gamma \lambda'^2_n t + \delta'_n$ $(1 \leqslant n \leqslant K')$.

In the following, we turn to study the asymptotic behavior of the function $\tau^{\diamond}_{N,M}$ as $t \to \pm\infty$, and define the dominant exponentials [5] in the expansion of $\tau^{\diamond}_{N,M}$ as follows.

Definition 4.1. Suppose that $\Theta_{K'} := \{2\sum_{n=1}^{N}\theta'_{i_n} | \{i_1,\ldots,i_N\} \subset [K']\}$. A given exponential $E(i_1,\ldots,i_N)$ is said to be dominant for the function $\tau^{\diamond}_{N,M}$ in some region $R \subset \mathbb{R}^2$ if $\sum_{n=1}^{N}\theta'_{e_n} \leqslant \sum_{n=1}^{N}\theta'_{i_n}$ for all $\{e_1,\ldots,e_N\} \subset [K']$ and for all $(x,t) \in R$. The region R is called the dominant region of the exponential $E(i_1,\ldots,i_N)$.

It should be noted that the phase combination $\sum_{n=1}^{N}\theta'_{i_n}$ is a linear function of x and t, and it is defined globally on $\mathbb{R}^2$. As a result, the xt plane can be partitioned into a finite number of convex dominant regions, intersecting only at points on the boundaries of each region (the details can be found in Ref. [5]). In the interior of any dominant region of the xt-plane, $(\ln \tau^{\diamond}_{N,M})_x$ approaches exponentially to a nonzero constant and $(\ln \tau^{\diamond}_{N,M})_{xx}$ remains exponentially small; but they are both localized at the boundaries of the dominant regions where a balance exists between two or more dominant exponentials. Eq. (4.2) suggests that the boundary between any two adjacent dominant regions can be identified by the equation $\sum_{n=1}^{N}\theta'_{i_n} = \sum_{n=1}^{N}\theta'_{e_n}$, which represents a straight line in the xt plane. Therefore, along such line the balance between two dominant exponentials generates an asymptotic soliton.

Similar to the tau function of the KPII soliton solutions, the function $\tau^{\diamond}_{N,M}$ also admits the single-phase transition theorem [5].

Proposition 4.1. *Asymptotically as $t \to \pm\infty$, the dominant exponentials for $\tau^{\diamond}_{N,M}$ in two adjacent regions of the xt plane are of the form $E(i,i_1,\ldots,i_{N-1})$ and $E(j,i_1,\ldots,i_{N-1})$, where $i \neq j$ and $i,j \in [K']$. That is, the two exponentials contain $N-1$ common phases and differ by only one phase.*

Proof. The above results can be proved by following the way of Proposition 5.1 in Ref. [8]. □

From the above proposition, one can see that the transition $E(i,i_1,\ldots,i_{N-1}) \mapsto E(j,i_1,\ldots,i_{N-1})$ between any two adjacent regions occurs along the line defined by $\mathscr{L}_{ij} : \theta'_i = \theta'_j$, where a single phase θ'_i in one dominant exponential is replaced by θ'_j in the other dominant one. In the neighborhood of the single-phase transition line $\mathscr{L}_{ij}$, the balance between two dominant exponentials yields

$$(\ln \tau^{\diamond}_{N,M})_x \sim u^{[i,j]}_{N,M} := (\lambda'_i - \lambda'_j)\tanh\left(\theta'_i - \theta'_j + \ln\sqrt{\frac{C_{ij}}{D_{ij}}}\right) + \Xi_{ij}, \tag{4.3a}$$

$$(\ln \tau^{\diamond}_{N,M})_{xx} \sim v^{[i,j]}_{N,M} := (\lambda'_i - \lambda'_j)^2\mathrm{sech}^2\left(\theta'_i - \theta'_j + \ln\sqrt{\frac{C_{ij}}{D_{ij}}}\right), \tag{4.3b}$$

asymptotically as $t \to \infty$ or $t \to -\infty$, where $(u^{[i,j]}_{N,M}, v^{[i,j]}_{N,M})$ defines a pair of asymptotic solitons generated by $\tau^{\diamond}_{N,M}$. The coefficients C_{ij}, D_{ij} and Ξ_{ij} have three different forms, that is,

$$C_{ij}^{(1)} = (-1)^i \beta_j' \prod_{1\leqslant n\leqslant N-1} |\lambda_i' - \lambda_{i_n}'| \prod_{1\leqslant m\leqslant M-1} |\lambda_j' - \lambda_{j_m}'|,$$
$$D_{ij}^{(1)} = (-1)^j \beta_i' \prod_{1\leqslant n\leqslant N-1} |\lambda_j' - \lambda_{i_n}'| \prod_{1\leqslant m\leqslant M-1} |\lambda_i' - \lambda_{j_m}'|,$$
$$C_{ij}^{(2)} = (-1)^i \beta_j' \frac{\lambda_i'}{\lambda_j'} \prod_{1\leqslant n\leqslant N-2} |\lambda_i' - \lambda_{i_n}'| \prod_{1\leqslant m\leqslant M} |\lambda_j' - \lambda_{j_m}'|,$$
$$D_{ij}^{(2)} = (-1)^j \beta_i' \frac{\lambda_j'}{\lambda_i'} \prod_{1\leqslant n\leqslant N-2} |\lambda_j' - \lambda_{i_n}'| \prod_{1\leqslant m\leqslant M} |\lambda_i' - \lambda_{j_m}'|,$$
$$C_{ij}^{(3)} = (-1)^i \prod_{1\leqslant n\leqslant N-1} |\lambda_i' - \lambda_{i_n}'|, \quad D_{ij}^{(3)} = \frac{1}{2} p_0 \frac{\prod_{n=1}^{N-1} \lambda_{i_n}'}{\lambda_i' \prod_{m=1}^{M} \lambda_{j_m}'} \beta_i' \prod_{1\leqslant m\leqslant M} |\lambda_i' - \lambda_{j_m}'|,$$
$$\Xi_{ij}^{(1)} = \lambda_i' + \lambda_j' + 2\sum_{n=1}^{N-1} \lambda_{i_n}',\ \Xi_{ij}^{(2)} = \lambda_i' + \lambda_j' + 2\sum_{n=1}^{N-2} \lambda_{i_n}',\ \Xi_{ij}^{(3)} = \lambda_i' + 2\sum_{n=1}^{N-1} \lambda_{i_n}',$$

where the superscripts (1), (2) and (3), respectively, correspond to three different cases for the two dominant exponentials in adjacent regions: (i) $E(i,i_1,\dots,i_{N-1})$ and $E(j,i_1,\dots,i_{N-1})$ with $i \neq j$, (ii) $E(i,i_1,\dots,i_{N-2},k)$ and $E(j,i_1,\dots,i_{N-2},k)$ with $k \neq i,j,i_1,\dots,i_{N-2}$, (iii) $E(i,i_1,\dots,i_{N-1})$ and $E(k,i_1,\dots,i_{N-1})$ with $i \neq k$.

Suggested by Eqs. (4.3a) and (4.3b), the amplitudes, wave number, frequency and velocity of the asymptotic solitons $(u_{N,M}^{[i,j]}, v_{N,M}^{[i,j]})$ can be given as follows:

$$(A_{ij}^u, A_{ij}^v) = \big(\lambda_i' - \lambda_j', (\lambda_i' - \lambda_j')^2\big),$$
$$K_{ij} = \lambda_i' - \lambda_j', \quad \Omega_{ij} = 2\gamma(\lambda_i'^2 - \lambda_j'^2), \quad s_{ij} = -2\gamma(\lambda_i' + \lambda_j'). \tag{4.4}$$

which shows that all the physical quantities are only dependent on the parameters λ_i' and λ_j' except that the background Ξ_{ij} that the filed u approaches as $t \to \pm\infty$ is related to more parameters in $\{\lambda_n'\}_{n=1}^{K'}$. In this regard, we use the index pair $[i,j]$ $(1 \leqslant i < j \leqslant K')$ to label the pair of asymptotic solitons $(u_{N,M}^{[i,j]}, v_{N,M}^{[i,j]})$.

In order to specify which exponentials in the expansion of $\tau_{N,M}^{\diamond}$ dominate in different regions of the xt plane as $t \to \pm\infty$ (or equivalently which index pairs can be used to identify the asymptotic solitons generated by $\tau_{N,M}^{\diamond}$), we first determine the dominance relations among $\theta_1',\dots,\theta_{K'}'$ along the lines $\mathscr{L}_{ij}$ ($i,j \in [K']$ and $i<j$) asymptotically as $t \to \pm\infty$.

Lemma 4.1. *[32] For any given $i,j \in [K']$ with $i<j$, let us define i^* as an integer in $[i,j-1]$ such that $\lambda_{i^*-1}' + \lambda_{i^*}' < \lambda_i' + \lambda_j'$ and $\lambda_{i^*}' + \lambda_{i^*+1}' \geqslant \lambda_i' + \lambda_j'$. Then, the phases $\theta_1',\dots,\theta_{K'}'$ along the line $\mathscr{L}_{ij}$ satisfy the following relations:*

(i) As $t \to \infty$, $\theta_1' > \cdots > \theta_i' > \cdots > \theta_{i^*}' \leqslant \theta_{i^*+1}' < \cdots < \theta_j' < \cdots < \theta_{K'}'$ and $\theta_i' = \theta_j'$;
(ii) As $t \to -\infty$, $\theta_1' < \cdots < \theta_i' < \cdots < \theta_{i^*}' \geqslant \theta_{i^*+1}' > \cdots > \theta_j' > \cdots > \theta_{K'}'$ and $\theta_i' = \theta_j'$.

Proof. Note that along the line $\mathscr{L}_{ij}:\theta_i'=\theta_j'$, we have $x=-2\gamma(\lambda_i'+\lambda_j')t-\frac{\delta_i'-\delta_j'}{\lambda_i'-\lambda_j'}$. Thus, the difference of θ_m' and θ_n' is explicitly given by

$$\theta_m'-\theta_n'=2\gamma(\lambda_m'-\lambda_n')(\lambda_m'+\lambda_n'-\lambda_i'-\lambda_j')t-\frac{(\lambda_m'-\lambda_n')(\delta_i'-\delta_j')}{\lambda_i'-\lambda_j'}+\delta_m'-\delta_n', \tag{4.5}$$

which implies that the sign of $\theta_m'-\theta_n'$ as $t\to\pm\infty$ is determined by $(\lambda_m'-\lambda_n')(\lambda_m'+\lambda_n'-\lambda_i'-\lambda_j')$. Then, the results can be proved on the basis of the ordering $\lambda_1'<\lambda_2'<\cdots<\lambda_{K'}'$. □

With the availability of Lemma 4.1, one can immediately obtain the dominant exponentials of $\tau_{N,M}^{\diamond}$ along the line $\mathscr{L}_{ij}$ as $t\to\pm\infty$ for any given $\lambda_1<\lambda_2<\cdots<\lambda_K$.

Proposition 4.2. *[32] Let us define that $\mathscr{S}_{N,M}^{[i,j]}:=\{n|n\leqslant i^*<\bar{n}$ and $n\in[N]\}$ and $\mathscr{T}_{N,M}^{[i,j]}:=\{m|m\leqslant i^*<\tilde{m}$ and $m\in[M+1]\}$ for any $i,j\in[K']$, where $\bar{n}=M+n+1$, $\tilde{m}=N+m$, i^* is an integer such that $i\leqslant i^*<j$, $\lambda_{i^*-1}'+\lambda_{i^*}'<\lambda_i'+\lambda_j'$ and $\lambda_{i^*}'+\lambda_{i^*+1}'\geqslant\lambda_i'+\lambda_j'$. Then, the function $\tau_{N,M}^{\diamond}$ along the line $\mathscr{L}_{ij}$ has the following asymptotic properties:*

(i) As $t\to\infty$, the asymptotic behavior of $\tau_{N,M}^{\diamond}$ is given by

$$\tau_{N,M}^{\diamond}\sim\begin{cases}C_{ij}^{+}E(1,\ldots,n,\bar{n}+1,\ldots,K')+D_{ij}^{+}E(1,\ldots,n-1,\bar{n},\ldots,K'),\\ \qquad\qquad \text{if}\,\lambda_i'+\lambda_j'=\lambda_n'+\lambda_{\bar{n}}'\,\text{for some}\,n\in\mathscr{S}_{N,M}^{[i,j]},\\ C_{ij}^{+}E(1,\ldots,n^*,\bar{n^*}+1,\ldots,K'),\quad \text{if}\,\lambda_i'+\lambda_j'\neq\lambda_n'+\lambda_{\bar{n}}'\,\text{for all}\,n\in\mathscr{S}_{N,M}^{[i,j]},\end{cases} \tag{4.6}$$

where n^* is defined as

$$n^*=\begin{cases}0,\quad \text{if}\,\lambda_i'+\lambda_j'<\lambda_1'+\lambda_{M+2}',\\ n,\quad \text{if}\,\lambda_i'+\lambda_j'>\lambda_n'+\lambda_{\bar{n}}'\,\text{and}\,\lambda_i'+\lambda_j'\\ \qquad <\lambda_{n+1}'+\lambda_{\bar{n}+1}'\text{for some}\,n\in[N-1],\\ N,\quad \text{if}\,\lambda_i'+\lambda_j'>\lambda_N'+\lambda_{K'}'.\end{cases} \tag{4.7}$$

(ii) As $t\to-\infty$, the asymptotic behavior of $\tau_{N,M}^{\diamond}$ is given by

$$\tau_{N,M}^{\diamond}\sim\begin{cases}C_{ij}^{-}E(m,\ldots,\tilde{m}-1)+D_{ij}^{-}E(m+1,\ldots,\tilde{m}),\\ \qquad\qquad \text{if}\,\lambda_i'+\lambda_j'=\lambda_m'+\lambda_{\tilde{m}}'\,\text{for some}\,m\in\mathscr{T}_{N,M}^{[i,j]},\\ D_{ij}^{-}E(m^*+1,\ldots,\tilde{m^*}),\quad \text{if}\,\lambda_i'+\lambda_j'\neq\lambda_m'+\lambda_{\tilde{m}}'\,\text{for all}\,m\in\mathscr{T}_{N,M}^{[i,j]},\end{cases} \tag{4.8}$$

where m^* is defined as

$$m^*=\begin{cases}0,\quad \text{if}\,\lambda_i'+\lambda_j'<\lambda_1'+\lambda_{N+1}',\\ m,\quad \text{if}\,\lambda_i'+\lambda_j'>\lambda_m'+\lambda_{\tilde{m}}'\,\text{and}\,\lambda_i'+\lambda_j'\\ \qquad <\lambda_{m+1}'+\lambda_{\tilde{m}+1}'\text{for some}\,m\in[M],\\ M+1,\quad \text{if}\,\lambda_i'+\lambda_j'>\lambda_{M+1}'+\lambda_{K'}'.\end{cases} \tag{4.9}$$

Proof. This proposition can be proved in the way of Lemma 3.4 in Ref. [32]. □

Remark 4.1.

(i) For any given $i,j\in[K']$ with $i<j$, the function $\tau_{N,M}^{\diamond}$ has the same asymptotic behavior along the lines $\mathscr{L}_{ij}$ and $\mathscr{L}_{n\bar{n}}$ as $t\to\infty$ if $\lambda_i'+\lambda_j'=\lambda_n'+\lambda_{\bar{n}}'$ for some $n\in\mathscr{S}_{N,M}^{[i,j]}$, and has the same asymptotic

behavior along the lines $\mathscr{L}_{ij}$ and $\mathscr{L}_{m\tilde{m}}$ as $t \to -\infty$ if $\lambda_i' + \lambda_j' = \lambda_m' + \lambda_{\tilde{m}}'$ for some $m \in \mathscr{T}_{N,M}^{[i,j]}$. By virtue of Eqs. (4.3a) and (4.3b), we have

$$(\ln \tau_{N,M}^{\diamond})_x \sim u_{N,M}^{[n,\bar{n}]}, \quad (\ln \tau_{N,M}^{\diamond})_{xx} \sim v_{N,M}^{[n,\bar{n}]}, \quad \text{as } t \to \infty, \tag{4.10a}$$

$$(\ln \tau_{N,M}^{\diamond})_x \sim u_{N,M}^{[m,\tilde{m}]}, \quad (\ln \tau_{N,M}^{\diamond})_{xx} \sim v_{N,M}^{[m,\tilde{m}]}, \quad \text{as } t \to -\infty, \tag{4.10b}$$

which shows that the $t \to \infty$ and $t \to -\infty$ asymptotic solitons can be labeled by the index pairs $[n,\bar{n}]$ $(1 \leqslant n \leqslant N)$ and $[m,\tilde{m}]$ $(1 \leqslant m \leqslant M+1)$, respectively.

(ii) If $\lambda_i' + \lambda_j' \neq \lambda_n' + \lambda_{\bar{n}}'$ for all $n \in \mathscr{S}_{N,M}^{[i,j]}$ and $\lambda_i' + \lambda_j' \neq \lambda_m' + \lambda_{\tilde{m}}'$ for all $m \in \mathscr{T}_{N,M}^{[i,j]}$, there is only one single dominant exponential along the line $\mathscr{L}_{ij}$ as $|t| \to \infty$. Then, we have

$$(\ln \tau_{N,M}^{\diamond})_x \sim \Delta_{N,M}^{[i,j]} := 2\left(\sum_{i=1}^{n^*} \lambda_i' + \sum_{i=\bar{n}^*+1}^{K'} \lambda_i'\right), \quad (\ln \tau_{N,M}^{\diamond})_{xx} \sim 0, \quad \text{as } t \to \infty, \tag{4.11}$$

$$(\ln \tau_{N,M}^{\diamond})_x \sim \Sigma_{N,M}^{[i,j]} := 2\sum_{i=m^*+1}^{\tilde{m}^*} \lambda_i', \quad (\ln \tau_{N,M}^{\diamond})_{xx} \sim 0, \quad \text{as } t \to -\infty. \tag{4.12}$$

As a consequence, we present the following proposition to characterize the asymptotic solitons generated by the function $\tau_{N,M}^{\diamond}$.

Proposition 4.3. *[32] With Condition (3.7) satisfied, the function $\tau_{N,M}^{\diamond}$ (or equivalently $\tau_{N,M}$) can generate an $(M+1,N)$-soliton configuration, in which there are N asymptotic solitons as $t \to \infty$, and each one is uniquely identified by an index pair $[n,\bar{n}]$ for $n \in [N]$; and there are $M+1$ asymptotic solitons as $t \to -\infty$, and each one is uniquely identified by an index pair $[m,\tilde{m}]$ for $m \in [M+1]$.*

Proof. The proof for this proposition can be finished in the way of Proposition 3.6 in Ref. [32]. □

Remark 4.2.

(i) According to the results in Ref. [32], we know that if $p_0 = 0$, the function $\tau_{N,M+1} = \tau_{N,M+1}^{\mathrm{I}}$ also generates an $(M+1,N)$-soliton configuration. It is natural to ask if $\tau_{N,M}$ is just a special case of $\tau_{N,M+1}^{\mathrm{I}}$ when some λ_k $(1 \leqslant k \leqslant K')$ is chosen as 0. By comparison, it can be found that $p_0 = 2$ is a necessary condition for the equivalence of $\tau_{N,M+1}^{\mathrm{I}}$ to $\tau_{N,M}$. Hence, $\tau_{N,M}$ is in general different from $\tau_{N,M+1}^{\mathrm{I}}$ due to the arbitrariness of p_0.

(ii) Because $\beta_n \neq 0$ for all $n \in [K]$, the function $\tau_{N,M}^{\diamond}$ contains all possible $\binom{K'}{N}$ terms. The multi-soliton solutions generated by $\tau_{N,M}^{\diamond}$ are of the fully-resonant type, since they are similar to the "T-type" soliton solutions to the KPII equation [5, 7]. In a like manner, we obtain that the function $\tau_{N+1,M-1}^{\diamond}$ can generate a fully-resonant $(M,N+1)$-soliton configuration with $N+1$ asymptotic solitons labeled by $[n,\bar{n}]$ $(1 \leqslant n \leqslant N+1)$ as $t \to \infty$ and M asymptotic solitons labeled by $[m,\tilde{m}]$ $(1 \leqslant m \leqslant M)$ as $t \to -\infty$.

To this stage, we have the following theorem which gives a complete characterization of the multi-soliton configurations for Solutions (2.21a) and (2.21b).

Theorem 4.1. *With Condition (3.7) satisfied, Solutions (2.21a) and (2.21b) are the linear superposition of one fully-resonant* $(M+1,N)$*-soliton configuration:*

$$(u,v) \sim \begin{cases} \left(2\gamma u_{N,M}^{[n,\bar{n}]} - 2\gamma \Delta_{N+1,M-1}^{[n,\bar{n}]}, 2(\gamma^2-\beta\gamma) v_{N,M}^{[n,\bar{n}]}\right) & \text{for } n \in [N] \text{ as } t \to \infty, \\ \left(2\gamma u_{N,M}^{[m,\tilde{m}]} - 2\gamma \Sigma_{N+1,M-1}^{[m,\tilde{m}]}, 2(\gamma^2-\beta\gamma) v_{N,M}^{[m,\tilde{m}]}\right) & \text{for } m \in [M+1] \text{ as } t \to -\infty, \end{cases} \tag{4.13}$$

and one fully-resonant $(M,N+1)$*-soliton configuration:*

$$(u,v) \sim \begin{cases} \left(2\gamma \Delta_{N,M}^{[n,\bar{n}-1]} - 2\gamma u_{N+1,M-1}^{[n,\bar{n}-1]}, 2(\beta\gamma+\gamma^2) v_{N+1,M-1}^{[n,\bar{n}-1]}\right) & \text{for } n \in [N+1] \text{ as } t \to \infty, \\ \left(2\gamma \Sigma_{N,M}^{[m,\tilde{m}+1]} - 2\gamma u_{N+1,M-1}^{[m,\tilde{m}+1]}, 2(\beta\gamma+\gamma^2) v_{N+1,M-1}^{[m,\tilde{m}+1]}\right) & \text{for } m \in [M] \text{ as } t \to -\infty, \end{cases} \tag{4.14}$$

where $u_{N,M}^{[i,j]}$, $u_{N+1,M-1}^{[i,j]}$, $v_{N,M}^{[i,j]}$, $v_{N+1,M-1}^{[i,j]}$, $\Delta_{N+1,M-1}^{[i,j]}$, $\Delta_{N,M}^{[i,j]}$, $\Sigma_{N+1,M-1}^{[i,j]}$ *and* $\Sigma_{N,M}^{[i,j]}$ *are defined by Eqs. (4.3a), (4.3b), (4.11) and (4.12), respectively.*

Based on Propositions 4.1 and 4.3 as well as Theorem 4.1, for any given values of N, M, $\lambda_1,\ldots,\lambda_K$, we can obtain the following general procedure on how to determine the asymptotic solitons and dominant exponentials in Solutions (2.21a) and (2.21b).

Procedure 4.2.

(i) According to the ordering $\lambda_1 < \cdots < \lambda_{k-1} < 0 < \lambda_k < \cdots < \lambda_K$, redefine the parameters $\{\lambda_n'\}_{n=1}^{K'}$, $\{\beta_n'\}_{n=1}^{K'}$ and $\{\delta_n'\}_{n=1}^{K'}$ in the way of Eqs. (4.1a) and (4.1b).

(ii) Determine the index pairs labeling the asymptotic solitons which are, respectively, generated by $\tau_{N,M}^{\diamond}$ and $\tau_{N+1,M-1}^{\diamond}$, and obtain their asymptotic expressions by Eqs. (4.13) and (4.14).

(iii) Due to the ordering $\lambda_1' < \cdots < \lambda_k' = 0 < \lambda_{k+1}' < \cdots < \lambda_{K'}'$ and the difference $\theta_i' - \theta_j' = (\lambda_i' - \lambda_j')x + 2\gamma(\lambda_i'^2 - \lambda_j'^2)t + \delta_i' - \delta_j'$, the dominant exponential of $\tau_{N,M}^{\diamond}$ (or $\tau_{N+1,M-1}^{\diamond}$) is $E(1,\ldots,N)$ [or $E(1,\ldots,N+1)$] as $x \to -\infty$ for finite values of t.

(iv) Note that the soliton velocity increases from the negative x-axis to the positive x-axis for $t > 0$, and decreases from the negative x-axis to the positive x-axis for $t < 0$. Since the solitons are sorted according to their velocities $s_{ij} = -2\gamma(\lambda_i' + \lambda_j')$, the dominant exponentials for the function $\tau_{N,M}^{\diamond}$ (or $\tau_{N+1,M-1}^{\diamond}$) in the xt plane can be determined by Proposition 4.1 starting from the dominant exponential $E(1,\ldots,N)$ [or $E(1,\ldots,N+1)$] as $x \to -\infty$.

5. Examples of soliton interactions

Based on the above analysis in Section 4, Solutions (2.21a) and (2.21b) can exhibit the linear superposition of one fully-resonant $(M+1,N)$-soliton configuration and one fully-resonant $(M,N+1)$-soliton configuration, which are generated by the functions $\tau_{N,M}^{\diamond}$ and $\tau_{N+1,M-1}^{\diamond}$, respectively. Since the case when $p_0 = 0$ has been discussed detailedly in Ref. [32], we will graphically demonstrate some examples of soliton interactions in Solutions (2.21a) and (2.21b) with $p_0 \neq 0$. For illustrative purpose, we plot the contour lines only for the field v to show the temporal-spatial structures of soliton interactions, and use "$*$" and "$\circ$" to distinguish the index pairs labeling the asymptotic solitons generated by $\tau_{N,M}^{\diamond}$ and $\tau_{N+1,M-1}^{\diamond}$, respectively.

5.1. $(2,1)$*- and* $(1,2)$*-soliton resonant interactions*

We first consider $N = 1, M = 1$ and particularly take $\alpha = 0$. Because $\gamma = \sqrt{\alpha + \beta^2}$, the soliton configuration in the field v depends only on the function $\tau_{1,1}^{\diamond}$ when $\beta < 0$ ($\beta\gamma + \gamma^2 = 0$) and on $\tau_{2,0}^{\diamond}$

when $\beta > 0$ ($\beta\gamma - \gamma^2 = 0$). Theorem 4.1 shows that there are one asymptotic soliton $[1,3]^*$ as $t \to \infty$ and two asymptotic solitons $[1,2]^*$ and $[2,3]^*$ as $t \to -\infty$ for $\beta < 0$, and there are two asymptotic solitons $[1,2]^\circ$ and $[2,3]^\circ$ as $t \to \infty$ and one asymptotic soliton $[1,3]^\circ$ as $t \to -\infty$ for $\beta > 0$. In both the two cases, the solution v displays a three-soliton resonance triad because their wave numbers and frequencies obey the resonant conditions:

$$K_{[1,2]} + K_{[2,3]} = K_{[1,3]}, \quad \Omega_{[1,2]} + \Omega_{[2,3]} = \Omega_{[1,3]}. \tag{5.1}$$

For example, with $\lambda_1 < 0 < \lambda_2$ and $\beta < 0$, from Eq. (4.4) the wave numbers and frequencies of three asymptotic solitons can be given as follows:

$$\begin{aligned} &K_{[1,2]^*} = \lambda_1, \quad K_{[2,3]^*} = -\lambda_2, \quad K_{[1,3]^*} = \lambda_1 - \lambda_2, \\ &\Omega_{[1,2]^*} = 2\gamma\lambda_1^2, \quad \Omega_{[2,3]^*} = -2\gamma\lambda_2^2, \quad \Omega_{[1,3]^*} = 2\gamma(\lambda_1^2 - \lambda_2^2), \end{aligned}$$

which implies that Condition (5.1) is exactly satisfied.

Based on Proposition 4.1, we can further identify the dominant exponentials of $\tau_{1,1}^{\diamond}$ and $\tau_{2,0}^{\diamond}$ in different regions of the *xt* plane. For $\tau_{1,1}^{\diamond}$, we first know that $E(1)$ is its dominant exponential as $x \to -\infty$ for finite t. Second, when $t > 0$ there is only one asymptotic soliton $[1,3]$, which corresponds to the dominant exponential transition $E(1) \xrightarrow{[1,3]^*} E(3)$. Third, when $t < 0$ the asymptotic solitons are sorted counterclockwise as $[1,2]^*$ and $[2,3]^*$. Thus, the dominant exponential transitions as $t \to -\infty$ can be given by $E(1) \xrightarrow{[1,2]^*} E(2) \xrightarrow{[2,3]^*} E(3)$. For $\tau_{2,0}^{\diamond}$, starting from the dominant exponential $E(1,2)$ as $x \to -\infty$, we can work out the transitions of dominant exponentials as $E(1,2) \xrightarrow{[2,3]^\circ} E(1,3) \xrightarrow{[1,2]^\circ} E(2,3)$ for $t > 0$ and $E(1,2) \xrightarrow{[1,3]^\circ} E(2,3)$ for $t < 0$. In Figs. 1 and 2, we present the (2,1)- and (1,2)-soliton resonant interactions, and mark the index pairs labeling asymptotic solitons and the dominant exponentials in three different regions of the *xt* plane.

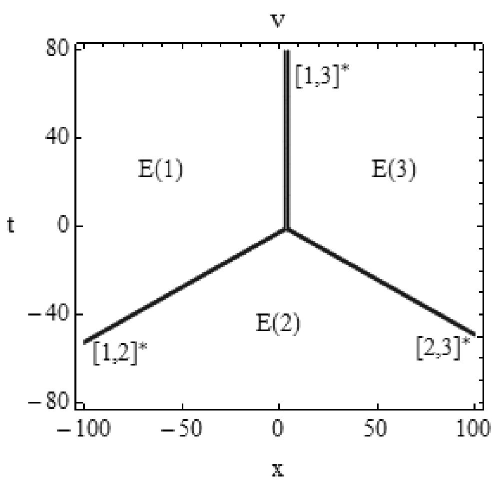

Fig. 1. $(2,1)$-soliton resonant interaction with $N = 1$, $M = 1$, $\beta_1 = 1$, $\beta_2 = -1$, $\alpha = 0$, $\beta = -1$, $\lambda_1 = -1$, $\lambda_2 = 1$, $\delta_1 = 5$, $\delta_2 = -2$ and $p_0 = 1$.

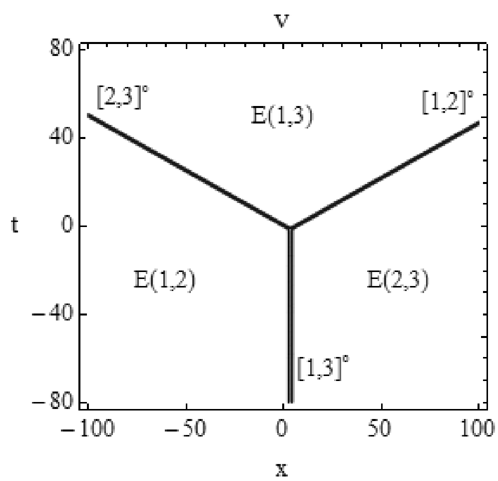

Fig. 2. $(1,2)$-soliton resonant interaction with $N = 1$, $M = 1$, $\beta_1 = 1$, $\beta_2 = -1$, $\alpha = 0$, $\beta = 1$, $\lambda_1 = -1$, $\lambda_2 = 1$, $\delta_1 = 5$, $\delta_2 = -2$ and $p_0 = 1$.

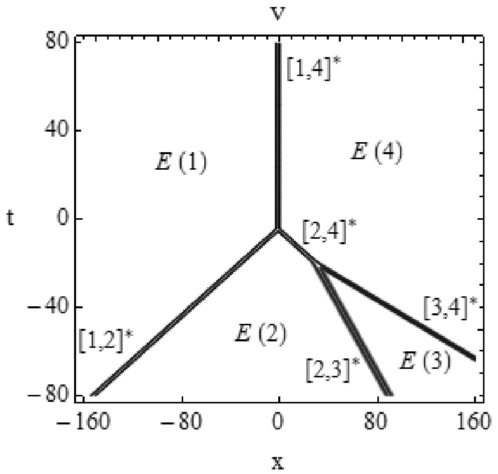

Fig. 3. $(3,1)$-soliton resonant interaction with $N=1$, $M=2$, $\beta_1=-1$, $\beta_2=1$, $\beta_3=-1$, $\alpha=0$, $\beta=-1$, $\lambda_1=-1$, $\lambda_2=0.5$, $\lambda_3=1$, $\delta_1=8$, $\delta_2=-5$, $\delta_3=10$ and $p_0=-1$.

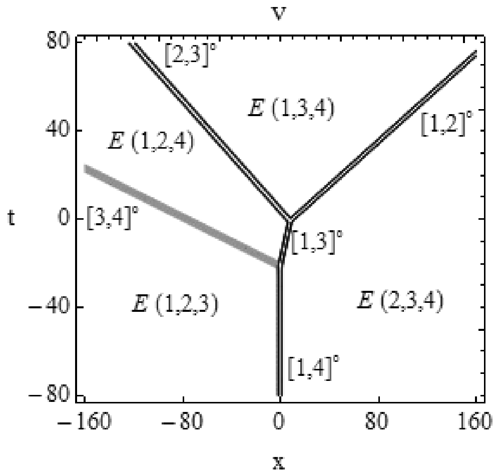

Fig. 4. $(1,3)$-soliton resonant interaction with $N=2$, $M=1$, $\beta_1=-1$, $\beta_2=1$, $\beta_3=-1$, $\alpha=0$, $\beta=1$, $\lambda_1=-1$, $\lambda_2=0.8$, $\lambda_3=1$, $\delta_1=8$, $\delta_2=-5$, $\delta_3=10$ and $p_0=-1$.

5.2. $(3,1)$*- and* $(1,3)$*-soliton resonant interactions*

Considering $N=1$, $M=2$ and $\alpha=0$, $\beta<0$, from Theorem 4.1 we can obtain one asymptotic soliton $[1,4]^*$ as $t\to\infty$ and three asymptotic solitons $[1,2]^*$, $[2,3]^*$ and $[3,4]^*$ as $t\to-\infty$ generated by $\tau^{\diamond}_{1,2}$. In this case, the dominant exponentials of $\tau^{\diamond}_{1,2}$ are determined as follows: First, $E(1)$ is the dominant exponential of $\tau^{\diamond}_{1,2}$ as $x\to-\infty$ for finite t. Second, for $t>0$ $E(1)$ becomes $E(4)$ across the transition line $[1,4]^*$, i.e., $E(1)\xrightarrow{[1,4]^*}E(4)$; while for $t<0$ the dominant exponential transitions occurs $E(1)\xrightarrow{[1,2]^*}E(2)\xrightarrow{[2,3]^*}E(3)\xrightarrow{[3,4]^*}E(4)$, which means that the three solitons are distributed counterclockwise as $[1,2]^*$, $[2,3]^*$ and $[3,4]^*$. In illustration, Fig. 3 presents a $(3,1)$-soliton interaction structure, from which one can find that there are two resonant interactions in the near-field regions, that is, the solitons $[2,3]^*$ and $[3,4]^*$ coalesce into an intermediate soliton $[2,4]^*$ via a three-soliton resonance, and then the solitons $[1,2]^*$ and $[1,4]^*$ interact with the intermediate soliton $[2,4]^*$ to form another three-soliton resonance triad.

If taking $N=2$, $M=1$ and $\alpha=0$, $\beta>0$, the solution v shows the $(1,3)$-soliton interaction, in which there are three asymptotic solitons $[1,2]^{\circ}$, $[2,3]^{\circ}$ and $[3,4]^{\circ}$ as $t\to\infty$ and one soliton $[1,4]^{\circ}$ as $t\to-\infty$. Similarly, the dominant exponential transitions of $\tau^{\diamond}_{3,0}$ can be obtained as $E(1,2,3)\xrightarrow{[3,4]^{\circ}}E(1,2,4)\xrightarrow{[2,3]^{\circ}}E(1,3,4)\xrightarrow{[1,2]^{\circ}}E(2,3,4)$ for $t>0$ and $E(1,2,3)\xrightarrow{[1,4]^{\circ}}E(2,3,4)$ for $t<0$. For this case, there are also two resonant interactions in the near-field regions, as shown in Fig. 4.

5.3. $(2,2)$*-soliton resonant interactions*

For $N=2$, $M=1$, $\alpha=0$ and $\beta<0$, Theorem 4.1 tells us that there are both two asymptotic solitons as $t\to\pm\infty$ generated by the function $\tau^{\diamond}_{2,1}$ and they are labeled by the same two index pairs $[1,3]^*$ and $[2,4]^*$. In this case, $E(1,2)$ is the dominant exponential as $x\to-\infty$ for finite t. The transitions of dominant exponentials take place in the way: $E(1,2)\xrightarrow{[2,4]^*}E(1,4)\xrightarrow{[1,3]^*}E(3,4)$ for $t>0$ and $E(1,2)\xrightarrow{[1,3]^*}E(2,3)\xrightarrow{[2,4]^*}E(3,4)$ for $t<0$. As displayed in Fig. 5, each asymptotic soliton interacts

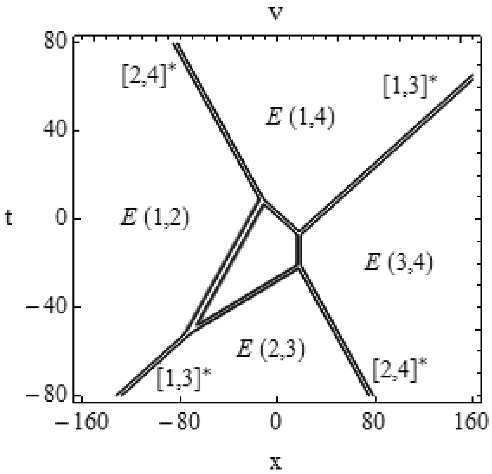

Fig. 5. $(2,2)$-soliton resonant interaction with phase shift, where $N=2$, $M=1$, $\beta_1=-1$, $\beta_2=1$, $\beta_3=-1$, $\alpha=0$, $\beta=-1$, $\lambda_1=-1$, $\lambda_2=-0.5$, $\lambda_3=1$, $\delta_1=30$, $\delta_2=-10$, $\delta_3=-5$ and $p_0=-1$.

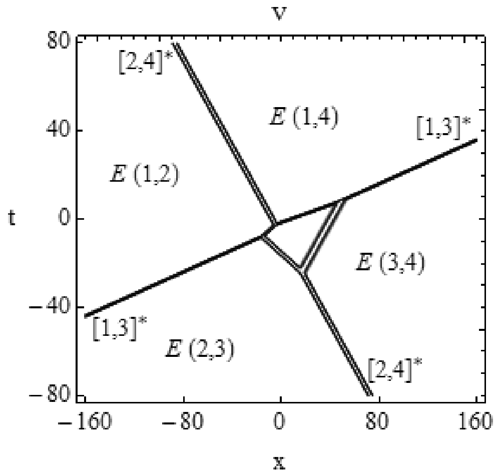

Fig. 6. $(2,2)$-soliton resonant interaction without phase shift, where $N=2$, $M=1$, $\beta_1=-1$, $\beta_2=1$, $\beta_3=-1$, $\alpha=0$, $\beta=-1$, $\lambda_1=-2$, $\lambda_2=-0.5$, $\lambda_3=1$, $\delta_1=30$, $\delta_2=20$, $\delta_3=30$ and $p_0=-1$.

with two intermediate solitons and there are in total four three-soliton resonance triads in the near-field regions. On the other hand, from the far-field view of point the two-soliton interaction obeys the usual elastic properties, that is, the solitons $[1,3]^*$ and $[2,4]^*$ retain their individual velocities and amplitudes after interaction. However, the two interacting solitons may not experience the phase shift upon their interactions. For example, with $\lambda_1 < 0 < \lambda_2 < \lambda_3$, the phase shift of the solitons $[1,3]^*$ and $[2,4]^*$ can be given by

$$\delta\phi_{[1,3]^*} = -\delta\phi_{[2,4]^*} = \ln\left|\frac{\lambda_1(\lambda_3-\lambda_2)}{\lambda_2(\lambda_3-\lambda_1)}\right|. \tag{5.2}$$

Here, if imposing $|\lambda_1(\lambda_3-\lambda_2)| = |\lambda_2(\lambda_3-\lambda_1)|$ (e.g., $\lambda_1=-1, \lambda_2=\frac{1}{2}, \lambda_3=2$), the phases of the solitons $[1,3]^*$ and $[2,4]^*$ keep unchanged when $t\to\pm\infty$ (see Fig. 6).

In addition, we note that for $N=1$, $M=2$, $\alpha=0$ and $\beta>0$, the solution v also describes the $(2,2)$-soliton resonant interactions between the $[1,3]^\circ$ and $[2,4]^\circ$ and they enjoy the same interaction properties.

5.4. $(3,2)$*- and* $(2,3)$*-soliton resonant interactions*

With $N=2, M=2$, $\alpha=0$ and $\beta<0$, the solution v displays the $(3,2)$-soliton interaction, in which there are two asymptotic solitons $[1,4]^*$ and $[2,5]^*$ as $t\to\infty$ and three asymptotic solitons $[1,3]^*$, $[2,4]^*$ and $[3,5]^*$ as $t\to-\infty$. In this case, $E(1,2)$ is the dominant exponential of $\tau^\diamond_{2,2}$ as $x\to-\infty$ for finite t. The asymptotic solitons are sorted clockwise as $[2,5]^*$ and $[1,4]^*$ for $t>0$, and are sorted counterclockwise as $[1,3]^*$, $[2,4]^*$ and $[3,5]^*$ for $t<0$. Accordingly, starting from $E(1,2)$, the transitions of dominant exponentials occur in the way $E(1,2) \xrightarrow{[2,5]^*} E(1,5) \xrightarrow{[1,4]^*} E(4,5)$ when $t>0$, and $E(1,2) \xrightarrow{[1,3]^*} E(2,3) \xrightarrow{[2,4]^*} E(3,4) \xrightarrow{[3,5]^*} E(4,5)$ when $t<0$. As shown in Fig. 7, the solution also shows the fully-resonant feature in the near-field regions, and there are in total seven three-soliton resonance triads.

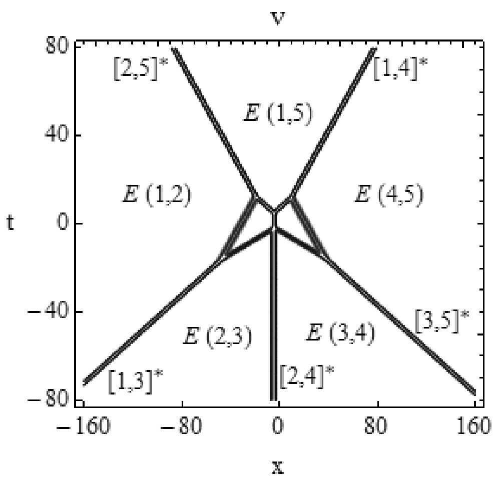

Fig. 7. $(3,2)$-soliton resonant interaction with $N=2$, $M=2$, $\beta_1=-1$, $\beta_2=1$, $\beta_3=-1$, $\beta_4=1$, $\alpha=0$, $\beta=-1$, $\lambda_1=-1$, $\lambda_2=-0.5$, $\lambda_3=0.5$, $\lambda_4=1$, $\delta_1=-15$, $\delta_2=-15$, $\delta_3=-10$, $\delta_4=-6$ and $p_0=1$

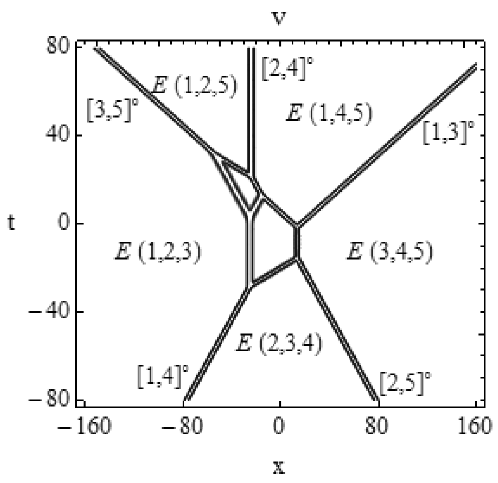

Fig. 8. $(2,3)$-soliton resonant interaction with $N=2$, $M=2$, $\beta_1=-1$, $\beta_2=1$, $\beta_3=-1$, $\beta_4=1$, $\alpha=0$, $\beta=1$, $\lambda_1=-1$, $\lambda_2=-0.5$, $\lambda_3=0.5$, $\lambda_4=1$, $\delta_1=-15$, $\delta_2=-14$, $\delta_3=10$, $\delta_4=-11$ and $p_0=1$.

With $\beta>0$, the function $\tau^{\diamond}_{3,1}$ generates the $(2,3)$-soliton resonant interaction. As illustrated in Fig. 8, the three asymptotic solitons as $t\to\infty$ are sorted clockwise as $[3,5]^\circ$, $[2,4]^\circ$ and $[1,3]^\circ$, and the two solitons as $t\to-\infty$ are sorted counterclockwise as $[1,4]^\circ$ and $[2,5]^\circ$. The dominant exponentials can be determined by the transitions $E(1,2,3)\xrightarrow{[3,5]^\circ}E(1,2,5)\xrightarrow{[2,4]^\circ}E(1,4,5)\xrightarrow{[1,3]^\circ}E(3,4,5)$ for $t>0$, and by the transitions $E(1,2,3)\xrightarrow{[1,4]^\circ}E(2,3,4)\xrightarrow{[2,5]^\circ}E(3,4,5)$ for $t<0$.

5.5. *Linear superposition of two resonant multi-soliton interactions*

With the parameter $\alpha\neq0$, the solution v can exhibit various linear combinations of two fully-resonant soliton interactions which are, respectively, generated by $\tau^{\diamond}_{N,M}$ and $\tau^{\diamond}_{N+1,M-1}$. With $N=M=1$, Fig. 9 shows a linearly-superposed soliton structure which includes one confluent soliton resonance of $[1,2]^*$ and $[2,3]^*$ merging into $[1,3]^*$ generated by $\tau^{\diamond}_{1,1}$, and one divergent soliton resonance of $[1,3]^\circ$ splitting into $[1,2]^\circ$ and $[2,3]^\circ$ generated by $\tau^{\diamond}_{2,0}$. With $N=1$ and $M=2$, Fig. 10 describes a linearly-superposed soliton structure which comprises of one $(3,1)$-soliton resonance of $[1,2]^*$, $[2,3]^*$, $[3,4]^*$ merging into $[1,4]^*$ generated by $\tau^{\diamond}_{1,2}$, and one $(2,2)$-soliton interaction between $[1,3]^\circ$ and $[2,4]^\circ$ generated by $\tau^{\diamond}_{2,1}$. For $N=2$ and $M=1$, Fig. 11 presents a linearly-superposed soliton structure which is the linear combination of one $(2,2)$-soliton interaction between $[1,3]^*$ and $[2,4]^*$ generated by $\tau^{\diamond}_{2,1}$, and one divergent resonance of $[1,4]^\circ$ splitting into $[1,2]^\circ$, $[2,3]^\circ$ and $[3,4]^\circ$ generated by $\tau^{\diamond}_{3,0}$. If choosing $N=2$ and $M=2$, in Fig. 12 we illustrate a linearly-superposed soliton structure which consists of one confluent resonance of $[1,3]^*$, $[2,4]^*$ and $[3,5]^*$ merging into $[1,4]^*$ and $[2,5]^*$ generated by $\tau^{\diamond}_{2,2}$, and one divergent resonance of $[1,4]^\circ$ and $[2,5]^\circ$ splitting into $[1,3]^\circ$, $[2,4]^\circ$ and $[3,5]^\circ$ generated by $\tau^{\diamond}_{3,1}$.

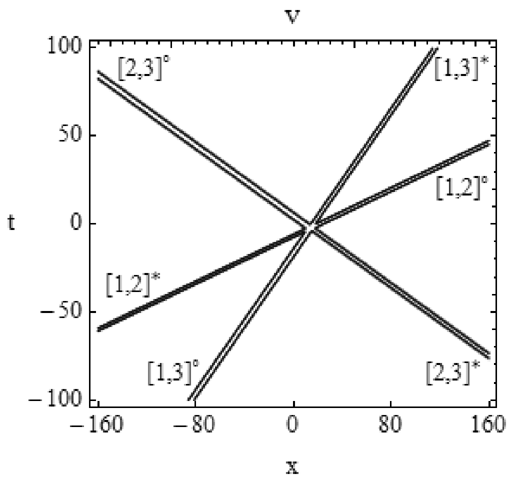

Fig. 9. Linear superposition of $(2,1)$ and $(1,2)$-soliton resonant interactions with $N = 1$, $M = 1$, $\beta_1 = 1$, $\beta_2 = -1$, $\alpha = 3$, $\beta = 1$, $\lambda_1 = -0.75$, $\lambda_2 = 0.5$, $\delta_1 = 15$, $\delta_2 = -5$ and $p_0 = 1$.

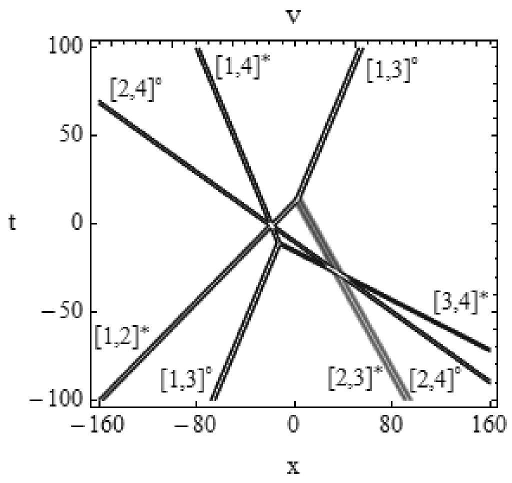

Fig. 10. Linear superposition of $(3,1)$ and $(2,2)$-soliton resonant interactions, where $N = 1$, $M = 2$, $\beta_1 = -1$, $\beta_2 = 1$, $\beta_3 = -1$, $\alpha = 3$, $\beta = 0.1$, $\lambda_1 = -0.5$, $\lambda_2 = 0.3$, $\lambda_3 = 0.7$, $\delta_1 = -11$, $\delta_2 = -5$, $\delta_3 = 10$ and $p_0 = -1$.

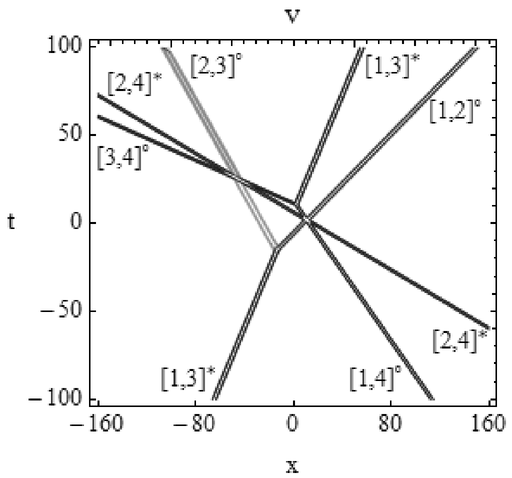

Fig. 11. Linear superposition of $(2,2)$ and $(1,3)$-soliton resonant interactions, where $N = 2$, $M = 1$, $\beta_1 = -1$, $\beta_2 = 1$, $\beta_3 = -1$, $\alpha = 1$, $\beta = 0.1$, $\lambda_1 = -0.5$, $\lambda_2 = 0.3$, $\lambda_3 = 1$, $\delta_1 = 15$, $\delta_2 = 10$, $\delta_3 = -12$ and $p_0 = -1$.

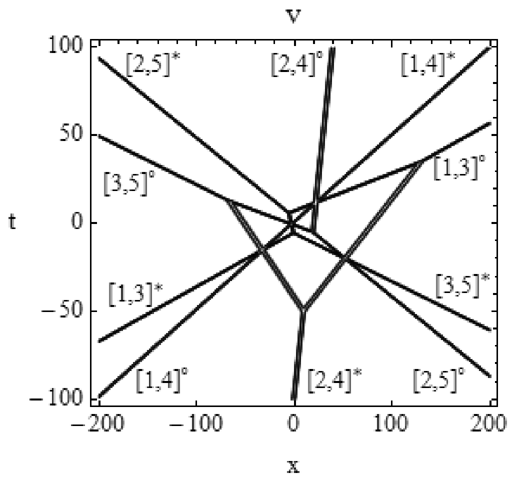

Fig. 12. Linear superposition of $(3,2)$ and $(2,3)$-soliton resonant interactions with $N = 2$, $M = 2$, $\beta_1 = -1$, $\beta_2 = 1$, $\beta_3 = -1$, $\beta_4 = 1$, $\alpha = 1$, $\beta = 1$, $\lambda_1 = -0.5$, $\lambda_2 = -0.25$, $\lambda_3 = 0.45$, $\lambda_4 = 0.85$, $\delta_1 = -8$, $\delta_2 = -11$, $\delta_3 = -20$, $\delta_4 = -12$ and $p_0 = 1$.

6. Conclusions

In this paper, we have constructed new double Wronskian solutions for the WBK system, and have revealed the asymptotic behavior and soliton interaction properties of the multi-soliton solutions. First, based on the transformation (2.1) from the WBK system to the AKNS system, we have derived new determinant solutions (2.21a) and (2.21b) in terms of the (N,M)-component double

Wronskians in Eqs. (2.15a)–(2.15c) by using the DT method together with the Wronskian technique. Such new double Wronskian solutions are more general than those obtained in Refs. [20, 32] because an additional free parameter is included. Second, by analyzing the algebraic properties of new double Wronskians, we have given the parametric condition for the non-singular, non-trivial and irreducible soliton solutions obtained from Eqs. (2.21a) and (2.21b). Third, via the asymptotic analysis method [5, 7], we have studied the asymptotic behavior of the double Wronskian functions as $t \to \pm\infty$, and have identified the index pairs to label all the asymptotic solitons of the non-singular, non-trivial and irreducible soliton solutions (Proposition 4.3). It has been shown that the solutions are linearly superposed of the fully-resonant $(M+1,N)$- and $(M,N+1)$-soliton configurations (Theorem 4.1), in each of which the amplitudes, velocities and numbers of asymptotic solitons are in general not equal as $t \to \pm\infty$. Also, we have presented an algebraic procedure to determine the asymptotic solitons and dominant exponentials for any given multi-soliton solution. Fourth, with some specific values of the involved parameters we have graphically demonstrated the $(2,1)$-, $(1,2)$-, $(3,1)$-, $(1,3)$-, $(2,2)$-, $(3,2)$-, $(2,3)$-soliton resonant interactions (Figs. 1–8) as well as some complex soliton structures linearly superposed of two resonant multi-soliton interactions (Figs. 9–12). Finally, we make a comparison of the above obtained results with those in Ref. [32] (see Table 1).

Table 1. Comparison of the results of this paper (the 2nd column) with those in Ref. [32] (the 3rd column).

Seed solutions / Properties	$p = p_0,\ q = 0$	$p = 0,\ q = 0$
Double Wronskians	$\tau_{N,M} = \tau^{\mathrm{I}}_{N,M} + \frac{1}{2}(-1)^{N-1} p_0 \tau^{\mathrm{II}}_{N-1,M+1}$	$\tau_{N,M} = \tau^{\mathrm{I}}_{N,M}$
Nonsingular, and irreducible condition	$\alpha_k\alpha_{k+1}\beta_k\beta_{k+1} \le 0 \ \ (1 \leqslant k \leqslant K-1)$ and $\sigma p_0 \alpha_1 \beta_1 \le 0$ with $\sigma = \mathrm{sign}(\prod_{n=1}^{K} \lambda_i)$	$\alpha_k\alpha_{k+1}\beta_k\beta_{k+1} \le 0 \ \ (1 \leqslant k \leqslant K-1)$
Total number of asymptotic solitons	$N+M+1$	$N+M$
Index pairs labeling the asymptotic solitons	$[n, M+n+1]$, as $t \to \infty$, $[m, N+m]$, as $t \to -\infty$	$[n, M+n]$, as $t \to \infty$, $[m, N+m]$, as $t \to -\infty$

Note: $\tau^{\mathrm{I}}_{N,M}$ and $\tau^{\mathrm{II}}_{N-1,M+1}$ are defined in Eqs. (2.14b) and (2.14c).

Acknowledgments

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Appendix A. Notations for W_x, U_x, V_x, W_{xx}, U_{xx}, V_{xx}, W_t, U_t and V_t.

$$\begin{aligned}W_x =&|\widehat{N-2},N;\widehat{M-1}|+|\widehat{N-1};\widehat{M-2},M|\\&+\frac{1}{2}(-1)^{N-1}p_0(|\overline{N-2},N;\widetilde{M-1}|+|\overline{N-1};\widetilde{M-2},M|),\end{aligned}\tag{A.1}$$

$$\begin{aligned}U_x =&2|\widehat{N-1},N+1;\widehat{M-2}|+2|\widehat{N};\widehat{M-3},M-1|\\&-(-1)^{N-1}p_0(|\overline{N-1},N+1;\widetilde{M-2}|+|\overline{N};\widetilde{M-3},M-1|),\end{aligned}\tag{A.2}$$

$$\begin{aligned}V_x =&2|\widehat{N-3},N-1;\widehat{M}|+2|\widehat{N-2};\widehat{M-1},M+1|\\&-(-1)^{N-1}p_0|\overline{N-3},N-1;\widetilde{M}|-(-1)^{N-1}p_0|\overline{N-2};\widetilde{M-1},M+1|,\end{aligned}\tag{A.3}$$

$$\begin{aligned}W_{xx} =&|\widehat{N-3},N-1,N;\widehat{M-1}|+|\widehat{N-2},N+1;\widehat{M-1}|\\&+2|\widehat{N-2},N;\widehat{M-2},M|+|\widehat{N-1};\widehat{M-2},M+1|\\&+|\widehat{N-1};\widehat{M-3},M-1,M|+\frac{1}{2}(-1)^{N-1}p_0(|\overline{N-3},N-1,N;\widetilde{M-1}|\\&+|\overline{N-2},N+1;\widetilde{M-1}|+2|\overline{N-2},N;\widetilde{M-2},M|\\&+|\overline{N-1};\widetilde{M-3},M-1,M|+|\overline{N-1};\widetilde{M-2},M+1|),\end{aligned}\tag{A.4}$$

$$\begin{aligned}U_{xx} =&2|\widehat{N-2},N,N+1;\widehat{M-2}|+2|\widehat{N-1},N+2;\widehat{M-2}|\\&+4|\widehat{N-1},N+1;\widehat{M-3},M-1|+2|\widehat{N};\widehat{M-4},M-2,M-1|\\&+2|\widehat{N};\widehat{M-3},M|+(-1)^{N-1}p_0(|\overline{N-2},N,N+1;\widetilde{M-2}|\\&+|\overline{N-1},N+2;\widetilde{M-2}|+2|\overline{N-1},N+1;\widetilde{M-3},M-1|\\&+|\overline{N};\widetilde{M-4},M-2,M-1|+|\overline{N};\widetilde{M-3},M|),\end{aligned}\tag{A.5}$$

$$\begin{aligned}V_{xx} =&2|\widehat{N-4},N-2,N-1;\widehat{M}|+2|\widehat{N-3},N;\widehat{M}|\\&+4|\widehat{N-3},N-1;\widehat{M-1},M+1|+2|\widehat{N-2};\widehat{M-2},M,M+1|\\&+2|\widehat{N-2};\widehat{M-1},M+2|-(-1)^{N-1}p_0(|\overline{N-4},N-2,N-1;\widetilde{M-1}|\\&+|\overline{N-3},N;\widetilde{M}|+2|\overline{N-3},N-1;\widetilde{M-1},M+1|\\&+|\overline{N-2};\widetilde{M-2},M,M+1|+|\overline{N-2};\widetilde{M-1},M+2|),\end{aligned}\tag{A.6}$$

$$\begin{aligned}W_t =&2\gamma(-|\widehat{N-3},N-1,N;\widehat{M-1}|+|\widehat{N-2},N+1;\widehat{M-1}|\\&+|\widehat{N-1};\widehat{M-3},M-1,M|-|\widehat{N-1};\widehat{M-2},M+1|)\\&+2\gamma\frac{(-1)^{N-1}}{2}p_0(-|\overline{N-3},N-1,N;\widetilde{M-1}|+|\overline{N-2},N+1;\widetilde{M-1}|\\&+|\overline{N-1};\widetilde{M-3},M-1,M|-|\overline{N-1};\widetilde{M-2},M+1|),\end{aligned}\tag{A.7}$$

$$\begin{aligned}U_t =&2\gamma[-2|\widehat{N-2},N,N+1;\widehat{M-2}|+2|\widehat{N-1},N+2;\widehat{M-2}|\\&+2|\widehat{N};\widehat{M-4},M-2,M-1|-2|\widehat{N};\widehat{M-3},M|\\&+(-1)^{N-1}p_0(-|\overline{N-2},N,N+1;\widetilde{M-2}|+|\overline{N-1},N+2;\widetilde{M-2}|\\&+|\overline{N};\widetilde{M-4},M-2,M-1|-|\overline{N};\widetilde{M-3},M|)],\end{aligned}\tag{A.8}$$

$$\begin{aligned}V_t =&2\gamma[-2|\widehat{N-4},N-2,N-1;\widehat{M}|+2|\widehat{N-3},N;\widehat{M}|\\&+2|\widehat{N-2};\widehat{M-2},M,M+1|-2|\widehat{N-2};\widehat{M-1},M+2|\\&-(-1)^{N-1}p_0(-|\overline{N-4},N-2,N-1;\widetilde{M-1}|+|\overline{N-3},N;\widetilde{M}|\\&+|\overline{N-2};\widetilde{M-2},M,M+1|-|\overline{N-2};\widetilde{M-1},M+2|)].\end{aligned} \tag{A.9}$$

Appendix B. Three lemmas for useful double Wronskian identities.

Lemma B.1. *[10,11] Suppose that Q is an $n\times(n-2)$ matrix, $\mathbf{h}_1$, $\mathbf{h}_2$, $\mathbf{h}_3$ and $\mathbf{h}_4$ are n-dimensional column vectors. Then, we have*

$$|Q,\mathbf{h}_1,\mathbf{h}_2||Q,\mathbf{h}_3,\mathbf{h}_4|-|Q,\mathbf{h}_1,\mathbf{h}_3||Q,\mathbf{h}_2,\mathbf{h}_4|+|Q,\mathbf{h}_1,\mathbf{h}_4||Q,\mathbf{h}_2,\mathbf{h}_3|=0. \tag{B.1}$$

Lemma B.2. *[10, 11] For a matrix $\Pi=(\pi_{ij})_{n\times n}=[\Pi_1,\dots,\Pi_n]$ and a column vector $\Upsilon=(r_1,\dots,r_n)^T$, we have the relation*

$$\sum_{j=1}^{n}\big|\Pi_1,\dots,\Pi_{j-1},\Upsilon\circ\Pi_j,\Pi_{j+1},\dots,\Pi_n\big|=|\Pi|\sum_{j=1}^{n}r_j, \tag{B.2}$$

where $\Upsilon\circ\Pi_j=(r_1\pi_{1j},r_2\pi_{2j},\dots,r_n\pi_{nj})^T$.

As a direct result of Lemma B.2, some useful double Wronskian identities are presented as follows:

$$W_x+W\sum_{n=1}^{K}\lambda_n=2|\widehat{N-2},N;\widehat{M-1}|+(-1)^{N-1}p_0|\overline{N-2},N;\widetilde{M-1}|, \tag{B.3}$$

$$U_x+U\sum_{n=1}^{K}\lambda_n=4|\widehat{N-1},N+1;\widehat{M-2}|-2(-1)^{N-1}p_0|\overline{N-1},N+1;\widetilde{M-2}|, \tag{B.4}$$

$$V_x+V\sum_{n=1}^{K}\lambda_n=4|\widehat{N-3},N-1;\widehat{M}|-2(-1)^{N-1}p_0|\overline{N-3},N-1;\widetilde{M}|, \tag{B.5}$$

$$W_{xx}-W\big(\sum_{n=1}^{K}\lambda_n\big)^2=4|\widehat{N-2},N;\widehat{M-2},M|-2(-1)^{N-1}p_0|\overline{N-2},N;\widetilde{M-2},M|, \tag{B.6}$$

$$\begin{aligned}U_{xx}-U\big(\sum_{n=1}^{K}\lambda_n\big)^2=&8|\widehat{N-1},N+1;\widehat{M-3},M-1|\\&-4(-1)^{N-1}p_0|\overline{N-1},N+1;\widetilde{M-3},M-1|,\end{aligned} \tag{B.7}$$

$$\begin{aligned}V_{xx}-V\big(\sum_{n=1}^{K}\lambda_n\big)^2=&8|\widehat{N-3},N-1;\widehat{M-1},M+1|\\&-4(-1)^{N-1}p_0|\overline{N-3},N-1;\widetilde{M-1},M+1|.\end{aligned} \tag{B.8}$$

Lemma B.3. *Suppose that Q is $n\times(n-3)$ matrix, $\mathbf{h}_1$, $\mathbf{h}_2$, $\mathbf{h}_3$, $\mathbf{h}_4$, $\mathbf{h}_5$ and $\mathbf{h}_6$ are n-dimensional column vectors. Then, we have*

$$|Q,\mathbf{h}_1,\mathbf{h}_3,\mathbf{h}_4||Q,\mathbf{h}_2,\mathbf{h}_5,\mathbf{h}_6|-|Q,\mathbf{h}_1,\mathbf{h}_3,\mathbf{h}_5||Q,\mathbf{h}_2,\mathbf{h}_4,\mathbf{h}_6|$$

$$
\begin{aligned}
&+|Q,\mathbf{h}_1,\mathbf{h}_3,\mathbf{h}_6||Q,\mathbf{h}_2,\mathbf{h}_4,\mathbf{h}_5|+|Q,\mathbf{h}_1,\mathbf{h}_4,\mathbf{h}_5||Q,\mathbf{h}_2,\mathbf{h}_3,\mathbf{h}_6|\\
&-|Q,\mathbf{h}_1,\mathbf{h}_4,\mathbf{h}_6||Q,\mathbf{h}_2,\mathbf{h}_3,\mathbf{h}_5|+|Q,\mathbf{h}_1,\mathbf{h}_5,\mathbf{h}_6||Q,\mathbf{h}_2,\mathbf{h}_3,\mathbf{h}_4|=0.
\end{aligned}
\tag{B.9}
$$

Proof. Let X be the following matrix

$$
X=\begin{pmatrix} Q & \mathbf{h}_1 & \mathbf{0} & \mathbf{0} & \mathbf{h}_3 & \mathbf{h}_4 & \mathbf{h}_5 & \mathbf{h}_6\\ \mathbf{0} & \mathbf{0} & Q & \mathbf{h}_2 & \mathbf{h}_3 & \mathbf{h}_4 & \mathbf{h}_5 & \mathbf{h}_6 \end{pmatrix}.
\tag{B.10}
$$

By virtue of the determinant properties, we have

$$
\det(X)=\begin{vmatrix} Q & \mathbf{h}_1 & \mathbf{0} & \mathbf{0} & \mathbf{h}_3 & \mathbf{h}_4 & \mathbf{h}_5 & \mathbf{h}_6\\ \mathbf{0} & -\mathbf{h}_1 & Q & \mathbf{h}_2 & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \end{vmatrix}=0.
\tag{B.11}
$$

On the other hand, by the Laplace technique one can expand $\det(X)$ as products of the $n\times n$ minors associated with $(Q,\mathbf{h}_1,\mathbf{0},\mathbf{0},\mathbf{h}_3,\mathbf{h}_4,\mathbf{h}_5,\mathbf{h}_6)$ and those of $(\mathbf{0},\mathbf{0},Q,\mathbf{h}_2,\mathbf{h}_3,\mathbf{h}_4,\mathbf{h}_5,\mathbf{h}_6)$, that is,

$$
\begin{aligned}
\det(X)=&|Q,\mathbf{h}_1,\mathbf{h}_3,\mathbf{h}_4||Q,\mathbf{h}_2,\mathbf{h}_5,\mathbf{h}_6|-|Q,\mathbf{h}_1,\mathbf{h}_3,\mathbf{h}_5||Q,\mathbf{h}_2,\mathbf{h}_4,\mathbf{h}_6|\\
&+|Q,\mathbf{h}_1,\mathbf{h}_3,\mathbf{h}_6||Q,\mathbf{h}_2,\mathbf{h}_4,\mathbf{h}_5|+|Q,\mathbf{h}_1,\mathbf{h}_4,\mathbf{h}_5||Q,\mathbf{h}_2,\mathbf{h}_3,\mathbf{h}_6|\\
&-|Q,\mathbf{h}_1,\mathbf{h}_4,\mathbf{h}_6||Q,\mathbf{h}_2,\mathbf{h}_3,\mathbf{h}_5|+|Q,\mathbf{h}_1,\mathbf{h}_5,\mathbf{h}_6||Q,\mathbf{h}_2,\mathbf{h}_3,\mathbf{h}_4|.
\end{aligned}
\tag{B.12}
$$

Hence, combination of Eqs. (B.11) and (B.12) yields the determinant identity in Eq. (B.9). □

Appendix C. Proof of Theorem 2.1.

Proof. Inserting the double Wronskian notations of W_x, U_x, V_x, W_t, U_t, V_t, W_{xx}, U_{xx} and V_{xx} in Eqs. (A.1)–(A.9) into the left-hand sides of Eqs. (2.18)–(2.20) and using the identities (B.3)–(B.8), we obtain

$$
\begin{aligned}
&WW_{xx}-W_x^2-UV\\
&\quad=-4\Big(|\widehat{N};\widehat{M-2}||\widehat{N-2};\widehat{M}|-|\widehat{N-1};\widehat{M-1}||\widehat{N-2},N;\widehat{M-2},M|\\
&\qquad+|\widehat{N-1};\widehat{M-2},M||\widehat{N-2},N;\widehat{M}|\Big)-p_0^2\Big(|\overline{N};\widetilde{M-2}||\overline{N-2};\widetilde{M}|\\
&\qquad-|\overline{N-1};\widetilde{M-1}||\overline{N-2},N;\widetilde{M-2},M|+|\overline{N-1};\widetilde{M-2},M||\overline{N-2},N;\widetilde{M}|\Big)\\
&\qquad+2(-1)^{N-1}p_0\Big(|\widehat{N};\widehat{M-2}||\overline{N-2};\widetilde{M}|+|\widehat{N-1};\widehat{M-1}||\overline{N-2},N;\widetilde{M-2},M|\\
&\qquad-|\widehat{N-1};\widehat{M-2},M||\overline{N-2},N;\widetilde{M-1}|-|\widehat{N-2},N;\widehat{M-1}||\overline{N-1};\widetilde{M-2},M|\\
&\qquad+|\widehat{N-2},N;\widehat{M-2},M||\overline{N-1};\widetilde{M-1}|+|\widehat{N-2};\widehat{M}||\overline{N};\widetilde{M-2}|\Big),
\end{aligned}
\tag{B.1}
$$

$$
\begin{aligned}
&UW_t-WU_t+\gamma(WU_{xx}+UW_{xx}-2W_xU_x)\\
&\quad=8\gamma\Big(|\widehat{N};\widehat{M-2}||\widehat{N-2},N+1;\widehat{M-1}|-|\widehat{N-1},N+1;\widehat{M-2}||\widehat{N-2},N;\widehat{M-1}|\\
&\qquad+|\widehat{N-1};\widehat{M-1}||\widehat{N-2},N,N+1;\widehat{M-2}|\Big)-2\gamma p_0^2\Big(|\overline{N};\widetilde{M-2}||\overline{N-2},N+1;\widetilde{M-1}|\\
&\qquad-|\overline{N-1},N+1;\widetilde{M-2}||\overline{N-2},N;\widetilde{M-1}|+|\overline{N-1};\widetilde{M-1}||\overline{N-2},N,N+1;\widetilde{M-2}|\Big)\\
&\qquad+8\gamma\Big(|\widehat{N};\widehat{M-2}||\widehat{N-1};\widehat{M-3},M-1,M|-|\widehat{N};\widehat{M-3},M-1||\widehat{N-1};\widehat{M-2},M|
\end{aligned}
$$

$$
\begin{aligned}
&+|\widehat{N};\widehat{M-3},M||\widehat{N-1};\widehat{M-1}|\Big)-2\gamma p_0^2\Big(|\overline{N};\widetilde{M-2}||\overline{N-1};\widetilde{M-3},M-1,M|\\
&-|\overline{N};\widetilde{M-3},M-1||\overline{N-1};\widetilde{M-2},M|+|\overline{N};\widetilde{M-3},M||\overline{N-1};\widetilde{M-1}|\Big)+4\gamma(-1)^{N-1}p_0\\
&\times\Big(|\widehat{N};\widehat{M-2}||\overline{N-2},N+1;\widetilde{M-1}|-|\widehat{N-1},N+1;\widehat{M-2}||\overline{N-2},N;\widetilde{M-1}|\\
&-|\widehat{N-1};\widehat{M-1}||\overline{N-2},N,N+1;\widetilde{M-2}|+|\widehat{N-2},N,N+1;\widehat{M-2}||\overline{N-1};\widetilde{M-1}|\\
&+|\widehat{N-2},N;\widehat{M-1}||\overline{N-1},N+1;\widetilde{M-2}|-|\widehat{N-2},N+1;\widehat{M-1}||\overline{N};\widetilde{M-2}|\Big)\\
&+4\gamma(-1)^{N-1}p_0\Big(|\widehat{N};\widehat{M-2}||\overline{N-1};\widetilde{M-3},M-1,M|+|\widehat{N};\widehat{M-3},M||\overline{N-1};\widetilde{M-1}|\\
&-|\widehat{N};\widehat{M-3},M-1||\overline{N-1};\widetilde{M-2},M|-|\widehat{N-1};\widehat{M-1}||\overline{N};\widetilde{M-3},M|\\
&+|\widehat{N-1};\widehat{M-2},M||\overline{N};\widetilde{M-3},M-1|-|\widehat{N-1};\widehat{M-3},M-1,M||\overline{N};\widetilde{M-2}|\Big),
\end{aligned}
\tag{B.2}
$$

$$
\begin{aligned}
&VW_t-WV_t-\gamma(WV_{xx}+VW_{xx}-2W_xV_x)\\
&=-8\gamma\Big(|\widehat{N-1};\widehat{M-1}||\widehat{N-3},N;\widehat{M-1}|-|\widehat{N-2},N,\widehat{M-1}||\widehat{N-3},N-1;\widehat{M}|\\
&+|\widehat{N-2};\widehat{M}||\widehat{N-3},N-1,N;\widehat{M-1}|\Big)-8\gamma\Big(|\widehat{N-1};\widehat{M-1}||\widehat{N-2},\widehat{M-2},M,M+1|\\
&-|\widehat{N-1};\widehat{M-2},M||\widehat{N-2};\widehat{M-1},M+1|+|\widehat{N-1};\widehat{M-2},M+1||\widehat{N-2};\widehat{M}|\Big)\\
&+2\gamma p_0^2\Big(|\overline{N-1};\widetilde{M-1}||\overline{N-3},N;\widetilde{M}|-|\overline{N-2},N;\widetilde{M-1}||\overline{N-3},N-1;\widetilde{M}|\\
&+|\overline{N-2};\widetilde{M}||\overline{N-3},N-1,N;\widetilde{M-1}|\Big)+2\gamma p_0^2\Big(|\overline{N-1};\widetilde{M-2},M+1||\overline{N-2};\widetilde{M}|\\
&-|\overline{N-1};\widetilde{M-2},M||\overline{N-2};\widetilde{M-1},M+1|+|\overline{N-1};\widetilde{M-1}||\overline{N-2};\widetilde{M-2},M,M+1|\Big)\\
&+4\gamma(-1)^{N-1}p_0\Big(|\widehat{N-1};\widehat{M-1}||\overline{N-3},N;\widetilde{M}|-|\widehat{N-2},N;\widehat{M-1}||\overline{N-3},N-1;\widetilde{M}|\\
&-|\widehat{N-2};\widehat{M}||\overline{N-3},N-1,N;\widetilde{M-1}|+|\widehat{N-3},N-1,N;\widehat{M-1}||\overline{N-2};\widetilde{M}|\\
&+|\widehat{N-3},N-1;\widehat{M}||\overline{N-2},N;\widetilde{M-1}|-|\widehat{N-3},N;\widehat{M}||\overline{N-1};\widetilde{M-1}|\Big)\\
&+4\gamma(-1)^{N-1}p_0\Big(|\widehat{N-1};\widehat{M-2},M+1||\overline{N-2};\widetilde{M}|-|\widehat{N-2};\widehat{M}||\overline{N-1};\widetilde{M-2},M+1|\\
&+|\widehat{N-1};\widehat{M-1}||\overline{N-2};\widetilde{M-2},M,M+1|-|\widehat{N-1};\widehat{M-2},M||\overline{N-2};\widetilde{M-1},M+1|\\
&+|\widehat{N-2};\widehat{M-1},M+1||\overline{N-1};\widetilde{M-2},M|-|\widehat{N-2};\widehat{M-2},M,M+1||\overline{N-1};\widetilde{M-1}|\Big).
\end{aligned}
\tag{B.3}
$$

Via the determinant identities (B.1) and (B.9), one can show that the right-hand sides of Eqs. (B.1)–(B.3) are all equal to zero. □

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