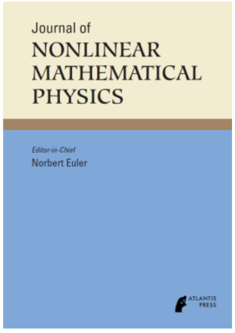

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P. Holba, I.S. Krasil'shchik, O.I. Morozov, P. Vojčák

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P. Holba
Mathematical Institute, Silesian University in Opava,
Na Rybníčku 1, 746 01 Opava, Czech Republic
M160016@math.slu.cz

I.S. Krasil′shchik
Independent University of Moscow, B. Vlasevsky 11, 119002 Moscow, Russia, Russian State University for Humanities, Miusskaya sq. 6, Moscow, GSP-3, 125993, Russia & Trapeznikov Institute of Control Sciences, 65 Profsoyuznaya street, Moscow 117997, Russia
josephkra@gmail.com

O.I. Morozov
Faculty of Applied Mathematics, AGH University of Science and Technology,
Al. Mickiewicza 30, Kraków 30-059, Poland
morozov@agh.edu.pl

P. Vojčák
Mathematical Institute, Silesian University in Opava,
Na Rybníčku 1, 746 01 Opava, Czech Republic
Petr.Vojcak@math.slu.cz

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We consider the 3D equation $u_{yy} = u_{tx} + u_y u_{xx} - u_x u_{xy}$ and its 2D symmetry reductions: (1) $u_{yy} = (u_y + y)u_{xx} - u_x u_{xy} - 2$ (which is equivalent to the Gibbons-Tsarev equation) and (2) $u_{yy} = (u_y + 2x)u_{xx} + (y - u_x)u_{xy} - u_x$. Using the corresponding reductions of the known Lax pair for the 3D equation, we describe nonlocal symmetries of (1) and (2) and show that the Lie algebras of these symmetries are isomorphic to the Witt algebra.

Keywords: Partial differential equations, Lax integrable equations, symmetry reductions, nonlocal symmetries, Gibbons-Tsarev equation.

2010 Mathematics Subject Classification: 35B06

1. Introduction

The equation

$$u_{yy} = u_{tx} + u_y u_{xx} - u_x u_{xy} \tag{1.1}$$

belongs to the class of linearly degenerate integrable equations [6] and was, as far as we know, introduced first in [9] and independently in [5, 10]. In [1], we described all two-dimensional symmetry reductions of this equation. All these reductions are either linearizable or exactly solvable, except for the two ones:

$$u_{yy} = (u_y + y)u_{xx} - u_x u_{xy} - 2 \tag{1.2}$$

and

$$u_{yy} = (u_y + 2x)u_{xx} + (y - u_x)u_{xy} - u_x. \tag{1.3}$$

Equation (1.2) is reduced to the Gibbons-Tsarev equation [7] by the simple transformation $u \mapsto u - y^2/2$.

Equation (1.1) admits the Lax pair

$$\begin{aligned} w_t &= (\lambda^2 - \lambda u_x - u_y)w_x, \\ w_y &= (\lambda - u_x)w_x \end{aligned} \tag{1.4}$$

with the non-removable spectral parameter λ. In [2], we, in particular, studied the behavior of this Lax pair under the symmetry reduction.

In this paper, we use system (1.4) to describe nonlocal symmetries of Equations (1.2) and (1.3) and prove that in both cases these symmetries form the Lie algebra isomorphic to the Witt algebra

$$\mathfrak{W} = \{\, e_i = z^{i+1}\tfrac{\partial}{\partial z} \mid i \in \mathbb{Z} \,\}.$$

In Section 2, we recall some necessary results obtained in the previous research and introduce the notions and constructions needed for the subsequent exposition. Section 3 is devoted to the proofs of the basic results. We discuss in detail Equation (1.2) and briefly repeat the main steps for the second equation, because the reasoning is quite similar in both cases. The obtained results, together with further perspectives, are discussed in Section 4.

2. Preliminaries

We recall here basic facts from the theory of nonlocal symmetries (see [8]) and previous results on Equation (1.1) and its reductions, see [1,2].

2.1. *Basics*

Consider a differential equation[a]

$$F^\alpha\left(x, \dots, \frac{\partial^{|\sigma|}u}{\partial x^\sigma}, \dots\right) = 0, \qquad \alpha = 1, \dots, r, \tag{2.1}$$

of order k in unknowns $u = (u^1, \dots, u^m)$, where $u^j = u^j(x)$, $x = (x^1, \dots, x^n)$. To the infinite prolongation of (2.1) there corresponds a locus $\mathscr{E} \subset J^\infty(\pi)$ in the space of infinite jets of the trivial bundle $\pi\colon \mathbb{R}^m \times \mathbb{R}^n \to \mathbb{R}^n$, see [4]. Consider jet coordinates u^j_σ on $J^\infty(\pi)$, $j = 1, \dots, m$, where $\sigma = i_1 \dots i_s$ is a symmetric multi-index, $i_\alpha = 1, \dots, n$, and choose a set $\mathbf{I} = \{u^j_\sigma\}$ of internal variables on $\mathscr{E}$.

[a]We do not distinguish between scalar and multi-component systems.

Denote by

$$D_{x^i} = \frac{\partial}{\partial x^i} + \sum_{\mathbf{I}} u_{\sigma i} \frac{\partial}{\partial u^j_\sigma}, \qquad i = 1, \ldots, n,$$

the operators of total derivatives on $\mathscr{E}$. For any function f on $J^\infty(\pi)$ we denote by ℓ_f the restriction of its linearization $\sum_{j,\sigma} \partial f / \partial u^j_\sigma D_\sigma$ to $\mathscr{E}$, where $D_\sigma = D_{x^{i_1}} \circ \cdots \circ D_{x^{i_s}}$. A (local) symmetry of $\mathscr{E}$ is an evolutionary vector field

$$\mathbf{E}_\varphi = \sum_{\mathbf{I}} D_\sigma(\varphi^j) \frac{\partial}{\partial u^j_\sigma},$$

where the generating function $\varphi = (\varphi^1, \ldots, \varphi^m)$ satisfies the equation $\ell_F(\varphi) = 0$ and $F = (F^1, \ldots, F^r)$ determines Equation (2.1). We identify symmetries with their generating functions and denote the Lie algebra of symmetries by $\operatorname{sym} \mathscr{E}$.

A (differential) covering over $\mathscr{E}$ is a bundle $\tau \colon \tilde{\mathscr{E}} = \mathscr{E} \times \mathbb{R}^l \to \mathscr{E}$, $l = 1, 2, \ldots, \infty$, endowed with vector fields

$$\tilde{D}_{x^i} = D_{x^i} + \sum_\alpha X^\alpha_i \frac{\partial}{\partial w^\alpha}, \qquad i = 1, \ldots, n,$$

that pair-wise commute, where w^α are coordinates in $\mathbb{R}^l$ called nonlocal variables. Equivalently, $\tilde{\mathscr{E}}$ may be regarded as an overdetermined system

$$w^\alpha_{x^i} = X^\alpha_i, \qquad i = 1, \ldots, n, \quad \alpha = 1, \ldots, l,$$

whose compatibility conditions are consequences of $\mathscr{E}$. For a linear differential operator $\Delta = \left(\sum_\sigma a^\sigma_{ij} D_\sigma\right)$ on $\mathscr{E}$ we denote by $\tilde{\Delta} = \left(\sum_\sigma a^\sigma_{ij} \tilde{D}_\sigma\right)$ its lift to $\tilde{\mathscr{E}}$.

Coverings τ_1 and τ_2 are said to be equivalent if there exists a diffeomorphism $g \colon \tilde{\mathscr{E}}_1 \to \tilde{\mathscr{E}}_2$ such that $\tau_2 \circ g = \tau_1$ and which takes $\tilde{\mathscr{C}}_1$ to $\tilde{\mathscr{C}}_2$, where $\tilde{\mathscr{C}}_i$ is the span of the fields $\tilde{D}_{x^i}$ on $\tilde{\mathscr{E}}_i$.

A nonlocal symmetry of $\mathscr{E}$ in the covering τ is a vector field

$$\mathbf{S}_\Phi = \sum_{\mathbf{I}} \tilde{D}_\sigma(\varphi^j) \frac{\partial}{\partial u^j_\sigma} + \sum_\alpha \psi^\alpha \frac{\partial}{\partial w^\alpha},$$

where $\varphi = (\varphi^1, \ldots, \varphi^m)$ and ψ^α are functions on $\tilde{\mathscr{E}}$ satisfying the system

$$\begin{aligned} &\tilde{\ell}_F(\varphi) = 0, \\ &\tilde{D}_{x^i}(\psi^\alpha) = \tilde{\ell}_{X^\alpha_i}(\varphi) + \sum_\beta \frac{\partial X^\alpha_i}{\partial w^\beta} \end{aligned}$$

for all i and α. Any nonlocal symmetry is identified with the collection $\Phi = (\varphi, \psi^1, \ldots, \psi^\alpha, \ldots)$. A symmetry is called a lift of a local symmetry if the component φ is a function on $\mathscr{E}$; it is called invisible if $\varphi = 0$. Nonlocal symmetries also form a Lie algebra denoted by $\operatorname{sym}_\tau \mathscr{E}$.

2.2. *Coverings over Equation* (1.1) *and symmetries*

Let us assume that $w = w(\lambda)$ in Equation (1.4) and consider the expansion

$$w = \sum_{i=-\infty}^{+\infty} \lambda^{-i} w_i.$$

This leads to the infinite-dimensional covering

$$\begin{aligned} w_{i,x} &= w_{i-1,y} + u_x w_{i-1,x}, \\ w_{i,y} &= w_{i-1,t} + u_y w_{i-1,x}, \end{aligned} \tag{2.2}$$

where $i \in \mathbb{Z}$.

Recall now that the space $\operatorname{sym}\mathscr{E}$ for Equation (1.1) is spanned by the functions

$$\theta_1 = 2x - yu_x, \qquad \theta_2 = 3u - 2xu_x - yu_y,$$
$$\theta_3(T) = Tu_y + T'(yu_x - x) - \frac{1}{2}T''y^2, \qquad \theta_4(T) = Tu_x - T'y,$$
$$\theta_5(T) = Tu_t + T'(xu_x + yu_y - u) + \frac{1}{2}T''(y^2u_x - 2xy) - \frac{1}{6}T'''y^3, \qquad \theta_6(T) = T,$$

where T is a function in t and 'prime' denotes the t-derivatives. In what follows, we shall need the following

Proposition 2.1. *The symmetries θ_1, θ_2, and $\theta_5 = \theta_5(1) = u_t$ can be lifted to the covering* (2.2).

Proof. Let us set

$$\begin{aligned} \theta_1^i &= -yw_{i,x} + (i+2)w_{i-1}, \\ \theta_2^i &= -2xw_{i,x} - yw_{i,y} + (i+3)w_i, \\ \theta_5^i &= w_{i,t}. \end{aligned}$$

Then

$$\Theta_j = (\theta_j, \dots, \theta_j^i, \dots), \qquad j = 1,2,5, \quad i \in \mathbb{Z},$$

are the desired lifts. □

2.3. *Reductions*

Using Proposition 2.1, we now state the following

Proposition 2.2. *Reduction of* (2.2) *with respect to the symmetry $\Theta_5 + \Theta_1$ leads to the covering*

$$\begin{aligned} w_{i,x} &= w_{i-1,y} + u_x w_{i-1,x}, \\ w_{i,y} &= (u_y + y)w_{i-1,x} - (i+1)w_{i-2} \end{aligned} \tag{2.3}$$

over Equation (1.2), *while reduction with respect to $\Theta_5 + \Theta_2$ leads to the covering*

$$\begin{aligned} w_{i,x} &= w_{i-1,y} + u_x w_{i-1,x}, \\ w_{i,y} &= (u_y + 2x)w_{i-1,x} + yw_{i-1,y} - (i+2)w_{i-1} \end{aligned} \tag{2.4}$$

over Equation (1.3).

Let us finally describe the algebras of local symmetries for Equations (1.2) and (1.3). Direct computations show that $\operatorname{sym}\mathscr{E}_1$ for the first equation is spanned by the functions

$$\varphi_{-4} = 1, \quad \varphi_{-3} = u_x, \quad \varphi_{-2} = -u_y - y, \quad \varphi_{-1} = -2x + yu_x, \quad \varphi_0 = 4u - 3xu_x - 2yu_y, \tag{2.5}$$

while in the second case we have the following generators of the symmetry algebra $\operatorname{sym}\mathscr{E}_2$:

$$\gamma_{-3} = 1, \quad \gamma_{-2} = -y - \frac{1}{2}u_x, \quad \gamma_{-1} = y^2 - 2x + 2yu_x - 2u_y, \quad \gamma_0 = 3u - 2xu_x - yu_y.$$

It is also easily seen that

for Equation (1.2)	for Equation (1.3)
$[\varphi_i, \varphi_j] = \begin{cases} (j-i)\varphi_{i+j} & \text{if } i+j \geq -4, \\ 0 & \text{otherwise,} \end{cases}$	$[\gamma_i, \gamma_j] = \begin{cases} (j-i)\gamma_{i+j} & \text{if } i+j \geq -3, \\ 0 & \text{otherwise.} \end{cases}$

Denote by $\mathfrak{W}_0 \subset \mathfrak{W}$ the subalgebra in the Witt algebra spanned by the fields e_i with $i \leq 0$ and by $\mathfrak{W}_k$ its ideal spanned by e_i, $i \leq k < 0$.

Proposition 2.3. *For Equation* (1.2), *one has* $\operatorname{sym}\mathscr{E}_1 = \mathfrak{W}_0/\mathfrak{W}_5$, *while* $\operatorname{sym}\mathscr{E}_2 = \mathfrak{W}_0/\mathfrak{W}_4$ *for Equation* (1.3).

2.4. *The coverings τ^p and σ^p*

Consider the covering (2.3) over Equation (1.2). Choose an integer $p \in \mathbb{Z}$ and assume that $w_i = 0$ for all $i < p$. To proceed further, it is convenient to relabel nonlocal variables by setting $w_{p+i} = r^p_{i-4}$. Then $r^p_i = 0$ for $i < -4$ and

$$r^p_{-4,x} = 0, \qquad r^p_{-4,y} = 0;$$
$$r^p_{-3,x} = 0, \qquad r^p_{-3,y} = 0,$$

and without loss of generality one can set $r^p_{-4} = 1$, $r^p_{-3} = 0$. Then

$$r^p_{-2} = -(p+3)y, \qquad r^p_{-1} = -(p+3)x, \qquad r^p_0 = -(p+3)u + \frac{1}{2}(p+3)(p+4)y^2,$$

while

$$\begin{aligned} r^p_{i,x} &= r^p_{i-1,y} + u_x r^p_{i-1,x}, \\ r^p_{i,y} &= (u_y + y)r^p_{i-1,x} - (p+i+5)r^p_{i-2}, \end{aligned} \tag{2.6}$$

for all $i \geq 1$. We denote this covering by τ^p.

In a similar way, for Equation (1.3) and its covering (2.4) we have $r^p_i = 0$ for $i < -3$, $r^p_{-3} = 1$, and

$$r^p_{-2} = -(p+3)y, \quad r^p_{-1} = -(p+3)x + \frac{1}{2}(p+3)^2y^2, \quad r^p_0 = -(p+3)u + (p+3)^2xy - \frac{1}{6}(p+3)^3y^3,$$

while

$$\begin{aligned} r^p_{i,x} &= r^p_{i-1,y} + u_x r_{i-1,x} \\ r^p_{i,y} &= (u_y + 2x)r^p_{i-1,x} + yr^p_{i-1,y} - (p+i+5)r^p_{i-1} \end{aligned} \tag{2.7}$$

for $i \geq 1$. This covering will be denoted by σ^p.

3. The main result

We prove the main result of the paper in this section, which states that in the coverings τ^p and σ^p, naturally associated with (2.3) and (2.4), respectively, the algebras of nonlocal symmetries for both Equations (1.2) and (1.3) are isomorphic to the Witt algebra $\mathfrak{W}$. The case of Equation (1.2) is considered in detail, while for Equation (1.3) we provide a sketch of proofs only.

3.1. *Equivalence of the coverings τ^p*

The first step of the proof is to establish the equivalence of different τ^p to each other.

Proposition 3.1. *The covering τ^p is equivalent to τ^{-4} for any p.*

Proof. We distinguish the two cases: $p \neq -3$ and $p = 3$.

The case $p \neq -3$**.** Introduce the notation $s_i = r_i^{-4}$ and consider the vector field

$$\mathscr{X} = x\frac{\partial}{\partial y} + 2u\frac{\partial}{\partial x} + 3s_1\frac{\partial}{\partial u} + \sum_{i\geq 1}(i+3)s_{i+1}\frac{\partial}{\partial s_i}.$$

Let us define the quantities $Q_{k,j}$, $k \geq 0$, by

$$Q_{k,0} = \frac{1}{(k+2)!}y^{k+2}, \qquad Q_{k,j+1} = \frac{1}{j}\mathscr{X}(Q_{k,j})$$

and assume $Q_{k,j} = 0$ for $j < 0$. Set

$$d_i = \sum_{k\geq 0}(-1)^k\frac{(p+k+4)!}{(p+4)!}Q_{k,i-2k}.$$

Then the transformation

$$r_i^p = -(p+3)\big(s_i - (p+4)d_i\big), \qquad i \geq 1,$$

is the desired equivalence.

The case $p = -3$**.** The first three pairs of defining equations in this case are

$$r_{i,x} = 0, \quad r_{i,y} = 0, \qquad i = -3, -2, -1,$$

while all the rest ones do not contain the variable r_{-3}. Thus r_{-3} is a constant whose value does not influence the subsequent computations. So, we may set $r_{-3} = 0$ and this reduces the case under consideration to $p > -3$. □

3.2. *The Lie algebra of nonlocal symmetries*

Let $\Phi = (\varphi, \varphi^1, \ldots, \varphi^i, \ldots)$ be a nonlocal symmetry of Equation (1.2) in the covering (2.6). The defining equations for the components of Φ are

$$\begin{aligned}
&\tilde{\ell}_F(\varphi) \equiv \tilde{D}_y^2(\varphi) - (u_y + y)\tilde{D}_x^2(\varphi) - u_{xx}\tilde{D}_y(\varphi) + u_x\tilde{D}_x\tilde{D}_y(\varphi) + u_{xy}\tilde{D}_x(\varphi) = 0,\\
&\tilde{D}_x(\varphi^i) = \tilde{D}_y(\varphi^{i-1}) + u_x\tilde{D}_x(\varphi^{i-1}) + r_{i-1,x}^p\tilde{D}_x(\varphi),\\
&\tilde{D}_y(\varphi^i) = (u_y + y)\tilde{D}_x(\varphi^{i-1}) - (p+i+1)\varphi^{i-2} + r_{i-1,x}^p\tilde{D}_y(\varphi).
\end{aligned} \tag{3.1}$$

We need a number of auxiliary results to describe the algebra $\mathrm{sym}_{\tau^p}\mathscr{E}_1$. As a first step, it is convenient to assign weights to the internal coordinates in $\tilde{\mathscr{E}}_1$ in such a way that all polynomial objects become homogeneous with respect to these weights. Let us set

$$|x| = 3, \qquad |y| = 2, \qquad |u| = 4.$$

Then

$$|u_x| = |u| - |x| = 1, \qquad |u_y| = |u| - |y| = 2,$$

etc., and

$$\left|r_i^p\right| = i + 4, \qquad i \geq 1.$$

The weight of a monomial is the sum of weights of its factors and the weight of a vector field $R\partial/\partial\rho$ is $|R| - |\rho|$. In particular, one has $\left|\mathbf{E}_{\varphi_k}\right| = k$, $k = -4, \ldots, 0$ for the local symmetries presented in (2.5).

Lemma 3.1. *The local symmetry φ_{-1} can be lifted to the covering τ^{-4}.*

Proof. Let us set $\Phi_{-1} = (\varphi_{-1}, \varphi_{-1}^1, \ldots, \varphi_{-1}^i, \ldots)$, where $\varphi_{-1}^i = y r_{i,x}^{-4} - (i+2) r_{i-1}^{-4}$. It is straightforward to check that Equations (3.1) are satisfied for $p = -4$. □

Lemma 3.2. *The local symmetry φ_{-2} can be lifted to the covering τ^{-5}.*

Proof. We set $\Phi_{-2} = (\varphi_{-2}, -r_{1,y}^{-5}, \ldots, -r_{i,y}^{-5}, \ldots)$. Then (3.1) are fulfilled in an obvious way for $p = -5$. □

Lemma 3.3. *There exists a nonlocal symmetry of weight 2 in the covering τ^{-1}.*

Proof. Let us set $\Phi_2 = (\varphi_2, \varphi_2^1, \ldots, \varphi_2^i, \ldots)$, where

$$\varphi_2 = -\frac{1}{2}\left(8r_2^{-1} + 42y(3y^2 - 2u) + (80xy - 7r_1^{-1})u_x + 2(7y^2 + 6u)u_y - 50x^2 - 96y^3\right)$$

and

$$\varphi_2^i = -\frac{1}{2}\left((80xy - 7r_1^{-1})r_{i,x}^{-1} + 2(7y^2 + 6u)r_{i,y}^{-1} - 2(i+8)r_{i+2}^{-1} - 8y(i+6)r_i^{-1} - 10(i+5)xr_{i-1}^{-1}\right)$$

for all $i \geq 1$. □

Proposition 3.2. *The symmetries Φ_{-2}, Φ_{-1}, and Φ_2 exist in any covering τ^p.*

Proof. A direct consequence of Lemmas 3.1– 3.3 and Proposition 3.1. □

Theorem 3.1. *The symmetries Φ_{-2}, Φ_{-1}, and Φ_2 generate the entire Lie algebra $\mathrm{sym}_{\tau^p}\mathscr{E}_1$ which is isomorphic to the Witt algebra $\mathfrak{W}$, i.e., there exists a basis Φ_k, $k \in \mathbb{Z}$, such that*

$$[\Phi_k, \Phi_l] = (l - k)\Phi_{k+l}$$

for all $k, l \in \mathbb{Z}$.

Proof. Let set

$$\Phi_0 = 4[\Phi_{-2}, \Phi_2], \qquad \Phi_1 = 3[\Phi_{-1}, \Phi_2]$$

and by induction

$$\Phi_{k+1} = (k-1)[\Phi_1, \Phi_k], \qquad \Phi_{-k-1} = (1-k)[\Phi_{-1}, \Phi_{-k}]$$

for all $k \geq 2$. □

Remark 3.1. The symmetries Φ_k are invisible for $k < -4$.

Remark 3.2. The symmetries Φ_{-4}, Φ_{-3}, and Φ_0 are lifts of local symmetries φ_{-4}, φ_{-3}, φ_0, respectively.

3.3. *Explicit formulas*

To obtain explicit formulas for nonlocal symmetries in τ^p for arbitrary[b] value of p consider the fields

$$\mathscr{Y}_m = \sum_{i=m}^{\infty} (i-m+1) r_{i-3}^p \frac{\partial}{\partial r_{i-4}^p}$$

on the space of τ^p, where $m = 1\dots,4$, and the quantities $P_{i,j}^m$ defined by induction as follows:

$$P_{i,0}^{(m)} = \frac{1}{(i+2)!} \left(r_{m-4}^p\right)^{i+2}, \qquad P_{i,j+1}^{(m)} = \frac{1}{j+1} \mathscr{Y}_m(P_{i,j}^{(m)}),$$

where $i, j \geq 0$, $m = 1,\dots,4$. We set $P_{i,j}^{(m)} = 0$ if at least one of the subscripts is < 0. In terms of these quantities, the lift of φ_{-2} to τ^p acquires the form

$$\Phi_{-2}^p = (\varphi_{-2}, \varphi_{-2}^{1,p}, \dots, \varphi_{-2}^{i,p}, \dots),$$

where

$$\varphi_{-2}^{i,p} = -r_{i,y}^p - (p+5) \left(r_{i-2}^p + \sum_{j=1}^{\infty} \left(-\frac{1}{p+3} \right)^j \cdot \prod_{l=0}^{j-1} ((p+3)l - 2) \cdot P_{j-1,i-2j}^{(2)} \right).$$

The lift

$$\Phi_{-1}^p = (\varphi_{-1}, \varphi_{-1}^{1,p}, \dots, \varphi_{-1}^{i,p}, \dots)$$

of φ_{-1} is given by

$$\varphi_{-1}^{i,p} = y r_{i,x}^p - (i+2) r_{i-1}^p + (p+4) \sum_{j=1}^{\infty} \left(-\frac{1}{p+3} \right)^j \cdot \prod_{l=0}^{j-1} ((p+3)l - 1) \cdot P_{j-1,i-2j+1}^{(2)}.$$

The nonlocal symmetry Φ_2, when passing to τ^p, acquires the form

$$\Phi_2^p = (\varphi_2^p, \varphi_2^{1,p}, \dots, \varphi_2^{i,p}, \dots)$$

with

$$\varphi_2^p = -\frac{1}{p+3} \left(8r_2^p + 2y(4p+25)r_0^p + \left((p+3)(7p+47)xy - 7r_1^p \right) u_x \right)$$
$$- \left((7y^2+6y)u_y - (4p+29)x^2 - \frac{1}{3} y^3 (8p^2+87p+223) \right)$$

[b]The case $p = -3$ is special, but, as it was indicated above, is reduced to the case $p = -2$ for example.

and

$$\varphi_2^{i,p} = -\frac{1}{p+3}\left((p+3)(7p+47)xy - 7r_1^p\right) r_{i,x}^p - (7y^2+6u)r_{i,y}^p$$
$$+ (i+8)r_{i+2}^p + 2y(p+2i+13)r_i^p + x(3p+5i+28)r_{i-1}^p$$
$$+ \frac{p+1}{p+3}\left(2P_{0,i-2}^{(4)} + \sum_{j=1}^{\infty}\left(-\frac{1}{p+3}\right)^j \cdot \prod_{l=0}^{j}(l(p+3)+2)\cdot P_{j,i-2j+2}^{(2)}\right).$$

Explicit formulas for other nonlocal symmetries can be obtained using commutator relations from the proof of Theorem 3.1. For example, the symmetry Φ_1 that generates the positive[c] part of $\mathrm{sym}_{\tau^p}\mathscr{E}_1$ is

$$\Phi_1 = (\varphi_1^p, \varphi_1^{1,p}, \dots, \varphi_1^{i,p}, \dots)$$

with

$$\varphi_1^p = -\frac{6}{p+3} r_1^p - \left((4y^2+5u)u_x + 4xu_y - 2(3p+16)xy\right)$$

and

$$\varphi_1^{i,p} = -\left((4y^2+5u)r_{i,x}^p + 4xr_{i,y}^p - (i+6)r_{i+1}^p - y(2p+3i+16)r_{i-1}^p\right)$$
$$+ \frac{p+2}{p+3}\left(P_{0,i-1}^{(3)} + \sum_{j=1}^{\infty}\left(-\frac{1}{p+3}\right)^j \cdot \prod_{l=0}^{j}(l(p+3)+1)\cdot P_{j,i-2j+1}^{(2)}\right).$$

3.4. *Equation* (1.3)

The results on this equation and their proofs are almost identical to those on Equation (1.2). So, we confine ourselves with the final description of $\mathrm{sym}_{\sigma^p}\mathscr{E}_2$. Introduce the weights

$$|x| = 2, \qquad |y| = 1, \qquad |u| = 3.$$

Consequently, the nonlocal variables in the covering σ^p acquire the weights $\left|r_i^p\right| = i+3$.

Then we have

Theorem 3.2. *The symmetries γ_{-2} and γ_{-1} can be lifted to symmetries Γ_{-2} and Γ_{-1} in any covering σ^p over Equation* (1.3). *In addition, there exists a nonlocal symmetry Γ_2 of weight* 2. *These three symmetries generate the entire Lie algebra $\mathrm{sym}_{\sigma^p}\mathscr{E}_2$ which is isomorphic to the Witt algebra $\mathfrak{W}$.*

Proof. The symmetry $\Gamma_{-2} = (\gamma_{-2}, \gamma_{-2}^{1,p}, \dots, \gamma_{-2}^{j,p}, \dots)$ is given by $\gamma_{-2} = -y - u_x/2$ and

$$\gamma_{-2}^{i,p} = -\frac{1}{2}\left(r_{i,x}^p + (p+5)\left(r_{i-2}^p + \sum_{j=1}^{\infty}\left(-\frac{1}{p+3}\right)^j \cdot \prod_{l=0}^{j-1}(l(p+3)-2)\cdot P_{j-1,i-j}^{(1)}\right)\right).$$

For $\Gamma_{-1} = (\gamma_{-1}, \gamma_{-1}^{1,p}, \dots, \gamma_{-1}^{j,p}, \dots)$ we have $\gamma_{-1} = y^2 - 2x + 2yu_x - 2u_y$ and

$$\gamma_{-1}^{j,p} = 2\left(yr_{i,x}^p - r_{i,y}^p - (p+4)\left(r_{i-1}^p + \sum_{j=1}^{\infty}\left(-\frac{1}{p+3}\right)^j \cdot \prod_{l=0}^{j-1}(l(p+3)-1)\cdot P_{j-1,i-j+1}^{(1)}\right)\right).$$

[c]With respect to weights.

Finally, we have

$$\begin{aligned}\gamma_2^p &= 7r_2^p + \big((7p+37)y - 6u_x\big) r_1^p \\ &+ (p+3)\Bigg((6p+31)yu + (3p+19)x^2 - (3p^2+18p+13)xy^2 + \frac{1}{12}(3p^3+27p^2+81p+89)y^4 \\ &+ \left(5u + 14xy + \frac{19}{3}y^3\right)u_y - \frac{1}{2}(7p^2+74p+197)y^2u - (7p+37)xu - 4(2p+7)x^2y \\ &+ \frac{1}{3}(7p^3+87p^2+333p+397)xy^3 - \frac{1}{30}(7p^4+104p^3+558p^2+1296p+1115)y^5\Bigg)\end{aligned}$$

and

$$\gamma_2^{i,p} = -(p+3)((i+7)r_{i+2}^p + (p+3i+19)yr_{i+1}^p) + Ar_{i,x}^p + Br_{i,y}^p + Cr_i^p + D$$

for $\Gamma_2 = (\gamma_2^p, \gamma_2^{1,p}, \dots, \gamma_2^{i,p}, \dots)$, where

$$\begin{aligned}A &= (p+3)\left((6p+31)uy + (3p+19)x^2 - (3p^2+18p+13)xy^2 + \frac{1}{12}(3p^3+27p^2+81p+89)y^4\right) \\ &\quad - 6r_1^p, \\ B &= (p+3)\left(5u + 14xy + \frac{19}{3}y^3\right), \\ C &= -(p+3)\Big(2(p+2i+11)x + (p^2+9p+5i+33)y^2\Big),\end{aligned}$$

and

$$D = -(p+1)\left(2P_{0,i-1}^{(3)} + \sum_{j=1}^{\infty}\left(-\frac{1}{p+3}\right)^j \cdot \prod_{l=0}^{j}(l(p+3)+2)\cdot P_{j,i-j+3}^{(1)}\right)$$

for $i \geq 1$. □

4. Discussion

We conclude with two remarks.

Remark 4.1. Consider the covering (2.6) with $p = 2$ and the generating series

$$R = \sum_{i=1}^{\infty} \lambda^i r_i$$

of the corresponding nonlocal variables. The function R satisfies the system

$$\begin{aligned}R_x &= \lambda(R_y + u_x R_x), \\ R_y &= \lambda(u_y + y)R_x - (\lambda^3 R)_\lambda,\end{aligned}$$

which is equivalent to

$$\begin{aligned}\tilde{R}_x &= \frac{\lambda^4 \tilde{R}_\lambda}{\lambda^2(u_y+y) + \lambda u_x - 1}, \\ \tilde{R}_y &= \frac{\lambda^3(1-\lambda u_x)\tilde{R}_\lambda}{\lambda^2(u_y+y) + \lambda u_x - 1},\end{aligned} \tag{4.1}$$

where $\tilde{R} = \lambda^3 R$. Let $\tilde{R} = \tilde{R}(x,y,\lambda)$ be a solution to (4.1) such that $\tilde{R}_\lambda \not\equiv 0$. Then equation $\tilde{R}(x,y,\lambda) =$ const defines λ as a fuction of x and y, that is, $\lambda = \psi(x,y)$, cf. [11]. Then

$$\tilde{R}_\lambda \cdot \psi_x + \tilde{R}_x = 0, \quad \tilde{R}_\lambda \cdot \psi_y + \tilde{R}_y = 0$$

and (4.1) transforms to

$$\psi_x = \frac{\psi^4}{1 - u_x\psi - (u_y + y)\psi^2}, \qquad \psi_y = \frac{\psi^3(1 - u_x\psi)}{1 - u_x\psi - (u_y + y)\psi^2},$$

which, by the gauge transformation $\psi \mapsto \psi^{-1}$, is equivalent to

$$\psi_x = -\frac{1}{\psi^2 - u_x\psi - (u_y + y)}, \qquad \psi_y = \frac{u_x - \psi}{\psi^2 - u_x\psi - (u_y + y)}.$$

This is the covering obtained in [2] by the direct reduction and coincides with the known covering over the Gibbons-Tsarev equation, see [7].

Remark 4.2. In [1], two other equations,

$$u_y u_{xy} - u_x u_{yy} = e^y u_{xx}$$

and

$$u_{yy} = (u_x + x)u_{xy} - u_y(u_{xx} + 2),$$

were obtained as symmetry reductions of the universal hierarchy and the 3D rdDym equations, respectively, were obtained. We plan study these equations by the methods similar to the used above in the forthcoming publications.

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