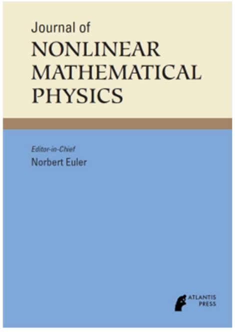

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The Multiplication of Distributions in the Study of a Riemann Problem in Fluid Dynamics

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The present paper concerns the study of a Riemann problem for the system $u_t + \left(\frac{1}{2}u^2 + \phi(v)\right)_x = 0, v_t + \left(uv\right)_x = 0$, with a one dimensional space variable. We consider ϕ an entire function that takes real values on the real axis. Under certain conditions, this system provides solutions to the pressureless gas dynamics and the isentropic fluid dynamics systems. We get all solutions of this problem within a convenient space of distributions that contains discontinuous functions and Dirac measures. For this purpose, we use a solution concept defined in the setting of a distributional product. This concept consistently extends the classical solution concept and can also be considered as an extension of the weak solution concept for nonlinear evolution equations. Our product, not defined by approximation processes, can be applied to several physical models.

Keywords: Products of distributions; Isentropic fluid dynamics system; Pressureless gas dynamics system; Shock waves; Delta waves; Delta shock waves.

2000 Mathematics Subject Classification: 46F10, 35D99, 35L67

1. Introduction and contents

Let us consider the system of nonlinear conservation laws

$$u_t + \left(\tfrac{1}{2}u^2 + \phi(v)\right)_x = 0, \tag{1.1}$$

$$v_t + \left(uv\right)_x = 0, \tag{1.2}$$

where $x \in \mathbb{R}$ is the space variable, $t \in \mathbb{R}$ is the time variable and $u(x,t)$, $v(x,t)$ are the unknown state variables. We consider ϕ an entire function taking real values on the real axis. The goal is to study the evolution of this system subjecting the variables u, v to the initial conditions

$$u(x,0) = u_1 + (u_2 - u_1)H(x), \tag{1.3}$$

$$v(x,0) = v_1 + (v_2 - v_1)H(x) \tag{1.4}$$

(H stands for the Heaviside function, $u_1, u_2, v_1, v_2 \in \mathbb{R}$, and $u_2 \neq u_1$) and to seek for solutions in the space W of pairs of distributions (u,v) defined by

$$u(x,t) = a(t) + b(t)H\big(x - \gamma(t)\big),$$
$$v(x,t) = f(t) + g(t)H\big(x - \gamma(t)\big) + h(t)\delta\big(x - \gamma(t)\big),$$

where $a, b, f, g, h, \gamma : \mathbb{R} \to \mathbb{R}$ are C^1-functions and δ stands for the Dirac measure supported at the origin.

The main result is the explicit solution of the Riemann problem (1.1), (1.2), (1.3), (1.4):

$$u(x,t) = u_1 + (u_2 - u_1)H\big(x - \gamma(t)\big), \tag{1.5}$$

$$v(x,t) = v_1 + (v_2 - v_1)H\big(x - \gamma(t)\big) + (A - B)t\delta\big(x - \gamma(t)\big), \tag{1.6}$$

where

$$\gamma(t) = \Big(\frac{u_1 + u_2}{2} + \frac{\phi(v_2) - \phi(v_1)}{u_2 - u_1}\Big)t, \tag{1.7}$$

$A = (v_2 - v_1)\big(\frac{u_1+u_2}{2} + \frac{\phi(v_2)-\phi(v_1)}{u_2-u_1}\big)$ and $B = u_2 v_2 - u_1 v_1$. These solutions when they exist are unique in W. Thus, when $A = B$, it arises a travelling shock wave propagating with constant speed $\frac{u_1+u_2}{2} + \frac{\phi(v_2)-\phi(v_1)}{u_2-u_1}$. When $A \neq B$, the emergence of a delta shock wave becomes possible.

System (1.1), (1.2) provides solutions for the isentropic fluid dynamics system in Eulerian coordinates

$$(vu)_t + \big(vu^2 + P(v)\big)_x = 0, \tag{1.8}$$

$$v_t + (vu)_x = 0, \tag{1.9}$$

if $P'(v) = v\phi'(v)$. Here u, v, P stand for the speed, density and pressure, respectively. The pressure P (a function of the density v) is determined from the constitutive thermodynamic relations of the fluid under consideration. To get (1.1), (1.2), it is sufficient to develop (1.8) using (1.9) and the relation $P'(v) = v\phi'(v)$. See [13] p. 100 and [3] p. 175 for details.

As examples, we consider three particular cases of (1.1), (1.2) with physical significance corresponding to

(a) $\phi = 0$, which provides solutions for the well known pressureless gas dynamics system (1.8), (1.9) with $P = 0$;

(b) $\phi(v) = \frac{k\gamma}{\gamma-1}v^{\gamma-1}$ ($k \neq 0$ is a real number and $\gamma \geq 2$ is an integer), which provides solutions for the isentropic fluid dynamics system (1.8), (1.9) with the so called polytropic pressure-density relation $P(v) = kv^{\gamma}$;

(c) $\phi(v) = \frac{4}{3}v^3 - 3v^2 + 2v$, which provides solutions for the isentropic system (1.8), (1.9) with the pressure-density relation $P(v) = v^4 - 2v^3 + v^2$.

Notice that ϕ can be replaced by $\phi + constant$ without any change in the Riemann problem (1.1), (1.2), (1.3), (1.4) (the same happens with the function P in (1.8)).

Often, in this kind of problems, the distributional solutions obtained by approximation processes depend on the chosen processes (in general asymptotic algorithms) and appear as weak limits. It may even happen that these weak limits cannot be substituted into equations or systems owing to the well known difficulties of multiplying distributions. Our α-products overcome those difficulties, as we will explain.

For the system (1.1), (1.2), we will adopt a solution concept defined within the framework of a product of distributions. This concept is a consistent extension of the classical solution concept and, in a sense explained at the end of Section 5, can also be seen as a new type of weak solution.

In our framework, the product of two distributions is a distribution that depends on the choice of a certain function α encoding the indeterminacy inherent to such products. This indeterminacy generally is not avoidable and in many cases it also has a physical meaning; concerning this point let us mention [1,2,4,19]. Thus, the solutions of differential equations or systems containing such products may depend (or not) of α. We call such solutions α-solutions. The possibility of their occurrence depends on the physical system: in certain cases we cannot previously know the behavior of the system, possibly due to physical features omitted in the formulation of the model with the goal of simplifying it. Thus, the mathematical indetermination sometimes observed may have this origin. It is an interesting fact that, for the present problem (1.1), (1.2), (1.3), (1.4), the solutions, when they exist, are independent of α.

To show the scope of these methods, let us recall some results we have obtained.

For the conservation law

$$u_t + [\phi(u)]_x = \psi(u),$$

where ϕ, ψ are entire functions taking real values on the real axis, we have established [22] necessary and sufficient conditions for the propagation of a travelling wave with a given distributional profile and we also have computed its speed. For C^1-wave profiles with one jump discontinuity, our methods easily lead to the well known Rankine – Hugoniot conditions.

Conditions for the propagation of travelling waves with profiles $\beta + m\delta$ and $\beta + m\delta'$ (where β is a continuous function, $m \in \mathbb{R}$ and $m \neq 0$) were also obtained, as well as their speeds [23].

Gas dynamics phenomena, known as “infinitely narrow soliton solutions”, discovered by Maslov and collaborators [5,9,14,15], can be obtained directly in distributional form [20].

For a Riemann problem concerning the generalized pressureless gas dynamics system,

$$\begin{aligned} u_t + [\phi(u)]_x &= 0, \\ v_t + [\psi(u)v]_x &= 0, \end{aligned}$$

only assuming $\phi, \psi : \mathbb{R} \to \mathbb{R}$ continuous, we were able to show the formation of a delta shock wave solution [25]. In this case we arrived, more easily and in a much more general setting, to the same result of Danilov and Mitrovic [6], which have employed the weak asymptotic method, and also to the same result of Mitrovic et al. [16], which have used a different approach, based on a linearization process.

In the Brio system

$$\begin{aligned} u_t + \tfrac{1}{2}\left(u^2 + v^2\right)_x &= 0, \\ v_t + \left(uv - v\right)_x &= 0, \end{aligned}$$

a simplified model for the study of plasmas, we got a delta shock wave as explicit solution for a Riemann problem [27]. This problem (suggested by Hayes and LeFloch in [10], p. 1558), was first studied by Kalisch and Mitrovic [12] who also constructed a delta shock wave using an extension of the weak asymptotic method. Their solution coincides with our solution (in [12], p. 718 there is a misprint in formula (3.8); the correct $\alpha(t)$ has the opposite sign and in [27] p. 522, formula (17), $-\frac{k_0}{c_0}$ must replace $-\frac{k_0}{b_0}$).

Also for the Brio system we have subjected $u(x,t)$ and $v(x,t)$ to the initial conditions

$$\begin{aligned} u(x,0) &= c_0\delta(x), \\ v(x,0) &= h_0\delta(x), \end{aligned}$$

with $c_0, h_0 \in \mathbb{R}\backslash\{0\}$. Under certain assumptions, we got, as solutions, travelling delta waves with speed $\frac{c_0^2+h_0^2}{c_0^2-h_0^2}$ and certain singular perturbations (which are not measures) propagating with speed 1 [26].

Regarding the interaction of singular waves, we have shown that delta waves under collision behave just as classical soliton collisions (as in the Korteweg – de Vries equation) in models ruled by a singular perturbation of Burgers conservative equation [21]. Also in a conservation law with singular flux, the interaction of a δ wave with a δ' wave was studied. Here, we were able to distinguish three distinct dynamics for that collision to which correspond phenomena of solitonic behavior, scattering, and merging [28].

Let us now summarize the present paper's contents. In Section 2, we present the main ideas of our method for multiplying distributions. In Section 3, we define powers of certain distributions. In Section 4, we define the composition of an entire function with a distribution. In Section 5, we define the concept of α-solution for the system (1.1), (1.2). In Section 6, we present the main result, that is, all the possible solutions of the Riemann problem (1.1), (1.2), (1.3), (1.4) that belong to W. In Section 7, we apply these results to the study of the examples (a), (b), (c) already referred.

2. The multiplication of distributions

Let $\mathscr{C}^\infty$ be the space of indefinitely differentiable real or complex-valued functions defined on $\mathbb{R}^N$, $N \in \{1,2,3,...\}$, and $\mathscr{D}$ the subspace of C^∞ consisting of those functions with compact support. Let $\mathscr{D}'$ be the space of Schwartz distributions and $L(\mathscr{D})$ the space of continuous linear maps $\phi : \mathscr{D} \to \mathscr{D}$, where we suppose $\mathscr{D}$ endowed with the usual topology. We will sketch the main ideas of our distributional product (the reader can look at (2.4), (2.7), and (2.9) as definitions, if he prefers to skip this presentation). For proofs and other details concerning this product see [18].

First, we define a product $T\phi \in \mathscr{D}'$ for $T \in \mathscr{D}'$ and $\phi \in L(\mathscr{D})$ by

$$\langle T\phi, \xi\rangle = \langle T, \phi(\xi)\rangle,$$

for all $\xi \in \mathscr{D}$; this makes $\mathscr{D}'$ a right $L(\mathscr{D})$-module. Next, we define an epimorphism $\tilde{\zeta} : L(\mathscr{D}) \to \mathscr{D}'$, where the image of ϕ is the distribution $\tilde{\zeta}(\phi)$ given by

$$\langle \tilde{\zeta}(\phi), \xi\rangle = \int \phi(\xi),$$

for all $\xi \in \mathscr{D}$ (when the domain of the integral is not specified, we consider that it is extended all over $\mathbb{R}^N$); given $S \in \mathscr{D}'$, we say that ϕ is a representative operator of S if $\tilde{\zeta}(\phi) = S$. For instance, if $\beta \in C^\infty$ is seen as a distribution, the operator $\phi_\beta \in L(\mathscr{D})$ defined by $\phi_\beta(\xi) = \beta\xi$, for all $\xi \in \mathscr{D}$, is a representative operator of β because, for all $\xi \in \mathscr{D}$, we have

$$\langle \tilde{\zeta}(\phi_\beta), \xi\rangle = \int \phi_\beta(\xi) = \int \beta\xi = \langle \beta, \xi\rangle.$$

For this reason $\tilde{\zeta}(\phi_\beta) = \beta$. If $T \in \mathscr{D}'$, we also have

$$\langle T\phi_\beta, \xi\rangle = \langle T, \phi_\beta(\xi)\rangle = \langle T, \beta\xi\rangle = \langle T\beta, \xi\rangle,$$

for all $\xi \in \mathscr{D}$. Hence,

$$T\beta = T\phi_\beta.$$

Thus, given $T,S \in \mathscr{D}'$, we are tempted to define a natural product by setting $TS := T\phi$, where $\phi \in L(\mathscr{D})$ is a representative operator of S, i.e., ϕ is such that $\tilde{\tilde{\zeta}}(\phi) = S$. Unfortunately, this product is not well defined, because TS depends on the representative $\phi \in L(\mathscr{D})$ of $S \in \mathscr{D}'$.

This difficulty can be overcome, if we fix $\alpha \in \mathscr{D}$ with $\int \alpha = 1$ and define $s_\alpha : L(\mathscr{D}) \to L(\mathscr{D})$ by

$$[(s_\alpha \phi)(\xi)](y) = \int \phi[(\tau_y \check{\alpha})\xi], \tag{2.1}$$

for all $\xi \in \mathscr{D}$ and all $y \in \mathbb{R}^N$, where $\tau_y \check{\alpha}$ is given by $(\tau_y \check{\alpha})(x) = \check{\alpha}(x-y) = \alpha(y-x)$ for all $x \in \mathbb{R}^N$. It can be proved that for each $\alpha \in \mathscr{D}$ with $\int \alpha = 1$, $s_\alpha(\phi) \in L(\mathscr{D})$, s_α is linear, $s_\alpha \circ s_\alpha = s_\alpha$ (s_α is a projector of $L(\mathscr{D})$), $\ker s_\alpha = \ker \tilde{\tilde{\zeta}}$, and $\tilde{\tilde{\zeta}} \circ s_\alpha = \tilde{\tilde{\zeta}}$.

Now, for each $\alpha \in \mathscr{D}$, we can define a general α-product $\underset{\alpha}{\odot}$ of $T \in \mathscr{D}'$ with $S \in \mathscr{D}'$ by setting

$$T \underset{\alpha}{\odot} S := T(s_\alpha \phi), \tag{2.2}$$

where $\phi \in L(\mathscr{D})$ is a representative operator of $S \in \mathscr{D}'$. This α-product is independent of the representative ϕ of S because if ϕ, ψ are such that $\tilde{\tilde{\zeta}}(\phi) = \tilde{\tilde{\zeta}}(\psi) = S$, then $\phi - \psi \in \ker \tilde{\tilde{\zeta}} = \ker s_\alpha$. Hence,

$$T(s_\alpha \phi) - T(s_\alpha \psi) = T[s_\alpha(\phi - \psi)] = 0.$$

Since ϕ in (2.2) satisfies $\tilde{\tilde{\zeta}}(\phi) = S$, we have $\int \phi(\xi) = \langle S, \xi \rangle$ for all $\xi \in \mathscr{D}$, and by (2.1)

$$[(s_\alpha \phi)(\xi)](y) = \langle S, (\tau_y \check{\alpha})\xi \rangle = \langle S\xi, \tau_y \check{\alpha} \rangle = (S\xi * \alpha)(y),$$

for all $y \in \mathbb{R}^N$, which means that $(s_\alpha \phi)(\xi) = S\xi * \alpha$. Therefore, for all $\xi \in \mathscr{D}$,

$$\begin{aligned} \langle T \underset{\alpha}{\odot} S, \xi \rangle &= \langle T(s_\alpha \phi), \xi \rangle = \langle T, (s_\alpha \phi)(\xi) \rangle = \langle T, S\xi * \alpha \rangle \\ &= [T * (S\xi * \alpha)\check{}\,](0) = [(S\xi)\check{}\, * (T * \check{\alpha})](0) = \langle (T * \check{\alpha})S, \xi \rangle, \end{aligned}$$

and we obtain an easier formula for the general product (2.2),

$$T \underset{\alpha}{\odot} S = (T * \check{\alpha})S. \tag{2.3}$$

In general, this α-product is neither commutative nor associative but it is bilinear and satisfies the Leibniz rule written in the form

$$D_k(T \underset{\alpha}{\odot} S) = (D_k T) \underset{\alpha}{\odot} S + T \underset{\alpha}{\odot} (D_k S),$$

where D_k is the usual k-partial derivative operator in distributional sense ($k = 1, 2, ..., N$).

Recall that the usual Schwartz products of distributions are not associative and the commutative property is a convention inherent to the definition of such products (see the classical monograph of Schwartz [29] pp. 117, 118, and 121, where these products are defined). Unfortunately, the α-product (2.3), in general, is not consistent with the classical Schwartz products of distributions with functions.

In order to obtain consistency with the usual product of a distribution with a C^∞-function, we are going to introduce some definitions and single out a certain subspace H_α of $L(\mathscr{D})$.

An operator $\phi \in L(\mathscr{D})$ is said to vanish on an open set $\Omega \subset \mathbb{R}^N$, if and only if $\phi(\xi) = 0$ for all $\xi \in \mathscr{D}$ with support contained in Ω. The support of an operator $\phi \in L(\mathscr{D})$ will be defined as the complement of the largest open set in which ϕ vanishes.

Let $\mathscr{N}$ be the set of operators $\phi \in L(\mathscr{D})$ whose support has Lebesgue measure zero, and $\rho(C^\infty)$ the set of operators $\phi \in L(\mathscr{D})$ defined by $\phi(\xi) = \beta\xi$ for all $\xi \in \mathscr{D}$, with $\beta \in C^\infty$. For each $\alpha \in \mathscr{D}$, with $\int \alpha = 1$, let us consider the space $H_\alpha = \rho(C^\infty) \oplus s_\alpha(\mathscr{N}) \subset L(\mathscr{D})$. It can be proved that $\zeta_\alpha := \tilde{\zeta}|_{H_\alpha} : H_\alpha \to C^\infty \oplus \mathscr{D}'_\mu$ is an isomorphism ($\mathscr{D}'_\mu$ stands for the space of distributions whose support has Lebesgue measure zero). Therefore, if $T \in \mathscr{D}'$ and $S = \beta + f \in C^\infty \oplus \mathscr{D}'_\mu$, a new α-product, $\dot{\alpha}$, can be defined by $T_{\dot{\alpha}}S := T\phi_\alpha$, where for each α, $\phi_\alpha = \zeta_\alpha^{-1}(S) \in H_\alpha$. Hence,

$$\begin{aligned} T_{\dot{\alpha}}S &= T\zeta_\alpha^{-1}(S) = T[\zeta_\alpha^{-1}(\beta+f)] \\ &= T[\zeta_\alpha^{-1}(\beta) + \zeta_\alpha^{-1}(f)] = T\beta + T \underset{\alpha}{\odot} f = T\beta + (T * \check{\alpha})f, \end{aligned}$$

and putting α instead of $\check{\alpha}$ (to simplify), we get

$$T_{\dot{\alpha}}S = T\beta + (T * \alpha)f. \tag{2.4}$$

Thus, the referred consistency is obtained when the C^∞-function is placed at the right-hand side: if $S \in C^\infty$, then $f = 0$, $S = \beta$, and $T_{\dot{\alpha}}S = T\beta$.

The α-product (2.4) can be easily extended for $T \in \mathscr{D}'^p$ and $S = \beta + f \in C^p \oplus \mathscr{D}'_\mu$, where $p \in \{0, 1, 2, ..., \infty\}$, $\mathscr{D}'^p$ is the space of distributions of order $\leq p$ in the sense of Schwartz ($\mathscr{D}'^\infty$ means $\mathscr{D}'$), $T\beta$ is the Schwartz product of a $\mathscr{D}'^p$-distribution with a C^p-function, and $(T * \alpha)f$ is the usual product of a C^∞-function with a distribution. This extension is clearly consistent with all Schwartz products of $\mathscr{D}'^p$-distributions with C^p-functions, if the C^p-functions are placed at the right-hand side. It also keeps the bilinearity and satisfies the Leibniz rule written in the form

$$D_k(T_{\dot{\alpha}}S) = (D_kT)_{\dot{\alpha}}S + T_{\dot{\alpha}}(D_kS),$$

clearly under certain natural conditions; for $T \in \mathscr{D}'^p$, we must suppose $S \in C^{p+1} \oplus \mathscr{D}'_\mu$. Moreover, these products are invariant by translations, that is,

$$\tau_a(T_{\dot{\alpha}}S) = (\tau_aT)_{\dot{\alpha}}(\tau_aS),$$

where τ_a stands for the usual translation operator in distributional sense. These products are also invariant for the action of any group of linear transformations $h : \mathbb{R}^N \to \mathbb{R}^N$ with $|\det h| = 1$, that leave α invariant .

Thus, for each $\alpha \in \mathscr{D}$ with $\int \alpha = 1$, formula (2.4) allows us to evaluate the product of $T \in \mathscr{D}'^p$ with $S \in C^p \oplus \mathscr{D}'_\mu$; therefore, we have obtained a family of products, one for each α.

From now on, we always consider the dimension $N = 1$. For instance, if β is a continuous function we have for each α by applying (2.4),

$$\begin{aligned} &\delta_{\dot{\alpha}}\beta = \delta_{\dot{\alpha}}(\beta+0) = \delta\beta + (\delta*\alpha)0 = \beta(0)\delta, \\ &\beta_{\dot{\alpha}}\delta = \beta_{\dot{\alpha}}(0+\delta) = \beta 0 + (\beta*\alpha)\delta = [(\beta*\alpha)(0)]\delta, \\ &\delta_{\dot{\alpha}}\delta = \delta_{\dot{\alpha}}(0+\delta) = \delta 0 + (\delta*\alpha)\delta = \alpha\delta = \alpha(0)\delta, \\ &\delta_{\dot{\alpha}}(D\delta) = (\delta*\alpha)D\delta = \alpha D\delta = \alpha(0)D\delta - \alpha'(0)\delta, \\ &(D\delta)_{\dot{\alpha}}\delta = (D\delta*\alpha)\delta = \alpha'\delta = \alpha'(0)\delta, \end{aligned} \tag{2.5}$$

$$H_{\dot{\alpha}}\delta = (H*\alpha)\delta = \left[\int_{-\infty}^{+\infty} \alpha(-\tau)H(\tau)\,d\tau\right]\delta = \left(\int_{-\infty}^{0} \alpha\right)\delta. \tag{2.6}$$

For each α, the support of the α-product (2.4) satisfies $\mathrm{supp}(T_{\dot\alpha}S) \subset \mathrm{supp}\, S$, as for usual functions, but it may happen that $\mathrm{supp}(T_{\dot\alpha}S) \not\subset \mathrm{supp}\, T$. For instance, if $a, b \in \mathbb{R}$, from (2.4) we have,

$$(\tau_a\delta)_{\dot\alpha}(\tau_b\delta) = [(\tau_a\delta) * \alpha](\tau_b\delta) = (\tau_a\alpha)(\tau_b\delta) = \alpha(b-a)(\tau_b\delta).$$

It is also possible to multiply many other distributions preserving the consistency with all Schwartz products of distributions with functions. For instance, using the Leibniz formula to extend the α-products, it is possible to write

$$T_{\dot\alpha}S = Tw + (T * \alpha)f, \tag{2.7}$$

with $T \in \mathscr{D}'^{-1}$ and $S = w + f \in L^1_{loc} \oplus \mathscr{D}'_\mu$, where $\mathscr{D}'^{-1}$ stands for the space of distributions $T \in \mathscr{D}'$ such that $DT \in \mathscr{D}'^0$ and Tw is the usual pointwise product of $T \in \mathscr{D}'^{-1}$ with $w \in L^1_{loc}$. Recall that, locally, T can be read as a function of bounded variation (see [24], Sec. 2 for details). For instance, since $H \in \mathscr{D}'^{-1}$ and $H = H + 0 \in L^1_{loc} \oplus \mathscr{D}'_\mu$, we have

$$H_{\dot\alpha}H = HH + (H * \alpha)0 = H. \tag{2.8}$$

More generally, if $T \in \mathscr{D}'^{-1}$ and $S \in L^1_{loc}$, then $T_{\dot\alpha}S = TS$ because by (2.7) we can write

$$T_{\dot\alpha}S = T_{\dot\alpha}(S+0) = TS + (T * \alpha)0 = TS.$$

Thus, in distributional sense, the α-products corresponding to functions that, locally, are of bounded variation coincide with the usual pointwise product of these functions seen as a distribution. We stress that in (2.4) or (2.7) the convolution $T * \alpha$ is not to be understood as an approximation of T. Those formulas are exact.

Another useful extension is given by the formula

$$T_{\dot\alpha}S = D(Y_{\dot\alpha}S) - Y_{\dot\alpha}(DS), \tag{2.9}$$

for $T \in \mathscr{D}'^0 \cap \mathscr{D}'_\mu$ and $S, DS \in L^1_{loc} \oplus \mathscr{D}'_c$, where $\mathscr{D}'_c \subset \mathscr{D}'_\mu$ is the space of distributions whose support is at most countable, and $Y \in \mathscr{D}'^{-1}$ is such that $DY = T$ (the products $Y_{\dot\alpha}S$ and $Y_{\dot\alpha}(DS)$ are supposed to be computed by (2.4) or (2.7)). The value of $T_{\dot\alpha}S$ given by (2.9) is independent of the choice of $Y \in \mathscr{D}'^{-1}$ such that $DY = T$ (see [24] p. 1004 for the proof). For instance, by (2.6) and (2.9), we have for any α,

$$\delta_{\dot\alpha}H = D(H_{\dot\alpha}H) - H_{\dot\alpha}(DH) = DH - H_{\dot\alpha}\delta = \delta - \left(\int_{-\infty}^{0} \alpha\right)\delta = \left(\int_{0}^{+\infty} \alpha\right)\delta. \tag{2.10}$$

so that $H_{\dot\alpha}\delta + \delta_{\dot\alpha}H = \delta$ for any α. The products (2.4), (2.7), and (2.9) are compatible, that is, if an α-product can be computed by two of them, the result is the same.

3. Powers of distributions

Let $M \subset \mathscr{D}'$ be a set of distributions such that, if $T_1, T_2 \in M$, then $T_1{}_{\dot\alpha}T_2$ is well defined and $T_1{}_{\dot\alpha}T_2 \in M$. For each $T \in M$ we define the α-power T^n_α by the recurrence relation

$$T^n_\alpha = (T^{n-1}_\alpha)_{\dot\alpha}T \text{ for } n \geq 1, \text{ with } T^0_\alpha = 1 \text{ for } T \neq 0; \tag{3.1}$$

naturally, if $0 \in M$, $0^n_\alpha = 0$ for all $n \geq 1$. Since our distributional products are consistent with all Schwartz products of distributions with functions, when the functions are placed at the right-hand side, we have $\beta^n_{\dot\alpha} = \beta^n$ for all $\beta \in C^0 \cap M$. Thus, this definition is consistent with the usual definition of powers of C^0-functions. Moreover, if M is such that $\tau_a T \in M$ for all $T \in M$ and all $a \in \mathbb{R}$, then we also have $(\tau_a T)^n_{\dot\alpha} = \tau_a(T^n_{\dot\alpha})$.

Taking, for instance, $M = C^p \oplus (\mathscr{D}'^p \cap \mathscr{D}'_\mu)$ and supposing $T_1, T_2 \in M$, we have $T_1 = \beta_1 + f_1$, $T_2 = \beta_2 + f_2$ and by (2.4), we can write

$$\begin{aligned} T_1 \dot\alpha T_2 &= T_1\beta_2 + (T_1 * \alpha) f_2 = (\beta_1 + f_1)\beta_2 + [(\beta_1 + f_1) * \alpha] f_2 \\ &= \beta_1\beta_2 + f_1\beta_2 + [(\beta_1 + f_1) * \alpha] f_2 \in M. \end{aligned}$$

Therefore, we can define α-powers $T^n_{\dot\alpha}$ of distributions $T \in C^p \oplus (\mathscr{D}'^p \cap \mathscr{D}'_\mu)$. For instance, if $m \in \mathbb{C}\backslash\{0\}$, we have $(m\delta)^0_{\dot\alpha} = 1$, $(m\delta)^1_{\dot\alpha} = m\delta$, and for $n \geq 2$, $(m\delta)^n_{\dot\alpha} = m^n[\alpha(0)]^{n-1}\delta$, as can be easily seen by induction applying (2.5).

Setting $M = \mathscr{D}'^{-1}$ and supposing $T_1, T_2 \in \mathscr{D}'^{-1}$, we have $T_1 \dot\alpha T_2 \in \mathscr{D}'^{-1}$. Thus, we also can define α-powers $T^n_{\dot\alpha}$ of distributions $T \in \mathscr{D}'^{-1}$ by the recurrence relation (3.1) and clearly we get,

$$T^n_{\dot\alpha} = T^n,$$

that is, in distributional sense the α-powers corresponding to functions that, locally, are of bounded variation, coincide with the usual powers of these functions seen as distributions. In the sequel we will write, in all cases, T^n instead of $T^n_{\dot\alpha}$, supposing α fixed. For instance, if $m \in \mathbb{C}$ we will write $(m\delta)^1 = m\delta$, and for $n \geq 2$, $(m\delta)^n = m^n[\alpha(0)]^{n-1}\delta$. More generally, taking $M = \{a + (b-a)H + m\delta : a, b, m \in \mathbb{C}\}$ we have:

Theorem 3.1. *Given α, let us suppose $a, b, m \in \mathbb{C}$, $p = \int_{-\infty}^0 \alpha$, $q = \int_0^{+\infty} \alpha$ and $\lambda = \alpha(0)m + (b-a)q$. Then we have*

$$\big[a + (b-a)H + m\delta\big]^n = a^n + (b^n - a^n)H + m[P_{n-1}(a+\lambda)]\delta, \tag{3.2}$$

where P_{n-1} is the polynomial defined by the recurrence relation $P_0(s) = 1$ and for $n \geq 1$, $P_n(s) = sP_{n-1}(s) + pb^n + qa^n$.

Proof. We will prove (3.2) by induction. For $n = 1$ the statement is clearly true. Let us now suppose that (3.2) is true for n. Then, taking into account the bilinearity of the α-products, formulas (2.5), (2.6), (2.8), (2.10), and also that $p + q = 1$, we have

$$\begin{aligned} \big[a + (b-a)H + m\delta\big]^{n+1} &= \big[a + (b-a)H + m\delta\big]^n_{\dot\alpha}\big[a + (b-a)H + m\delta\big] \\ &= \big[a^n + (b^n - a^n)H + mP_{n-1}(a+\lambda)\delta\big]_{\dot\alpha}\big[a + (b-a)H + m\delta\big] \\ &= a^{n+1} + a^n(b-a)H + a^n m\delta + (b^n - a^n)aH + (b^n - a^n)(b-a)H + \\ &\quad + (b^n - a^n)mp\delta + maP_{n-1}(a+\lambda)\delta + m(b-a)P_{n-1}(a+\lambda)q\delta + \\ &\quad + m^2 P_{n-1}(a+\lambda)\alpha(0)\delta \\ &= a^{n+1} + (b^{n+1} - a^{n+1})H + m\big[a^n + (b^n - a^n)p + (a+\lambda)P_{n-1}(a+\lambda)\big]\delta \\ &= a^{n+1} + (b^{n+1} - a^{n+1})H + m\big[pb^n + qa^n + (a+\lambda)P_{n-1}(a+\lambda)\big]\delta \\ &= a^{n+1} + (b^{n+1} - a^{n+1})H + mP_n(a+\lambda), \end{aligned}$$

which proves the statement. □

4. Composition of entire functions with distributions

Let $\phi : \mathbb{C} \to \mathbb{C}$ be an entire function. Then we have,

$$\phi(s) = a_0 + a_1 s + a_2 s^2 + \cdots \tag{4.1}$$

for the sequence $a_n = \frac{\phi^{(n)}(0)}{n!}$ of complex numbers and all $s \in \mathbb{C}$. If $T \in M$, we define the composition $\phi \circ T$ by formula

$$\phi \circ T = a_0 + a_1 T + a_2 T^2 + \cdots \tag{4.2}$$

whenever this series converge in $\mathscr{D}'$. Clearly, this definition is consistent with the usual meaning of $\phi \circ T$, if $T \in M$ is a function. Moreover, if M is such that $\tau_a T \in M$ for all $T \in M$ and all $a \in \mathbb{R}$, we have $\tau_a(\phi \circ T) = \phi \circ (\tau_a T)$, if $\phi \circ T$ or $\phi \circ (\tau_a T)$ are well defined. Remember that, in general, $\phi \circ T$ depends on α. For instance, taking $M = \{m\delta : m \in \mathbb{C}\}$ we have

Theorem 4.1. *Let $m \in \mathbb{C}$ and let $\phi : \mathbb{C} \to \mathbb{C}$ be an entire function. Then, given α, we have*

$$\phi \circ (m\delta) = \begin{cases} \phi(0) + \phi'(0) m\delta & \text{if } \alpha(0) = 0, \\ \phi(0) + \frac{\phi(m\alpha(0)) - \phi(0)}{\alpha(0)} \delta & \text{if } \alpha(0) \neq 0. \end{cases} \tag{4.3}$$

Proof. If $m = 0$ the result is obvious. Suppose $m \neq 0$ and let ϕ be defined by (4.1). Then, by definition (4.2)

$$\begin{aligned} \phi \circ (m\delta) &= a_0 + a_1 m\delta + a_2 (m\delta)^2 + a_3 (m\delta)^3 + \cdots \\ &= a_0 + a_1 m\delta + a_2 m^2 \alpha(0)\delta + a_3 m^3 \alpha(0)^2 \delta + \cdots \\ &= a_0 + [a_1 + a_2 m\alpha(0) + a_3 m^2 \alpha(0)^2 + \cdots](m\delta), \end{aligned} \tag{4.4}$$

if this series converges in $\mathscr{D}'$. Thus, if $\alpha(0) = 0$ we have $\phi \circ (m\delta) = a_0 + a_1 m\delta$. If $\alpha(0) \neq 0$, setting

$$S = a_1 + a_2 m\alpha(0) + a_3 m^2 \alpha(0)^2 + \cdots$$

it follows

$$m\alpha(0) S = a_1 [m\alpha(0)] + a_2 [m\alpha(0)]^2 + a_3 [m\alpha(0)]^3 + \cdots = \phi\big(m\alpha(0)\big) - a_0,$$

and the result stems from (4.4) because $a_0 = \phi(0)$ and $a_1 = \phi'(0)$. $\square$

With the goal of computing $\phi \circ \big(a + (b-a)H + m\delta\big)$, we need the following statements.

Lemma 4.1. *Let $a, b, p, q \in \mathbb{C}$ and let P_n be the sequence of polynomials defined by $P_0(s) = 1$, $P_n(s) = sP_{n-1}(s) + pb^n + qa^n$. Then, there exists $c \geq 1$ such that for any $n \geq 0$,*

$$\big|P_n(s)\big| \leq \big(|s| + 2c^2\big)^n. \tag{4.5}$$

Proof. By induction. Let $c = \max\{|a|,|b|,|p|,|q|,1\}$. For $n = 0$ (4.5) is clearly true. Now, suppose that (4.5) is satisfied for a certain $n \geq 0$. Then

$$\begin{aligned}
\left|P_{n+1}(s)\right| &= \left|sP_n(s) + pb^{n+1} + qa^{n+1}\right| \\
&\leq |s|\left|P_n(s)\right| + |p||b|^{n+1} + |q||a|^{n+1} \\
&\leq |s|\left(|s| + 2c^2\right)^n + 2c^{n+2}.
\end{aligned}$$

However, $2c^{n+2} = 2c^2c^n \leq 2c^2\left(|s| + 2c^2\right)^n$ and so

$$\left|P_{n+1}(s)\right| \leq |s|\left(|s| + 2c^2\right)^n + 2c^2\left(|s| + 2c^2\right)^n = \left(|s| + 2c^2\right)^{n+1}.$$

□

Lemma 4.2. *With the assumptions of lemma 4.1 let us suppose ϕ an entire function defined by (4.1). Then, the function $W_\phi : \mathbb{C} \to \mathbb{C}$ defined by $W_\phi(s) = \sum_{n=1}^{\infty} a_n P_{n-1}(s)$ is well defined and satisfies the following conditions:*

(a) W_ϕ is an entire function;

(b) if $p + q = 1$ then $W_\phi(pa + qb) = \begin{cases} \frac{\phi(b)-\phi(a)}{b-a} & \text{if } b \neq a \\ \phi'(a) & \text{if } b = a \end{cases}$;

(c) if $p + q = 1$ then $W_\phi = 0$ if and only if $\phi' = 0$.

Proof. (a) In fact, $W_\phi(s)$ is a power series of s and it is absolutely convergent for each s because by lemma 4.1 $\left|a_n P_{n-1}(s)\right| \leq |a_n|\left(|s| + 2c^2\right)^{n-1}$ and the series

$$\sum_{n=1}^{\infty} |a_n|\left(|s| + 2c^2\right)^{n-1} = \frac{1}{|s| + 2c^2} \sum_{n=1}^{\infty} |a_n|\left(|s| + 2c^2\right)^n$$

converges, since $\phi(s)$ is absolutely convergent.

(b) It is easy to prove by induction that if $p + q = 1$, then for any $n \geq 2$ entire we have

$$P_{n-1}(pa + qb) = \begin{cases} \frac{b^n - a^n}{b-a} & \text{if } b \neq a \\ na^{n-1} & \text{if } b = a \end{cases}.$$

Thus, if $b \neq a$ we have

$$\begin{aligned}
W_\phi(pa + qb) &= \sum_{n=1}^{\infty} a_n P_{n-1}(pa + qb) = a_1 + \sum_{n=2}^{\infty} a_n P_{n-1}(pa + qb) \\
&= a_1 + \sum_{n=2}^{\infty} a_n \frac{b^n - a^n}{b - a} = \frac{\phi(b) - \phi(a)}{b - a};
\end{aligned}$$

if $b = a$ we have

$$\begin{aligned}
W_\phi(pa + qb) &= \sum_{n=1}^{\infty} a_n P_{n-1}(pa + qb) = a_1 + \sum_{n=2}^{\infty} a_n P_{n-1}(pa + qb) \\
&= a_1 + \sum_{n=2}^{\infty} a_n n a^{n-1} = \phi'(a).
\end{aligned}$$

(c) Let us suppose that $\phi' \neq 0$, that is, let us suppose that ϕ is not a constant function. Then, there exist $a, b \in \mathbb{C}$ such that $a \neq b$ and $\phi(a) \neq \phi(b)$. Applying (b) we conclude that $W_\phi(pa + qb) \neq 0$, that is, $W_\phi \neq 0$. Conversely, if $\phi' = 0$ then $\phi'(s) = a_1 + 2a_2 s + 3a_3 s^2 + \cdots = 0$, for all $s \in \mathbb{C}$. Therefore, $a_1 = a_2 = a_3 = \cdots = 0$ and $W_\phi = 0$ follows. □

Theorem 4.2. *Given α, let $a,b,m \in \mathbb{C}$, $q = \int_0^\infty \alpha$ and $\lambda = \alpha(0)m + q(b-a)$. Suppose also that $T = a + (b-a)H + m\delta$ and ϕ is an entire function defined by (4.1). Then,*

$$\phi \circ T = \phi(a) + [\phi(b) - \phi(a)]H + mW_\phi(a+\lambda)\delta,$$

where W_ϕ is defined in lemma 4.2.

Proof. By (4.2) and theorem 3.1 we have

$$\begin{aligned}
\phi \circ T &= a_0 + \sum_{n=1}^{\infty} a_n[a^n + (b^n - a^n)H + mP_{n-1}(a+\lambda)\delta] \\
&= a_0 + \sum_{n=1}^{\infty} a_n a^n + \sum_{n=1}^{\infty} a_n(b^n - a^n)H + m\sum_{n=1}^{\infty} a_n P_{n-1}(a+\lambda)\delta \\
&= a_0 + [\phi(a) - a_0] + [\phi(b) - a_0 - \phi(a) + a_0]H + mW_\phi(a+\lambda)\delta \\
&= \phi(a) + [\phi(b) - \phi(a)]H + mW_\phi(a+\lambda)\delta
\end{aligned}$$

because $\sum_{n=1}^{\infty} a_n a^n$, $\sum_{n=1}^{\infty} a_n(b^n - a^n)$, and $\sum_{n=1}^{\infty} a_n P_{n-1}(a+\lambda)$ converge. □

5. The α-solution concept

Let I be an interval of $\mathbb{R}$ with more that one point, and let $\mathscr{F}(I)$ be the space of continuously differentiable maps $\tilde{u} : I \to \mathscr{D}'$ in the sense of the usual topology of $\mathscr{D}'$. For $t \in I$, the notation $[\tilde{u}(t)](x)$ is sometimes used for emphasizing that the distribution $\tilde{u}(t)$ acts on functions $\xi \in \mathscr{D}$ depending on x.

Let $\Sigma(I)$ be the space of functions $u : \mathbb{R} \times I \to \mathbb{R}$ such that:

(a) for each $t \in I$, $u(x,t) \in L^1_{loc}(\mathbb{R})$;
(b) $\tilde{u} : I \to \mathscr{D}'$, defined by $[\tilde{u}(t)](x) = u(x,t)$ is in $\mathscr{F}(I)$.

The natural injection $u \mapsto \tilde{u}$ from $\Sigma(I)$ into $\mathscr{F}(I)$ identifies any function in $\Sigma(I)$ with a certain map in $\mathscr{F}(I)$. Since $C^1(\mathbb{R}\times I) \subset \Sigma(I)$, we can write the inclusions

$$C^1(\mathbb{R}\times I) \subset \Sigma(I) \subset \mathscr{F}(I).$$

Thus, identifying u with $\tilde{u}$ and v with $\tilde{v}$ the system (1.1), (1.2) can be read as follows:

$$\frac{d\tilde{u}}{dt}(t) + D\big[\tfrac{1}{2}\tilde{u}(t)^2 + \phi \circ \big(\tilde{v}(t)\big)\big] = 0, \tag{5.1}$$

$$\frac{d\tilde{v}}{dt}(t) + D\big[\tilde{u}(t)\dot{\alpha}\tilde{v}(t)\big] = 0. \tag{5.2}$$

Definition 5.1. Given α, the pair $(\tilde{u},\tilde{v}) \in \mathscr{F}(I) \times \mathscr{F}(I)$ will be called an α- solution of the system (5.1), (5.2) on I, if the α-products appearing in this system are well defined, and if both equations are satisfied for all $t \in I$.

We have the following results:

Theorem 5.1. *If (u,v) is a classical solution of (1.1), (1.2) on $\mathbb{R}\times I$ then, for any α, the pair $(\tilde{u},\tilde{v})\in\mathscr{F}(I)\times\mathscr{F}(I)$ defined by $[\tilde{u}(t)](x)=u(x,t)$, $[\tilde{v}(t)](x)=v(x,t)$ is an α-solution of (5.1), (5.2).*

Note that, by a classical solution of (1.1), (1.2) on $\mathbb{R}\times I$, we mean a pair $\big(u(x,t),v(x,t)\big)$ of C^1-functions that satisfies (1.1), (1.2) on $\mathbb{R}\times I$.

Theorem 5.2. *If $u,v\colon \mathbb{R}\times I\to\mathbb{R}$ are C^1-functions and, for a certain α, the pair $(\tilde{u},\tilde{v})\in\mathscr{F}(I)\times\mathscr{F}(I)$ defined by $[\tilde{u}(t)](x)=u(x,t)$, $[\tilde{v}(t)](x)=v(x,t)$ is an α-solution of (5.1), (5.2), then the pair (u,v) is a classical solution of (1.1), (1.2) on $\mathbb{R}\times I$.*

For the proof, it is enough to observe that C^1-functions $u(x,t)$, $v(x,t)$ can be read as continuously differentiable functions $\tilde{u},\tilde{v}\in\mathscr{F}(I)$ defined by $[\tilde{u}(t)](x)=u(x,t)$, $[\tilde{v}(t)](x)=v(x,t)$ and to use the consistency of the α-products with the classical Schwartz products. As a consequence, an α-solution $(\tilde{u},\tilde{v})$ in this sense, read as a usual distributional solution (u,v) affords a general consistent extension of the concept of a classical solution of the system (1.1), (1.2).

If in equation (5.2) we replace $\tilde{u}(t)\,\dot{\alpha}\,\tilde{v}(t)$ by $\tilde{v}(t)\,\dot{\alpha}\,\tilde{u}(t)$, we get the equation

$$\frac{d\tilde{v}}{dt}(t)+D\big[\tilde{v}(t)\,\dot{\alpha}\,\tilde{u}(t)\big]=0 \tag{5.3}$$

which is not equivalent to (5.2), since our α-products are not, in general, commutative. However, all we have said for the systems (1.1), (1.2) and (5.1), (5.2) is also valid for the systems (1.1), (1.2) and (5.1), (5.3). Thus, taking advantage of this situation, we will introduce the following definition, which further extends the concept of a classical solution.

Definition 5.2. Given α, we call α-solution of the system (1.1), (1.2) on I, to any α-solution of the system (5.1), (5.2), or of the system (5.1), (5.3), on I.

As it is well known, a weak solution of a differential equation is a function for which the derivatives may not exist but satisfies the equation in some precise sense.

One of the most important definitions is based in the classical theory of distributions. In this theory, the study of differential equations is, of course, restricted to linear equations, owing to the well known difficulties of multiplying distributions. The classical setting usually considers linear partial differential equations with C^∞-coefficients, and a weak solution is defined as satisfying the equation in the sense of distributions.

Linear partial differential evolution equations in the unknown u can be re-interpreted as evolution equations with α-products, in the unknown $\tilde{u}(t)$, if the (re-interpreted) C^∞-coefficients are placed at the right-hand side of $\tilde{u}(t)$ and its derivatives. Actually, in this case, our α-products are consistent with the products of distributions with C^∞-functions. Thus, if $u\in\Sigma(I)$ is a weak solution, then, for any α, the corresponding map $\tilde{u}\in\mathscr{F}(I)$ is an α-solution. Conversely, if $u\in\Sigma(I)$ and for a certain α the corresponding map $\tilde{u}\in\mathscr{F}(I)$ is an α-solution, then u is a weak solution. In this sense, the α-solution concept can be identified with the weak solution concept. Meanwhile, an advantage arises: the coefficients of such equations can now be considered as distributions, if the α-products involved are well defined and the solutions are considered as elements of $\mathscr{F}(I)$.

Thus, in the framework of evolution equations, the α-solution concept is an extension of the classical solution concept, and may be also considered as a new type of weak solution provided by distribution theory in the nonlinear setting.

6. The Riemann problem (1), (2), (3), (4)

Let us consider the system (1.1), (1.2) with $(x,t) \in \mathbb{R} \times \mathbb{R}$ (we could also have considered $(x,t) \in \mathbb{R} \times [0,+\infty[$), ϕ an entire function taking real values on the real axis, and the unknowns u, v subjected to the initial conditions (1.3), (1.4) with $u_1, u_2, v_1, v_2 \in \mathbb{R}$ and $u_1 \neq u_2$. When we read this problem in $\mathscr{F}(\mathbb{R})$ having in mind the identifications $u \mapsto \tilde{u}$, $v \mapsto \tilde{v}$, we must replace the system (1.1), (1.2) by the system (5.1), (5.2) and the initial conditions (1.3), (1.4) by the following ones

$$\tilde{u}(0) = u_1 + (u_2 - u_1)H, \tag{6.1}$$

$$\tilde{v}(0) = v_1 + (v_2 - v_1)H. \tag{6.2}$$

Theorem (6.1) concerns the α-solutions $(\tilde{u}, \tilde{v})$ of the problem (5.1), (5.2), (6.1), (6.2) in the interval of time $I = \mathbb{R}$, which belong to a convenient space $\tilde{W} \subset \mathscr{F}(\mathbb{R}) \times \mathscr{F}(\mathbb{R})$, defined the following way: $(\tilde{u}, \tilde{v}) \in \tilde{W}$ if and only if

$$\tilde{u}(t) = a(t) + b(t)\tau_{\gamma(t)}H, \tag{6.3}$$

$$\tilde{v}(t) = f(t) + g(t)\tau_{\gamma(t)}H + h(t)\tau_{\gamma(t)}\delta, \tag{6.4}$$

for certain C^1-functions $a, b, f, g, h, \gamma\colon \mathbb{R} \to \mathbb{R}$ and all $t \in \mathbb{R}$.

Theorem 6.1. *Let A, B be such that*

$$A = (v_2 - v_1)\Big(\frac{u_1 + u_2}{2} + \frac{\phi(v_2) - \phi(v_1)}{u_2 - u_1}\Big) \text{ and } B = u_2 v_2 - u_1 v_1.$$

Then, given α, the problem (5.1), (5.2), (6.1), (6.2) has an α-solution $(\tilde{u}, \tilde{v}) \in \tilde{W}$ defined by

$$\tilde{u}(t) = u_1 + (u_2 - u_1)\tau_{\gamma(t)}H, \tag{6.5}$$

$$\tilde{v}(t) = v_1 + (v_2 - v_1)\tau_{\gamma(t)}H + (A - B)t\tau_{\gamma(t)}\delta, \tag{6.6}$$

with

$$\gamma(t) = \Big(\frac{u_2 + u_1}{2} + \frac{\phi(v_2) - \phi(v_1)}{u_2 - u_1}\Big)t, \tag{6.7}$$

in the following conditions:

- *(I) if $A = B$ then (6.5), (6.6) is an α-solution;*
- *(II) if $A \neq B$ and $\phi' = 0$, then (6.5), (6.6) is an α-solution if and only if $\int_{-\infty}^{0} \alpha = \frac{1}{2}$;*
- *(III) if $A \neq B$, $\phi' \neq 0$, and $v_2 \neq v_1$, then we have necessarily $v_2 \neq -v_1$ and (6.5), (6.6) is an α-solution if and only if $\phi(v_2) = \phi(v_1)$, $\alpha(0) = 0$, and $\int_{-\infty}^{0} \alpha = \frac{1}{2}$;*
- *(IV) if $A \neq B$, $\phi' \neq 0$, and $v_2 = v_1$, then we have necessarily $v_2 = v_1 \neq 0$ and (6.5), (6.6) is an α-solution if and only if $\phi'(v_1) = 0$, $\alpha(0) = 0$, and $\int_{-\infty}^{0} \alpha = \frac{1}{2}$.*

When it exists, the α-solution (6.5), (6.6) is unique in $\tilde{W}$.

Proof. Let us suppose $(\tilde{u},\tilde{v}) \in \tilde{W}$. Then from (6.3) and (6.4) we must have for each $t \in \mathbb{R}$,

$$\begin{aligned}
\frac{d\tilde{u}}{dt}(t) &= a'(t) + b'(t)\tau_{\gamma(t)}H - b(t)\gamma'(t)\tau_{\gamma(t)}\delta, \\
\frac{d\tilde{v}}{dt}(t) &= f'(t) + g'(t)\tau_{\gamma(t)}H - g(t)\gamma'(t)\tau_{\gamma(t)}\delta + h'(t)\tau_{\gamma(t)}\delta - h(t)\gamma'(t)\tau_{\gamma(t)}(D\delta) \\
&= f'(t) + g'(t)\tau_{\gamma(t)}H + \big[h'(t) - g(t)\gamma'(t)\big]\tau_{\gamma(t)}\delta - h(t)\gamma'(t)\tau_{\gamma(t)}(D\delta).
\end{aligned}$$

Therefore, taking $p = \int_{-\infty}^{0}\alpha$ and $q = \int_{0}^{+\infty}\alpha$, we have $p+q=1$, and using the bilinearity of the α-products and the formulas (2.5), (2.6), (2.8), (2.10) we can write,

$$\begin{aligned}
\tilde{u}^2(t) &= a^2(t) + 2a(t)b(t)\tau_{\gamma(t)}H + b^2(t)\tau_{\gamma(t)}H \\
&= a^2(t) + [b^2(t) + 2a(t)b(t)]\tau_{\gamma(t)}H,
\end{aligned}$$

$$\begin{aligned}
\tilde{u}(t)\,\dot{\alpha}\,\tilde{v}(t) &= a(t)f(t) + a(t)g(t)\tau_{\gamma(t)}H + a(t)h(t)\tau_{\gamma(t)}\delta + \\
&\quad + b(t)f(t)\tau_{\gamma(t)}H + b(t)g(t)\tau_{\gamma(t)}H + b(t)h(t)p\tau_{\gamma(t)}\delta, \\
&= a(t)f(t) + \big[a(t)g(t) + b(t)f(t) + b(t)g(t)\big]\tau_{\gamma(t)}H + \\
&\quad + h(t)\big[a(t) + pb(t)\big]\tau_{\gamma(t)}\delta.
\end{aligned} \tag{6.8}$$

On the one hand, we have

$$\phi \circ \big(\tilde{v}(t)\big) = \phi \circ \Big(\tau_{\gamma(t)}\big(f(t) + g(t)H + h(t)\delta\big)\Big) = \tau_{\gamma(t)}\phi \circ \big(f(t) + g(t)H + h(t)\delta\big).$$

On the other hand, applying theorem 4.2 with $a = f(t)$, $b = f(t) + g(t)$ and $m = h(t)$, we have

$$\begin{aligned}
\phi \circ \big(f(t) + g(t)H + h(t)\delta\big) = \phi\big(f(t)\big) + \big[\phi\big(f(t) + g(t)\big) - \phi\big(f(t)\big)\big]H + \\
+ h(t)W_\phi\big(f(t) + \alpha(0)h(t) + qg(t)\big)\delta.
\end{aligned}$$

Thus,

$$\phi \circ \tilde{v}(t) = \phi\big(f(t)\big) + \big[\phi\big(f(t) + g(t)\big) - \phi\big(f(t)\big)\big]\tau_{\gamma(t)}H + h(t)W_\phi\big(f(t) + \alpha(0)h(t) + qg(t)\big)\tau_{\gamma(t)}\delta.$$

As a consequence, equations (5.1), (5.2) turn out to be

$$\begin{aligned}
&a'(t) + b'(t)\tau_{\gamma(t)}H + \\
&\qquad + \big[-b(t)\gamma'(t) + \tfrac{1}{2}b^2(t) + a(t)b(t) + \phi\big(f(t) + g(t)\big) - \phi\big(f(t)\big)\big]\tau_{\gamma(t)}\delta + \\
&\qquad\qquad + h(t)W_\phi\big(f(t) + \alpha(0)h(t) + qg(t)\big)\tau_{\gamma(t)}(D\delta) = 0,
\end{aligned}$$

$$\begin{aligned}
&f'(t) + g'(t)\tau_{\gamma(t)}H + \\
&\qquad + \big[h'(t) - g(t)\gamma'(t) + a(t)g(t) + b(t)f(t) + b(t)g(t)\big]\tau_{\gamma(t)}\delta + \\
&\qquad\qquad + h(t)\big[-\gamma'(t) + a(t) + pb(t)\big]\tau_{\gamma(t)}(D\delta) = 0,
\end{aligned}$$

and we conclude that $(\tilde{u},\tilde{v})$ defined by (6.3), (6.4) is an α-solution of (5.1), (5.2) if and only if the following eight equations hold

$$a'(t)=0, \quad b'(t)=0, \quad f'(t)=0, \quad g'(t)=0, \tag{6.9}$$

$$-b(t)\gamma'(t)+\tfrac{1}{2}b^2(t)+a(t)b(t)+\phi\big(f(t)+g(t)\big)-\phi\big(f(t)\big)=0, \tag{6.10}$$

$$h(t)W_\phi\big(f(t)+\alpha(0)h(t)+qg(t)\big)=0, \tag{6.11}$$

$$h'(t)-g(t)\gamma'(t)+a(t)g(t)+b(t)f(t)+b(t)g(t)=0, \tag{6.12}$$

$$h(t)\big[-\gamma'(t)+a(t)+pb(t)\big]=0. \tag{6.13}$$

From (6.1) and (6.3), we have $a(0)+b(0)\tau_{\gamma(0)}H=u_1+(u_2-u_1)H$ and $a(0)=u_1$, $b(0)=u_2-u_1$ follow. Moreover, $\gamma(0)=0$ because $u_2\neq u_1$. Likewise, from (6.2) and (6.4) we have $f(0)+g(0)H+h(0)\delta=v_1+(v_2-v_1)H$ and $f(0)=v_1$, $g(0)=v_2-v_1$, $h(0)=0$ also follow. As a consequence, (6.9), (6.10), (6.11), (6.12), (6.13) turn out to be

$$a(t)=u_1, \quad b(t)=u_2-u_1, \quad f(t)=v_1, \quad g(t)=v_2-v_1,$$

$$-(u_2-u_1)\gamma'(t)+\tfrac{1}{2}\big(u_2^2-u_1^2\big)+\phi(v_2)-\phi(v_1)=0, \tag{6.14}$$

$$h(t)W_\phi\big(\alpha(0)h(t)+pv_1+qv_2\big)=0, \tag{6.15}$$

$$h'(t)-(v_2-v_1)\gamma'(t)+B=0, \tag{6.16}$$

$$h(t)\big[-\gamma'(t)+u_1+p(u_2-u_1)\big]=0, \tag{6.17}$$

where W_ϕ is the function defined in lemma 4.2 with $a=v_1$ and $b=v_2$. From (6.14) we get

$$\gamma'(t)=\frac{u_1+u_2}{2}+\frac{\phi(v_2)-\phi(v_1)}{u_2-u_1}$$

and (6.7) follows. From (6.16) we get

$$h'(t)=(v_2-v_1)\Big(\frac{u_1+u_2}{2}+\frac{\phi(v_2)-\phi(v_1)}{u_2-u_1}\Big)-B=A-B,$$

and then $h(t)=(A-B)t$. Thus,

(I) If $A=B$, then $h(t)=0$, (6.15), (6.16), (6.17) are satisfied and (6.5), (6.6) follow for any α, with $\gamma(t)$ given by (6.7).

(II) If $A\neq B$ and $\phi'=0$, then by lemma 4.2(c) $W_\phi=0$, the equations (6.15), (6.16) are satisfied and from (6.17), since ϕ is a constant function , we get

$$(A-B)t\big[-\tfrac{1}{2}(u_1+u_2)+u_1+p(u_2-u_1)\big]=0,$$

which is satisfied for all $t\in\mathbb{R}$ if and only if $[\cdots]=0$, that is, if and only if $p=\frac{1}{2}$.

(III) If $A\neq B$, $\phi'\neq 0$, and $v_1\neq v_2$ then, by lemma 4.2(c), $W_\phi\neq 0$ and (6.15) turns out to be

$$(A-B)tW_\phi\big(\alpha(0)(A-B)t+pv_1+qv_2\big)=0,$$

which is satisfied for all t if and only if for all $t\neq 0$

$$W_\phi\big(\alpha(0)(A-B)t+pv_1+qv_2\big)=0. \tag{6.18}$$

Thus, we have necessarily $\alpha(0)=0$ because if $\alpha(0)\neq 0$, then for each $t\neq 0$ the number $\alpha(0)(A-B)t+pv_1+qv_2$ would be a zero of W_ϕ, which is impossible because the zeros of the entire function

$W_\phi \neq 0$ are isolated points. Furthermore, from (6.18), we have necessarily $W_\phi(pv_1+qv_2)=0$ and applying lemma 4.2(b) with $a=v_1$ and $b=v_2$ we conclude that $\phi(v_2)=\phi(v_1)$ so that

$$A-B=(v_2-v_1)\tfrac{1}{2}(u_1+u_2)-u_2v_2+u_1v_1=\tfrac{1}{2}(v_1+v_2)(u_1-u_2),$$

and $v_2 \neq -v_1$ follows because $A \neq B$. Since (6.16) is satisfied and (6.17) implies $p=\frac{1}{2}$, the necessity of the statement is proved. Conversely, if $\phi(v_2)=\phi(v_1)$, then by lemma 4.2(b) with $a=v_1$ and $b=v_2$ we conclude that $W_\phi(pv_1+qv_2)=0$ and (6.15) is satisfied for all $t\in\mathbb{R}$ because $\alpha(0)=0$. Moreover, (6.17) is satisfied because $p=\frac{1}{2}$. Hence, (III) is proved.

(IV) Clearly, in this case, we cannot have $v_1=v_2=0$ because it would follows $A=B$. Furthermore, from $\phi'\neq 0$ we conclude, as in the proof of (III), that $W_\phi(pv_1+qv_2)=0$ and $\alpha(0)=0$. By lemma 4.2(b) with $a=b=v_1=v_2$ it follows $\phi'(v_1)=0$. Conversely, if $\phi'(v_1)=0$, from lemma 4.2(b) with $a=b=v_1=v_2$ we have $W_\phi(pv_1+qv_2)=0$ and since $\alpha(0)=0$, (6.15) is satisfied for all $t\in\mathbb{R}$. Moreover (6.16) is satisfied, and also (6.17) because $p=\frac{1}{2}$.

Finally, it is easy to see that in anyone of the four cases the α-solution $(\tilde{u},\tilde{v})$ defined by (6.5), (6.6) is unique in $\tilde{W}$. □

If we replace (5.2) by (5.3), that is, if we replace $\tilde{u}(t)\dot{\alpha}\tilde{v}(t)$ by $\tilde{v}(t)\dot{\alpha}\tilde{u}(t)$, then the value of $\tilde{v}(t)\dot{\alpha}\tilde{u}(t)$ is still given by the same formula (6.8), with q in the place of p, as a result of the product $\delta\dot{\alpha}H$ given by (2.10). Thus, for the problem (5.1), (5.3), (6.1), (6.2), theorem 6.1 must be replaced by another theorem where $p=\int_{-\infty}^{0}\alpha$ must be replaced by $q=\int_{0}^{+\infty}\alpha$. However, since in this theorem $p=\frac{1}{2}$ we have $q=1-p=\frac{1}{2}$ and both theorems coincide !

As a consequence of definition 5.2, these considerations allows us to conclude that when the output of the problem (1.1), (1.2), (1.3), (1.4) is conditioned to the space W, we get:

- if $A=B$, a travelling shock wave given by

$$u(x,t)=u_1+(u_2-u_1)H\big(x-\gamma(t)\big), \tag{6.19}$$
$$v(x,t)=v_1+(v_2-v_1)H\big(x-\gamma(t)\big), \tag{6.20}$$

- if $A\neq B$, the possible emergence of a delta shock wave given by (1.5), (1.6).

In both cases, $\gamma(t)$ is given by (1.7).

7. Examples

By applying theorem 6.1, we now work out the three examples referred in the introduction.

(a) If $\phi=0$ we have $A-B=\frac{1}{2}(v_1+v_2)(u_1-u_2)$ and if $A=B$, that is, if $v_2=-v_1$, we get the travelling shock α-solution (6.19), (6.20) with $\gamma(t)=\frac{1}{2}(u_1+u_2)t$, for any α. The appearing of the delta shock α-solution (1.5), (1.6) is also possible but only if $v_2\neq -v_1$ and $\int_{-\infty}^{0}\alpha=\frac{1}{2}$. This coincides with the result obtained by the first author for the generalized pressureless gas dynamics, taking in [25] p. 1450007-7, $\phi(u)=\frac{1}{2}u^2$ and $\psi(u)=u$. This setting and some generalizations have been extensively studied (see [6–8, 11, 16, 17]). We stress that the referred delta shock wave was obtained in [6, 16] under less general conditions than the ones obtained in [25] (see [6], Corollary 11, p. 3723 and Theorem 13, p. 3729). Thus, for the pressureless gas dynamics system the arising of shock waves and delta shock waves may happen.

(b) If $\phi(v)=\frac{k\gamma}{\gamma-1}v^{\gamma-1}$ ($k\neq 0$ is a real number and $\gamma\geq 2$ is an integer), only case (I) of theorem 6.1 can be applied. Actually, we cannot apply

- (II) because $\phi' \neq 0$;
- (III) because $\phi(v_2) = \phi(v_1)$ is equivalent to $v_2^{\gamma-1} = v_1^{\gamma-1}$ and γ cannot be even because $v_2 \neq v_1$; however, if γ is odd we would have $v_2 = v_1$, which is impossible.
- (IV) because $v_1 \neq 0$ implies $\phi'(v_1) \neq 0$.

Thus, for the isentropic fluid dynamics system with the polytropic pressure-density relation $P(v) = kv^\gamma$, we can only assure the possible emergence of the shock wave (6.19), (6.20) with $\gamma(t) = [\frac{u_2+u_1}{2} + \frac{v_2^{\gamma-1} - v_1^{\gamma-1}}{u_2-u_1}]t$.

(c) If $\phi(v) = \frac{4}{3}v^3 - 3v^2 + 2v$, taking for short $v_1 = \frac{1}{2}$ and $v_2 = \frac{5}{4}$, only case (III) of theorem 10 can be applied. Actually, we cannot apply

- (I) because, $\phi(v_1) = \phi(v_2)$ and $u_1 \neq u_2$ implies $A \neq B$;
- (II) because $\phi' \neq 0$;
- (IV) because $v_2 \neq v_1$.

As a consequence, we get the delta shock α-solution

$$\begin{aligned} u(x,t) &= u_1 + (u_2 - u_1)H\big(x - \gamma(t)\big), \\ v(x,t) &= \tfrac{1}{2} + \tfrac{3}{4}H\big(x - \gamma(t)\big) + \tfrac{7}{8}(u_1 - u_2)t\delta\big(x - \gamma(t)\big), \end{aligned}$$

where $\gamma(t) = \frac{1}{2}(u_1 + u_2)t$, if and only if $\alpha(0) = 0$ and $\int_{-\infty}^{0} \alpha = \frac{1}{2}$.

This example shows that for the isentropic fluid dynamics system (1.8), (1.9) with the pressure-density relation $P(v) = v^4 - 2v^3 + v^2$ the formation of a delta shock wave is possible when we subject the state variables u, v to the initial conditions

$$\begin{aligned} u(x,0) &= u_1 + (u_2 - u_1)H(x), \\ v(x,0) &= \tfrac{1}{2} + \tfrac{3}{4}H(x). \end{aligned}$$

If we want to take into account the usual physical domain of the density v, we may consider, this evolution problem in the time interval $[0, +\infty[$ and also $u_1 > u_2$ in order that the moving mass corresponding to the density $\frac{7}{8}(u_1 - u_2)t\delta(x - \frac{u_1+u_2}{2}t)$ be positive at each instant $t > 0$. Notice also that the pressure $P(v) = v^4 - 2v^3 + v^2 \geq 0$ for all values of v.

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References

[1] A. Bressan and F. Rampazzo, On differential systems with vector valued impulsive controls, *Bull. Un. Mat. Ital.* 2B **7** (1988), 641–656.

[2] J.F. Colombeau and A. Le Roux, Multiplication of distributions in elasticity and hydrodynamics, *J. Math. Phys.* **29**, (1988), 315–319.

[3] C. Dafermos, *Hyperbolic Conservation Laws in Continuum Physics,* 2nd, Springer (2005).

[4] G. Dal Maso, P. LeFloch and F. Murat, Definitions and weak stability of nonconservative products, *J. Math. Pure Appl.* **74** (1995), 483–548.

[5] V.G. Danilov, V.P. Maslov and V.M. Shelkovich, Algebras of singularities of singular solutions to first-order quasi-linear strictly hyperbolic systems, *Teoret. Mat. Fiz.* **114**(1) (1998), 3–55 (in Russian); *Theor. Math. Phys+* **114**(1) (1998), 1–42.

[6] V.G. Danilov and D. Mitrovic, Delta shock wave formation in the case of triangular hyperbolic system of conservation laws, *J. Differ. Equations* **245** (2008), 3704–3734.

[7] V.G. Danilov and V.M. Shelkovich, Delta-shock wave type solution of hyperbolic systems of conservation laws, *Quart. Appl. Math.* **29** (2005), 401–427.

[8] V.G. Danilov and V.M. Shelkovich, Dynamics of propagation and interaction of δ-shock waves in conservation law systems, *J. Differ. Equations* **211**(2) (2005), 333–381.

[9] Yu. V. Egorov, On the theory of generalized functions, *Usp. Mat. Nauk* **45**(5) (1990), 3–40 (in Russian); *Russ. Math. Surv+* **45**(5) (1990), 1–49.

[10] B.T. Hayes and Ph.G. LeFloch, Measure solutions to a strictly hyperbolic system of conservation laws, *Nonlinearity* **9**(6) (1996), 1547–1563.

[11] F. Huang, Weak solution to pressureless type system, *Commun. Part. Diff. Eq.* **30**(3) (2005), 283–304.

[12] H. Kalisch and D. Mitrovic, Singular solutions of a Fully Nonlinear 2×2 System of Conservation Laws, *Proc. Edinb. Math. Soc.* (2) **55**(3) (2012), 711–729.

[13] B.L. Keyfitz, Conservation laws, delta shocks and singular shocks, in: M. Oberguggenberger et al. (Eds.), *Nonlinear Theory of Generalized Functions*, Chapman & Hall/CRC Res. Notes Math., vol. 401, Chapman & Hall/CRC, Boca Raton, FL (1999), 99–111.

[14] V.P. Maslov, Nonstandard characteristics in asymptotical problems, *Usp. Mat. Nauk* **38**(6) (1983), 3–36 (in Russian); *Russ. Math. Surv+* **38**(6)(1983), 1–42.

[15] V.P. Maslov and G. A. Omel'yanov, Asymptotic soliton-form solutions of equations with small dispersion, *Usp. Mat. Nauk* **36**(3) (1981), 63–126 (in Russian); *Russ. Math. Surv+* **36**(3) (1981), 73–149.

[16] D. Mitrovic, V. Bojkovic and V.G. Danilov, Linearization of the Riemann problem for a triangular system of conservation laws and delta shock wave formation process, *Math. Method. Appl. Sci.* **33** (2010), 904–921.

[17] M. Nedeljkov, Delta and singular delta locus for one-dimensional systems of conservation laws, *Math. Method. Appl. Sci.* **27**(8) (2004), 931–955.

[18] C.O.R. Sarrico, About a family of distributional products important in the applications, *Port. Math.* **45** (1988), 295–316.

[19] C.O.R. Sarrico, Distributional products and global solutions for nonconservative inviscid Burgers equation, *J. Math. Anal. Appl.* **281** (2003), 641–656.

[20] C.O.R. Sarrico, New solutions for the one-dimensional nonconservative inviscid Burgers equation, *J. Math. Anal. Appl.* **317** (2006), 496–509.

[21] C.O.R. Sarrico, Collision of delta-waves in a turbulent model studied via a distributional product, *Nonlinear Anal-Theor.* **73** (2010), 2868–2875.

[22] C.O.R. Sarrico, Products of distributions and singular travelling waves as solutions of advection-reaction equations, *Russ. J. Math. Phys.* **19**(2) (2012), 244–255.

[23] C.O.R. Sarrico, Products of distributions, conservation laws and the propagation of δ'-shock waves, *Chin. Ann. Math.* Ser. B **33**(3) (2012), 367–384.

[24] C.O.R. Sarrico, The multiplication of distributions and the Tsodyks model of synapses dynamics, *Int. J. Math. Anal.* **6**(21), (2012), 999–1014.

[25] C.O.R. Sarrico, A distributional product approach to δ-shock wave solutions for a generalized pressureless gas dynamics system, *Int. J. Math.* **25**(1) (2014), 1450007 (12 pages).

[26] C.O.R. Sarrico, The Brio system with initial conditions involving Dirac masses: a result afforded by a distributional product, *Chin. Ann. Math.* **35B**(6) (2014), 941–954. DOI:10.1007/s11401-014-0862-8.

[27] C.O.R. Sarrico, The Riemann problem for the Brio system: A solution containing a Dirac mass obtained via a distributional product *Russ. J. Math. Phys.* **22**(4) (2015), 518–527.

[28] C.O.R. Sarrico and A. Paiva, Products of distributions and collision of a δ-wave with a δ'-wave in a turbulent model *J. Nonlinear Math. Phy.* **22** (3) (2015), 381–394.

[29] L. Schwartz, *Théorie des distributions* Hermann, Paris (1965).