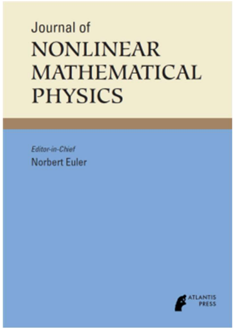

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Periodic solutions of symmetric Kepler perturbations and applications

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We investigate the existence of several families of symmetric periodic solutions as continuation of circular orbits of the Kepler problem for certain symmetric differentiable perturbations using an appropriate set of Poincaré-Delaunay coordinates which are essential in our approach. More precisely, we try separately two situations in an independent way, namely, when the unperturbed part corresponds to a Kepler problem in inertial cartesian coordinates and when it corresponds to a Kepler problem in rotating coordinates on $\mathbb{R}^3$. Moreover, the characteristic multipliers of the symmetric periodic solutions are characterized. The planar case arises as a particular case. Finally, we apply these results to study the existence and stability of periodic orbits of the Matese-Whitman Hamiltonian and the generalized Størmer model.

Keywords: Perturbation theory; Symmetries; Continuation method; Delaunay-Poincaré variables; Circular Solutions.

2000 Mathematics Subject Classification: 34C25, 34C14

1. Introduction and statement of the main results

The objective of this paper is to show analytically the existence of several families of symmetric periodic solutions close to circular orbits of differentiable systems which are symmetric perturbations of the Kepler problem. More precisely, our purpose is to study separately the existence of periodic solutions of Hamiltonian systems in two situations, namely: the first case is associated to the Hamiltonian function of the form

$$H^*(\mathbf{q},\mathbf{p},\varepsilon) = H_0(\mathbf{q},\mathbf{p}) + \varepsilon^\alpha H_1(\mathbf{q},\mathbf{p}) + H_R(\mathbf{q},\mathbf{p},\varepsilon), \quad \mathbf{q},\mathbf{p} \in \mathbb{R}^3, \alpha \in \mathbb{N}, \tag{1.1}$$

where $H_0(\mathbf{q},\mathbf{p}) = \frac{\|\mathbf{p}\|^2}{2} - \frac{1}{\|\mathbf{q}\|}$ is the spatial Kepler problem, and the perturbed functions $H_1(\mathbf{q},\mathbf{p})$ and $H_R(\mathbf{q},\mathbf{p},\varepsilon)$ are both differentiable and H_R is of order $\mathscr{O}(\varepsilon^{\alpha+1})$. Furthermore, the characteristic multipliers of the symmetric periodic solutions are characterized, so its type of stability can be deduced.

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There exists extensive literature where the existence of periodic solutions in Celestial Mechanics is studied using the process of continuation and the symmetry of the problem e.g., [2], [16], [20], [21]. The continuation method goes back to Poincaré who in [18] studied the existence of periodic solutions in the three-body problem using the method that now is called the Poincaré continuation method [20]. Another geometric method was introduced by Duistermaat in two papers [6], [7] which studied the existence of periodic solutions of periodic systems of ordinary differential equations containing a parameter and near equilibrium points, respectively. In our work we obtain periodic solutions of Hamiltonian function which are perturbation of the integrable Kepler problem with 3 degrees of freedom. Our results combine the discrete symmetries of the Hamiltonian and the Continuation Poincaré method, using strongly the first approximation of the solutions of the full Hamiltonian system given by a variational system, although these ideas have been used in other works (see for example, [1], [2], [4], [9], [16], [20], [21], [22], [23], etc.) for specific perturbations or under a different point of view. We point out that our results are interesting, because we try several situations (for different type of perturbations, symmetries, rotating and non-rotating perturbations of the Kepler problem) and we compare them in different theorems. Moreover, the set of coordinates (appropriate Delaunay-Poincaré coordinates) that we use in our approach permits us to give necessary conditions in order to continue different circular Keplerian solutions which are in: the polar plane (third component of angular momentum equal to zero); in an inclined plane and in the equatorial plane (see the applications for more details).

In order to enunciate our main results, under our approach, we consider conveniently modified Poincaré-Delaunay variables (see details for example in [1], [3], [15], [17], [21]). In fact, the main reasons to consider these types of variables can be summarized as follows: Firstly, in these coordinates the characterization of ”reflection” symmetries is simpler, secondly because the periodicity equation (equation that characterizes the initial conditions of symmetric periodic solutions) can be reduced to a minimum number of equations, and in third place the elimination of the degeneracy due to periodicity (maximal rank). We will consider the following Poincaré-Delaunay variables

$$\begin{array}{ll} Q_1 = l+g, & P_1 = L, \\ Q_2 = -\sqrt{2(L-G)}\sin(g), & P_2 = \sqrt{2(L-G)}\cos(g), \\ Q_3 = h, & P_3 = H. \end{array} \qquad \text{(PD-1)} \qquad (1.2)$$

See more details in Section 2. It is verified that in the (PD-1) variables the Hamiltonian (1.1) takes the form

$$\mathscr{H}(\mathbf{Q},\mathbf{P},\varepsilon) = \mathscr{H}_0(\mathbf{Q},\mathbf{P}) + \varepsilon^{\alpha}\mathscr{H}_1(\mathbf{Q},\mathbf{P}) + \mathscr{H}_R(\mathbf{Q},\mathbf{P},\varepsilon), \qquad (1.3)$$

where $\mathscr{H}_R(\mathbf{Q},\mathbf{P},\varepsilon) = \mathscr{O}(\varepsilon^{\alpha+1})$, and the Hamiltonian of the Kepler problem assumes the form

$$\mathscr{H}_0(\mathbf{P}) = -\frac{1}{2P_1^2}. \qquad (1.4)$$

The Hamiltonian system associated to the Hamiltonian (1.3) is written as

$$\dot{\mathbf{Q}} = \mathscr{H}_{\mathbf{P}}, \quad \dot{\mathbf{P}} = -\mathscr{H}_{\mathbf{Q}}. \qquad (1.5)$$

We will use the notation $\varphi(t,\mathbf{Q},\mathbf{P};\varepsilon)$ to denote the flow of the Hamiltonian system associated to (1.5). Now, we are going to assume that the Hamiltonian function H^* in (1.1) is invariant under the anti-symplectic reflection:

$$S_1 : (x,y,z,p_x,p_y,p_z) \longrightarrow (-x,y,z,p_x,-p_y,-p_z).$$

with $\mathbf{q} = (x,y,z)$, $\mathbf{p} = (p_x,p_y,p_z)$. Thus, if $\varphi(t,\mathbf{q},\mathbf{p}) = (x(t),y(t),z(t),p_x(t),p_y(t),p_z(t))$ is a solution associated to the Hamiltonian (1.1), then $S_1 \circ \varphi(t,\mathbf{q},\mathbf{p}) = (-x(-t),y(-t),z(-t),p_x(-t),-p_y(-t),-p_z(-t))$ is also a solution. Now we observe that the fixed set of the symmetry S_1 is the Lagrangian subspace, $\mathscr{L}_1 = \{(0,y,z,p_x,0,0); y,z,p_x \in \mathbb{R}\}$. In particular, if we consider an initial condition $(\mathbf{q},\mathbf{p}) \in \mathscr{L}_1$ such that $\varphi(T/2,\mathbf{q},\mathbf{p}) \in \mathscr{L}_1$, then the solution $\varphi(t,\mathbf{q},\mathbf{p})$ is T-periodic and S_1-symmetric. Next, we consider a convenient circular solution of the Kepler problem (1.4), denoted by $\varphi_{kep}(t,\mathbf{Y}_0)$ with initial condition

$$\mathbf{Y}_0 = (Y_1^{(0)},Y_2^{(0)},Y_3^{(0)},Y_4^{(0)},Y_5^{(0)},Y_6^{(0)}) \equiv (\pi/2,0,0,s^{-1/3},0,p_3) \in \mathscr{L}_1, \tag{1.6}$$

in Section 2, we give the conditions such that the solution hits $\mathscr{L}_1$. After that, we take a small and convenient perturbation of the initial condition $\mathbf{Y}_0$ of the previous circular solution in the form

$$\mathbf{Y} = (\pi/2,\Delta Q_2,0,s^{-1/3}+\Delta P_1,0,p_3+\Delta P_3) \in \mathscr{L}_1, \tag{1.7}$$

that is, we perturb the initial condition of the circular Keplerian solution only in three convenient directions, namely Q_2, P_1 and P_3. Then, we take the solution of the Kepler problem $\varphi_{kep}(t,\mathbf{Y})$ with initial condition $\mathbf{Y}$ which is given in Poincaré-Delaunay (PD-1) variables by

$$\begin{array}{lll} Q_1(t) = st+\pi/2, & Q_2(t) = \Delta Q_2, & Q_3(t) = 0, \\ P_1(t) = s^{-1/3}+\Delta P_1, & P_2(t) = 0, & P_3(t) = p_3+\Delta P_3, \end{array} \tag{1.8}$$

with $s \in \mathbb{R}^+$ and $p_3,\Delta Q_2,\Delta P_1,\Delta P_3 \in \mathbb{R}$. In order to enunciate our first main result we set the number

$$\begin{aligned} \Omega_1 = & \int_0^{T/2} \tfrac{\partial^2 \mathscr{H}_1}{\partial P_3 \partial \Delta Q_2}(\varphi_{kep}(\tau,\mathbf{Y}))\Big|_{\mathbf{Y}=\mathbf{Y}_0} d\tau \cdot \int_0^{T/2} \tfrac{\partial^2 \mathscr{H}_1}{\partial Q_2 \partial \Delta P_3}(\varphi_{kep}(\tau,\mathbf{Y}))\Big|_{\mathbf{Y}=\mathbf{Y}_0} d\tau - \\ & \int_0^{T/2} \tfrac{\partial^2 \mathscr{H}_1}{\partial P_3 \partial \Delta P_3}(\varphi_{kep}(\tau,\mathbf{Y}))\Big|_{\mathbf{Y}=\mathbf{Y}_0} d\tau \cdot \int_0^{T/2} \tfrac{\partial^2 \mathscr{H}_1}{\partial Q_2 \partial \Delta Q_2}(\varphi_{kep}(\tau,\mathbf{Y}))\Big|_{\mathbf{Y}=\mathbf{Y}_0} d\tau. \end{aligned} \tag{1.9}$$

Let $X = (X_1,X_2,X_3,X_4) = (Q_2,Q_3,P_2,P_3)$ and $\varphi(t,(Y_1^{(0)},Y_2^{(0)}+X_1,Y_3^{(0)}+X_2,Y_4^{(0)},Y_5^{(0)}+X_3,Y_6^{(0)}+X_4))$ be a solution of the Kepler problem with initial condition $(Y_1^{(0)},Y_2^{(0)}+X_1,Y_3^{(0)}+X_2,Y_4^{(0)},Y_5^{(0)}+X_3,Y_6^{(0)}+X_4)$,

$$\bar{H}(X) = \int_0^T H_1(\varphi_{kep}(t,(Y_1^{(0)},Y_2^{(0)}+X_1,Y_3^{(0)}+X_2,Y_4^{(0)},Y_5^{(0)}+X_3,Y_6^{(0)}+X_4)))dt, \tag{1.10}$$

be the average function of H_1 and the matrix

$$A = J\left(\frac{\partial^2 \bar{H}}{\partial X_i \partial X_j}\right)_{X=0}, \tag{1.11}$$

where J denotes the standard skew-symmetric matrix. Now we are ready to state our first result which provides sufficient conditions for the existence of S_1-symmetric periodic solutions of (1.1) with fixed period and its characteristic multipliers.

Theorem 1.1. *(First kind S_1-symmetric periodic solutions) Suppose that the Hamiltonian function H^* in (1.1) is S_1-symmetric and let $\varphi_{kep}(t,\mathbf{Y})$ be a solution of the Kepler problem as in (1.8). Assume the following conditions*

$$\begin{aligned}
&(a)\ \int_0^{T/2} \frac{\partial \mathscr{H}_1}{\partial P_3}(\varphi_{kep}(\tau,\mathbf{Y}))\Big|_{\mathbf{Y}=\mathbf{Y}_0} d\tau = 0,\\
&(b)\ \int_0^{T/2} \frac{\partial \mathscr{H}_1}{\partial Q_2}(\varphi_{kep}(\tau,\mathbf{Y}))\Big|_{\mathbf{Y}=\mathbf{Y}_0} d\tau = 0,\\
&(c)\ \Omega_1 \neq 0.
\end{aligned} \tag{1.12}$$

Then for ε sufficiently small there are initial by ε, $\mathbf{Y}_\varepsilon = \mathbf{Y}_0 + \mathbf{Y}(\varepsilon)$ with $\mathbf{Y}(\varepsilon) = (0,\Delta Q_2(\varepsilon),0,\Delta P_1(\varepsilon),0,\Delta P_3(\varepsilon))$ such that $\varphi(t,\mathbf{Y}_\varepsilon;\varepsilon) = \varphi_{kep}(t,\mathbf{Y}_0) + \mathscr{O}(\varepsilon)$ is a S_1-symmetric periodic solution of the Hamiltonian system associated to the Hamiltonian (1.1) or (1.3) with period $T = 2\pi/s$. Moreover, if $\lambda_1,\lambda_2,\lambda_3,\lambda_4$ are the eigenvalues of A in (1.11), then the characteristic multipliers of the periodic solution $\varphi(t,\mathbf{Y}_\varepsilon;\varepsilon)$ are $1,1,1+\varepsilon^\alpha\lambda_1+\mathscr{O}(\varepsilon^{\alpha+1}),1+\varepsilon^\alpha\lambda_2+\mathscr{O}(\varepsilon^{\alpha+1}),1+\varepsilon^\alpha\lambda_3+\mathscr{O}(\varepsilon^{\alpha+1}),1+\varepsilon^\alpha\lambda_4+\mathscr{O}(\varepsilon^{\alpha+1})$.

Theorem 1.1 will be proved in Section 3.

In order to simplify the form of the equations that characterize the T-periodic solution as a continuation of T-circular Keplerian solution (see details in [18]) and for future applications, we are going to assume that in addition to the existence of the reflection S_1, the Hamiltonian function H^* in (1.1) is also invariant under the reflection

$$S_2 : (x,y,z,p_x,p_y,p_z) \longrightarrow (x,-y,-z,-p_x,p_y,p_z),$$

where the fixed set of the symmetry S_2 is the Lagrangian subspace $\mathscr{L}_2 = \{(x,0,0,0,p_y,p_z);x,p_y,p_z \in \mathbb{R}\}$. Thus, if a solution starts in one of these Lagrangian planes at time $t=0$ and hits the other Lagrangian plane at a later time $t = T/4$, then the solution is T-periodic and the orbit of this solution is carried into itself by both symmetries. We call such a periodic solution doubly-symmetric. Geometrically, an orbit intersects $\mathscr{L}_2$ if it hits the x-axis perpendicularly and it intersects $\mathscr{L}_1$ if it hits the yz-plane perpendicularly.

Theorem 1.2. *(First kind $S_1 - S_2$ doubly-symmetric periodic solutions) Suppose that the Hamiltonian function H^* in (1.1) is S_1 and S_2-symmetric and let $\varphi_{kep}(t,\mathbf{Y})$ be a solution of the Kepler problem as in (1.8). Assume that*

$$\begin{aligned}
&(a)\ \int_0^{T/4} \frac{\partial \mathscr{H}_1}{\partial P_3}(\varphi_{kep}(\tau,\mathbf{Y}))\Big|_{\mathbf{Y}=\mathbf{Y}_0} d\tau = 0,\\
&(b)\ \int_0^{T/4} \frac{\partial^2 \mathscr{H}_1}{\partial P_3 \partial \Delta P_3}(\varphi_{kep}(\tau,\mathbf{Y}))\Big|_{\mathbf{Y}=\mathbf{Y}_0} d\tau \neq 0,
\end{aligned} \tag{1.13}$$

hold. Then for ε sufficiently small there are initial by ε, $\mathbf{Y}_\varepsilon = \mathbf{Y}_0 + \mathbf{Y}(\varepsilon)$ with $\mathbf{Y}(\varepsilon) = (0,\Delta Q_2(\varepsilon),0,\Delta P_1(\varepsilon),0,\Delta P_3(\varepsilon))$ such that $\varphi(t,\mathbf{Y}_\varepsilon;\varepsilon) = \varphi_{kep}(t,\mathbf{Y}_0) + \mathscr{O}(\varepsilon)$ is a doubly-symmetric periodic solution of the Hamiltonian system associated to the Hamiltonian (1.1) or (1.3) with period $T = 2\pi/s$. Moreover, if $\lambda_1,\lambda_2,\lambda_3,\lambda_4$ are the eigenvalues of A in (1.11), then the characteristic multipliers of the periodic solution $\varphi(t,\mathbf{Y}_\varepsilon;\varepsilon)$ are $1,1,1+\varepsilon^\alpha\lambda_1+\mathscr{O}(\varepsilon^{\alpha+1}),1+\varepsilon^\alpha\lambda_2+\mathscr{O}(\varepsilon^{\alpha+1}),1+\varepsilon^\alpha\lambda_3+\mathscr{O}(\varepsilon^{\alpha+1}),1+\varepsilon^\alpha\lambda_4+\mathscr{O}(\varepsilon^{\alpha+1})$.

The proof of this theorem can be found in Section 4. We point out that while in Theorem 1.1 we must verify three conditions for the existence of S_1-symmetric periodic solutions (besides the

Hamiltonian possesses one symmetry of reflection), in the case of doubly symmetric periodic solutions (besides the Hamiltonian function H has two symmetries of reflection) in Theorem 1.1 we observe that we need to check only two conditions in a simplified way. This difference will be clear during the process of the proof. We want to make clear the role of the symmetries of the Hamiltonian function H and the role of the sufficient conditions which guarantee the existence of periodic solutions in suitable coordinates. In particular, on one hand this justifies why we try different situations, and on the other hand it is due to future applications.

In order to get results for a planar case, we will assume that the "plane" or set $y = p_y = 0$ (which we call the polar plane) is invariant under the flow defined by the Hamiltonian associated to H^* in (1.1). In Poincaré-Delaunay (PD-1) variables this planar case is equivalent to putting $Q_3 = P_3 = 0$. Since in the spatial case the inclination of the orbital plane was arbitrary, we can fix a circular periodic solution of the Kepler problem on the xz-plane, so we must take $p_3 = 0$ and in this way we do not need to perturb the initial condition $\mathbf{Y}$ in the direction P_3.

Therefore, as an immediate consequence of Theorems 1.1 and 1.2, we have the following result.

Corollary 1.1. *(First kind symmetric periodic solutions on the polar plane) Suppose that the set $y = p_y = 0$ is invariant under the flow defined by the Hamiltonian associated to H^* in (1.1) and let $\varphi_{kep}(t,\mathbf{Y})$ be a solution of the Kepler problem as in (1.8) with $p_3 = 0$ and $\Delta P_3 = 0$.*

(i) If the Hamiltonian function H^* in (1.1) is S_1-symmetric and the assumptions

$$\begin{aligned} &(a) \int_0^{T/2} \frac{\partial \mathscr{H}_1}{\partial Q_2}(\varphi_{kep}(\tau,\mathbf{Y}))\Big|_{\mathbf{Y}=\mathbf{Y}_0} d\tau = 0,\\ &(b) \int_0^{T/2} \frac{\partial^2 \mathscr{H}_1}{\partial Q_2 \partial \Delta Q_2}(\varphi_{kep}(\tau,\mathbf{Y}))\Big|_{\mathbf{Y}=\mathbf{Y}_0} d\tau \neq 0, \end{aligned} \tag{1.14}$$

are satisfied, then for ε sufficiently small there are initial by ε, $\mathbf{Y}_\varepsilon = \mathbf{Y}_0 + \mathbf{Y}(\varepsilon)$ with $\mathbf{Y}(\varepsilon) = (0,\Delta Q_2(\varepsilon),0,\Delta P_1(\varepsilon),0,0)$ such that $\varphi(t,\mathbf{Y}_\varepsilon;\varepsilon) = \varphi_{kep}(t,\mathbf{Y}_0) + \mathscr{O}(\varepsilon)$ is a S_1-symmetric periodic solution of the Hamiltonian system associated to the Hamiltonian (1.1) or (1.3).

(ii) If the Hamiltonian function H^* in (1.1) is S_1 and S_2-symmetric then for ε sufficiently small there are initial by ε, $\mathbf{Y}_\varepsilon = \mathbf{Y}_0 + \mathbf{Y}(\varepsilon)$ with $\mathbf{Y}(\varepsilon) = (0,\Delta Q_2(\varepsilon),0,\Delta P_1(\varepsilon),0,0)$ such that $\varphi(t,\mathbf{Y}_\varepsilon;\varepsilon) = \varphi_{kep}(t,\mathbf{Y}_0) + \mathscr{O}(\varepsilon)$ is a doubly-symmetric periodic solution of the Hamiltonian system associated to the Hamiltonian (1.1) or (1.3).

Moreover, if $\lambda_1 = 0, \lambda_2 = 0, \lambda_3, \lambda_4$ are the eigenvalues of A in (1.11), then the characteristic multipliers of the periodic solution $\varphi(t,\mathbf{Y})$, obtained in each case (i) and (ii), are $1,1,1+\mathscr{O}(\varepsilon^{\alpha+1}), 1+\mathscr{O}(\varepsilon^{\alpha+1}), 1+\varepsilon^\alpha\lambda_3+\mathscr{O}(\varepsilon^{\alpha+1}), 1+\varepsilon^\alpha\lambda_4+\mathscr{O}(\varepsilon^{\alpha+1})$. The solutions obtained in item (i) and (ii) are on the xz-plane and have period $T = 2\pi/s$.

If the Hamiltonian (1.1) is invariant under rotations about the z-axis, then we can simplify the study of the stability of the periodic solutions whether they exist by Theorems 1.1 or 1.2. In fact, it is known that two eigenvalues of A in (1.11) are null since there is an additional first integral.

Corollary 1.2. *Suppose that the Hamiltonian H^* in (1.1) is invariant under rotations about the z-axis. Then the characteristic multipliers of the T-periodic solutions $\varphi(t,\mathbf{Y}_\varepsilon;\varepsilon)$ given by Theorems 1.1 or 1.2 are*

$$1,1,1,1,1+\varepsilon^\alpha\lambda_1+\mathscr{O}(\varepsilon^{\alpha+1}), 1+\varepsilon^\alpha\lambda_2+\mathscr{O}(\varepsilon^{\alpha+1}),$$

where λ_1 and λ_2 are the roots of the polynomial

$$p(\lambda) = \lambda^2 + a_{11}a_{33} - a_{13}a_{31},$$

with $a_{11} = \int_0^T \frac{\partial^2 H_1}{\partial X_1 \partial X_3}(\varphi_{kep}(t,\bar{Y}_\varepsilon))dt\ |_{X=0}$, $a_{13} = \int_0^T \frac{\partial^2 H_1}{\partial^2 X_3^2}(\varphi_{kep}(t,\bar{Y}_\varepsilon))dt\ |_{X=0}\ dt\ |_{X=0}$, $a_{31} = -\int_0^T \frac{\partial^2 H_1}{\partial^2 X_1^2}(\varphi_{kep}(t,\bar{Y}_\varepsilon))dt\ |_{X=0})dt\ |_{X=0}$ *and* $a_{33} = -\int_0^T \frac{\partial^2 H_1}{\partial X_3 \partial X_1}$ $(\varphi_{kep}(t,\bar{Y}_\varepsilon))dt\ |_{X=0}\ dt\ |_{X=0}$, *where* $\bar{Y}_\varepsilon = (\pi/2, \Delta Q_2(\varepsilon), 0, s^{-1/3}, 0, p_3 + \Delta P_3(\varepsilon))$ *with* $X = (\Delta Q_2(\varepsilon), 0, 0, \Delta P_3(\varepsilon))$. *In particular the periodic solutions* $\varphi(t, \mathbf{Y}_\varepsilon; \varepsilon)$ *are unstable if* $a_{11}a_{33} - a_{13}a_{31} > 0$ *and are linearly stable if* $a_{11}a_{33} - a_{13}a_{31} < 0$.

It is known in the literature that there are some Hamiltonians which are perturbation of the Kepler problem in rotating coordinates (see for example [16]). In some cases these Hamiltonians are obtained by the Hamiltonian function (1.1) because they are invariant under rotation around the z-axis, but in other cases they appear as a particular problem which is not obtained by (1.1) (for a concrete problem see [8] and [19], these situation can be extended to other problems of massive bodies such as rings or disks). We have some advantage if we consider symmetric perturbations of the rotating Kepler problem. The first one is that we can reduce the number of periodicity equations which depend on the perturbing function, or even if we consider both symmetries, we can get the non-dependence of the periodicity equations in relation to the perturbed function (see Theorem 1.4). The second case will be clear when we deal with the applications in Section 11, it is related with the choosing of the initial condition p_3, that is in this frame, it can be chosen arbitrary or excluding a specific value. If we have an arbitrary value for p_3, we can obtain symmetric periodic solutions of the perturbed problem close to circular Keplerian orbits in a arbitrary orbital plane. Because of this we are motivated to consider a second type of Hamiltonian system associated to the Hamiltonian function of the form

$$K = \frac{p_x^2+p_y^2+p_z^2}{2} - (xp_y - yp_x) - \frac{1}{\sqrt{x^2+y^2+z^2}} + \varepsilon^\alpha K_1(\mathbf{p},\mathbf{q}) + \mathscr{O}(\varepsilon^{\alpha+1}), \quad \alpha \in \mathbb{N}, \tag{1.15}$$

where of course $K_0 = \frac{p_x^2+p_y^2+p_z^2}{2} - (xp_y - yp_x) - \frac{1}{\sqrt{x^2+y^2+z^2}}$ represents the Kepler problem in rotating coordinates. Next, we write the Hamiltonian (1.15) in Poincaré-Delaunay (PD-1) variables (1.2), so we get

$$K(\mathbf{Q},\mathbf{P},\varepsilon) = -\frac{1}{2P_1^2} - P_3 + \varepsilon^\alpha K_1(\mathbf{Q},\mathbf{P}) + \mathscr{O}(\varepsilon^{\alpha+1}). \tag{1.16}$$

To obtain our results, we need to introduce time as a new independent variable in order to eliminate the degeneracy of the system defined by periodicity equations. The necessity of this condition will be clear during the proof of Theorems 1.3 and 1.4. On the other hand, the periodic solutions obtained in these theorems for the Hamiltonian system associated to (1.15) do not have necessarily the same period as the circular Keplerian solution.

We denote by $\phi(t,\cdot)$ the flow of the Hamiltonian system associated to the Hamiltonian (1.16). Now, we look for a circular solution of the Kepler problem $\phi_{kep}(t,\mathbf{Y}_0)$ with initial condition $\mathbf{Y}_0$ as in (1.6) and we consider the small perturbation $\mathbf{Y}$ of the initial condition as in (1.7). Let $\phi_{kep}(t,\mathbf{Y})$ be the solution of the rotating Kepler problem in Poincaré-Delaunay (PD-1) variables with initial condition $\mathbf{Y}$ given by

$$\begin{array}{lll} Q_1(t) = st + \pi/2, & Q_2(t) = \Delta Q_2, & Q_3(t) = -t, \\ P_1(t) = s^{-1/3} + \Delta P_1, & P_2(t) = 0, & P_3(t) = p_3 + \Delta P_3, \end{array} \tag{1.17}$$

with $s \in \mathbb{R}^+$ and $p_3, \Delta Q_2, \Delta P_1, \Delta P_3 \in \mathbb{R}$. It is verified that the energy of the circular Keplerian solution is given by $K = -\frac{1}{2s^{-1/3}} - p_3 = k_0$. Then writing $p_3 = -(2k_0 s^{-1/3} + 1)/(2s^{-2/3})$, for each k_0 such that $\frac{1}{2}\left(-\frac{2}{s} - 1\right)s^{2/3} < k_0 < \frac{1}{2}\left(\frac{2}{s} - 1\right)s^{2/3}$, we have associated to one circular solution of the Kepler problem.

Let $X = (X_1, X_2, X_3, X_4) = (Q_2, Q_3, P_2, P_3)$ and $\phi(t, (Y_1^{(0)}, Y_2^{(0)} + X_1, Y_3^{(0)} + X_2, Y_4, Y_5^{(0)} + X_3, Y_6^{(0)} + X_4))$ be a solution of the rotating Kepler rotating problem with initial condition $(Y_1^{(0)}, Y_2^{(0)} + X_1, Y_3^{(0)} + X_2, Y_4, Y_5^{(0)} + X_3, Y_6^{(0)} + X_4)$ on the level $K = -\frac{1}{2P_1^2} - P_3 = k_0$ where $Y_4 := Y_4(k_0, X_4) = \frac{1}{\sqrt{-2(k_0 + Y_6^{(0)} + X_4)}}$ with $Y_4(k_0, 0) = Y_4^{(0)}$. Define

$$\bar{K}_1(X) = \int_0^T K_1(\phi_{kep}(t, (Y_1^{(0)}, Y_2^{(0)} + X_1, Y_3^{(0)} + X_2, Y_4, Y_5^{(0)} + X_3, Y_6^{(0)} + X_4)))dt, \tag{1.18}$$

be the average function of K_1 and the matrix

$$B = J\left(\frac{\partial^2 \bar{K}_1}{\partial X_i \partial X_j}\right)_{X=0}, \tag{1.19}$$

where J denotes the standard skew-symmetric matrix and as before $T = 2\pi/s$. In this case we have the following result.

Theorem 1.3. *(First kind S_1-symmetric periodic solutions in the rotating case) Suppose that the Hamiltonian function K in (1.15) is S_1-symmetric and let $\phi_{kep}(t, \mathbf{Y})$ be a solution of the rotating Kepler problem as in (1.17). Assume that*

$$\begin{array}{l} (a) \displaystyle\int_0^{T/2} \frac{\partial K_1}{\partial Q_2}(\phi_{kep}(\tau, \mathbf{Y}))\Big|_{\mathbf{Y}=\mathbf{Y}_0} d\tau = 0, \\ (b) \displaystyle\int_0^{T/2} \frac{\partial^2 K_1}{\partial Q_2 \partial \Delta Q_2}(\phi_{kep}(t, \mathbf{Y}))\Big|_{\mathbf{Y}=\mathbf{Y}_0} d\tau \neq 0, \end{array} \tag{1.20}$$

hold. Then for ΔP_3 and ε sufficiently small there are initial conditions parametrized by two parameters ε and ΔP_3, $\mathbf{Y}_{\Delta P_3, \varepsilon} = \mathbf{Y}_0 + \mathbf{Y}(\Delta P_3, \varepsilon)$ with $\mathbf{Y}(\Delta P_3, \varepsilon) = (0, \Delta Q_2(\Delta P_3, \varepsilon), 0, \Delta P_1(\Delta P_3, \varepsilon), 0, \Delta P_3)$ such that $\phi(t, \mathbf{Y}_\varepsilon; \varepsilon) = \phi_{kep}(t, \mathbf{Y}_0) + \mathscr{O}(\varepsilon)$ is a S_1-symmetric periodic solution of the Hamiltonian system associated to (1.15) or (1.16) with period $\overline{T}(\Delta P_3, \varepsilon) = 2\pi/s + \mathscr{O}(\varepsilon^\alpha)$. Moreover, if $\lambda_1, \lambda_2, \lambda_3, \lambda_4$ are the eigenvalues of B in (1.19), then the characteristic multipliers of the periodic solution $\phi(t, \mathbf{Y}_\varepsilon; \varepsilon)$ are $1, 1, 1 + \varepsilon^\alpha \lambda_1 + \mathscr{O}(\varepsilon^{\alpha+1}), 1 + \varepsilon^\alpha \lambda_2 + \mathscr{O}(\varepsilon^{\alpha+1}), 1 + \varepsilon^\alpha \lambda_3 + \mathscr{O}(\varepsilon^{\alpha+1}), 1 + \varepsilon^\alpha \lambda_4 + \mathscr{O}(\varepsilon^{\alpha+1})$.

This theorem is proved in Section 6.
Of course, the periodic solutions obtained in Theorem 1.3 correspond to quasi-periodic solutions of the Hamiltonian (1.1). But, if the relation of commensurability between the period of the continued orbit and the rotation period (2π) is satisfied, these solutions are in fact periodic solutions of the system associated to the Hamiltonian (1.1).

In the doubly symmetric case we have the next result.

Theorem 1.4. *(First kind $S_1 - S_2$-symmetric periodic solutions in the rotating case) Assume that the Hamiltonian function K in (1.15) is $S_1 - S_2$-symmetric and let $\phi_{kep}(t,\mathbf{Y})$ be a solution of the rotating Kepler problem as in (1.17). Then for ΔP_3 and ε sufficiently small there are initial conditions parametrized by the two parameters ε and ΔP_3 as $\mathbf{Y}_{\Delta P_3,\varepsilon} = \mathbf{Y}_0 + \mathbf{Y}(\Delta P_3,\varepsilon)$ with $\mathbf{Y}(\Delta P_3,\varepsilon) = (0,\Delta Q_2(\Delta P_3,\varepsilon),0,\Delta P_1(\Delta P_3,\varepsilon),0,\Delta P_3)$ such that $\phi(t,\mathbf{Y}_\varepsilon;\varepsilon) = \phi_{kep}(t,\mathbf{Y}_0) + \mathscr{O}(\varepsilon^\alpha)$ gives us a doubly-symmetric periodic solution of the Hamiltonian system associated to (1.15) with period $\overline{T}(\Delta P_3,\varepsilon) = 2\pi/s + \mathscr{O}(\varepsilon)$. Moreover, if $\lambda_1,\lambda_2,\lambda_3,\lambda_4$ are the eigenvalues of B in (1.19), then the characteristic multipliers of the periodic solution $\phi(t,\mathbf{Y}_\varepsilon;\varepsilon)$ are $1,1,1+\varepsilon^\alpha\lambda_1+\mathscr{O}(\varepsilon^{\alpha+1}),1+\varepsilon^\alpha\lambda_2+\mathscr{O}(\varepsilon^{\alpha+1}),1+\varepsilon^\alpha\lambda_3+\mathscr{O}(\varepsilon^{\alpha+1}),1+\varepsilon^\alpha\lambda_4+\mathscr{O}(\varepsilon^{\alpha+1})$.*

This theorem is proved in Section 7.

Analogously to Corollary 1.2, if we assume that the Hamiltonian (1.15) is invariant under rotations about the z-axis, and note that in Theorems 1.3 and 1.4 we can consider the small perturbation $\mathbf{Y}$ of the initial condition $\mathbf{Y}_0$ in (1.12) perturbing only in the Q_2 and P_1 directions, i.e., to consider the rotating Kepler solution (1.17) with $\Delta P_3 = 0$. Since the third component $H = p_3$ of the angular momentum is an integral of motion, we shall restrict our attention to the level $H = p_3$. We enunciate this situation in the following corollary.

Corollary 1.3. *Assume that the Hamiltonian K in (1.15) is invariant under rotations about the z-axis and suppose that the third component of the angular momentum $H = p_3$ is fixed. Then for the Hamiltonian system associated to the Hamiltonian (1.15) or (1.16) the following statement holds:*

(i) If the Hamiltonian function (1.15) is S_1-symmetric and the assumptions

$$\begin{aligned} &(a)\ \int_0^{T/2} \frac{\partial K_1}{\partial Q_2}(\phi_{kep}(\tau,\mathbf{Y}))\Big|_{\mathbf{Y}=\mathbf{Y}_0} d\tau = 0, \\ &(b)\ \int_0^{T/2} \frac{\partial^2 K_1}{\partial Q_2 \partial \Delta Q_2}(\phi_{kep}(\tau,\mathbf{Y}))\Big|_{\mathbf{Y}=\mathbf{Y}_0} d\tau \neq 0, \end{aligned} \tag{1.21}$$

are satisfied, then for ε sufficiently small there are initial conditions parametrized by ε, $\mathbf{Y}_\varepsilon = \mathbf{Y}_0 + \mathbf{Y}(\varepsilon)$ with $\mathbf{Y}(\varepsilon) = (0,\Delta Q_2(\varepsilon),0,\Delta P_1(\varepsilon),0,0)$ such that $\phi(t,\mathbf{Y}_\varepsilon;\varepsilon) = \phi_{kep}(t,\mathbf{Y}_0) + \mathscr{O}(\varepsilon)$ is a S_1-symmetric periodic solution.

(ii) If the Hamiltonian function (1.15) is S_1 and S_2-symmetric, then for ε sufficiently small there are initial conditions parametrized by ε, $\mathbf{Y}_\varepsilon = \mathbf{Y}_0 + \mathbf{Y}(\varepsilon)$ with $\mathbf{Y}(\varepsilon) = (0,\Delta Q_2(\varepsilon),0,\Delta P_1(\varepsilon),0,0)$ such that $\phi(t,\mathbf{Y}_\varepsilon;\varepsilon) = \varphi_{kep}(t,\mathbf{Y}_0) + \mathscr{O}(\varepsilon)$ is a doubly-symmetric periodic solution.

Moreover, if $\lambda_1,\lambda_2,\lambda_3 = 0,\lambda_4 = 0$ are the eigenvalues of B in (1.19), then the characteristic multipliers of the periodic solution $\phi(t,\mathbf{Y}_\varepsilon;\varepsilon)$ in (i) or (ii), are $1,1,1,1,1+\varepsilon^\alpha\lambda_1+\mathscr{O}(\varepsilon^{\alpha+1}),1+\varepsilon^\alpha\lambda_2+\mathscr{O}(\varepsilon^{\alpha+1})$.

In the literature (see details in [15] and [16]) we found a second set of convenient Poincaré-Delaunay variables

$$\begin{aligned} &Q_1 = l+g+h, && P_1 = L, \\ &Q_2 = -\sqrt{2(L-G)}\sin(g+h), && P_2 = \sqrt{2(L-G)}\cos(g+h), \quad \text{(PD-2)} \\ &Q_3 = -\sqrt{2(G-H)}\sin(h), && P_3 = \sqrt{2(G-H)}\cos(h), \end{aligned} \tag{1.22}$$

We introduce these variables to get symmetric periodic solutions for an arbitrary problem associated to the Hamiltonian (1.15). In fact, we observe that using (PD-2) variables and considering the problem in rotating coordinates, we obtain periodic solutions for any differentiable perturbation (only maintaining the symmetry) of the rotating Kepler problem. Moreover, the variables (PD-2) are defined in the reference plane (xy-plane) but the variables $(PD-1)$ that are not defined. But, with these set of coordinates, we can only continue circular Keplerian orbits in the reference plane, i.e., the perturbed symmetric periodic solutions are close or are contained in the xy-plane. Contrary to what occurs with Theorems 1.3 and 1.4 the continued symmetric period solutions have the same period of the Keplerian circular orbit. In the $(PD-2)$ variables, the circular orbits correspond to $Q_2 = P_2 = 0$, and the orbits lying on the reference plane to $Q_3 = P_3 = 0$. The Kepler problem in rotating coordinates in $(PD-2)$ variables assumes the form

$$K_0(\mathbf{Q},\mathbf{P}) = -\frac{1}{2P_1^2} - P_1 + \frac{Q_2^2+P_2^2+Q_3^2+P_3^2}{2}. \tag{1.23}$$

Thus solutions of the rotating Kepler problem in Poincaré-Delaunay (PD-2) variables with initial condition $\mathbf{Z}_0 = (Q_1^0, Q_2^0, Q_3^0, P_1^0, P_2^0, P_2^0)$ are given by

$$\begin{aligned} Q_1(t) &= (s-1)t + Q_1^0, & P_1(t) &= P_1^0, \\ Q_2(t) &= Q_2^0\cos t + P_2^0\sin t, & P_2(t) &= -Q_2^0\sin t + P_2^0\cos t, \\ Q_3(t) &= Q_3^0\cos t + P_3^0\sin t, & P_3(t) &= -Q_3^0\sin t + P_3^0\cos t. \end{aligned} \tag{1.24}$$

Moreover, the Hamiltonian (1.15) in Poincaré-Delaunay (PD-2) variables takes the form

$$\mathscr{K}(\mathbf{Q},\mathbf{P},\varepsilon) = -\frac{1}{2P_1^2} - P_1 + \frac{Q_2^2+P_2^2+Q_3^2+P_3^2}{2} + \varepsilon^\alpha \mathscr{K}_1(\mathbf{Q},\mathbf{P}) + \mathscr{O}(\varepsilon^{\alpha+1}). \tag{1.25}$$

We denote by $\Phi_{kep}(t,\cdot)$ the flow of the Hamiltonian system associated to the Hamiltonian (1.25). Now, we will look for a circular solution of the Kepler problem $\Phi_{kep}(t,\mathbf{Z}_0)$ given in Poincaré-Delaunay (PD-2) variables such that $\Phi_{kep}(0,\mathbf{Z}_0) \in \mathscr{L}_1$ and at the instant $t = T/2$ satisfies $\Phi_{kep}(T/2, \mathbf{Z}_0) \in \mathscr{L}_1$. It is enough to take $\mathbf{Z}_0 = (Z_1^{(0)}, Z_2^{(0)}, Z_3^{(3)}, Z_4^{(0)}, Z_5^{(0)}, Z_6^{(0)}) \equiv (\pi/2, 0, 0, s^{-1/3}, 0, 0)$ and the Keplerian solution has period $T = 2\pi/(s-1)$ and is lying on the xy plane. It is verified that the energy of the circular Keplerian solution of rotating Kepler problem in Poincaré-Delaunay (PD-2) variables is given by $\mathscr{K} = -\frac{1}{2s^{-2/3}} - s^{-1/3} = k_0$. Since $s > 1$, by the expression of the period T, then for each negative $k_0 < -3/2$ we have associated to one circular solution of the Kepler problem in rotating coordinates.

Next we will look for the solution $\Phi_{kep}(t,\mathbf{Z})$ of the rotating Kepler problem in Poincaré-Delaunay (PD-2) of the form

$$\begin{aligned} Q_1(t) &= (s-1)t + \pi/2, & P_1(t) &= s^{-1/3} + \Delta P_1, \\ Q_2(t) &= \Delta Q_2 \cos t, & P_2(t) &= -\Delta Q_2 \sin t, \\ Q_3(t) &= \Delta P_3 \sin t, & P_3(t) &= \Delta P_3 \cos t, \end{aligned} \tag{1.26}$$

$\Delta Q_2, \Delta P_1, \Delta P_3 \in \mathbb{R}$. Let $X = (X_1, X_2, X_3, X_4) = (Q_2, Q_3, P_2, P_3)$ and $\Phi_{kep}(t, (Z_1^{(0)}, Z_2^{(0)} + X_1, Z_3^{(0)} + X_2, Z_4, Z_5^{(0)} + X_3, Z_6^{(0)} + X_4))$ be a solution of the Kepler problem with initial condition $(Z_1^{(0)}, Z_2^{(0)} + X_1, Z_3^{(0)} + X_2, Z_4^{(0)}, Z_5^{(0)} + X_3, Z_6^{(0)} + X_4)$, where $Z_4 := Z_4(k_0, X_1, X_2, X_3, X_4)$ is solution of $-\frac{1}{2Z_4^2} - Z_4 +$

$\frac{1}{2}(Q_2^2+P_2^2+Q_3^2+P_3^2)-k_0=0$ and $Z_4(k_0,0,0,0,0)=s^{-1/3}$. Define

$$\bar{\mathscr{K}_1}(X)=\int_0^T \mathscr{K}_1(\Phi_{kep}(t,(Z_1^{(0)},Z_2^{(0)}+X_1,Z_3^{(0)}+X_2,Z_4,Z_5^{(0)}+X_3,Z_6^{(0)}+X_4)))dt. \tag{1.27}$$

to be the average function of H_1 with $T=\frac{2\pi}{s-1}$ and the matrix

$$C=J\left(\frac{\partial^2\bar{\mathscr{K}_1}}{\partial X_i\partial X_j}\right)_{X=0}. \tag{1.28}$$

Theorem 1.5. *(First kind S_1-symmetric periodic solutions close to the equatorial plane) Suppose that the Hamiltonian function K in (1.15) is S_1-symmetric and let $\Phi_{kep}(t,\mathbf{Z})$ be a solution of the rotating Kepler problem as in (1.26) such that $\frac{1}{s-1}\notin\mathbb{N}$. Then for ε sufficiently small there are initial conditions parametrized by ε, $\mathbf{Z}_\varepsilon=\mathbf{Z}_0+\mathbf{Z}(\varepsilon)$ with $\mathbf{Z}(\varepsilon)=(0,\Delta Q_2(\varepsilon),\Delta P_1(\varepsilon),0,\Delta P_3(\varepsilon))$ such that $\Phi(t,\mathbf{X}_\varepsilon;\varepsilon)=\Phi_{kep}(t,\mathbf{Z}_0)+\mathscr{O}(\varepsilon)$ is a S_1-symmetric periodic solution of the Hamiltonian system associated to (1.15) or (1.16), with period $T=2\pi/(s-1)$ and close to the xy-plane with period $T=\frac{2\pi}{s-1}$. Moreover, if $\lambda_1,\lambda_2,\lambda_3,\lambda_4$ are the eigenvalues of C in (1.28), then the characteristic multipliers of the periodic solution $\Phi(t,\mathbf{Z}_\varepsilon;\varepsilon)$ are $1,1,\cos T-i\sin T+\varepsilon^\alpha\lambda_1+\mathscr{O}(\varepsilon^{\alpha+1}),\cos T-i\sin T+\varepsilon^\alpha\lambda_2+\mathscr{O}(\varepsilon^{\alpha+1}),\cos T+i\sin T+\varepsilon^\alpha\lambda_3+\mathscr{O}(\varepsilon^{\alpha+1}),\cos T+i\sin T+\varepsilon^\alpha\lambda_4+\mathscr{O}(\varepsilon^{\alpha+1})$.*

This result is proved in Section 8.

To obtain doubly symmetric periodic solutions, we consider a circular solution of the Kepler problem $\Phi_{kep}(0,\mathbf{Z}_0)$ in Poincaré-Delaunay (PD-2) variables, such that $\Phi_{kep}(0,\mathbf{Z}_0)\in\mathscr{L}_1$ and at the instant $t=T/4$ satisfies $\Phi_{kep}(T/4,\mathbf{Z}_0)\in\mathscr{L}_2$. It is verified that the circular Keplerian solution in the inertial frame has period $T=2\pi/(s-1)$, radius $s^{-2/3}$ and it is on the xy- plane with initial condition $\mathbf{Z}_0$.

Theorem 1.6. *(First kind S_1-S_2-symmetric periodic solutions close to equatorial plane) Suppose that the Hamiltonian function K in (1.15) is S_1 and S_2-symmetric and let $\Phi_{kep}(t,\mathbf{Z})$ be a solution of the rotating Kepler problem as in (1.26). Then for ε sufficiently small there are initial conditions $\mathbf{Z}_\varepsilon=\mathbf{Z}_0+\mathbf{Z}(\varepsilon)$ with $\mathbf{Z}(\varepsilon)=(0,\Delta Q_2(\varepsilon),0,\Delta P_1(\varepsilon),0,\Delta P_3(\varepsilon))$ such that $\Phi(t,\mathbf{X}_\varepsilon;\varepsilon)=\Phi_{kep}(t,\mathbf{Z}_0)+\mathscr{O}(\varepsilon)$ is a doubly-symmetric periodic solution of the Hamiltonian system associated to (1.15) with period $T=2\pi/(s-1)$ whether $1/(2s-2)\neq k$ for all $k\in\mathbb{N}$. This periodic solution is close to $\Phi_{kep}(t,\mathbf{Z}_0)$ which is on the $(x,y)-$ plane with radius $s^{-2/3}$. Moreover, if $\lambda_1,\lambda_2,\lambda_3,\lambda_4$ are the eigenvalues of C in (1.28), then the characteristic multipliers of the periodic solution $\Phi(t,\mathbf{X}_\varepsilon;\varepsilon)$ are $1,1,\cos T-i\sin T+\varepsilon^\alpha\lambda_1+\mathscr{O}(\varepsilon^{\alpha+1}),\cos T-i\sin T+\varepsilon^\alpha\lambda_2+\mathscr{O}(\varepsilon^{\alpha+1}),\cos T+i\sin T+\varepsilon^\alpha\lambda_3+\mathscr{O}(\varepsilon^{\alpha+1}),\cos T+i\sin T+\varepsilon^\alpha\lambda_4+\mathscr{O}(\varepsilon^{\alpha+1})$.*

This theorem is proved in Section 9.

We will assume that the "plane" $z=p_z=0$ is invariant under the flow defined by the Hamiltonian associated to $\mathscr{H}$ in (1.5), i.e., we have diminished the number of degrees of freedom by one dimension. Here we have the following result.

Corollary 1.4. *(First kind symmetric periodic solutions on the equatorial plane) Assume that the set $z=p_z=0$ is invariant and let $\Phi_{kep}(t,\mathbf{Z})$ be a solution of the rotating Kepler problem as in (1.26) with $\Delta P_3=0$. Then, there exists a family of initial conditions $\mathbf{Z}_\varepsilon$ as in Theorem 1.5, or* *respectively, Theorem 1.6 with* $\Delta P_3(\varepsilon) = 0$*, such that* $\Phi(t, \mathbf{Z}_\varepsilon; \varepsilon) = \varphi_{kep}(t, \mathbf{Z}_0) + \mathscr{O}(\varepsilon)$ *give us a* S_1*-symmetric, or respectively, doubly-symmetric contained in the xy-plane with period* $T = 2\pi/(s-1)$*. Moreover, if* $\lambda_1 = 0, \lambda_2 = 0, \lambda_3, \lambda_4$ *are the eigenvalues of C in (1.28), then the characteristic multipliers of the periodic solution* $\Phi(t, \mathbf{Z}_\varepsilon; \varepsilon)$ *(for the* S_1*-symmetric or doubly-symmetric case) are* $1, 1, 1 + \mathscr{O}(\varepsilon^{\alpha+1}), 1 + \mathscr{O}(\varepsilon^{\alpha+1}), 1 + \varepsilon^\alpha \lambda_3 + \mathscr{O}(\varepsilon^{\alpha+1}), 1 + \varepsilon^\alpha \lambda_4 + \mathscr{O}(\varepsilon^{\alpha+1})$.

This theorem is proved in Section 10.

We observe that for $(PD-2)$ variables there are some restrictions on the period and radius of the Keplerian solutions that can be continued but there are no restrictions on the perturbed part. Moreover, the use of the $(PD-2)$ variables is interesting because permits us to get a family of periodic solutions close to circular Kepler orbits but with fixed period, this is an important difference when we compare for example with Theorem 1.3.

The planar case also is considered, and it arises as a particular case. Our results give us explicit conditions associated to the first order perturbation term, which must be satisfied in order to obtain our results under the use of appropriate coordinates. We call the attention to the fact that the periodic orbits obtained in Theorems 1.3-1.4-1.5-1.6 associated to the Hamiltonian function K in (1.15) obtained from H^* in (1.1) under a time-periodic symplectic change of variables are in general quasi-periodic solutions. Finally, in Section 11 we apply our results to study the existence of periodic orbits of the Matese-Whitman Hamiltonian and the generalized Størmer model and we give some important information of these models. In fact, for the Matese-Whitman Hamiltonian we obtain new families of periodic solutions, while for the generalized Størmer model according to our information is a new result for this kind of dynamics.

2. Preliminary results

The elements (l, g, h, L, G, H) of the Poincaré-Delaunay variables defined in (1.2) or (1.22) are as usual, l is the mean anomaly, g the argument of the perigee measured from the ascending node, h is the longitude of the ascendent node measure from the x-axis, $L = \mathtt{a}^{1/2}$ is the semi-major axis of the ellipse, $G = [\mathtt{a}(1-e^2)]^{1/2}$ is the angular momentum, $H = [\mathtt{a}(1-e^2)]^{1/2} \cos \iota$ is the component of angular momentum in the direction of the z-axis and ι is the inclination of the orbital plane. The variables l, g are angular variables modulus 2π, and L, G, H are radial variables. The Poincaré-Delaunay coordinates are well defined in a neighborhood of a circular orbit, and such a solution ($e = 0$) occurs when $L = G$. This last condition in the Poincaré-Delaunay variables can be expressed by the condition $Q_2 = P_2 = 0$. In any type of variables (PD-1) or (PD-2), we verify that the solutions of the Hamiltonian system associated to the Hamiltonian equation in (1.1), (1.16) or (1.25) with initial condition $\mathbf{Y} = (\mathbf{Q}, \mathbf{P})$ are given by $\Psi(t, \mathbf{Y}; \varepsilon) = (Q_1(t, \mathbf{Y}; \varepsilon), Q_2(t, \mathbf{Y}; \varepsilon), Q_3(t, \mathbf{Y}; \varepsilon), P_1(t, \mathbf{Y}; \varepsilon), P_2(t, \mathbf{Y}; \varepsilon), P_3(t, \mathbf{Y}; \varepsilon))$ with

$$
\begin{aligned}
Q_1(t) &= Q_1^{(0)}(t) + \varepsilon^\alpha Q_1^{(1)}(t) + \mathscr{O}(\varepsilon^{\alpha+1}), \quad P_1(t) = P_1^{(0)}(t) + \varepsilon^\alpha P_1^{(1)}(t) + \mathscr{O}(\varepsilon^{\alpha+1}), \\
Q_2(t) &= Q_2^{(0)}(t) + \varepsilon^\alpha Q_2^{(1)}(t) + \mathscr{O}(\varepsilon^{\alpha+1}), \quad P_2(t) = P_2^{(0)}(t) + \varepsilon^\alpha P_2^{(1)}(t) + \mathscr{O}(\varepsilon^{\alpha+1}), \\
Q_3(t) &= Q_3^{(0)}(t) + \varepsilon^\alpha Q_3^{(1)}(t) + \mathscr{O}(\varepsilon^{\alpha+1}), \quad P_3(t) = P_3^{(0)}(t) + \varepsilon^\alpha P_3^{(1)}(t) + \mathscr{O}(\varepsilon^{\alpha+1}),
\end{aligned}
\tag{2.1}
$$

and the expressions $Q_j^{(0)}(t) = Q_j^{(0)}(t, \mathbf{Y})$, $P_j^{(0)}(t) = P_j^{(0)}(t, \mathbf{Y})$, $Q_j^{(1)}(t) = Q_j^{(1)}(t, \mathbf{Y})$ and $P_j^{(1)}(t) = P_j^{(1)}(t, \mathbf{Y})$, $j = 1, 2, 3$ are given by

$$\begin{aligned} Q_j^{(1)}(t,\mathbf{Y}) &= \int_0^t \frac{\partial \mathscr{H}_1}{\partial P_j}(\Psi_{kep}(\tau,\mathbf{Y}))d\tau \\ P_j^{(1)}(t,\mathbf{Y}) &= -\int_0^t \frac{\partial \mathscr{H}_1}{\partial Q_j}(\Psi_{kep}(\tau,\mathbf{Y}))d\tau, \end{aligned} \tag{2.2}$$

and

$$\Psi_{kep}(t,\mathbf{Y}) = (Q_1^{(0)}(t,\mathbf{Y}),Q_2^{(0)}(t,\mathbf{Y}),Q_3^{(0)}(t,\mathbf{Y}),P_1^{(0)}(t,\mathbf{Y}),P_2^{(0)}(t,\mathbf{Y}),P_3^{(0)}(t,\mathbf{Y})), \tag{2.3}$$

is any solution of the Kepler problem with initial condition $\mathbf{Y}$ (see details in [5]).

Now we enunciate the following results that characterize symmetric periodic solutions in (PD-1) variables and whose proofs are very simple.

Lemma 2.1.
i) An orbit hits $\mathscr{L}_1$ at time $t = T$ if it is perpendicular to the yz-plane and in Poincaré-Delaunay (PD-1) variables is defined by

$$Q_1(T) = \frac{\pi}{2}\,(mod\,\pi), \quad Q_3(T) = 0\,(mod\,\pi), \quad P_2(T) = 0. \tag{2.4}$$

ii) An orbit hits $\mathscr{L}_2$ at time $t = T$ if it is perpendicular to the x-axis and in Poincaré-Delaunay (PD-1) variables is defined by

$$Q_1(T) = 0\,(mod\,\pi), \quad Q_2(T) = 0, \quad Q_3(T) = 0\,(mod\,\pi). \tag{2.5}$$

In the Poincaré-Delaunay variables (PD-2), the characterizacion of symmetric solutions is given by the next lemma.

Lemma 2.2.
i) An orbit hits $\mathscr{L}_1$ at time $t = T$ if in Poincaré-Delaunay (PD-2) variables satisfies

$$Q_1(T) = \frac{\pi}{2}\,(mod\,\pi), \quad Q_3(T) = 0, \quad P_2(T) = 0. \tag{2.6}$$

ii) An orbit hits $\mathscr{L}_2$ at time $t = T$ if in Poincaré-Delaunay (PD-2) variables satisfies

$$Q_1(T) = 0\,(mod\,\pi), \quad Q_2(T) = 0, \quad Q_3(T) = 0. \tag{2.7}$$

3. Proof of Theorem 1.1

Let $\varphi_{kep}(\tau,\mathbf{Y})$ be a solution of the Kepler problem as in (1.8), then the solution $\varphi(t,\mathbf{Y};\varepsilon)$ of the system (1.5) in Poincaré-Delaunay (PD-1) variables is S_1-symmetric if at the instant $t = T/2$ it intercepts orthogonally the subspaces $\mathscr{L}_1$. So by Lemma 2.1 it is necessary to verify that at $t = T/2$ the following three equations in Poincaré-Delaunay (PD-1) variables must be satisfy the relations (2.4). From the equation for $Q_j(t)$ for $j = 1,3$ and $P_2(t)$ in (2.1) it follows that the periodicity equations or equations (2.1) which must be satisfied are

$$\begin{aligned} f_1(\Delta Q_2,\Delta P_1,\Delta P_3,\varepsilon) &= (s^{-1/3}+\Delta P_1)^{-3}\,T/2-\pi/2+\mathscr{O}(\varepsilon^{\alpha})=0,\\ f_2(\Delta Q_2,\Delta P_1,\Delta P_3,\varepsilon) &= Q_3^{(1)}(T/2,\mathbf{Y}) \qquad +\mathscr{O}(\varepsilon)=0,\\ f_3(\Delta Q_2,\Delta P_1,\Delta P_3,\varepsilon) &= P_2^{(1)}(T/2,\mathbf{Y}) \qquad +\mathscr{O}(\varepsilon)=0. \end{aligned} \tag{3.1}$$

Under the choosing of T and the hypotheses (a) and (b) it is clear that $f_1(0,0,0,0)=f_2(0,0,0,0)=f_3(0,0,0,0)=0$. Moreover, by differentiating the system (3.1) with respect to $(\Delta Q_2,\Delta P_1,\Delta P_3)$ and evaluating at $\mathbf{Y}=Y_0$ and $\varepsilon=0$, we obtain that the Jacobian matrix satisfies

$$\left.\frac{\partial(f_1,f_2,f_3)}{\partial(\Delta Q_2,\Delta P_1,\Delta P_3)}\right|_{\mathbf{Y}=Y_0,\varepsilon=0} = \begin{pmatrix} 0 & -3s^{4/3}\,T/2 & 0\\ \frac{\partial Q_3^{(1)}}{\partial \Delta Q_2} & \frac{\partial Q_3^{(1)}}{\partial \Delta P_1} & \frac{\partial Q_3^{(1)}}{\partial \Delta P_3}\\ \frac{\partial P_2^{(1)}}{\partial \Delta Q_2} & \frac{\partial P_2^{(1)}}{\partial \Delta P_1} & \frac{\partial P_2^{(1)}}{\partial \Delta P_3} \end{pmatrix}_{\mathbf{Y}=Y_0,\varepsilon=0}.$$

Since by hypothesis (c), $\Omega_1=\det\left.\frac{\partial(f_1,f_2,f_3)}{\partial(\Delta Q_2,\Delta P_1,\Delta P_3)}\right|_{\mathbf{Y}=Y_0,\varepsilon=0}\neq 0$ by the Implicit Function Theorem we obtain that there are unique differentiable functions $\Delta Q_2=\Delta Q_2(\varepsilon)$, $\Delta P_1=\Delta P_1(\varepsilon)$ and $\Delta P_3=\Delta P_3(\varepsilon)$ defined for $|\varepsilon|<\varepsilon_0$ where ε_0 is sufficiently small, such that $\Delta Q_2(0)=0$, $\Delta P_1(0)=0$, $\Delta P_3(0)=0$ and $f_j(\Delta Q_2(\varepsilon),\Delta P_1(\varepsilon),\Delta P_3(\varepsilon),\varepsilon)=0$ for $j=1,2$ and 3. When $\Delta Q_2=\Delta P_1=\Delta P_3=0$, it is clearly verified that the circular periodic solutions in (1.8) are of course S_1-symmetric, have period $T=2\pi/s$ radius $s^{-2/3}$ and are on any orbital plane (arbitrary inclination) on $\mathbb{R}^3$ and note that $\mathbf{Y}\in\mathscr{L}_1$.

In order to study the type of stability of the previous periodic solutions, we are going to calculate its multipliers characteristic. Let $\Sigma=\{(Q,P)\ /\ \mathscr{H}(Q,P)=h_0,\,Q_1=Y_1^{(0)}\}$, be a local cross section on the level $\mathscr{H}=-\frac{1}{2P_1^2}=h_0$ in a neighborhood of the point $(Y_1^{(0)},Y_2^{(0)},Y_3^{(0)},Y_4^{(0)},Y_5^{(0)},Y_6^{(0)})$. We denote by $X=(X_1,X_2,X_3,X_4)$ the points in Σ. Thus, considering $\overline{Y}=(Y_1^{(0)},Y_2^{(0)}+X_1,Y_3^{(0)}+X_2,Y_4^{(0)},Y_5^{(0)}+X_3,Y_6^{(0)}+X_4)$ the Poincaré map P on Σ is given by $P(X,\varepsilon)=\big(Q_2(\mathscr{T},\overline{Y},\varepsilon),Q_3(\mathscr{T},\overline{Y},\varepsilon),P_2(\mathscr{T},\overline{Y},\varepsilon),P_3(\mathscr{T},\overline{Y},\varepsilon)\big)$, where Q_j and P_j were characterized in (2.1) and $\mathscr{T}$ is the return time which is close to $T+\mathscr{O}(\varepsilon^{\alpha})$. Using the form of Y^0 and (2.3)-(2.2) we arrive to

$$\begin{aligned} P(X,\varepsilon) &= (X_1,X_2,X_3,p_3+X_4)+\varepsilon^{\alpha}\left(\int_0^{\mathscr{T}}\frac{\partial\mathscr{H}_1}{\partial P_2}(\varphi_{kep}(t,\overline{Y}))dt,\right.\\ &\left.\int_0^{\mathscr{T}}\frac{\partial\mathscr{H}_1}{\partial P_3}(\varphi_{kep}(t,\overline{Y}))dt,-\int_0^{\mathscr{T}}\frac{\partial\mathscr{H}_1}{\partial Q_2}(\varphi_{kep}(t,\overline{Y}))dt,-\int_0^{\mathscr{T}}\frac{\partial\mathscr{H}_1}{\partial Q_3}(\varphi_{kep}(t,\overline{Y}))dt\right)+\\ &\mathscr{O}(\varepsilon^{\alpha+1})\\ &= (X_1,X_2,X_3,p_3+X_4)+\varepsilon^{\alpha}\left(\int_0^{T}\frac{\partial\mathscr{H}_1}{\partial P_2}(\varphi_{kep}(t,\overline{Y}))dt,\right.\\ &\left.\int_0^{T}\frac{\partial\mathscr{H}_1}{\partial P_3}(\varphi_{kep}(t,\overline{Y}))dt,-\int_0^{T}\frac{\partial\mathscr{H}_1}{\partial Q_2}(\varphi_{kep}(t,\overline{Y}))dt,-\int_0^{T}\frac{\partial\mathscr{H}_1}{\partial Q_3}(\varphi_{kep}(t,\overline{Y}))dt\right)+\\ &\mathscr{O}(\varepsilon^{\alpha+1}). \end{aligned} \tag{3.2}$$

Remembering (1.10), we have that the differential of P has the form

$$DP(X,\varepsilon)=I+\varepsilon^{\alpha}D_X\overline{H}(X)+\mathscr{O}(\varepsilon^{\alpha+1}). \tag{3.3}$$

Since the initial conditions of the T-symmetric periodic solutions are $Y_{\varepsilon}=Y_0+(0,\Delta Q_2(\varepsilon),0,\Delta P_1(\varepsilon),0,\Delta P_3(\varepsilon))$, then the respective points on the section cross Σ will be $X_{\varepsilon}=$

$(\Delta Q_2(\varepsilon), 0, 0, \Delta P_3(\varepsilon))$ with $X_0 = (0,0,0,0)$. Therefore,

$$DP(X_\varepsilon, \varepsilon) = I + \varepsilon^\alpha A + \mathscr{O}(\varepsilon^{\alpha+1}), \tag{3.4}$$

with A as in (1.11). Since, the nontrivial characteristic multipliers of the T-symmetric periodic solutions are the eigenvalues of $DP(X_\varepsilon, \varepsilon)$, it follows the proof of the theorem follows.

4. Proof of Theorem 1.2

Let $\varphi_{kep}(\tau, \mathbf{Y})$ be a solution of the Kepler problem as in (1.8) with initial condition in $\mathscr{L}_1$. Then the solution $\varphi(t, \mathbf{Y}; \varepsilon)$ of the system (1.5) in Poincaré-Delaunay (PD-1) variables is doubly-symmetric if at the instant $t = T/4$ it intercepts orthogonally the subspaces $\mathscr{L}_2$. So by Lemma 2.1 it is necessary to verify that at $t = T/4$ the following three equations must be satisfied

$$Q_1(T/4) = 0\,(mod\,\pi), \quad Q_2(T/4) = 0, \quad Q_3(T/4) = 0\,(mod\,\pi). \tag{4.1}$$

From the equation for $Q_j(t)$ for $j = 1,3$ and $P_2(t)$ in (2.1), it sufficient to solve the periodicity equations

$$\begin{aligned} f_1(\Delta Q_2, \Delta P_1, \Delta P_3, \varepsilon) &= (s^{-1/3} + \Delta P_1)^{-3}\, T/4 - \pi/2 + \mathscr{O}(\varepsilon^\alpha) = 0, \\ f_2(\Delta Q_2, \Delta P_1, \Delta P_3, \varepsilon) &= \Delta Q_2 \qquad\qquad\qquad\qquad\qquad + \mathscr{O}(\varepsilon^\alpha) = 0, \\ f_3(\Delta Q_2, \Delta P_1, \Delta P_3, \varepsilon) &= Q_3^{(1)}(T/4, \mathbf{Y}) \qquad\qquad\qquad\quad + \mathscr{O}(\varepsilon) = 0. \end{aligned} \tag{4.2}$$

Under the choice of T and the hypothesis (a) we obtain that $f_1(0,0,0,0) = f_2(0,0,0,0) = f_3(0,0,0,0) = 0$. Moreover, by differentiating the system (4.2) with respect to $(\Delta Q_2, \Delta P_1,\ \Delta P_3)$ and evaluating at $\mathbf{Y} = Y_0$ and $\varepsilon = 0$ we obtain that the Jacobian matrix satisfies

$$\left.\frac{\partial(f_1,f_2,f_3)}{\partial(\Delta Q_2,\Delta P_1,\Delta P_3)}\right|_{\mathbf{Y}=Y_0,\varepsilon=0} = \left.\begin{pmatrix} 0 & -3s^{4/3}\,T/4 & 0 \\ 1 & 0 & 0 \\ \frac{\partial Q_3^{(1)}}{\partial \Delta Q_2} & \frac{\partial Q_3^{(1)}}{\partial \Delta P_1} & \frac{\partial Q_3^{(1)}}{\partial \Delta P_3} \end{pmatrix}\right|_{\mathbf{Y}=Y_0,\varepsilon=0}.$$

Since $\det \left.\frac{\partial(f_1,f_2,f_3)}{\partial(\Delta Q_2,\Delta P_1,\Delta P_3)}\right|_{\mathbf{Y}=Y_0,\varepsilon=0} = -3s^{4/3}\ T/4 \left.\frac{\partial Q_3^{(1)}}{\partial \Delta P_3}\right|_{\mathbf{Y}=Y_0,\varepsilon=0} \neq 0$ by hypothesis (b), it follows by the Implicit Function Theorem that there are unique differentiable functions $\Delta Q_2 = \Delta Q_2(\varepsilon)$, $\Delta P_1 = \Delta P_1(\varepsilon)$ and $\Delta P_3 = \Delta P_3(\varepsilon)$ defined for $|\varepsilon| < \varepsilon_0$ where ε_0 is sufficiently small, such that $\Delta Q_2(0) = 0$, $\Delta P_1(0) = 0$, $\Delta P_3(0) = 0$ and $f_j(\Delta Q_2(\varepsilon), \Delta P_1(\varepsilon),\ \Delta P_3(\varepsilon), \varepsilon) = 0$ for $j = 1,2$ and 3. The proof of the computation of the characteristic multipliers is analogous to the previous theorem. Thus, we have proved the theorem. □

5. Proof of Corollary 1.2

Suppose that the Hamiltonian (1.1) is invariant about rotations around the z-axis. In this case the matrix A has always two zero eigenvalues. In fact, by symmetry of rotation, the perturbed function H_1 does not depend on $Q_3 = Z_2$. Therefore, with the same notation that appears in the proof of

Theorem 3, we obtain that the matrix A has the form:

$$A = \begin{pmatrix} a_{11} & 0 & a_{13} & a_{14} \\ a_{21} & 0 & a_{23} & a_{24} \\ a_{31} & 0 & a_{33} & a_{34} \\ 0 & 0 & 0 & a_{34} \end{pmatrix}.$$

It is easy to see that the eigenvalues of the matrix A are the roots of the polynomial equation $p(\lambda) = \lambda^2(\lambda^2 - (a_{11} + a_{33})\lambda + a_{11}a_{33} - a_{13}a_{31})$, with a_{11}, a_{13}, a_{31} and a_{33} given in the statement of the corollary. Since $a_{11} = -a_{33}$, it follows that the eigenvalues of A are the roots of

$$p(\lambda) = \lambda^2(\lambda^2 + a_{11}a_{33} - a_{13}a_{31}).$$

Therefore, we conclude de proof. □

6. Proof of Theorem 1.3

Let $\phi(t, \mathbf{Y}, \varepsilon)$ be a solution of the Hamiltonian system associated to the Hamiltonian (1.16) as in (1.17). By Lemma 2.1 a necessary condition in order to have a S_1-symmetric solution in Poincaré-Delaunay (PD-1) variables is that the equations (2.4) must be satisfied. In this case, from the equation for $Q_j(t)$ for $j = 1, 3$ and $P_2(t)$ in (2.1) it follows that the periodicity equations are given by

$$\begin{aligned} Q_1(T/2, \Delta Q_2, \Delta P_1, \Delta P_3, \varepsilon) &= (s^{-1/3} + \Delta P_1)^{-3} T/2 - \pi/2 + \mathscr{O}(\varepsilon^\alpha) = 0, \\ \tfrac{1}{\varepsilon^\alpha} Q_2(T/2, \Delta Q_2, \Delta P_1, \Delta P_3, \varepsilon) &= Q_2^{(1)}(T/2, \mathbf{Y}) + \mathscr{O}(\varepsilon) = 0, \\ Q_3(T/2, \Delta Q_2, \Delta P_1, \Delta P_3, \varepsilon) &= -T/2 + n\pi + \mathscr{O}(\varepsilon^\alpha) = 0. \end{aligned}$$

In order to increase the rank of the Jacobian matrix, we introduce the time as a dependent variable, thus we define the following periodicity system

$$\begin{aligned} f_1(\tau, \Delta Q_2, \Delta P_1, \Delta P_3, \varepsilon) &= (s^{-1/3} + \Delta P_1)^{-3} \tau - \pi/2 + \mathscr{O}(\varepsilon^\alpha), \\ f_2(\tau, \Delta Q_2, \Delta P_1, \Delta P_3, \varepsilon) &= Q_2^{(1)}(\tau, \mathbf{Y}) + \mathscr{O}(\varepsilon), \\ f_3(\tau, \Delta Q_2, \Delta P_1, \Delta P_3, \varepsilon) &= -\tau + n\pi + \mathscr{O}(\varepsilon^\alpha). \end{aligned} \tag{6.1}$$

Under the choose of T it is clear that $f_1(T/2, 0, 0, 0, 0) = f_3(T/2, 0, 0, 0, 0) = 0$ and by hypothesis (a) it follows that $f_2(T/2, 0, 0, 0, 0) = 0$. Moreover, differentiating the system (6.1) with respect to $(\tau, \Delta Q_2, \Delta P_1, \Delta P_3)$ and evaluating at $\tau = T/2$, $\mathbf{Y} = Y_0$ and $\varepsilon = 0$, we obtain that the Jacobian matrix $\left.\frac{\partial(f_1, f_2, f_3)}{\partial(\tau, \Delta Q_2, \Delta P_1, \Delta P_3)}\right|_{\tau = T/2, \mathbf{Y} = \mathbf{Y}_0, \varepsilon = 0}$ is

$$\begin{pmatrix} s & 0 & -3s^{4/3}T/2 & 0 \\ \frac{\partial Q_2^{(1)}}{\partial \tau} & \frac{\partial Q_2^{(1)}}{\partial \Delta Q_2} & \frac{\partial Q_2^{(1)}}{\partial \Delta P_1} & \frac{\partial Q_2^{(1)}}{\partial \Delta P_3} \\ -1 & 0 & 0 & 0 \end{pmatrix}_{\tau = T/2, \mathbf{Y} = \mathbf{Y}_0, \varepsilon = 0}.$$

If in the previous analysis we eliminate the variable ΔP_3 we obtain that the determinant of the Jacobian matrix is reduced to

$$det \begin{pmatrix} s & -3s^{4/3}T/2 & 0 \\ \frac{\partial Q_2^{(1)}}{\partial \tau} & \frac{\partial Q_2^{(1)}}{\partial \Delta Q_2} & \frac{\partial Q_2^{(1)}}{\partial \Delta P_1} \\ -1 & 0 & 0 \end{pmatrix}_{\tau = T/2, \mathbf{Y} = Y_0, \varepsilon = 0} = 3s^{4/3}T/2 \left.\frac{\partial Q_2^{(1)}}{\partial \Delta P_2}\right|_{\tau = T/2, \mathbf{Y} = \mathbf{Y}_0} \neq 0. \tag{6.2}$$

This last condition is equivalent to the hypothesis (b), then by the Implicit Function Theorem there are unique differentiable functions $\Delta Q_2(\Delta P_3,\varepsilon)$, $\Delta P_1(\Delta P_3,\varepsilon)$, and $\tau(\Delta P_3,\varepsilon) = T/2+\mathscr{O}(\varepsilon^{\alpha})$ defined for ε and ΔP_3 sufficiently small, such that, $\Delta P_2(0,0) = 0$, $\Delta P_1(0,0) = 0$, and $\tau(0,0) = T/2$. Thus, we obtain a periodic S_1-symmetric solution of the perturbed system associated to the Hamiltonian function (1.16) with initial condition $\mathbf{Y}_{\Delta P_3,\varepsilon} = (\pi/2,\Delta Q_2(\Delta P_3,\varepsilon),0,s^{-1/3}+\Delta P_1(\Delta P_3,\varepsilon),0,p_3+\Delta P_3)$ which is 2τ periodic and it is close to $T = 2\pi/s$. Therefore, we have concluded the proof.

We remark that we can consider $\Delta P_3 = 0$ in equations (6.1) and differentiating this system only with respect to $(\tau,\Delta Q_2,\Delta P_1)$, maintaining fixed the variable in the direction P_3. In this way, we shall restrict our attention to the invariant set $H = p_3$. Thus, we obtain a periodic S_1-symmetric solution of the perturbed system associated to the Hamiltonian function (1.16) with initial condition $\mathbf{Y}_{\varepsilon} = (\pi/2,\Delta Q_2(\varepsilon),0,s^{-1/3}+\Delta P_1(\varepsilon),0,p_3)$ which is 2τ periodic and it is close to $T = 2\pi/s$.

In order to compute the characteristic multipliers of the previous symmetric periodic solutions, we consider a local cross section $\Sigma = \{(\mathbf{Q},\mathbf{P})\ /\ K = k_0,\ Q_1 = Y_1^0 = \pi/2\}$, We label the coordinates in Σ by $X = (X_1,X_2,X_3,X_4) = (Q_2,Q_3,P_2,P_3)$, and let P be the Poincaré map on Σ. Considering $\overline{Y} = (Y_1^{(0)},Y_2^{(0)}+X_1,Y_3^{(0)}+X_2,Y_4,Y_5^{(0)}+X_3,Y_6^{(0)}+X_4)$, where Y_4 is like in the statement of Theorem 1.3 and following the same ideas as in Theorem 1.1, we have that P is given by $P(X,\varepsilon) = \big(Q_2(\mathscr{T},\overline{Y},\varepsilon),Q_3(\mathscr{T},\overline{Y},\varepsilon),P_2(\mathscr{T},\overline{Y},\varepsilon),P_3(\mathscr{T},\overline{Y},\varepsilon)\big))$, where $\mathscr{T}$ is the return time which is close to $\overline{T}(\Delta P3,\varepsilon) = \frac{2\pi}{s-1}+\mathscr{O}(\varepsilon^{\alpha})$. Using the form of Y^0 and (2.3)-(2.2), we arrive to

$$\begin{aligned}
P(X,\varepsilon) &= (X_1,-\mathscr{T}+X_2,X_3,p_3+X_4)+\varepsilon^{\alpha}\left(\int_0^{\mathscr{T}}\frac{\partial K_1}{\partial P_2}(\phi_{kep}(t,\overline{Y}))dt,\right.\\
&\left.\int_0^{\mathscr{T}}\frac{\partial K_1}{\partial P_3}(\phi_{kep}(t,\overline{Y}))dt,-\int_0^{\mathscr{T}}\frac{\partial K_1}{\partial Q_2}(\phi_{kep}(t,\overline{Y}))dt,-\int_0^{\mathscr{T}}\frac{\partial K_1}{\partial Q_3}(\phi_{kep}(t,\overline{Y}))dt\right)+\\
&\mathscr{O}(\varepsilon^{\alpha+1})\\
&= (X_1,-\overline{T}+X_2,X_3,p_3+X_4)+\varepsilon^{\alpha}\left(\int_0^{\overline{T}}\frac{\partial K_1}{\partial P_2}(\phi_{kep}(t,\overline{Y}))dt,\right.\\
&\left.\int_0^{\overline{T}}\frac{\partial K_1}{\partial P_3}(\phi_{kep}(t,\overline{Y}))dt,-\int_0^{\overline{T}}\frac{\partial K_1}{\partial Q_2}(\phi_{kep}(t,\overline{Y}))dt,-\int_0^{\overline{T}}\frac{\partial K_1}{\partial Q_3}(\phi_{kep}(t,\overline{Y}))dt\right)+\\
&\mathscr{O}(\varepsilon^{\alpha+1})\\
&= (X_1,-T+X_2,X_3,p_3+X_4)+\varepsilon^{\alpha}\left(\int_0^{T}\frac{\partial K_1}{\partial P_2}(\phi_{kep}(t,\overline{Y}))dt,\right.\\
&\left.\int_0^{T}\frac{\partial K_1}{\partial P_3}(\phi_{kep}(t,\overline{Y}))dt,-\int_0^{T}\frac{\partial K_1}{\partial Q_2}(\phi_{kep}(t,\overline{Y}))dt,-\int_0^{T}\frac{\partial K_1}{\partial Q_3}(\phi_{kep}(t,\overline{Y}))dt\right)+\\
&\mathscr{O}(\varepsilon^{\alpha+1}).
\end{aligned} \tag{6.3}$$

Recalling that the initial condition of the $\bar{T}$-symmetric periodic solutions is $X_{\varepsilon} = Y_0+(0,\Delta Q_2(\Delta P_3,\varepsilon),0,\Delta P_1(\Delta P_3,\varepsilon),0,\Delta P_3)$, then the respective points on the cross section Σ will be $X_{\varepsilon} = (\Delta Q_2(\Delta P_3,\varepsilon)),0,0,\Delta P_3)$ with $X_0 = (0,0,0,p_3)$. Then, the nontrivial characteristic multipliers associated to the symetric $\overline{T}$-periodic solutions $\phi(t,\mathbf{Y}_{\varepsilon};\varepsilon)$ given in Theorem 1.5 are the eigenvalues of

$$D_X P(X_{\varepsilon},\varepsilon) = I+\varepsilon^{\alpha}C+\mathscr{O}(\varepsilon^{\alpha+1}), \tag{6.4}$$

where the matrix C is as in (1.28). Thus, we have proved the theorem. □

7. Proof of Theorem 1.4

We proceed in a similar way as in the proof in Section 6. Here, we need to consider the circular solution of the Kepler problem in (PD-1) variables as

$$\begin{array}{lll} Q_1(t) = st + \pi/2, & Q_2(t) = 0, & Q_3(t) = -t, \\ P_1(t) = s^{-1/3}, & P_2(t) = 0, & P_3(t) = p_3, \end{array} \tag{7.1}$$

with $p_3 \in \mathbb{R}$. In order to have a doubly-symmetric periodic solution of this Kepler problem, by Lemma 1.12 it is necessary to solve, for $t = T/4$, the following periodicity equations

$$\begin{aligned} Q_1(T/4, \mathbf{Y}_0) &= sT/4 + \pi/2 = \pi, \\ Q_3(T/4, \mathbf{Y}_0) &= -T/4 = -n\pi, \end{aligned}$$

with $n \in \mathbb{N}$. Note that from (7.1) the equation $Q_2(T/4, \mathbf{Y}_0) = 0$ is trivially satisfied. It is verified that this solution, in inertial frame, is a circular periodic solution doubly-symmetric, and has period $T = 2\pi/s$, with the choice $s = 1/2n$, and it is on any orbital plane (arbitrary inclination) on $\mathbb{R}^3$ with initial condition $\mathbf{Y}_0 = (\pi/2, 0, 0, s^{-1/3}, 0, p_3) \in \mathscr{L}_1$.

We consider now the solution $\phi_{kep}(\tau, \mathbf{Y})$ of the Kepler problem as in (1.17). This solution is doubly symmetric if at the instant $t = T/4$ it intercepts orthogonally the subspaces $\mathscr{L}_2$. So by Lemma 2.1 and the equation for $Q_j(t)$ for $j = 1, 3$ and $P_2(t)$ in (2.1) it follows that the following periodicity equations must be satisfied

$$\begin{aligned} f_1(\tau, \Delta Q_2, \Delta P_1, \Delta P_3, \varepsilon) &= (s^{-1/3} + \Delta P_1)^{-3}\tau - \pi/2 + \mathscr{O}(\varepsilon^\alpha), \\ f_2(\tau, \Delta Q_2, \Delta P_1, \Delta P_3, \varepsilon) &= \Delta Q_2 \qquad\qquad\qquad\qquad + \mathscr{O}(\varepsilon^\alpha), \\ f_3(\tau, \Delta Q_2, \Delta P_1, \Delta P_3, \varepsilon) &= -\tau + n\pi \qquad\qquad\qquad + \mathscr{O}(\varepsilon^\alpha). \end{aligned} \tag{7.2}$$

Under the choice of T it is clear that $f_1(T/4, 0, 0, 0, 0) = f_2(T/4, 0, 0, 0, 0) = f_3(T/4, 0, 0, 0, 0) = 0$. Moreover differentiating the system (7.2) with respect to $(\tau, \Delta Q_2, \Delta P_1, \Delta P_3)$ and evaluating at $\tau = T/4$, $\mathbf{Y} = Y_0$ and $\varepsilon = 0$, we obtain that the Jacobian matrix $\left.\frac{\partial(f_1, f_2, f_3)}{\partial(\tau, \Delta Q_2, \Delta P_1, \Delta P_3)}\right|_{\tau = T/4, \mathbf{Y} = \mathbf{Y}_0, \varepsilon = 0}$ is

$$\begin{pmatrix} s & 0 & -3s^{4/3}T/4 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix}_{\tau = T/4, \mathbf{Y} = \mathbf{Y}_0, \varepsilon = 0}.$$

If in the previous analysis we eliminate the variable ΔP_3 we obtained that the determinant of the Jacobian matrix is reduced to

$$det \begin{pmatrix} s & 0 & -3s^{4/3}T/4 \\ 0 & 1 & 0 \\ -1 & 0 & 0 \end{pmatrix}_{\tau = T/4, \mathbf{Y} = Y_0, \varepsilon = 0} = 3s^{4/3}T/4 \neq 0. \tag{7.3}$$

Again as in the S_1 periodic symmetric case we can to consider $\Delta P_3 = 0$, next we differentiate the system (7.2) only with respect to $(\tau, \Delta Q_2, \Delta P_1)$ maintaining fixed the variable in the direction P_3. In this way, we shall restrict our attention to the invariant set $H = p_3$. The study of the stability is analogous to Theorem 1.3. Thus, we conclude the proof. □

8. Proof of Theorem 1.5

Now we will consider the Hamiltonian system associated to the Hamiltonian function (1.25) but in this case we will write it in Poincaré-Delaunay (PD-2) variables. Thus the Kepler solution must satisfy

$$\begin{aligned} Q_1(T/2,\mathbf{Z}_0) &= (s-1)T/2+\pi/2 = (1+1/2)\pi, \\ Q_3(T/2,\mathbf{Z}_0) &= p_3 \sin T/2 \qquad\qquad = 0, \end{aligned} \tag{8.1}$$

where $s \in \mathbb{R}-\{1\}$. Note that the equation $Q_2(T/2,\mathbf{Z}_0) = 0$ is trivially satisfied. Then we must have two possibilities:

(i) $p_3 = 0$ and $T = 2\pi/(s-1)$ or
(ii) $T = 2k\pi$ and $s-1 = 1/k$ with p_3 arbitrary.

We use the Poincaré-Delaunay (PD-2) variables and in this case the initial condition is on the subspace $\mathscr{L}_1$ and at time $T/2$ is also on the subspace $\mathscr{L}_1$. In this case, from (2.6), the periodicity equations assume the form

$$\begin{aligned} f_1(\Delta Q_2,\Delta P_1,\Delta P_3,\varepsilon) &= ((s^{-1/3}+\Delta P_1)^{-3}-1)T/2-\pi +\mathscr{O}(\varepsilon^\alpha) = 0, \\ f_2(\Delta Q_2,\Delta P_1,\Delta P_3,\varepsilon) &= (p_3+\Delta P_3)\sin T/2 \qquad\qquad +\mathscr{O}(\varepsilon^\alpha) = 0, \\ f_3(\Delta Q_2,\Delta P_1,\Delta P_3,\varepsilon) &= -\Delta Q_2 \sin T/2 \qquad\qquad\quad +\mathscr{O}(\varepsilon^\alpha) = 0. \end{aligned} \tag{8.2}$$

Observe that (i) or (ii) imply $f_1(0,0,0,0) = f_2(0,0,0,0) = f_3(0,0,0,0) = 0$. Now, differentiating the system (8.2) with respect to $(\Delta Q_2,\Delta P_1,\Delta P_3)$ and evaluating at $\tau = T/2$, $\mathbf{Z} = Z_0$ and $\varepsilon = 0$, we obtain that the Jacobian matrix is

$$\left.\frac{\partial(f_1,f_2,f_3)}{\partial(\Delta Q_2,\Delta P_1,\Delta P_3)}\right|_{\mathbf{Z}=\mathbf{Z}_0,\varepsilon=0} = \begin{pmatrix} 0 & -3s^{4/3}\,T/2 & 0 \\ 0 & 0 & p_3\sin T/2 \\ \sin T/2 & 0 & 0 \end{pmatrix}_{\mathbf{Z}=\mathbf{Z}_0,\varepsilon=0}.$$

Therefore to have $\det \left.\frac{\partial(f_1,f_2,f_3)}{\partial(\Delta Q_2,\Delta P_1,\Delta P_3)}\right|_{\mathbf{Z}=\mathbf{Z}_0,\varepsilon=0} = p_3(-3s^{4/3}\,T/2)\sin^2 T/2 \neq 0$ we must assume that $\sin T/2 \neq 0$ and then $p_3 = 0$. Since $p_3 = 0$ the solution (1.26) with $\Delta Q_2 = \Delta P_1 = \Delta P_3 = 0$ is a circular solution of Kepler problem, that in inertial frame is in the xy-plane.

To compute the characteristic multipliers of the previous symmetric periodic solutions, we consider a local cross section $\Sigma = \{(\mathbf{Q},\mathbf{P})\ /\ \mathscr{K} = k_0,\ Q_1 = Z_1^0 = \pi/2\}$, on the level set in the level $\mathscr{K} = -\frac{1}{2P_1^2} - P_1 + \frac{1}{2}(Q_2^2+P_2^2+Q_3^2+P_3^2) = k_0$. Again, we label the coordinates in Σ by $X = (X_1,X_2,X_3,X_4) = (Q_2,Q_3,P_2,P_3)$, and let P be the Poincaré map on Σ. Considering $\overline{X} = (Z_1^{(0)}, Z_2^{(0)}+X_1, Z_3^{(0)}+X_2, Z_4, Z_5^{(0)}+X_3, Z_5^{(0)}+X_4)$ and following the same ideas as in Theorem 1.1, we have that P is given by $P(X,\varepsilon) = \big(Q_2(\mathscr{T},\overline{X},\varepsilon), Q_3(\mathscr{T},\overline{X},\varepsilon), P_2(\mathscr{T},\overline{X},\varepsilon), P_3(\mathscr{T},\overline{X},\varepsilon)\big)$, where $\mathscr{T}$ is the return time which is close to $T = \frac{2\pi}{s-1}$. Using the form of Z^0 and (2.3)-(2.2), we arrive to

$$\begin{aligned} Q_i(\mathscr{T},X,\varepsilon) = {} & (Z_i^{(0)}+X_{i-1})\cos T + (Z_{i+3}^{(0)}+X_{i+1})\sin T + \varepsilon^\alpha \int_0^T \frac{\partial \mathscr{K}_1}{\partial P_i}\big(\Phi_{kep}(t,\overline{X})\big)dt + \\ & \mathscr{O}(\varepsilon^{\alpha+1}), \\ P_i(\mathscr{T},X,\varepsilon) = {} & -(Z_i^{(0)}+X_{i-1})\sin T + (Z_{i+3}^{(0)}+X_{i+1})\cos T - \varepsilon^\alpha \int_0^T \frac{\partial \mathscr{K}_1}{\partial Q_i}\big(\Phi_{kep}(t,\overline{X})\big)dt + \\ & \mathscr{O}(\varepsilon^{\alpha+1}), \end{aligned}$$

for $i = 2,3$. Since the initial condition of the T-symmetric periodic solutions is $X_\varepsilon = Z_0 + (0,\Delta Q_2(\varepsilon),0,\Delta P_1(\varepsilon),0,\Delta P_3(\varepsilon))$, then the respective points on the cross section Σ will be $X_\varepsilon =$

$(\Delta Q_2(\varepsilon), 0, 0, \Delta P_3(\varepsilon))$ with $X_0 = (0,0,0,0)$. Then, the nontrivial characteristic multipliers associated to the symmetric T-periodic solutions $\Phi(t, \mathbf{Z}_\varepsilon; \varepsilon)$ given in Theorem 1.5 are the eigenvalues of

$$D_X P(X_\varepsilon, \varepsilon) = \cos T\, I + \sin T\, J + \varepsilon^\alpha C + \mathscr{O}(\varepsilon^{\alpha+1}), \tag{8.3}$$

where the matrix C is as in (1.28). Thus, we have proved the theorem. □

9. Proof of Theorem 1.6

The proof follows the same ideas as in Theorem 1.5. Consider the solution $\Phi(t, \mathbf{Z})$ of the Kepler problem as in (1.26) with initial condition on the subspace $\mathscr{L}_1$. Let $\Phi(t, \mathbf{Z}, \varepsilon)$ be a solution of the Hamiltonian system associated to the Hamiltonian (1.25). To obtain a doubly symmetric solution is necessary that at time $T/4$ the solution is on the subspace $\mathscr{L}_2$. We observe that from (2.6) and (2.7) of Lemma 2.2, the periodicity equations in this situation assume the form

$$\begin{aligned} f_1(\Delta Q_2, \Delta P_1, \Delta P_3, \varepsilon) &= ((s^{-1/3} + \Delta P_1)^{-3} - 1)T/4 - \pi + \mathscr{O}(\varepsilon^\alpha) = 0, \\ f_2(\Delta Q_2, \Delta P_1, \Delta P_3, \varepsilon) &= \Delta Q_2 \cos T/4 \qquad\qquad\qquad\quad + \mathscr{O}(\varepsilon^\alpha) = 0, \\ f_3(\Delta Q_2, \Delta P_1, \Delta P_3, \varepsilon) &= \Delta P_3 \sin T/4 \qquad\qquad\qquad\quad\; + \mathscr{O}(\varepsilon^\alpha) = 0. \end{aligned} \tag{9.1}$$

Observe that under the hypotheses about T and s, we have $f_1(0,0,0,0) = f_2(0,0,0,0) = f_3(0,0,0,0) = 0$. Moreover

$$\left.\frac{\partial(f_1, f_2, f_3)}{\partial(\Delta Q_2, \Delta P_1, \Delta P_3)}\right|_{\mathbf{Z}=\mathbf{Z}_0, \varepsilon=0} = \begin{pmatrix} 0 & -3s^{4/3}\, T/4 & 0 \\ \cos T/4 & 0 & 0 \\ 0 & 0 & \sin T/4 \end{pmatrix}_{\mathbf{Z}=\mathbf{Z}_0, \varepsilon=0}.$$

Therefore to have $\det \left.\frac{\partial(f_1, f_2, f_3)}{\partial(\Delta Q_2, \Delta P_1, \Delta P_3)}\right|_{\mathbf{Z}=\mathbf{Z}_0, \varepsilon=0} \neq 0$, we must assume that $\sin T/4 \neq 0$ and $\cos T/4 \neq 0$. The proof of stability is analogous to the proof in Theorem 1.5 and so the proof is concluded.

10. Proof of Corollary 1.4

We proceed in a similar way as in the proof of Theorems 1.5 and 1.6. If the set $z = p_z = 0$ is invariant then in terms of (PD-2) variables $Q_3(t) = P_3(t) = 0$ for all t. For S_1 symmetric solutions, it follows from Section 8 that the periodicity equations (8.2) must be

$$\begin{aligned} f_1(\Delta Q_2, \Delta P_1, \Delta P_3, \varepsilon) &= ((s^{-1/3} + \Delta P_1)^{-3} - 1)T/2 - \pi + \mathscr{O}(\varepsilon^\alpha) = 0, \\ f_3(\Delta Q_2, \Delta P_1, \Delta P_3, \varepsilon) &= -\Delta Q_2 \sin T/2 \qquad\qquad\qquad\quad + \mathscr{O}(\varepsilon^\alpha) = 0. \end{aligned} \tag{10.1}$$

In this particular case, we have

$$\left.\frac{\partial(f_1, f_3)}{\partial(\Delta Q_2, \Delta P_1)}\right|_{\mathbf{Z}=\mathbf{Z}_0, \varepsilon=0} = \begin{pmatrix} 0 & -3s^{4/3}\, T/2 \\ \sin T/2 & 0 \end{pmatrix}_{\mathbf{Z}=\mathbf{Z}_0, \varepsilon=0}.$$

Therefore $\det \left.\frac{\partial(f_1, f_3)}{\partial(\Delta Q_2, \Delta P_1)}\right|_{\mathbf{Z}=\mathbf{Z}_0, \varepsilon=0} = (-3s^{4/3}\, T/2) \sin T/2 \neq 0$. The planar case of the doubly symmetric solution is verified in a similar way. To compute the characteristic multipliers it is enough to follow the same ideas used in the proof of Theorem 1.5 and to observe that $X = (X_1, X_3)$, and $\overline{X} = (Z_1^{(0)}, Z_2^{(0)} + X_1, Z_3^{(0)}, Z_4, Z_5^{(0)} + X_3, Z_5^{(0)})$. □

11. Applications

11.1. *The Matese-Whitman Hamiltonian*

In what follows we shall study the *Matese-Whitman Hamiltonian*, or shortly, MWH, whose Hamiltonian models the galactic tidal interaction with the Oort coment cloud. The MWH Hamiltonian is given by

$$H_{MWH} = \frac{1}{2}(p_x^2 + p_y^2 + p_z^2) + \frac{1}{\sqrt{x^2+y^2+z^2}} + \varepsilon z^2, \tag{11.1}$$

and it was proposed in the seminal papers of Matese and Whitman [13,14] for studying the dynamics of the Oort cloud. See also [9]. Of course, this Hamiltonian function H_{MWH} in (11.1) is a particular case of the Hamiltonian (1.1) and is invariant under the anti-symplectic reflections S_1 and S_2.

We verify that the Hamiltonian (11.1) in (PD-1) variables assumes the form

$$\mathscr{H}(\mathbf{P},\mathbf{Q},\varepsilon) = \mathscr{H}_0 + \varepsilon \mathscr{H}_1, \tag{11.2}$$

where the perturbed function $\mathscr{H}_1$ is given by

$$\begin{aligned}
\mathscr{H}_1 = {} & \frac{-1}{8\left(-P_2^2-Q_2^2+2P_1\right)^2\left(P_2^2+Q_2^2\right)^2} P_1^2\Big(-P_2^2-Q_2^2+2P_1-2P_3\Big)\Big(-P_2^2-Q_2^2+2P_1+2P_3\Big) \\
& (-(P_2^2+Q_2^2)^2((-2P_1+P_2^2)^2+(16P_1-3P_2^2)Q_2^2-4Q_2^4)-2P_1P_2(P_2^2-11Q_2^2)(P_2^2+Q_2^2) \\
& \sqrt{4P_1-P_2^2-Q_2^2}\cos(Q1)+(4P_1^2(P_2^4-6P_2^2Q_2^2+Q_2^4)-4P_1(P_2^2+Q_2^2)(2P_2^4-15P_2^2Q_2^2 \\
& +3Q_2^4)+(P_2^2+Q_2^2)^2(2P_2^4-15P_2^2Q_2^2+3Q_2^4))\cos(2Q_1)-20P_1P_2^3Q_2^2\sqrt{4P_1-P_2^2-Q_2^2} \\
& \cos(3Q_1)+2P_1P_2^5\sqrt{4P_1-P_2^2-Q_2^2}\cos(3Q_1)+10P_1P_2Q_2^4\sqrt{(}4P_1-P_2^2-Q_2^2)\cos(3Q_1) \\
& -(-4P_1+P_2^2+Q_2^2)(P_2^6-15P_2^4Q_2^2+15P_2^2Q_2^4-Q_2^6)\cos(4Q_1)-14P_1\sqrt{4P_1-P_2^2-Q_2^2} \\
& P_2^4Q_2\sin(Q_1)-4P_1P_2^2Q_2^3\sqrt{4P_1-P_2^2-Q_2^2}\sin(Q_1)+10P_1Q_2^5\sqrt{4P_1-P_2^2-Q_2^2}\sin(Q_1) \\
& +P_2Q_2(16P_1^2(P_2-Q_2)(P_2+Q_2)+(9P_2^2-11Q_2^2)(P_2^2+Q_2^2)^2+(8P_2^2Q_2^2-36P_2^4+44Q_2^4)) \\
& P_1\sin(2Q_1)+10P_1P_2^4Q_2\sqrt{4P_1-P_2^2-Q_2^2}\sin(3Q_1)+2P_1Q_2^5\sqrt{4P_1-P_2^2-Q_2^2}\sin(3Q_1) \\
& -20P_1P_2^2Q_2^3\sqrt{4P_1-P_2^2-Q_2^2}\sin(3Q_1)-2P_2Q_2(P_2^2+Q_2^2-4P_1)(3P_2^4-10P_2^2Q_2^2+3Q_2^4) \\
& \sin(4Q_1])).
\end{aligned} \tag{11.3}$$

We do not enter into the details of the algebraic operations involved in constructing the perturbed function $\mathscr{H}_1$. They were executed with the symbolic processor Mathematica. We have the following result on the periodic orbits for the Hamiltonian system associated to the Hamiltonian (11.1).

Theorem 11.1. *Given $s \in \mathbb{R}^+$ and $T = 2\pi/s$. Then for the MWH problem associated to the Hamiltonian (11.1) or (11.2) the following statements hold:*

(a) If ε is sufficiently small, then there exists a family of initial conditions $\mathbf{Y}_\varepsilon = (\pi/2, \Delta Q_2(\varepsilon), 0, s^{-1/3} + \Delta P_1(\varepsilon), 0, \Delta P_3(\varepsilon))$ in (PD-1) variables parametrized by ε, such that each of them gives us a S_1-symmetric T-periodic solution.

(b) If ε is sufficiently small, then there exists a family of initial conditions $\mathbf{Y}_\varepsilon = (\pi/2, \Delta Q_2(\varepsilon), 0, s^{-1/3} + \Delta P_1(\varepsilon), 0, \Delta P_3(\varepsilon))$ in (PD-1) variables parametrized by ε, such that each of them gives us a doubly-symmetric T- periodic solution.

Moreover, the periodic solutions obtained in item (a) and item (b) have characteristic multipliers $1,1,1,1,1+\varepsilon\frac{4\pi}{s^2}+\mathscr{O}(\varepsilon^2), 1-\varepsilon\frac{4\pi}{s^2}+\mathscr{O}(\varepsilon^2)$ and therefore these periodic solutions are unstable. The *periodic solutions are close to a circular Keplerian solution which has radius $s^{-2/3}$, period T and lies on the xz-plane.*

Proof. In the proof, we omit the details of algebraic operations involved that were executed with Mathematica. To prove item (a), we need to verify the conditions (1.12) of Theorem 1.1. We maintain the same notation of Theorem 1.1 and after some algebraic manipulations, from the expression (11.3) we obtain

$$i)\int_0^{T/2}\frac{\partial\mathscr{H}_1}{\partial P_3}(\varphi_{kep}(\tau,\mathbf{Y}))\Big|_{\mathbf{Y}=\mathbf{Y}_0}d\tau=-2p_3\int_0^{\pi/s}\frac{\cos^2(s\tau)d\tau}{s^{2/3}}=-p_3\pi/s^{5/3}.$$

To verify item (a) in (1.12), we need that $p_3\pi/s^{5/3}=0$, therefore we must take $p_3=0$, thus the Keplerian circular orbit is on the vertical plane xz. For $p_3=0$ it follows

$$\begin{aligned} ii)&\int_0^{T/2}\tfrac{\partial\mathscr{H}_1}{\partial Q_2}(\varphi_{kep}(\tau,\mathbf{Y}))\Big|_{\mathbf{Y}=\mathbf{Y}_0}d\tau=\int_0^{T/2}\tfrac{\cos(3s\tau)-5\cos(s\tau)}{2s^{7/6}}d\tau=0,\\ iii)&\ \Omega_1=-\tfrac{1}{256s^{5/6}}\int_0^{\pi/s}\tfrac{768\cos(2s\tau)}{\sqrt[6]{s}}-\tfrac{256\cos(4s\tau)}{\sqrt[6]{s}}-\tfrac{1024}{\sqrt[6]{s}}d\tau\cdot\int_0^{\pi/s}\tfrac{-4\cos(2s\tau)-4}{4s^{2/3}}d\tau\\ &=-\tfrac{4\pi}{s^2}\cdot\tfrac{\pi}{s^{5/3}}\neq 0.\end{aligned}$$

Therefore, from *i*), *ii*) and *iii*) it follows that (1.12) holds and we have proved item (a).

Now, we prove item (b). Maintaining the notation of Theorem 1.2, we obtain from (11.3) the following statements

$$\begin{aligned} iv)&\int_0^{T/4}\tfrac{\partial\mathscr{H}_1}{\partial P_3}(\varphi_{kep}(\tau,\mathbf{Y}))\Big|_{\mathbf{Y}=\mathbf{Y}_0}d\tau=-\tfrac{\pi p_3}{2s^{5/3}},\\ v)&\int_0^{T/4}\tfrac{\partial^2\mathscr{H}_1}{\partial P_3\partial\Delta P_3}(\varphi_{kep}(\tau,\mathbf{Y}))\Big|_{\mathbf{Y}=\mathbf{Y}_0}d\tau=-\tfrac{\pi}{2s^{5/3}}\neq 0,\end{aligned}\tag{11.4}$$

To verify first the condition in (1.13), we need $\frac{\pi p_3}{2s^{5/3}}=0$ and as in the previous item, we must put $p_3=0$. The second condition in (1.13) follows directly from v. We now study the stability. After some algebraic manipulations, the matrix A in (1.11) for the S_1-symmetric and doubly-symmetric solutions obtained for the MWH problem is given by

$$A=\begin{pmatrix}0&0&-\frac{2\pi}{s^2}&0\\0&0&0&-\frac{2\pi}{s^{5/3}}\\-\frac{8\pi}{s^2}&0&0&0\\0&0&0&0\end{pmatrix}.$$

Thus the eigenvalues of A are $0,0,4\pi/s^2,-4\pi/s^2$. Therefore we have proved the theorem. □

Since the set $y=p_y=0$ is invariant under the flow defined by the Hamiltonian associated to H_{MWH} in (11.1), we prove analogous versions of items (a) and (b) of Theorem 11.1 to the polar plane, i.e. we obtain symmetric periodic solutions contained in the vertical plane xz close to circular Keplerian solutions on the same plane.

The Hamiltonian (11.1) is invariant under the group $SO(2)$ relative to the xy-plane. Therefore, we apply a $2\pi-$time sympletic transformation given by a rotation around the z-axis to the Hamiltonian function H_{MWH} in (11.1), then the new Hamiltonian function assumes the form as in (1.16). We define which will be called the *critical inclination* as the value of the inclination ι such that $4-5\cos^2\iota=0$. Then we can apply Theorems 1.3-1.4-1.5 and 1.6, to obtain periodic orbits for the Hamiltonian system associated to the Hamiltonian (11.1).

Of course, when we apply Theorems 1.3-1.6, the periodic solutions obtained in these theorems correspond to quasi-periodic solutions of the Hamiltonian (11.1). Particulary, in Theorem 1.3 we can obtain an analytic function $\overline{T}(\Delta P_3,\varepsilon) = 2\tau(\Delta P_3,\varepsilon)$, for ΔP_3 and ε sufficiently small, close to $T = 2n\pi$. Now, for all values of the parameter ε, sufficiently small, we can take ΔP_3 such that $\overline{T}(\Delta P_3(\varepsilon),\varepsilon) = 2\pi p/q$, $p,q \in \mathbb{N}$, and $\frac{\overline{T}(\Delta P_3(\varepsilon),\varepsilon)}{2\pi} = \frac{p}{q} \in \mathbb{Q}$, that is, the relation of commensurability between the period of the continued orbit and the rotation period is satisfied. Thus, in fact, we can get periodic solutions, from Theorem 1.3, to the problem associated to the Hamiltonian (11.1). Similarly, we obtain the following conclusion to Theorems 1.4-1.5 and 1.6.

Theorem 11.2. *Given $s \in \mathbb{R}^+$ and $m,n \in \mathbb{N}$ prime to each other. Then for the MWH problem associated to the Hamiltonian (11.1) the following statements hold:*

(a) If $s = 1/n$, and the inclination ι of the plane of the circular Keplerian orbit satisfies $4-5\cos^2\iota \neq 0$, then for ε and ΔP_3 sufficiently small, there exists a family of periodic S_1-symmetric solutions $\gamma(t,\Delta P_3,\varepsilon) = (Q_1(t,\varepsilon),Q_2(t,\Delta P_3,\varepsilon),Q_3(t,\varepsilon),P_1(t,\Delta P_3,\varepsilon),P_2(t,\varepsilon),\ P_3(t,\varepsilon))$ in (PD-1) variables parametrized by ε and ΔP_3.

(b) If $s = \frac{1}{2n}$, then for ε and ΔP_3 sufficiently small, there exists a family of periodic doubly-symmetric solutions $\gamma(t,\Delta P_3,\varepsilon) = (Q_1(t,\varepsilon),Q_2(t,\Delta P_3,\varepsilon),Q_3(t,\varepsilon),P_1(t,\Delta P_3,\varepsilon),P_2(t,\varepsilon),\ P_3(t,\varepsilon))$ in (PD-1) variables parametrized by ε and ΔP_3.

(c) If $s = \frac{m+n}{m}$, then for ε sufficiently small, there exists a family of periodic S_1-symmetric solutions $\gamma(t,\varepsilon) = (Q_1(t,\varepsilon),Q_2(t,\varepsilon),Q_3(t,\varepsilon),P_1(t,\varepsilon),P_2(t,\varepsilon),P_3(t,\varepsilon))$ in (PD-2) variables parametrized by ε.

(d) If $s = \frac{m+n}{m}$, then for ε sufficiently small, there exists a family of periodic doubly-symmetric solutions $\gamma(t,\varepsilon) = (Q_1(t,\varepsilon),Q_2(t,\varepsilon),\ Q_3(t,\varepsilon),P_1(t,\varepsilon),P_2(t,\varepsilon),P_3(t,\varepsilon))$ in (PD-2) variables parametrized by ε.

Moreover, in item (a) and (b) for ε and ΔP_3 sufficiently small, there exist unique analytical functions $\Delta Q_2(\Delta P_3,\varepsilon)$ and $\Delta P_1(\Delta P_3,\varepsilon)$ for the parameters ε and ΔP_3, such that

$$\gamma(0,\Delta P_3,\varepsilon) = (\pi/2,\Delta Q_2(\Delta P_3,\varepsilon),0,s^{-1/3}+\Delta P_1(\Delta P_3,\varepsilon),0,p_3+\Delta P_3),$$

and each symmetric periodic solution has period $\overline{T} = 2\pi p/q,\ p,q \in \mathbb{N}$ close to $T = 2\pi/s$ and is close to a circular Keplerian solution that in inertial frame has radius $s^{-2/3}$ and period T. In items (c) and (d), for ε sufficiently small, there exist analytical functions $\Delta Q_2(\varepsilon),\Delta P_1(\varepsilon)$ and $\Delta P_3(\varepsilon)$, such that

$$\gamma(0,\varepsilon) = (\pi/2,\Delta Q_2(\varepsilon),0,s^{-1/3}+\Delta P_1(\varepsilon),0,\Delta P_3(\varepsilon)),$$

and each symmetric periodic solution has period $T = 2\pi m/n = 2\pi/(s-1)$ and they are close to a circular Keplerian solution that in inertial frame has radius $s^{-2/3}$ and period T.

The periodic solutions obtained in items (a) and (b) have characteristic multipliers $1,1,1,1,1+\varepsilon\frac{2\pi\sqrt{4-5p_3{}^2s^{2/3}}}{s^2}+\mathscr{O}(\varepsilon^2),1-\varepsilon\frac{2\pi\sqrt{4-5p_3{}^2s^{2/3}}}{s^2}+\mathscr{O}(\varepsilon^2)$. Thus, for $4-5p_3{}^2s^{2/3}>0$, or equivalently $|\cos\iota| < \sqrt{4/5}$, these solutions are unstable and for $4-5p_3{}^2s^{2/3}<0$, or equivalently, $|\cos\iota| > \sqrt{4/5}$, these solutions are linearly stable.

Proof. Again, we use the processor Mathematica to execute the algebraic manipulations. In the proof of item (a), we maintain the notation of Theorem 1.3 and we observe that the perturbed function $\mathscr{K}_1$ is given by $\mathscr{H}_1$ in (11.3), because the problem is invariant by rotations about z axis. Therefore, from the expression of $\mathscr{H}_1$ in (11.3), we obtain the following expressions:

$$\int_0^{T/2} \frac{\partial K_1}{\partial Q_2}(\phi_{kep}(\tau,\mathbf{Y}))\Big|_{\mathbf{Y}=\mathbf{Y}_0} d\tau = -\int_0^{\pi/s} \frac{\left(p_3{}^2 s^{2/3}-1\right)(\cos(3s\tau)-5\cos(s\tau))}{2s^{7/6}} d\tau = 0,$$
$$\int_0^{T/2} \frac{\partial^2 K_1}{\partial Q_2 \partial \Delta Q_2}(\phi_{kep}(t,\mathbf{Y}))\Big|_{\mathbf{Y}=\mathbf{Y}_0} d\tau = -\int_0^{\pi/s} \frac{s^{5/6}}{256}\Big(-\frac{8\left(64 p_3{}^2 s^{2/3}-96\right)\cos(2s\tau)}{s^{1/6}} +$$
$$256 p_3{}^2 s^{1/2}\cos(4s\tau) + 1280 p_3{}^2 s^{1/2} - \frac{256\cos(4s\tau)}{s^{1/6}} - \frac{1024}{s^{1/6}}\Big) d\tau = \frac{\pi\left(4-5p_3{}^2 s^{2/3}\right)}{s^2},$$

If $4-5p_3{}^2 s^{2/3} \neq 0$, or equivalently, $4-5\cos^2\iota \neq 0$, we have that the conditions (1.20) of Theorem 1.3 hold. Therefore, we have proved item (a).

Now, we observe that items (b), (c) and (d) follow directly from Theorems 1.4, 1.5 and 1.6, respectively. To determinate the characteritic multipliers, observe that the matrix B in (1.19) for the S_1-symmetric or doubly-symmetric periodic solutions obtained in items (a) and (b) respectively, is given by

$$B = \begin{pmatrix} 0 & 0 & -\frac{2\pi}{s^2} & 0 \\ 0 & 0 & 0 & -\frac{2\pi}{s^{5/3}} \\ \frac{2\pi\left(5\mathrm{p}3^2 s^{2/3}-4\right)}{s^2} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

and the respective eigenvalues are $0, 0, -\frac{2\pi\sqrt{4-5\mathrm{p}3^2 s^{2/3}}}{s^2}, \frac{2\pi\sqrt{4-5\mathrm{p}3^2 s^{2/3}}}{s^2}$. Therefore, we have concluded the proof. □

The circular Keplerian solution which gives rise to the periodic solutions of item (a) and (b) of Theorem 11.2 has inclination $\iota \in (0,\pi)$ arbitrary (in item (b), the inclination must be different from the critical inclination) while in the items (c) and (d) the inclination is $\iota = 0$.

Since the set $z = p_z = 0$ is invariant under the flow defined by the Hamiltonian associated to H in (11.1), from items (c) and (d) of Theorem 11.2, we can obtain symmetric periodic solutions contained in the xy-plane.

11.2. *The Generalized Størmer model*

The generalized Størmer model, which will be denoted from now by GS, is associated to the Hamiltonian function

$$H_{GS} = \frac{1}{2}(p_x^2+p_y^2+p_z^2) - \frac{1}{||\mathbf{q}||} - \frac{\varepsilon}{||\mathbf{q}||^3}(xp_y - yp_x) + \varepsilon\beta\frac{x^2+y^2}{||\mathbf{q}||^3} + \frac{\varepsilon^2}{2}\frac{x^2+y^2}{||\mathbf{q}||^6}. \qquad (11.5)$$

where $||\mathbf{q}|| = \sqrt{x^2+y^2+z^2}$ stands for the distance of the particle to the center of the planet, $H = (xp_y - yp_x)$ is the third component of the angular momentum and ε and β are external parameters depending on the planet and on the charge mass ratio of the particle respectively. Note that the perturbation part of the Hamiltonian H_{GS} depends explicitly on $\mathbf{q} = (x,y,z)$ and $\mathbf{p} = (p_x,p_y,p_z)$. The GS model has been revisited in a series of recent papers [10–12]. The authors use a GS model that includes Keplerian gravity, a magnetic dipole aligned along the axis of rotation of the planet and a corotational electric field. The Hamiltonian (11.5) is of the form (1.1) and in (PD-1) variables is given by

$$\mathscr{H}(\mathbf{Q},\mathbf{P}) = -\frac{1}{P_1^2} + \varepsilon \mathscr{H}_1(\mathbf{Q},\mathbf{P}) + \mathscr{O}(\varepsilon^2) \tag{11.6}$$

with

$$\mathscr{H}_1(\mathbf{Q},\mathbf{P}) = -\frac{P_3}{\|\mathbf{q}\|^3} + \frac{\beta}{\|\mathbf{q}\|}(\cos^2\psi + \sin^2\psi \frac{P_3^2}{G^2}). \tag{11.7}$$

where $G = P_1 - \frac{Q_2^2+P_2^2}{2}$ and $\|\mathbf{q}\|$ is given in the new variables by

$$\|\mathbf{q}\| = P_1^2(1 - [e\cos g\cos Q_1 + e\sin g\sin Q_1] + [e\cos g\sin Q_1 - e\sin g\cos Q_1]^2) + O(e^3),$$

here e is the eccentricity of the orbit, $\psi = g + f$, f is the true anomaly and g the argument of the perigee of the unperturbed elliptic orbit measured in the invariant plane. Due to the long expression of perturbed function $\mathscr{H}_1$ in terms of Poincaré-Delaunay variables (PD-1), we put it in the Appendix. Of course that the Hamiltonian function (11.5) is invariant under the anti-symplectic reflections S_1 and S_2. Our main result on the existence of symmetric periodic solutions of the Hamiltonian system associated to the GS model is the following.

Theorem 11.3. *Given $s \in (0,1)$, $\beta \in \mathbb{R}$ such that $|\beta| > s$ and $T = 2\pi/s$. If ε is sufficiently small, then for the GS problem associated to the Hamiltonian (11.5) or (11.6) the following statements hold:*

(a) There exists a family of T-periodic S_1-symmetric solutions $\gamma(t,\varepsilon) = (Q_1(t,\varepsilon), Q_2(t,\varepsilon), Q_3(t,\varepsilon), P_1(t,\varepsilon), P_2(t,\varepsilon), P_3(t,\varepsilon))$ in (PD-1) variables parametrized by ε.

(b) There exists a family of T-periodic doubly-symmetric solutions $\gamma(t,\varepsilon) = (Q_1(t,\varepsilon), Q_2(t,\varepsilon), Q_3(t,\varepsilon), P_1(t,\varepsilon), P_2(t,\varepsilon), P_3(t,\varepsilon))$ in (PD-1) variables parametrized by ε. Moreover, in each case, there exist analytic functions $\Delta Q_2(\varepsilon), \Delta P_1(\varepsilon)$ and $\Delta P_3(\varepsilon)$ such that

$$\gamma(0,\varepsilon) = (0, \Delta Q_2(\varepsilon), 0, s^{-1/3} + \Delta P_1(\varepsilon), 0, s^{2/3}/\beta + \Delta P_3(\varepsilon)),$$

i.e., the symmetric periodic solutions are close to a circular Keplerian solution which has radius $s^{-2/3}$, period T and orbital plane with inclination $\iota = \arccos(s/\beta)$.

The periodic solutions obtained in item (a) and item (b) have characteristic multipliers $1,1,1,1,1-\varepsilon\frac{\pi\sqrt{(\beta^2-9s^2)(\beta^2+7s^2)}}{2\beta} + \mathscr{O}(\varepsilon^2), 1+\varepsilon\frac{\pi\sqrt{(\beta^2-9s^2)(\beta^2+7s^2)}}{2\beta} + \mathscr{O}(\varepsilon^2)$. Then, for $0 < s < \beta/3$ these periodic solutions are unstable and for $s > \beta/3$ these periodic solutions are linearly stable.

Proof. To prove item (a), we maintain the same notation of Theorem 1.1. After some algebraic manipulations that we have executed with Mathematica, from the expression (A.1) in Appendix, it follows that

$$\begin{aligned} i)\int_0^{T/2} \frac{\partial \mathscr{H}_1}{\partial P_3}(\varphi_{kep}(\tau,\mathbf{Y}))\Big|_{\mathbf{Y}=\mathbf{Y}_0} d\tau &= s^{4/3}\int_0^{\pi/s} [\beta p_3(1+\cos(2s\tau)) - s^{2/3}]d\tau \\ &= -s(1 - p_3\beta/s^{2/3}), \end{aligned}$$

and to verify item (a) in (1.12), we need that $1 - p_3\beta/s^{2/3} = 0$, or equivalently, $p_3 = s^{2/3}/\beta$ and the Keplerian circular orbit is in a plane with inclination $\iota = \arccos(s/\beta)$. Fixing $p_3 = s^{2/3}/\beta$, it follows

$$
\begin{aligned}
ii)\ & \int_0^{T/2} \tfrac{\partial \mathscr{H}_1}{\partial Q_2}(\varphi_{kep}(\tau,\mathbf{Y}))\Big|_{\mathbf{Y}=\mathbf{Y}_0} d\tau \\
&= \int_0^{\pi/s} -\tfrac{1}{2}s^{5/6}\cos(s\tau)\left(-5\beta+3\beta p_3^2 s^{2/3}-5\beta\left(p_3^2 s^{2/3}-1\right)\cos(2s\tau)+6p_3 s^{4/3}\right)d\tau \\
&= 0. \\
iii)\ & \Omega_1 = -\tfrac{s^{1/6}}{1024}\int_0^{\pi/s} 256\beta s^{5/6}+\tfrac{1792 s^{17/6}}{\beta}+4864\beta s^{5/6}\cos(4s\tau)-\tfrac{4864 s^{17/6}\cos(4s\tau)}{\beta} \\
&-32 s^{5/6}\left(160\beta-\tfrac{352 s^2}{\beta}\right)\cos(2s\tau)d\tau\cdot\tfrac{s^{4/3}\beta}{16}\int_0^{\pi/s}(16+16\cos(2s\tau))d\tau \\
&= -\tfrac{\pi\left(\beta^2+7s^2\right)}{4\beta}\pi\beta s^{1/3}.
\end{aligned}
$$

Note that if $\beta \neq 0$, then from *iii*) we have that $\Omega_1 \neq 0$. From *i*), *ii*) and *iii*) above it follows conditions (1.12) and thus we have concluded the proof of item (a).

Now we use Theorem 1.2, with the same notation of it, to prove item (b). Using the expression (A.1) in the Appendix, after some algebraic manipulations, we arrive to

$$
\begin{aligned}
iv)\ & \int_0^{T/4} \tfrac{\partial \mathscr{H}_1}{\partial P_3}(\varphi_{kep}(\tau,\mathbf{Y}))\Big|_{\mathbf{Y}=\mathbf{Y}_0} d\tau = -\tfrac{s}{2}\pi\left(1-\beta p_3 s^{2/3}\right), \\
v)\ & \int_0^{T/4} \tfrac{\partial^2 \mathscr{H}_1}{\partial P_3 \partial \Delta P_3}(\varphi_{kep}(\tau,\mathbf{Y}))\Big|_{\mathbf{Y}=\mathbf{Y}_0} d\tau = \int_0^{\pi/2s} \beta s^{4/3}(\cos(2s\tau)+1)d\tau = \tfrac{1}{2}\pi\beta s^{1/3}.
\end{aligned}
$$

In equation *iv*) we need that $1-\beta p_3 s^{2/3}=0$ and it is sufficient to take $p_3 = s^{2/3}/\beta$. On the other hand, we need that the right hand side of equation *v*) is nonzero and it is enough to take $\beta \neq 0$. Therefore, from *iv*) and *v*) we verify the conditions (1.13) and we conclude the proof of item (b). To study the stability, observe that, after some algebraic manipulations, the matrix C in (1.28) for the symmetric periodic solutions obtained in items (a) and (b) respectively, is given by

$$
C = \begin{pmatrix} 0 & 0 & \frac{\pi\left(\beta^2-9s^2\right)}{2\beta} & 0 \\ 0 & 0 & 0 & 2\pi s^{1/3}\beta \\ \frac{\pi\left(7s^2+\beta^2\right)}{2\beta} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},
$$

and the respective eigenvalues are $0, 0, -\frac{\pi\sqrt{(\beta^2-9s^2)(\beta^2+7s^2)}}{2\beta}, \frac{\pi\sqrt{(\beta^2-9s^2)(\beta^2+7s^2)}}{2\beta}$. Therefore, we have concluded the proof. □

We remark that the choice of $s \in (0,1)$ was given in order to obtain realistic orbits, because in the coordinates of formulation of the Hamiltonian (11.5) the planet has radius 1 and consequently the radius of the circular Keplerian solution needs to be greater than 1. For more details about the formulation of the problem and the physical interpretation see [10].

Appendix A.

We shall present the expression for the perturbed function $\mathscr{H}_1$ in (11.7) for the GS model, in terms of Poincaré-Delaunay variables (PD-1). We use the Mathematica processor to obtain it.

$$
\begin{aligned}
\mathscr{H}_1 = {} & \frac{-1}{32P_1^8\left(P_2^2+Q_2^2\right)^2\left(-2P_1+P_2^2+Q_2^2\right)^2}\Big(-20P_1^5\delta\beta\cos(3Q_1)P_2^9+28P_1^5Q_2\delta\beta\sin(Q_1)P_2^8+48P_1 \\
& P_3Q_2\delta\sin(Q_1)P_2^8-100P_1^5Q_2\delta\beta\sin(3Q_1)P_2^8+80P_1^6\delta\beta\cos(3Q_1)P_2^7+160P_1^5Q_2^2\beta\cos(3Q_1) \\
& \delta P_2^7+64P_1^5Q_2^3\delta\beta\sin(Q_1)P_2^6-112P_1^6Q_2\delta\beta\sin(Q_1)P_2^6+192P_1P_3Q_2^3\delta\sin(Q_1)P_2^6-192P_1^2 \\
& P_3Q_2\delta\sin(Q_1)P_2^6+400P_1^6Q_2\delta\beta\sin(3Q_1)P_2^6-80P_1^7\delta\beta\cos(3Q_1)P_2^5+280P_1^5Q_2^4\beta\cos(3Q_1) \\
& \delta P_2^5+80P_1^5P_3^2\delta\beta\cos(3Q_1)P_2^5-720P_1^6Q_2^2\delta\beta\cos(3Q_1)P_2^5+24P_1^5Q_2^5\delta\beta\sin(Q_1)P_2^4-144P_1^6 \\
& Q_2^3\delta\beta\sin(Q_1)P_2^4+112P_1^7Q_2\delta\beta\sin(Q_1)P_2^4-176P_1^5P_3^2Q_2\delta\beta\sin(Q_1)P_2^4+288P_1P_3Q_2^5\delta P_2^4 \\
& \sin(Q_1)-576P_1^2P_3Q_2^3\delta\sin(Q_1)P_2^4+192P_1^3P_3Q_2\delta\sin(Q_1)P_2^4+280P_1^5\beta Q_2^5\delta\sin(3Q_1)P_2^4 \\
& -400P_1^6Q_2^3\delta\beta\sin(3Q_1)P_2^4-400P_1^7Q_2\delta\beta\sin(3Q_1)P_2^4+400P_1^5P_3^2Q_2P_2^4\beta\delta\sin(3Q_1)-400\beta \\
& \delta P_1^6Q_2^4\cos(3Q_1)P_2^3+800P_1^7Q_2^2\delta\beta\cos(3Q_1)P_2^3-800P_1^5P_3^2Q_2^2\beta\cos(3Q_1)\delta P_2^3-32P_1^5Q_2^7\beta \\
& \delta\sin(Q_1)P_2^2+48P_1^6Q_2^5\delta\beta\sin(Q_1)P_2^2+32P_1^7Q_2^3\delta\beta\sin(Q_1)P_2^2-160P_1^5P_3^2Q_2^3\delta\beta\sin(Q_1)P_2^2 \\
& +192P_1P_3Q_2^7\delta\sin(Q_1)P_2^2-576P_1^2P_3Q_2^5\delta\sin(Q_1)P_2^2+384P_1^3P_3Q_2^3\delta P_2^2\sin(Q_1)+160P_1^5\beta \\
& \delta Q_2^7\sin(3Q_1)P_2^2-720P_1^6Q_2^5\delta\beta\sin(3Q_1)P_2^2+800P_1^7Q_2^3\delta\beta\sin(3Q_1)P_2^2-800P_1^5P_3^2Q_2^3P_2^2\beta \\
& \delta\sin(3Q_1)+4P_1\delta\left(P_2^2+Q_2^2\right)\Big(\Big(-11Q_2^6+\left(44P_1-21P_2^2\right)Q_2^4+\big(-9P_2^4+40P_1P_2^2-44P_1^2 \\
& +28P_3^2\big)Q_2^2+P_2^2\left(\left(P_2^2-2P_1\right)^2-20P_3^2\right)\Big)\beta P_1^4+12P_3\left(P_2^2+Q_2^2-2P_1\right)^2\left(P_2^2+Q_2^2\right)\Big)P_2 \\
& \cos(Q_1)-100P_1^5Q_2^8\delta\beta\cos(3Q_1)P_2+400P_1^6Q_2^6\delta\beta\cos(3Q_1)P_2-400P_1^7Q_2^4\beta\delta\cos(3Q_1)P_2 \\
& +400P_1^5P_3^2Q_2^4\delta\beta\cos(3Q_1)P_2-8Q_2\Big(P_1^4\big(32(P_2-Q_2)(P_2+Q_2)P_1^4+16P_1^3\big(2Q_2^2P_2^2-9P_2^4 \\
& +11Q_2^4\big)+4\big(37P_2^6+27Q_2^2P_2^4-\left(57Q_2^4+8P_3^2\right)P_2^2-47Q_2^6+8P_3^2Q_2^2\big)P_1^2-8\left(P_2^2+Q_2^2\right) \\
& \big(7P_2^6+5Q_2^2P_2^4-\left(11Q_2^4+18P_3^2\right)P_2^2-9Q_2^6+14P_3^2Q_2^2\big)P_1+\left(P_2^2+Q_2^2\right)^2\big(7P_2^6+5Q_2^2P_2^4 \\
& -\left(11Q_2^4+36P_3^2\right)P_2^2-9Q_2^6+28P_3^2Q_2^2\big)\big)\beta-9P_3\left(-P_2^2-Q_2^2+4P_1\right)\left(P_2^2+Q_2^2\right)^2\big(P_2^2+Q_2^2 \\
& -2P_1\big)^2\Big)\sin(2Q_1)P_2+\left(P_2^2+Q_2^2\right)^2\Big(\Big(-64P_1^4+16\left(3P_2^2+5Q_2^2\right)P_1^3+4P_1^2\big(P_2^4-8Q_2^2P_2^2 \\
& -9Q_2^4-16P_3^2\big)-8(P_2-Q_2)(P_2+Q_2)\left(\left(P_2^2+Q_2^2\right)^2-2P_3^2\right)P_1+\left(P_2^4-Q_2^4\right)\Big(\left(P_2^2+Q_2^2\right)^2 \\
& -4P_3^2\Big)\Big)\beta P_1^4+4P_3\left(P_2^2+Q_2^2-2P_1\right)^2\left(8P_1^2+12\left(P_2^2+Q_2^2\right)P_1-3\left(P_2^2+Q_2^2\right)^2\right)\Big)-4\Big(16\big(P_2^4 \\
& -6Q_2^2P_2^2+Q_2^4\big)\beta P_1^8-32\left(P_2^2+Q_2^2\right)\left(2P_2^4-15Q_2^2P_2^2+3Q_2^4\right)\beta P_1^7+8\big(8P_2^8-47Q_2^2P_2^6-P_2^4 \\
& \left(105Q_2^4+2P_3^2\right)+\left(12P_3^2Q_2^2-37Q_2^6\right)P_2^2+13Q_2^8-2P_3^2Q_2^4\big)\beta P_1^6-8\left(P_2^2+Q_2^2\right)\big(3P_2^8-18Q_2^2 \\
& P_2^6-10\left(4Q_2^4+P_3^2\right)P_2^4+2\left(24P_3^2Q_2^2-7Q_2^6\right)P_2^2+5Q_2^8-6P_3^2Q_2^4\big)\beta P_1^5+\left(P_2^2+Q_2^2\right)^2\big(3P_2^8 \\
& -18Q_2^2P_2^6-20\left(2Q_2^4+P_3^2\right)P_2^4+2\left(48P_3^2Q_2^2-7Q_2^6\right)P_2^2+5Q_2^8-12P_3^2Q_2^4\big)\beta P_1^4-144P_3(P_2 \\
& -Q_2)(P_2+Q_2)\left(P_2^2+Q_2^2\right)^2P_1^3+180P_3(P_2-Q_2)(P_2+Q_2)\left(P_2^2+Q_2^2\right)^3P_1^2-72P_3P_1(P_2 \\
& -Q_2)(P_2+Q_2)\left(P_2^2+Q_2^2\right)^4+9P_3(P_2-Q_2)(P_2+Q_2)\left(P_2^2+Q_2^2\right)^5\Big)\cos(2Q_1)+19P_1^4\big(2P_1 \\
& -P_2^2-Q_2^2+2P_3\big)\left(-P_2^2-Q_2^2+2P_1-2P_3\right)\left(P_2^2+Q_2^2-4P_1\right)\big(P_2^6-15Q_2^2P_2^4+15Q_2^4P_2^2 \\
& -Q_2^6\big)\beta\cos(4Q_1)-20P_1^5Q_2^9\delta\beta\sin(Q_1)+80P_1^6Q_2^7\delta\beta\sin(Q_1)-80P_1^7Q_2^5\delta\beta\sin(Q_1)+16P_1^5\beta \\
& P_3^2Q_2^5\delta\sin(Q_1)+48P_1P_3Q_2^9\delta\sin(Q_1)-192P_1^2P_3Q_2^7\delta\sin(Q_1)+192P_1^3P_3Q_2^5\delta\sin(Q_1)-20 \\
& P_1^5Q_2^9\delta\beta\sin(3Q_1)+80P_1^6Q_2^7\delta\beta\sin(3Q_1)-80P_1^7Q_2^5\delta\beta\sin(3Q_1)+80P_1^5P_3^2Q_2^5\delta\beta\sin(3Q_1) \\
& +38P_1^4Q_2\left(-P_2^2-Q_2^2+2P_1+2P_3\right)\left(-P_2^2-Q_2^2+2P_1-2P_3\right)\left(P_2^2+Q_2^2-4P_1\right)\big(3Q_2^4P_2 \\
& -10Q_2^2P_2^3+3P_2^5\big)\beta\sin(4Q_1)\Big),
\end{aligned}
\tag{A.1}
$$

where $\delta=\delta(Q_2,P_1,P_2)=\sqrt{4P_1-Q_2^2-P_2^2}$.

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References

[1] A. Alberti, C. Vidal, New families of symmetric periodic solutions of the spatial anisotropic Manev problem, *J. Math. Phys.* **56** (2015) 012901.
[2] R. Arenstorf, New periodic solutions of the planar three-body problem corresponding to elliptic motion in the lunar theory, *J. Diff. Eqs.* **4** (1993) 202–256.
[3] D. Boccaletti, G. Pucacco, *Theory of orbits* **vol 1** (Springer-Verlag, Berlin, Heidelberg, New York, 2004).
[4] H. Cabral, C. Vidal, Periodic solutions of symmetric perturbations of the Kepler problem, *J. Diff. Eqs.* **163** (2000) 76–88.
[5] J. Cors, C. Pinyol, J. Soler, Analytic continuation in the case of non-regular dependency on a small parameter with an application to celestial mechanics, *J. Diff. Eqs.* **219** (2005) 1–19.
[6] J. Duistermaat, Periodic solutions of periodic systems of ordinary differential equations containing a parameter, *Arch. Rat. Mech. Anal.* **38** (1970) 59–80.
[7] J. Duistermaat, On periodic solutions near equilibrium points of conservative systems, *Arch. Rat. Mech. Anal.* **45** (1970) 143–160.
[8] A. Elipe, M. Lara, A simple model for the chaotic motion around (433) Eros, *J. Astron. Sci.* **51**(4) (2003) 391–404.
[9] J.L.G. Guirao, J. Llibre, J.A. Vera, Periodic orbits of Hamiltonian systems: Applications to perturbed Kepler problems, *Chaos, solitons & fractals* **57** (2013) 105–111.
[10] M. Iñarrea, V. Lanchares, J.F. Palacián, A.I. Pascual, J.P. Salas, P. Yanguas, The Keplerian regime of charged particle in planetary magnetospheres, *Physica D* **197** (2004) 242–268.
[11] M. Iñarrea, V. Lanchares, J.F. Palacián, A.I. Pascual, J.P. Salas, P. Yanguas, Global dynamics of dust grains in magnetic planets, *Physics Letters A* **338** (2005) 247–252.
[12] M. Iñarrea, V. Lanchares, J.F. Palacián, A.I. Pascual, J.P. Salas, P. Yanguas, Symplectic coordinates on $S^2 \times S^2$ for perturbed Keplerian problems: Application to the dynamics of a generalised Stormer problem, *J. Diff. Eqs.* **250** (2011) 1386–1407.
[13] J.J. Matese, P.G. Whitman, The galactic disk tidal field and nonrandom distribution of observed Oort cloud comets, *Icarus* **82** (1998) 389–401.
[14] J.J. Matese, P.G. Whitman, A model of galactic tidal interaction tidal with the Oort comet cloud, *Cel. Mech. dyn. Astron. 54* (1992) 13–35.
[15] K.R. Meyer, G.R. Hall, D. Offin, *Introduction to Hamiltonian Dynamical System and the N-Body Problem* (Second edition. Applied Mathematical Sciences, **90**. Springer, New York, 2009).
[16] K. Meyer, R. Howison Clarissa, Doubly-Symmetric Periodic Solutions in the Spatial Restricted Three-Body Problem, *J. Diff. Eqs.* **163** (2000) 174–197.
[17] K. Meyer, *Periodic solutions of the N-Body Problem*, Lectures Notes in Math. **1719**, Springer-Verlag, Berlin, 1999).
[18] H. Poincaré, *Les Méthodes Nouvelles de la Mecánique Céleste*, (Dover Publ., 1987).
[19] A. Riaguas, A. Elipe, M. Lara, Periodic orbits around a massive straight segment, *Celest. Mech. Dyn. Astron.* **73**(1-4) (1999) 169–178.
[20] C. Siegel, J. Moser, *Lectures on Celestial Mechanics*, (Springer-Verlag, New York, 1971).
[21] V. Szebehely, *Theory of Orbits*, (New York, Academic Press, 1967).
[22] C. Vidal, Periodic solutions for any planar symmetric perturbation of the Kepler problem, *Celest. Mech. Dyn. Astron.* **80** (2001) 119–132.
[23] C. Vidal, Periodic solutions of symmetric perturbations of gravitational problems, *J. Dynam. Differential Equations* **17**(1) (2005) 85–114.