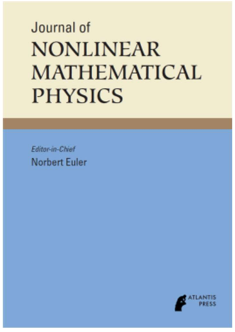

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Darboux transformations of the Supersymmetric BKP hierarchy

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In this paper, we construct Darboux transformations of the supersymmetric BKP(SBKP) hierarchy. These Darboux transformations can generate new solutions from seed solutions by using bosonic eigenfunctions.

Keywords: Darboux transformation, supersymmetric BKP hierarchy, bosonic eigenfunctions.

2000 Mathematics Subject Classification: 37K05, 37K10, 37K20

1. Introduction

In the study of integrable hierarchies, KP hierarchy is an interesting object. Two sub-hierarchies as the CKP hierarchy, BKP hierarchy [2] and their generalizations [6] have been shown to possess many nice integrable properties [7], with consideration of the reductions on the Lax operators.

Various generalizations and supersymmetric extensions [17] of the KP hierarchy have deep implications in mathematical physics, particularly in the theory of Lie algebras. In [5], the theory of the super Lie algebras was surveyed by considering super Boson-Fermion Correspondences. One important supersymmetric extension is the supersymmetric Manin-Radul Kadomtsev-Petviashvili (MR-SKP) hierarchy [11] which contains a lot of integrable super solitary equations equipped with the super-pseudodifferential operators. Apart from the Manin-Radul one, Mulase supersymmetrize the KP hierarchy by constructing a hierarchy called the Jacobian SKP hierarchy which does not possess a standard Lax formulation [13]. This hierarchy has strict Jacobian flows, i.e. it preserves the super Riemman surface about which one can also see [16].

The Darboux transformation of the supersymmetric KP hierarchy was studied in [1, 10]. The supersymmetric BKP (SBKP) hierarchy was constructed in [15]. After that this series of super hierarchies were seldom studied in mathematical physics partly because of their extreme complexities till the appearance of our paper [8].

It is well known that the Darboux transformation is one efficient method to generate the soliton solutions for integrable systems [9, 12, 14]. Darboux transformations for the constrained BKP hierarchy and constrained CKP hierarchy were given in [3]. Two types of Darboux transformation operators for the constrained KP hierarchy were given in [4].

This paper is arranged as follows. In the next section we recall some necessary facts of the SBKP hierarchy. In Sections 3, we will give the Darboux transformations for the SBKP hierarchy.

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2. The supersymmetric BKP hierarchy

Let us firstly recall some basic facts [15] on the supersymmetric BKP hierarchy. $\mathscr{S}$ is assumed as an algebra of smooth functions of a spatial variable x, a grassmann variable θ and their super-derivation $D = \partial_\theta + \theta\partial$. This algebra $\mathscr{S}$ has the following multiplying rule

$$D^n \circ f = \sum_{i=0}^{\infty} \begin{bmatrix} n \\ n-i \end{bmatrix} (-1)^{|f|(n-i)} f^{[i]} D^{n-i}, \tag{2.1}$$

$$\begin{bmatrix} n \\ n-i \end{bmatrix} = \begin{cases} 0 & i<0 \ \ or \ \ (n,i)=(0,1) \pmod 2, \\ \begin{pmatrix} [\frac{n}{2}] \\ [\frac{n-i}{2}] \end{pmatrix} & i \geq 0, (n,i) \neq (0,1) \pmod 2. \end{cases} \tag{2.2}$$

Here the value $|f|$ means the super degree of the operator f which shows the operator f is fermionic or bosonic and $f^{[i]}$ means the i-th order super derivatives $D^i(f)$ of the function f. The supersymmetric derivative D satisfies the supersymmetric analog of the Leibniz rule

$$D(ab) = D(a)b + (-1)^{|a|} aD(b), \tag{2.3}$$

where a is a homogeneous element of $\mathscr{S}$. The even and odd time variables $(t_2, t_3, t_6, t_7, \cdot)$ and the even and odd flows can be defined as

$$D_{4i-2} = \frac{\partial}{\partial t_{4i-2}}, \ \ D_{4i-1} = \frac{\partial}{\partial t_{4i-1}} + \sum_{j=1}^{\infty} t_{4j-1} \frac{\partial}{\partial t_{4i+4j-2}}. \tag{2.4}$$

The supercommutator will be defined as the Lie bracket $[X,Y] = XY - (-1)^{|X||Y|} YX$. The bracket has a Leibniz property as $[X,YZ] = [X,Y]Z + (-1)^{|X||Y|} Y[X,Z]$. Then the ordinary derivative and the super derivative have a relation as $D^2 = \frac{1}{2}[D,D] = \partial$. The infinite odd and even flows satisfy a nonabelian Lie superalgebra whose commutation relations are

$$[D_{4i-2}, D_{4j-2}] = 0\ , \quad [D_{4i-2}, D_{4j-1}] = 0\ , \quad [D_{4i-1}, D_{4j-1}] = -2D_{4i+4j-2}\ ,$$

$$[D_{4i-2}, D] = 0\ , \quad [D_{4i-1}, D] = 0. \tag{2.5}$$

For any operator $A = \sum_{i\in\mathbb{Z}} f_i D^i \in \mathscr{S}$ and homogeneous operators P, Q, its nonnegative projection, negative projection, adjoint operator are respectively defined as

$$A_+ = \sum_{i\geq 0} f_i D^i, \quad A_- = \sum_{i<0} f_i D^i,\ f_i^* = f_i,\ (D^k)^* = (-1)^{\frac{k(k+1)}{2}} D^k, \tag{2.6}$$

$$(PQ)^* = (-1)^{|P||Q|} Q^* P^*,\ \ (P^{-1})^* = (-1)^{|P|} (P^*)^{-1}. \tag{2.7}$$

The Lax operator of the supersymmetric BKP hierarchy basing on definitions in [15], has a form as

$$L = D + \sum_{i\geq 1} u_i D^{1-i}, \quad u_2 = -\frac{1}{2} u_1^{[1]},\ \ L^* = -DLD^{-1}. \tag{2.8}$$

The supersymmetric BKP hierarchy is defined by the following Lax equations [15]

$$D_{4k-2} L = [(L^{4k-2})_+, L], \qquad D_{4k-1} L = [(L^{4k-1})_+, L] - 2L^{4k},\ \ k \geq 1. \tag{2.9}$$

The operator L has a dressing form as

$$L = WDW^{-1}, \tag{2.10}$$

where

$$W = 1 + \sum_{i\geq 1} a_i D^{-i}, \tag{2.11}$$

satisfy

$$W^* = DW^{-1}D^{-1}. \tag{2.12}$$

The eq.(2.12) will be called the B type condition of the supersymmetric BKP hierarchy. For a Lax operator L, the dressing operator W is determined uniquely up to a multiplication to the right by operators with constant coefficients. The supersymmetric BKP hierarchy (3.4) can also be redefined as the following Sato equations

$$\frac{\partial W}{\partial t_{4k-2}} = -(B_{4k-2})_- W, \quad \frac{\partial W}{\partial t_{4k-1}} = -(B_{4k-1})_- W, \;\; B_j = L^j, \tag{2.13}$$

with $k \geq 1$.

3. Darboux transformations of the supersymmetric BKP hierarchy

In this section, we will construct Darboux transformations of the supersymmetric BKP hierarchy which generate a new Lax operator

$$L^{(1)} = D + \sum_{i\geq 1} u_i^{(1)} D^{-i} = WLW^{-1}, \tag{3.1}$$

where W plays a role as the Darboux transformation operator.

The new Lax operator $L^{(1)}$ should satisfy the B type condition

$$(L^{(1)})^* = -DL^{(1)}D^{-1}, \tag{3.2}$$

therefore W satisfies the following B type condition

$$W^* = DW^{-1}D^{-1}. \tag{3.3}$$

To keep the Lax equation of the supersymmetric BKP hierarchy invariant, i.e.,

$$D_{t_n} L^{(1)} = [-(B_n^{(1)})_-, L^{(1)}], \;\; B_n^{(1)} = (L^{(1)})^n, \; n = 2,3,6,7,\cdots, \tag{3.4}$$

the Darboux transformation operator W should satisfy the following dressing equation

$$D_{t_n} W = (-1)^{n+1} W(B_n)_- - (WB_nW^{-1})_- W. \tag{3.5}$$

Definition 3.1. If the operator B is a super differential operator and has form $B := \sum_{n=0}^{\infty} D^n a_n$, then we define $B^* g(x) = \sum_{m=0}^{\infty} (-1)^{m|a_m|} g^{[m]} a_m$.

The super eigenfunction ϕ and the adjoint super eigenfunction ψ of the supersymmetric BKP hierarchy can be defined respectively by

$$\frac{\partial \phi}{\partial t_n} = (B_n)_+\phi, \frac{\partial \psi}{\partial t_n} = -(B_n)_+^*\psi, \tag{3.6}$$

where $\phi = \phi(\lambda,\eta;t)$ and $\psi = \psi(\lambda,\eta;t)$ and $t = (t_2,t_3,t_6,t_7...)$. To give Darboux transformations, we need the following lemma.

Lemma 3.1. *The operator $B := \sum_{n=0}^{\infty} b_n D^n (B := \sum_{n=0}^{\infty} D^n a_n)$ is a super differential operator and f,g (short for $f(x),g(x)$) are two supersymmetric functions of spatial parameter x, following identities hold true*

$$(BfD^{-1}g)_- = B(f)D^{-1}g, \quad (fD^{-1}gB)_- = fD^{-1}B^*(g). \tag{3.7}$$

Proof. Here we give the proof of the eq.(3.7) by the following direct calculation

$$\begin{aligned}
(BfD^{-1}g)_- &= \sum_{m=0}^{\infty}(b_m D^m f(x)D^{-1}g)_- \\
&= \sum_{m=0}^{\infty}(b_m \sum_{i=0}^{m}\begin{bmatrix} m \\ m-i \end{bmatrix}(-1)^{|f|(m-i)}f^{[i]}D^{m-i}D^{-1}g)_- \\
&= \sum_{m=0}^{\infty} b_m(\sum_{i=0}^{m}\begin{bmatrix} m \\ m-i \end{bmatrix}(-1)^{|f|(m-i)}f^{[i]}D^{m-i}D^{-1})_- g \\
&= \sum_{m=0}^{\infty} b_m f^{[m]}(x)D^{-1}g \\
&= B(f)D^{-1}g,
\end{aligned}$$

$$\begin{aligned}
(fD^{-1}gD^m a_m)_- &= ((-1)^{\sum_{a,b=|f|,|g|,|a_m|,m}ab+\sum_{a=|f|,|g|,|a_m|,m}a}a_m D^m gD^{-1}f)_-^* \\
&= ((-1)^{\sum_{a,b=|f|,|g|,|a_m|,m}ab+\sum_{a=|f|,|g|,|a_m|,m}a}a_m D^m(g)D^{-1}f)^* \\
&= (-1)^{m|g|}fD^{-1}D^m(g)a_m \\
&= fD^{-1}(-1)^{m|g|}D^m(g)a_m \\
&= fD^{-1}(D^m a_m)^*(g),
\end{aligned} \tag{3.8}$$

which further leads to

$$\begin{aligned}
(fD^{-1}gB)_- &= \sum_{m=0}^{\infty} fD^{-1}(-1)^{m|g|}D^m(g)a_m \\
&= fD^{-1}B^*(g).
\end{aligned}$$

□

The following lemma will tell us the Darboux transformation of the SBKP hierarchy can have some certain forms.

Lemma 3.2. *The operators $T_D = \phi \circ D \circ \phi^{-1}$ and $T_I = \psi^{-1} \circ D^{-1} \circ \psi$ and their multiplication $T_D T_I = \phi \circ D \circ \phi^{-1} \circ \psi^{-1} \circ D^{-1} \circ \psi$ satisfy eq.*(3.5).

Now we consider two Darboux transformation operators of the supersymmetric BKP hierarchy as following

$$T_D(\phi) = \phi \circ D \circ \phi^{-1}, T_I(\psi) = \psi^{-1} \circ D^{-1} \circ \psi, \tag{3.9}$$

which satisfy

$$(T_D^{-1}(\phi))^* = (-1)^{|\phi|} T_I(\phi), (T_I^{-1}(\psi))^* = (-1)^{|\psi|+1} T_D(\psi). \tag{3.10}$$

One can find

$$T_D(\phi) \cdot \phi = 0, (T_I^{-1}(\psi))^* \cdot \psi = 0. \tag{3.11}$$

Now we consider two sets of supersymmetric functions $\{\phi_i^{(0)}, i = 1, 2, ...n; \phi^{(0)}\}$ and $\{\psi_i^{(0)}, i = 1, 2 \cdots n; \psi^{(0)}\}$. For $T_D(\phi) = \phi \circ D \circ \phi^{-1}$, we consider

$$T_D^{(1)} = T_D^{(1)}(\phi_1^{(0)}) = \phi_1^{(0)} \circ D \circ (\phi_1^{(0)})^{-1}, \tag{3.12}$$

and new eigenfunctions $\phi_i^{(1)}, \psi_i^{(1)}$

$$\phi_i^{(1)} = T_D^{(1)}(\phi_1^{(0)}) \cdot \phi_i^{(0)}, \quad \psi_i^{(1)} = (T_D^{(1)}(\phi_1^{(0)}))^{*^{-1}} \cdot \psi_i^{(0)} = (-1)^{|\phi_1^{(0)}|} T_I(\phi_1^{(0)}) \cdot \psi_i^{(0)},$$

where $i \geqslant 2$. For the second step, we consider

$$T_D^{(2)} = T_D^{(2)}(\phi_2^{(1)}) = \phi_2^{(1)} \circ D \circ (\phi_2^{(1)})^{-1}, \tag{3.13}$$

and new supersymmetric eigenfunctions as

$$\phi_i^{(2)} = T_D^{(2)}(\phi_2^{(1)}) \cdot \phi_i^{(1)}, \quad \psi_i^{(2)} = (T_D^{(2)}(\phi_2^{(1)}))^{*^{-1}} \cdot \psi_i^{(1)} = (-1)^{|\phi_2^{(1)}|} T_I((\phi_2^{(1)})) \cdot \psi_i^{(1)},$$

where $i \geqslant 3$.

For $T_I(\psi) = \psi^{-1} \circ D^{-1} \circ \psi$, we may consider:

$$T_I^{(1)} = T_I^{(1)}(\psi_1^{(0)}) = (\psi_1^{(0)})^{-1} \circ D^{-1} \circ (\psi_1^{(0)}), \tag{3.14}$$

$$\phi_i^{(1)} = T_I^{(1)}(\psi_1^{(0)}) \cdot \phi_i^{(0)}, \quad \psi_i^{(1)} = (T_I^{(1)}(\psi_1^{(0)}))^{*^{-1}} \cdot \psi_i^{(0)} = (-1)^{|\psi_1^{(0)}|+1} T_D(\psi_1^{(0)}) \cdot \psi_i^{(0)},$$

where $i \geqslant 2$.

Secondly, we consider

$$T_I^{(2)} = T_I^{(2)}(\psi_2^{(1)}) = (\psi_2^{(1)})^{-1} \circ D^{-1} \circ (\psi_2^{(1)}), \tag{3.15}$$

$$\phi_i^{(2)} = T_I^{(2)}(\psi_2^{(1)}) \cdot \phi_i^{(1)}, \quad \psi_i^{(2)} = (T_I^{(2)}(\psi_2^{(1)}))^{*^{-1}} \cdot \psi_i^{(1)} = (-1)^{|\psi_2^{(1)}|+1} T_D((\psi_2^{(1)})) \cdot \psi_i^{(1)},$$

where $i \geqslant 3$.

It is obvious that T_D or T_I can not keep the restriction of the B type condition, therefore we use

$$W_1 = T_{1+1} = T_I(\psi_1^{(1)})T_D(\phi_1^{(0)}), \tag{3.16}$$

as the Darboux transformation operator and $L^{(1)} = W_1 L W_1^{-1}$. Let us check whether it satisfies the required constraint

$$(L^{(1)})^* = -DL^{(1)}D^{-1}, \tag{3.17}$$

We can do a direct calculation as

$$\begin{aligned}(L^{(1)})^* &= ((\psi_1^{(1)})^{-1}\circ D^{-1}\circ\psi_1^{(1)}\circ\phi_1^{(0)}\circ D\circ(\phi_1^{(0)})^{-1}\circ\\ &\quad L\circ\phi_1^{(0)}\circ D^{-1}\circ(\phi_1^{(0)})^{-1}\circ(\psi_1^{(1)})^{-1}\circ D\circ\psi_1^{(1)})^*\\ &= -(\psi_1^{(1)})\circ D\circ(\psi_1^{(1)})^{-1}(\phi_1^{(0)})^{-1}\circ D^{-1}\circ(\phi_1^{(0)})\circ D\circ L\\ &\quad\circ D^{-1}\circ(\phi_1^{(0)})^{-1}\circ D\circ(\phi_1^{(0)})\circ(\psi_1^{(1)})\circ D^{-1}\circ(\psi_1^{(1)})^{-1},\end{aligned} \tag{3.18}$$

and

$$\begin{aligned}-DL^{(1)}D^{-1} &= -D\circ(\psi_1^{(1)})^{-1}\circ D^{-1}\circ\psi_1^{(1)}\circ\phi_1^{(0)}\circ D\circ(\phi_1^{(0)})^{-1}\\ &\quad\circ L\circ\phi_1^{(0)}\circ D^{-1}\circ(\phi_1^{(0)})^{-1}\circ(\psi_1^{(1)})^{-1}\circ D\circ\psi_1^{(1)}\circ D^{-1},\end{aligned} \tag{3.19}$$

which means in order to keep the constraint $(L^{(1)})^* = -DL^{(1)}D^{-1}$, the following equation should hold

$$T_D(\psi_1^{(1)})T_I(\phi_1^{(0)})D = DT_I(\psi_1^{(1)})T_D(\phi_1^{(0)}), \tag{3.20}$$

where $\psi_1^{(1)} = (\phi_1^{(0)})^{-1}D^{-1}(\phi_1^{(0)}\psi_1^{(0)} + (-1)^{|\phi_1^{(0)}|}\psi_1^{(0)}\phi_1^{(0)})$. The following further calculations

$$(\psi_1^{(1)})\circ D\circ(\psi_1^{(1)})^{-1}(\phi_1^{(0)})^{-1}\circ D^{-1}\circ(\phi_1^{(0)})\circ D$$

$$= (-1)^{|\phi_1^{(0)}|+|\psi_1^{(1)}|}D + (-1)^{|\phi_1^{(0)}|}\psi_1^{(1)}(\psi_1^{(1)-1})^{[1]} \tag{3.21}$$

$$-(\psi_1^{(1)})[(\psi_1^{(1)})^{-1}(\phi_1^{(0)})^{-1}]^{[1]}\circ D^{-1}\circ(\phi_1^{(0)})^{[1]}, \tag{3.22}$$

$$D\circ(\psi_1^{(1)})^{-1}\circ D^{-1}\circ\psi_1^{(1)}\circ\phi_1^{(0)}\circ D\circ(\phi_1^{(0)})^{-1} \tag{3.23}$$

$$= (-1)^{|\phi_1^{(0)}|+|\psi_1^{(1)}|}D + (-1)^{|\phi_1^{(0)}|+|\psi_1^{(0)}|}(\psi_1^{(1)-1})^{[1]}\psi_1^{(1)} \tag{3.24}$$

$$-[(\psi_1^{(1)})^{-1}]^{[1]}\circ D^{-1}\circ[\psi_1^{(1)}\circ\phi_1^{(0)}]^{[1]}(\phi_1^{(0)})^{-1}, \tag{3.25}$$

can help us to get the following identity

$$\begin{aligned}&(\psi_1^{(1)})[(\psi_1^{(1)})^{-1}(\phi_1^{(0)})^{-1}]^{[1]} = [(\psi_1^{(1)})^{-1}]^{[1]},\\ &(\phi_1^{(0)})^{[1]} = [\psi_1^{(1)}\circ\phi_1^{(0)}]^{[1]}(\phi_1^{(0)})^{-1}.\end{aligned} \tag{3.26}$$

To get nontrivial results, here we consider the case when $\phi_1^{(0)}$ is bosonic, the B type condition of the SBKP hierarchy implies $\psi_1^{(0)}$ and $\phi_1^{(0)}$ have the following relation:

$$(\phi_1^{(0)})^{[1]} = [(-1)^{|\phi_1^{(0)}|(|\phi_1^{(0)}|+|\psi_1^{(0)}|)} + 1]\psi_1^{(0)} = 2\psi_1^{(0)},$$
$$\psi_1^{(1)} = \frac{1}{2}\phi_1^{(0)}. \tag{3.27}$$

So in order to keep the B type restriction of the Lax operator, we do iterations of the Darboux transformation to get the n-th Darboux transformation operator $W_n = T_{n+n}$ as

$$W_n = T_{n+n} = T_I(\psi_n^{(2n-1)})T_D(\phi_n^{(2n-2)})...T_I(\psi_1^{(1)})T_D(\phi_1^{(0)}), \tag{3.28}$$

where $(\phi_i^{(0)}, \psi_i^{(0)}) = (\phi(\lambda_i;t), \psi(\mu_i;t))$. One can find the supersymmetric eigenfunctions $\psi_n^{(2n-1)}, \phi_n^{(2n-2)}$ are bosonic functions. These supersymmetric property can keep the Darboux transformations go on step by step.

It can be easily checked that $W_n \cdot \phi_i^{(0)} |_{i\leq n}= 0, ((W_n)^{-1})^* \cdot (\psi_i^{(0)}) |_{i\leq n}= 0$. In particular,

$$W_2 = T_{2+2} = T_I(\psi_2^{(3)})T_D(\phi_2^{(2)})T_I(\psi_1^{(1)})T_D(\phi_1^{(0)}). \tag{3.29}$$

Now, we will give the following important theorem which will be used to generate new solutions from seed solutions.

Theorem 3.1. *If the super eigenfunction ϕ and the adjoint super eigenfunction ψ satisfy eq.*(3.6)*, the one-fold Darboux transformation operator of the supersymmetric BKP hierarchy*

$$W_1 = (\psi_1^{(1)})^{-1} \circ D^{-1} \circ \psi_1^{(1)} \circ \phi_1^{(0)} \circ D \circ (\phi_1^{(0)})^{-1}, \tag{3.30}$$

satisfies $W_1\phi_1^{(0)} = 0$ and $(W_1^{-1})^*(\psi_1^{(0)}) = 0$.

W_1 will generate new solutions $u_i^{[1]}$ of the supersymmetric BKP hierarchy from seed solutions u_i through $\bar{L}^{[1]} = W_1\bar{L}(W_1)^{-1}$,

$$u_1^{[1]} = u_1 + D^2(\ln\phi_1^{(0)}), \tag{3.31}$$

$$u_2^{[1]} = u_2 + D^2(\ln\phi_1^{(0)})D(\ln\phi_1^{(0)}) - D(\frac{D(\psi_1^{(0)})}{\phi_1^{(0)}}), \tag{3.32}$$

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