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Wolfram Koepf, Predrag M. Rajković, Sladjana D. Marinković


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Wolfram Koepf

Department of Mathematics and Computer Science, University of Kassel
Heinrich-Plett-Str. 40, 34132 Kassel, Germany
koepf@mathematik.uni-kassel.de

Predrag M. Rajković

Department of Mathematics, Faculty of Mechanical Engineering, University of Niš
A. Medvedeva 14, 18 000 Niš, Serbia
pedja.rajk@yahoo.com

Sladjana D. Marinković

Department of Mathematics, Faculty of Electronic Engineering, University of Niš
A. Medvedeva 14, 18 000 Niš, Serbia
sladjana.marinkovic@elfak.ni.ac.rs

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In this short communication, we want to pay attention to a few wrong formulas which are unfortunately cited and used in a dozen papers afterwards. We prove that the provided relations and asymptotic expansion about the \( q \)-gamma function are not correct. This is illustrated by numerous concrete counterexamples. The error came from the wrong assumption about the existence of a parameter which does not depend on anything. Here, we apply a similar procedure and derive a correct formula for the \( q \)-gamma function.

Keywords: \( q \)-Gamma function; asymptotic expansion; boundary functions.

2000 Mathematics Subject Classification: 33D05, 11A67

1. Introduction

Since J. Thomae (1869) and F. H. Jackson (1904) defined the \( q \)-gamma function, it plays an important role in the theory of the basic hypergeometric series [5] and its applications [8]. Its properties and different representations were discussed in numerous papers, such as in [4], [12] and [11]. A few successful algorithms for its numerical evaluation are introduced in [7] and [6] and [1]. An asymptotic expansion of the \( q \)-gamma function was provided in [3].

Here, we will make observations on the asymptotic expansions given in [9, 10]. Let \( q \in [0, 1) \). A \( q \)-number \( [a]_q \) is

\[
[a]_q := \frac{1-q^a}{1-q}, \quad a \in \mathbb{R}.
\]

The factorial of a positive integer number \( [n]_q \) is given by

\[
[0]_q! := 1, \quad [n]_q! := [n]_q[n-1]_q \cdots [1]_q, \quad (n \in \mathbb{N}).
\]
An important role in $q$–calculus plays the $q$-Pochhammer symbol defined by
\[(a; q)_0 = 1, \quad (a; q)_n = \prod_{i=0}^{n-1} (1 - aq^i) \quad (n \in \mathbb{N} \cup \{+\infty\}),\]
and
\[(a; q)_{\lambda} = \frac{(a; q)_\infty}{(aq^\lambda; q)_\infty} \quad (|q| < 1, \ \lambda \in \mathbb{C}).\]

The $q$-gamma function
\[\Gamma_q(z) = (q; q)_{z-1} (1 - q)^{1-z} = \frac{(q; q)_\infty}{(q^z; q)_\infty} (1 - q)^{1-z} \quad (0 < q < 1, \ z \not\in \mathbb{Z}^-) \quad (1.1)\]
has the following properties:
\[\Gamma_q(z+1) = [z]_q \Gamma_q(z) \quad (z \in \mathbb{C}), \quad \Gamma_q(n+1) = [n]_q! \quad (n \in \mathbb{N}_0).\]

In particular,
\[\lim_{q \to 1-} \Gamma_q(z) = \Gamma(z).\]

The exact $q$–Gauss multiplication formula can be found in [5] or [4]:
\[\Gamma_q(nx) \prod_{k=1}^{n-1} \Gamma_q^n \left( \frac{k}{n} \right) = [n]_q^{nx-1} \prod_{k=0}^{n-1} \Gamma_q \left( x + \frac{k}{n} \right) \quad (x > 0; \ n \in \mathbb{N}).\]
Equivalently, substituting $z = nx$, it can be written in the form
\[\Gamma_q(z) \prod_{k=1}^{n-1} \Gamma_q^n \left( \frac{k}{n} \right) = [n]_q^{z-1} \prod_{k=0}^{n-1} \Gamma_q \left( \frac{z+k}{n} \right) \quad (z > 0; \ n \in \mathbb{N}). \quad (1.2)\]

2. Our corrections to the paper [9]

Starting from the definition
\[\Gamma_q(x) = (q; q)_\infty (1 - q)^{1-x} (q^x; q)_\infty^{-1},\]
we can write
\[\Gamma_q(x) = (q; q)_\infty (1 - q)^{-1/2} (1 - q)^{1-x} e^{-\log(q^x; q)_\infty}.\]

Hence the function $\Gamma_q(x)$ can be written in the form
\[\Gamma_q(x) = a(q) \cdot (1 - q)^{1/2-x} e^{\mu(x)} \quad (a(q) \in \mathbb{R}), \quad (2.1)\]
where
\[0 < a(q) = (q; q)_\infty (1 - q)^{1/2} < 1, \quad \mu(x, q) = -\log(q^x; q)_\infty. \quad (2.2)\]

Let
\[\psi(x, q) = \frac{q^x}{(1-q)(1-q^x)}.\]
From the estimate
\[ 0 < \mu(x, q) < \psi(x, q) \quad (0 < q < 1, \ x > 0), \]
it exists \( \theta(x, q) \in (0, 1) \) such that
\[ \mu(x, q) = \theta(x, q) \cdot \psi(x, q). \]

Therefore, relation (2.1) becomes
\[ \Gamma_q(x) = a(q) \cdot (1 - q)^{1/2 - x} e^{\theta(x, q) \cdot \psi(x, q)}. \tag{2.3} \]

On the other hand, formula (1.2) can be written in the form
\[ a_p(q) \Gamma_q(x) = \left[ p \right]_q \prod_{k=0}^{p-1} \Gamma_q \left( \frac{x+k}{p} \right) \quad (x > 0; \ p \in \mathbb{N}), \tag{2.4} \]

where
\[ a_p(q) = \left[ p \right]_q \prod_{k=0}^{p-1} \Gamma_q \left( \frac{k}{p} \right) \Gamma_q \left( \frac{2}{p} \right) \cdots \Gamma_q \left( \frac{p}{p} \right). \]

Substituting \( q \to q^p \) and \( x \to k/p \) into the definition (1.1) of the \( q \)-gamma function, we have
\[ \Gamma_{q^p} \left( \frac{k}{p} \right) = \frac{(q^p; q^p)_n}{(q; q^p)_n} (1 - q^p)^{1-k/p} = (1 - q^p)^{1-k/p} \lim_{n \to \infty} \frac{(q^p; q^p)_n}{(q^k; q^p)_n}. \]

Moreover, using
\[ \prod_{k=1}^{p} (1 - q^p)^{1-k/p} = (1 - q^p)^{p^{n-1}}, \]

the following holds:
\[ a_p(q) = \left[ p \right]_q \prod_{k=1}^{p} \Gamma_q \left( \frac{k}{p} \right) = \left[ p \right]_q \prod_{k=1}^{p} (1 - q^p)^{1-k/p} \lim_{n \to \infty} \frac{(q^p; q^p)_n}{(q^k; q^p)_n} \]
\[ = \left[ p \right]_q \prod_{k=1}^{p} (1 - q^p)^{1-k/p} \lim_{n \to \infty} \prod_{k=1}^{p} \frac{(q^p; q^p)_n}{(q^k; q^p)_n} \]
\[ = \left[ p \right]_q (1 - q^p)^{p^{n-1}} \lim_{n \to \infty} \prod_{k=1}^{p} \frac{(q^p; q^p)_n}{(q^k; q^p)_n}. \]

The following identity is valid
\[ \prod_{k=1}^{p} (q^k; q^p)_n = (q; q)_n p. \]

Using estimate (2.3), we get
\[ \Gamma_{q^p}(n+1) = a(q^p) \cdot (1 - q^p)^{-n-1/2} e^{\theta(n+1,q^p) \cdot \psi(n+1,q^p)} \]

Since
\[ \frac{(q^p; q^p)_n}{(1 - q^p)^{np}} = \Gamma_{q^p} (n+1) = a_p(q^p) \cdot (1 - q^p)^{p(-1/2-n)} e^{\theta(n+1,q^p) \cdot \psi(n+1,q^p)}, \]
and \[
\prod_{k=1}^{p} \frac{(q^k; q^p)_n}{(1 - q)^{np}} = \frac{(q; q)_n}{(1 - q)^{np}} = \Gamma_q(np + 1) = a(q) \cdot (1 - q)^{-1/2 - np} \cdot e^{\theta(np + 1, q)} \psi(np + 1, q),
\]
we have
\[
a_p(q) = a_p(q^p) [p]_q^{1/2} \lim_{n \to \infty} e^{p \theta(n + 1, q^p)} \psi(n + 1, q^p).
\]

From
\[
\lim_{n \to \infty} \psi(n + 1, q^p) = \lim_{n \to \infty} \psi(np + 1, q) = 0 \quad (0 < q < 1; \ p \in \mathbb{N}),
\]
we find
\[
a_p(q) = [p]_q^{1/2} \frac{a_p(q^p)}{a(q)}.
\]
In that manner, the parameter \(a_p(q)\) from formula (2.4) is expressed via the parameter \(a(q)\) from formula (2.3).

3. Faults in paper [9]

In the very beginning of this section, we wish to express our opinion that in [9] an excellent approach was exposed, but a few mistakes were made in its realization. So, we have decided to refer to them.

In [9], the author has supposed that
\[
\Gamma_q(x) = a \cdot (1 - q)^{1/2 - x} e^{\mu(x)} \quad (a \in \mathbb{R}),
\]
where
\[
\mu(x) = - \log(q^x; q) > 0.
\]
His efforts in looking for \(\mu(x)\) we shortened a lot by starting from the definition of \(\Gamma_q(x)\). From the fact that
\[
0 < \mu(x) < \frac{q^x}{(1 - q)(1 - q^x)},
\]
and
\[
(1 - q)(1 - q^x) = 1 - q - q^x + q^{x+1} > 1 - q - q^x,
\]
the author in [9] concluded wrongly that
\[
0 < \mu(x) < \frac{q^x}{(1 - q) - q^x}.
\]
But, expression \(1 - q - q^x\) is not positive for all \(q \in (0, 1)\) and \(x > 0\). Indeed,
\[
1 - q - q^x \leq 0 \iff 1 - q \leq q^x \iff x \cdot \log q \geq \log(1 - q) \iff x \leq \frac{\log(1 - q)}{\log q}.
\]

**Example 3.1.** We examined the sign changes of the function \(h_q(x) \equiv 1 - q - q^x\) for different \(q\) and \(x\). Notice that \(x \to +\infty\) if \(q \to 1^+\).
Table 1. Unique real zero of the function $h_q(x)$ and the sign changes for random values of $q$ and $x$

<table>
<thead>
<tr>
<th>$q$</th>
<th>$x$ : $1 - q - q^x = 0$</th>
<th>$x$</th>
<th>$q$</th>
<th>$1 - q - q^x$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.1</td>
<td>0.045758</td>
<td>1.10500</td>
<td>0.592727</td>
<td>−0.15378</td>
</tr>
<tr>
<td>0.3</td>
<td>0.296248</td>
<td>2.27287</td>
<td>0.752038</td>
<td>−0.275286</td>
</tr>
<tr>
<td>0.5</td>
<td>1.0000</td>
<td>6.47584</td>
<td>0.816692</td>
<td>−0.0861563</td>
</tr>
<tr>
<td>0.7</td>
<td>3.37555</td>
<td>43.2362</td>
<td>0.946066</td>
<td>−0.0370453</td>
</tr>
<tr>
<td>0.9</td>
<td>21.8543</td>
<td>60.1635</td>
<td>0.954814</td>
<td>−0.0167368</td>
</tr>
</tbody>
</table>

This estimate should be written in the form

$$0 < \mu(x) < \frac{q^x}{(1-q) - q^x} \quad \left(0 < q < 1; \ x > \frac{\log(1-q)}{\log q}\right).$$

Furthermore, from the estimate

$$0 < \mu(x) < \frac{q^x}{(1-q) - q^x},$$

the author in [9] concluded wrongly that

$$\mu(x) = \frac{\theta q^x}{(1-q) - q^x},$$

where $\theta$ is a number independent of $x$ between 0 and 1.

**Example 3.2.** We find counterexamples which show that $\theta$ depends on $x$ and $q$. In the first table, we fixed $q = 0.9$ and take a few random values for $x$. In another we changed the rule of variables.

Table 2. The dependence of parameter $\theta$ from $x$ and $q$

<table>
<thead>
<tr>
<th>$x$</th>
<th>$q$</th>
<th>$\theta$</th>
<th>$x$</th>
<th>$q$</th>
<th>$\theta$</th>
</tr>
</thead>
<tbody>
<tr>
<td>3.78377</td>
<td>0.9</td>
<td>−7.27980</td>
<td>10.5</td>
<td>0.063920</td>
<td>1.00000</td>
</tr>
<tr>
<td>13.2554</td>
<td>0.9</td>
<td>−1.58344</td>
<td>10.5</td>
<td>0.234682</td>
<td>1.00000</td>
</tr>
<tr>
<td>20.6473</td>
<td>0.9</td>
<td>−0.139893</td>
<td>10.5</td>
<td>0.494904</td>
<td>0.99898</td>
</tr>
<tr>
<td>25.7471</td>
<td>0.9</td>
<td>0.342512</td>
<td>10.5</td>
<td>0.618621</td>
<td>0.98504</td>
</tr>
<tr>
<td>32.2948</td>
<td>0.9</td>
<td>0.673069</td>
<td>10.5</td>
<td>0.806515</td>
<td>0.473541</td>
</tr>
<tr>
<td>43.8850</td>
<td>0.9</td>
<td>0.904181</td>
<td>10.5</td>
<td>0.915828</td>
<td>−0.49862</td>
</tr>
</tbody>
</table>

In continuation, the author in [9] got the wrong formulas (2.21)-(2.27). He concluded that

$$a_p = \sqrt{[2]_q \Gamma_p(1/2)},$$

and

$$\Gamma_q(x) = \sqrt{[2]_q \Gamma_p(1/2)(1-q)^{1/2-x} e^{\theta \frac{q^x}{x-q} - \theta}} \quad (0 < \theta < 1).$$

The following wrong version of the $q$–Gauss multiplication formula was provided

$$[n]_{q}^{1/2-x} [2]_{q}^{(n-1)/2} \frac{\Gamma_{q}^{n-1}(1/2)\Gamma_q(x)}{\Gamma_{q}^{n}(1/2)} \prod_{k=0}^{n-1} \Gamma_q \left( \frac{x+k}{n} \right) \quad (x > 0; \ n \in \mathbb{N}).$$

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In a special case, for \( n = 2 \), it agrees with the exact \( q \)-Legendre relation. Also, when \( q \to 1 \), it reduces to well-known formulas for the gamma function.

4. Bounds of the \( q \)-gamma function

Let

\[
g(x) = \ln \Gamma_q(x)
\]

Since

\[
g(x + 1) = \ln \Gamma_q(x + 1) = \ln ([x]_q \Gamma_q(x)) = \ln [x]_q + g(x),
\]

by induction, we get

\[
g(x + n) = \sum_{k=0}^{n-1} \ln[x + k]_q + g(x) \quad (n \in \mathbb{N}).
\]

In the paper [2] it was proven that \( g(x) \) is a convex function.

**Lemma 4.1.** If \( x \in (0, 1) \) and \( n \in \mathbb{N} \), then

\[
g(n) + x \ln [x + n - 1]_q \leq g(x + n) \leq (1 - x)g(n) + xg(n + 1)
\]

**Proof.** Since

\[
x + n = (1 - x)n + x(n + 1),
\]

we can write

\[
g(x + n) = g((1 - x)n + x(n + 1)) \leq (1 - x)g(n) + xg(n + 1).
\]

Let us find a lower bound for \( \Gamma_q(x) \). Since

\[
n = (1 - x)(x + n) + x(x + n - 1),
\]

and because of the convexity of the function \( g(x) \), we have

\[
g(n) \leq (1 - x)g(x + n) + xg(x + n - 1).
\]

Applying (4.1), for \( x \to x + n - 1 \), we can write

\[
g(x + n) = \ln [x + n - 1]_q + g(x + n - 1),
\]

wherefrom

\[
g(n) \leq (1 - x)g(x + n) + x(g(x + n) - \ln [x + n - 1]_q) = g(x + n) - x \ln [x + n - 1]_q,
\]

i.e.,

\[
g(n) + x \ln [x + n - 1]_q \leq g(x + n). \square
\]

**Theorem 4.1.** The following bounds are valid:

\[
[n - 1]_q! [n - 1 + x]_q^* \leq \Gamma_q(n + x) \leq [n - 1]_q! [n]_q^* \quad (n \in \mathbb{N}; 0 \leq x < 1).
\]
Proof. According to the upper bound for $g(x)$, we get e. g.

$$\ln \Gamma_q(x+n) \leq (1-x) \ln \Gamma_q(n) + x \ln \Gamma_q(n+1).$$

Since the real logarithm is an increasing and continuous function, we have

$$\Gamma_q(x+n) \leq ([n-1]_q!)^{1-x} ([n]_q!)^x,$$

wherefrom

$$\Gamma_q(x+n) \leq [n-1]_q! [n]_q^x.$$  

According to the lower bound for $g(x)$, we get

$$\ln \Gamma_q(n) + x \ln [x+n-1]_q \leq \ln \Gamma_q(x+n),$$

i.e.,

$$\Gamma_q(n) [n+x-1]_q^x \leq \Gamma_q(n+x). \square$$

Theorem 4.2.

$$[n-(1-x)]_q \leq \left( \frac{\Gamma_q(n+x)}{[n-1]_q!} \right)^{1/x} \leq [n]_q \quad (n \in \mathbb{N}_0; \ 0 \leq x < 1).$$

Fig. 1. $\Gamma_q(x)$ and its boundary functions (green and blue) for $q = 0.5$.

Theorem 4.3. For any $n \in \mathbb{N}$ and $x \in (0, 1)$ there exists $\theta = \theta(n,x,q) \in (0, 1)$ such that

$$\Gamma_q(n+x) = [n-1]_q! [n-\theta(1-x)]_q^x.$$  

Introducing $y = n+x \ (n \in \mathbb{N}_0; \ 0 \leq x < 1)$ and denoting $n = \lfloor y \rfloor$, we can write

$$[[y]_q - 1]_q! [y-1]_q^{-[y]} \leq \Gamma_q(y) \leq [[y]_q - 1]_q! [[y]_q]_q^{-[y]} \quad (y > 1).$$

Theorem 4.4. For any $y \in (1, +\infty) \setminus \mathbb{N}$, there exists $\theta = \theta(y,q) \in (0, 1)$ such that

$$\Gamma_q(y) = [[y]_q - 1]_q! [[y]_q - \theta(1-(y-[y]))]_q^{-[y]}.$$
\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{fig2.png}
\caption{$\Gamma_q(x)$ and its boundary functions (green and blue) for $q = 0.9$.}
\end{figure}

\textbf{Example 4.1.} For $y = 15.5$ and $q = 0.1(0.1)0.9$, we have got the following values for $\theta$:
\[ \hat{\theta} = \{0.9851, 0.4021, 0.4259, 0.4432, 0.4681, 0.47762, 0.4855, 0.4917\}. \]
Also, for $q = 0.5$ and $y = 2.31(2)22.31$, we have got $\theta \in (0.4468, 0.4623)$.

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\textbf{References}