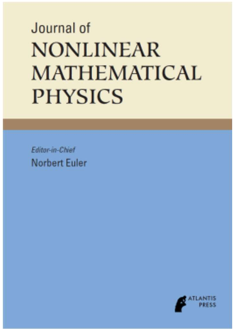

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Novel *solvable* many-body problems

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Novel classes of dynamical systems are introduced, including many-body problems characterized by nonlinear equations of motion of Newtonian type ("acceleration equals forces") which determine the motion of points in the *complex* plane. These models are *solvable*, namely their configuration at any time can be obtained from the initial data by *algebraic* operations, amounting to the determination of the *zeros* of a known time-dependent polynomial in the independent variable z. Some of these models are *multiply periodic*, *isochronous* or *asymptotically isochronous*; others display *scattering* phenomena.

Keywords: New solvable many-body problems; zeros and coefficients of monic polynomials; generations of monic polynomials.

2000 Mathematics Subject Classification: 12D99, 70F10, 70K42.

1. Introduction

Notation 1.1. Unless otherwise indicated, hereafter N is an *arbitrary positive integer*, $N \geq 2$, indices such as n, m, ℓ, ... run over the *integers* from 1 to N, and superimposed arrows denote N-vectors: for instance the vector $\vec{c}$ has the N components c_m. We use instead a superimposed tilde to denote an *unordered* set of N numbers: for instance the notation $\tilde{z}$ denotes the *unordered* set of N numbers z_n, say, the N zeros of a polynomial of degree N in z. Upper-case **boldface** letters denote $N \times N$ matrices: for instance the matrix $\mathbf{M}$ features the N^2 elements M_{nm}. The numbers we use are generally assumed to be *complex*; except for those restricted to be *positive integers* (see above), which generally play the role of indices; and except for the "time" variable, see below. The *imaginary unit* is hereafter denoted as $\mathbf{i}$, implying of course $\mathbf{i}^2 = -1$. For quantities depending on the *real* independent variable t ("time"), superimposed dots indicate differentiation with respect to it: so, for instance, $\dot{z}_n(t) \equiv dz_n(t)/dt$, $\ddot{z}_n \equiv d^2 z_n/dt^2$; but often the t-dependence is not explicitly indicated, whenever this is unlikely to cause any misunderstanding (as, for instance, in the second formula we just wrote and below in (1.3)). The Kronecker symbol δ_{nm} has the usual meaning: $\delta_{nm} = 1$ if $n = m$, $\delta_{nm} = 0$ if $n \neq m$; and we denote below as $\mathbf{I}$ the *unit* $N \times N$ matrix the elements of which are δ_{nm}. We adopt throughout the usual convention according to which a void sum vanishes and a void product equals unity: $\sum_{j=J}^{K} f_j = 0$, $\prod_{j=J}^{K} f_j = 1$ if $J > K$. Moreover we introduce the following convenient notations:

$$\sigma_m(\vec{z}) = \sum_{1 \le s_1 < s_2 < \ldots < s_m \le N} \left(z_{s_1} z_{s_2} \cdots z_{s_m} \right) , \tag{1.1a}$$

$$\sigma_{n,m}(\vec{z}) = \delta_{1m} + \sum_{1 \le s_1 < s_2 < \ldots < s_{m-1} \le N \, ; \, s_j \neq n, \; j=1,\ldots m-1} \left(z_{s_1} z_{s_2} \cdots z_{s_{m-1}} \right) , \tag{1.1b}$$

$$\sigma_{n_1 n_2,m}(\vec{z}) = \delta_{2m} + \sum_{\substack{1 \le s_1 < s_2 < \ldots < s_{m-2} \le N \, ; \\ s_j \neq n_1, \; s_j \neq n_2, \; j=1,\ldots m-2}} \left(z_{s_1} z_{s_2} \cdots z_{s_{m-2}} \right) , \tag{1.1c}$$

$$\sigma_{n_1 n_2 n_3,m}(\vec{z}) = \delta_{3m} + \sum_{\substack{1 \le s_1 < s_2 < \ldots < s_{m-3} \le N \, ; \\ s_j \neq n_1, \; s_j \neq n_2, \; s_j \neq n_3, \; j=1,\ldots m-3}} \left(z_{s_1} z_{s_2} \cdots z_{s_{m-3}} \right) , \tag{1.1d}$$

where of course the symbol $\sum_{1 \le s_1 < s_2 < \ldots < s_m \le N}$ denotes the sum from 1 to N over the m integer indices $s_1, s_2, \ldots, s_m$ with the restriction that $s_1 < s_2 < \ldots < s_m$, while the symbol $\sum_{1 \le s_1 < s_2 < \ldots < s_{m-1} \le N \, ; \, s_j \neq n, \; j=1,\ldots m-1}$ denotes the sum from 1 to N over the $m-1$ indices $s_1, s_2, \ldots, s_{m-1}$ with the restriction $s_1 < s_2 < \ldots < s_{m-1}$ and moreover the requirement that *all these indices be different from n*; and likewise for the symbols $\sum_{1 \le s_1 < s_2 < \ldots < s_{m-2} \le N \, ; \, s_j \neq n_1, \; s_j \neq n_2, \; j=1,\ldots m-2}$ and $\sum_{1 \le s_1 < s_2 < \ldots < s_{m-3} \le N \, ; \, s_j \neq n_1, \; s_j \neq n_2, \; s_j \neq n_3, \; j=1,\ldots m-3}$. Note that—according to the convention (see above) that a sum over an empty set of indices equals zero—these definitions imply $\sigma_{n,1}(\vec{z}) = 1$, $\sigma_{n_1 n_2,1}(\vec{z}) = 0$ and $\sigma_{n_1 n_2,2}(\vec{z}) = 1$, and $\sigma_{n_1 n_2 n_3,m}(\vec{z}) = 0$ for $m \le 2$ while $\sigma_{n_1 n_2 n_3,3}(\vec{z}) = 1$. Finally, the prime appended to a sum (see for instance below (1.3) and also note the simplification it would imply for (1.1b)) indicates that the sum runs—over the indicated indices, in the identified range—with the additional restrictions that these indices be *all different among themselves* and moreover *all different from the "outside" index* (which is for instance n in (1.3)); note that this sum becomes void hence vanishes identically if N is small enough, so for instance the last sum in the left-hand side of (1.3d) vanishes for $N \le 3$, and more generally the "primed" sum from 1 to N over k indices ℓ_1, ℓ_2,..., ℓ_k vanishes identically if $N \le k$. ■

Remark 1.1. Note that the notation $\sigma_m(\tilde{z})$ (instead of $\sigma_m(\vec{z})$) is equally meaningful, since this quantity, see (1.1a), only depends on *symmetrical sums* of the N components z_m of the N-vector $\vec{z}$, hence it is independent of the ordering of the N elements z_n of the *unordered* set $\tilde{z}$. The notations $\sigma_{n,m}(\tilde{z})$, $\sigma_{n_1 n_2,m}(\tilde{z})$, $\sigma_{n_1 n_2 n_3,m}(\tilde{z})$, see (1.1), are instead ill-defined and cannot therefore be used; except in the context of expressions which remain valid for *any* ordering of the N numbers z_n, i. e., for any assignments of the N different integer labels n (in the range $1 \le n \le N$) to the N elements of the *unordered* set $\tilde{z}$; provided of course that assignment is maintained throughout that expression (in which case the relevant expression amounts in fact to $N!$ different formulas; assuming, as we generally do, that the N numbers z_n are *all different among themselves*). This remark is of course equally valid for any function $f(\tilde{z})$. ■

The main protagonists of this paper are formulas relating the time-evolution of the N zeros $z_n(t)$ of a time-dependent monic polynomial of degree N in the independent variable z,

$$p_N(z;\, \vec{c}(t)\,,\, \tilde{z}(t)) = \prod_{n=1}^{N} [z - z_n(t)] \;, \tag{1.2a}$$

to the time-evolution of its N coefficients $c_m(t)$,

$$p_N(z;\, \vec{c}(t)\,,\, \tilde{z}(t)) = z^N + \sum_{m=1}^{N} \left[c_m(t)\; z^{N-m}\right] \;. \tag{1.2b}$$

The first two of these formulas read as follows [1]:

$$\dot{z}_n = -\left[\prod_{\ell=1,\; \ell\neq n}^{N} (z_n - z_\ell)^{-1}\right] \sum_{m=1}^{N} \left[\dot{c}_m\; (z_n)^{N-m}\right] \;, \tag{1.3a}$$

$$\ddot{z}_n - {\sum_{\ell=1}^{N}}' \left(\frac{2\,\dot{z}_n\,\dot{z}_\ell}{z_n - z_\ell}\right) = -\left[\prod_{\ell=1,\; \ell\neq n}^{N} (z_n - z_\ell)^{-1}\right] \sum_{m=1}^{N} \left[\ddot{c}_m\; (z_n)^{N-m}\right] \;. \tag{1.3b}$$

In the present paper we report two additional formulas of this kind:

$$\dddot{z}_n - 3\,{\sum_{\ell=1}^{N}}' \left(\frac{\ddot{z}_n\,\dot{z}_\ell + \ddot{z}_\ell\,\dot{z}_n}{z_n - z_\ell}\right) + 3\,{\sum_{\ell_1,\ell_2=1}^{N}}' \left[\frac{\dot{z}_n\,\dot{z}_{\ell_1}\,\dot{z}_{\ell_2}}{(z_n - z_{\ell_1})\;(z_n - z_{\ell_2})}\right]$$
$$= -\left[\prod_{\ell=1,\; \ell\neq n}^{N} (z_n - z_\ell)^{-1}\right] \sum_{m=1}^{N} \left[\dddot{c}_m\; (z_n)^{N-m}\right] \;, \tag{1.3c}$$

$$\ddddot{z}_n - {\sum_{\ell=1}^{N}}' \left(\frac{4\,\dddot{z}_n\,\dot{z}_\ell + 4\,\dddot{z}_\ell\,\dot{z}_n + 6\,\ddot{z}_n\,\ddot{z}_\ell}{z_n - z_\ell}\right)$$
$$+6\,{\sum_{\ell_1,\ell_2=1}^{N}}' \left[\frac{\ddot{z}_n\,\dot{z}_{\ell_1}\,\dot{z}_{\ell_2} + 2\,\ddot{z}_{\ell_1}\,\dot{z}_{\ell_2}\,\dot{z}_n}{(z_n - z_{\ell_1})\;(z_n - z_{\ell_2})}\right]$$
$$-4\,{\sum_{\ell_1,\ell_2,\;\ell_3=1}^{N}}' \left[\frac{\dot{z}_n\,\dot{z}_{\ell_1}\,\dot{z}_{\ell_2}\,\dot{z}_{\ell_3}}{(z_n - z_{\ell_1})\;(z_n - z_{\ell_2})\;(z_n - z_{\ell_3})}\right]$$
$$= -\left[\prod_{\ell=1,\; \ell\neq n}^{N} (z_n - z_\ell)^{-1}\right] \sum_{m=1}^{N} \left[\ddddot{c}_m\; (z_n)^{N-m}\right] \;. \tag{1.3d}$$

A terse outline of the proof of these identities is reported in Appendix A.

The first two of the formulas (1.3) have recently allowed the identification of (endless sequences of) new *solvable* many-body problems characterized by nonlinear equations of motion of Newtonian type ("acceleration equals forces") determining the motion of N points in the *complex* z-plane [1–4]. In the present paper we show how the last two of the formulas (1.3) allow the identification of additional endless sequences of new *solvable* dynamical systems determining the motion of points in the *complex* z-plane—also including many-body problems characterized by nonlinear equations of motion of Newtonian type ("acceleration equals forces").

Note that the notation (1.2), which we employ for polynomials, is somewhat redundant, since they are equally well defined by the (time-dependent) N-vector $\vec{c}(t)$ the N components of which are the N *coefficients* $c_m(t)$ of the polynomial (see (1.2b)), as by the (time-dependent) unordered set $\tilde{z}(t)$ the N elements of which are the N *zeros* $z_n(t)$ of the polynomial (see (1.2a)). Indeed the N *coefficients* $c_m(t)$ can be *explicitly* expressed in terms of the N *zeros* $z_n(t)$ as follows:

$$c_m = (-1)^m \ \sigma_m(\vec{z}) \equiv (-1)^m \ \sigma_m(\tilde{z}) \tag{1.4}$$

(see **Notation 1.1** and **Remark 1.1**). While the N *zeros* $z_n(t)$ are likewise uniquely determined (up to permutations) by the N *coefficients* $c_m(t)$, but of course *explicit* expressions to this effect are generally available only for $N \leq 4$.

There holds moreover the following identity:

$$(z_n)^N + \sum_{m=1}^{N} \left[c_m \ (z_n)^{N-m} \right] = 0 \ , \tag{1.5a}$$

which is an obvious consequence of (1.2), and via (1.4) it implies

$$(z_n)^N + \sum_{m=1}^{N} \left[(-1)^m \ \sigma_m(\tilde{z}) \ (z_n)^{N-m} \right] = 0 \ . \tag{1.5b}$$

Note that, while the formula (1.5a) is an identity valid for the N *coefficients* c_m and the N *zeros* z_n of any polynomial, see (1.2), the identity (1.5b) is clearly valid for *any arbitrary* assignment of the N elements z_n of the unordered set $\tilde{z}$.

Likewise, there holds the following formula that is also clearly valid for *any* assignment of the N elements z_n of the unordered set $\tilde{z}$ (see **Notation 1.1** and **Remark 1.1**):

$$-\left[\prod_{\ell=1,\ \ell\neq n}^{N} (z_n - z_\ell)^{-1} \right] \ \sum_{j=1}^{N} \left[(-1)^j \ (z_n)^{N-j} \ \sigma_{m,j}(\tilde{z}) \right] = \delta_{nm} \ ; \tag{1.6a}$$

and note that this formula can also be rewritten in the following $(N \times N)$-matrix version:

$$[\mathbf{R}(\tilde{z})]_{nm} \equiv R_{nm}(\tilde{z}) = -\left[\prod_{\ell=1,\ \ell\neq n}^{N} (z_n - z_\ell)^{-1} \right] \ (z_n)^{N-m} \ , \tag{1.6b}$$

$$\left[\mathbf{R}^{-1}(\tilde{z}) \right]_{nm} \equiv \left[R^{-1}(\tilde{z}) \right]_{nm} = (-1)^n \ \sigma_{n,m}(\tilde{z}) \ , \tag{1.6c}$$

implying of course (see **Notation 1.1** and **Remark 1.1**)

$$\mathbf{R}(\tilde{z}) \ \mathbf{R}^{-1}(\tilde{z}) = \mathbf{R}^{-1}(\tilde{z}) \ \mathbf{R}(\tilde{z}) = \mathbf{I} \ . \tag{1.6d}$$

Finally let us report 3 additional identities which are obvious consequences of the definitions (1.4) and (1.1) (see **Notation 1.1** and **Remark 1.1**):

$$\dot{c}_m = (-1)^m \ \dot{\sigma}_m(\vec{z}) \equiv (-1)^m \ \sum_{n=1}^{N} \left[\sigma_{n,m}(\tilde{z}) \ \dot{z}_n \right] \ , \tag{1.7a}$$

$$\ddot{c}_m = (-1)^m \ \ddot{\sigma}_m(\vec{z})$$
$$\equiv (-1)^m \ \left\{ \sum_{n=1}^{N} \left[\sigma_{n,m}(\tilde{z}) \ \ddot{z}_n \right] + \sum_{n_1,n_2=1, n_1\neq n_2}^{N} \left[\sigma_{n_1 n_2,m}(\tilde{z}) \ \dot{z}_{n_1} \ \dot{z}_{n_2} \right] \right\} \ , \tag{1.7b}$$

$$\dddot{c}_m = (-1)^m \, \dddot{\sigma}_m(\vec{z}) \equiv (-1)^m \, \left\{ \sum_{n=1}^{N} [\sigma_{n,m}(\tilde{z}) \, \dddot{z}_n] \right.$$

$$+3 \sum_{n_1,n_2=1,n_1\neq n_2}^{N} [\sigma_{n_1 n_2,m}(\tilde{z}) \, \ddot{z}_{n_1} \, \dot{z}_{n_2}]$$

$$\left. + \sum_{n_1,n_2,n_3=1,n_1\neq n_2\neq n_3}^{N} [\sigma_{n_1 n_2 n_3,m}(\tilde{z}) \, \dot{z}_{n_1} \, \dot{z}_{n_2} \, \dot{z}_{n_3}] \right\} , \tag{1.7c}$$

with the indices n_1, n_2, n_3 in the last sum all different among themselves.

In Section 3 it is indicated how these polynomial properties are instrumental to identify endless classes of *solvable* dynamical systems including many-body problems of Newtonian type, one of which is immediately reported in the following Section 2, while some of its solutions are displayed in Appendix B. The paper is then concluded by a section entitled “Outlook”, where further investigations are tersely outlined.

2. Display and discussion of a novel *solvable* many-body problem

In this section we provide and discuss an instance of the novel *solvable* many-body problems of Newtonian type identified in this paper. Its equations of motion, characterizing the time-evolution of the $2N$ complex dependent variables $z_n \equiv z_n(t)$ and $w_n \equiv w_n(t)$, read as follows:

$$\ddot{z}_n = w_n , \tag{2.8a}$$

$$\ddot{w}_n = \sum_{\ell=1}^{N}{}' \left(\frac{4 \, \dot{w}_n \, \dot{z}_\ell + 4 \, \dot{w}_\ell \, \dot{z}_n + 6 \, w_n \, w_\ell}{z_n - z_\ell} \right)$$

$$-6 \sum_{\ell_1,\ell_2=1}^{N}{}' \left[\frac{w_n \, \dot{z}_{\ell_1} \, \dot{z}_{\ell_2} + 2 \, w_{\ell_1} \, \dot{z}_n \, \dot{z}_{\ell_2}}{(z_n - z_{\ell_1}) \, (z_n - z_{\ell_2})} \right]$$

$$+4 \sum_{\ell_1,\ell_2,\,\ell_3=1}^{N}{}' \left[\frac{\dot{z}_n \, \dot{z}_{\ell_1} \, \dot{z}_{\ell_2} \, \dot{z}_{\ell_3}}{(z_n - z_{\ell_1}) \, (z_n - z_{\ell_2}) \, (z_n - z_{\ell_3})} \right] - \left[\prod_{\ell=1,\,\ell\neq n}^{N} (z_n - z_\ell)^{-1} \right] \cdot$$

$$\cdot \sum_{m=1}^{N} \left[(\alpha_m \, \dddot{c}_m + \beta_m \, \ddot{c}_m + \gamma_m \, \dot{c}_m + \delta_m \, c_m) \, (z_n)^{N-m} \right] , \tag{2.8b}$$

with c_m, $\dot{c}_m$ expressed in terms of z_n and $\dot{z}_n$ by (1.7a) and (1.4) and $\ddot{c}_m$, $\dddot{c}_m$ expressed in terms of the dependent variables z_n, w_n and their time derivatives $\dot{z}_n$, $\dot{w}_n$ as follows (see **Notation 1.1** and **Remark 1.1**),

$$\ddot{c}_m = (-1)^m \left\{ \sum_{n=1}^{N} [\sigma_{n,m}(\tilde{z}) \, w_n] + \sum_{n_1,n_2=1}^{N}{}' [\sigma_{n_1 n_2,m}(\tilde{z}) \, \dot{z}_{n_1} \, \dot{z}_{n_2}] \right\} , \tag{2.8c}$$

$$\dddot{c}_m = (-1)^m \left\{ \sum_{n=1}^{N} [\sigma_{n,m}(\tilde{z}) \, \dot{w}_n] + 3 \sum_{n_1,n_2=1}^{N}{}' [\sigma_{n_1 n_2,m}(\tilde{z}) \, w_{n_1} \, \dot{z}_{n_2}] \right.$$

$$\left. + \sum_{n_1,n_2,n_3=1}^{N}{}' [\sigma_{n_1 n_2 n_3,m}(\tilde{z}) \, \dot{z}_{n_1} \, \dot{z}_{n_2} \, \dot{z}_{n_3}] \right\} . \tag{2.8d}$$

In (2.8b) the parameters $\alpha_m,\ \beta_m,\ \gamma_m,\ \delta_m$ are $4N$ arbitrary *complex* numbers, which may be conveniently related to the $8N$ *real* parameters $a_m^{(1)}, a_m^{(2)}, a_m^{(3)}, a_m^{(4)}, \omega_m^{(1)}, \omega_m^{(2)}, \omega_m^{(3)}, \omega_m^{(4)}$ (for their role see below eq. (2.10), (3.2) and (3.3b)) by the following formulas

$$\alpha_m = -a_m^{(1)} - a_m^{(2)} - a_m^{(3)} - a_m^{(4)} + \mathbf{i}\left[\omega_m^{(1)} + \omega_m^{(2)} + \omega_m^{(3)} + \omega_m^{(4)}\right] , \tag{2.9a}$$

$$\begin{aligned}\beta_m = &-a_m^{(1)}a_m^{(2)} - a_m^{(1)}a_m^{(3)} - a_m^{(2)}a_m^{(3)} - a_m^{(1)}a_m^{(4)} - a_m^{(2)}a_m^{(4)} - a_m^{(3)}a_m^{(4)} \\ &+\omega_m^{(1)}\omega_m^{(2)} + \omega_m^{(1)}\omega_m^{(3)} + \omega_m^{(2)}\omega_m^{(3)} + \omega_m^{(1)}\omega_m^{(4)} + \omega_m^{(2)}\omega_m^{(4)} + \omega_m^{(3)}\omega_m^{(4)} \\ &+\mathbf{i}\Big[a_m^{(2)}\omega_m^{(1)} + a_m^{(3)}\omega_m^{(1)} + a_m^{(4)}\omega_m^{(1)} + a_m^{(1)}\omega_m^{(2)} + a_m^{(3)}\omega_m^{(2)} + a_m^{(4)}\omega_m^{(2)} \\ &+a_m^{(1)}\omega_m^{(3)} + a_m^{(2)}\omega_m^{(3)} + a_m^{(4)}\omega_m^{(3)} + a_m^{(1)}\omega_m^{(4)} + a_m^{(2)}\omega_m^{(4)} + a_m^{(3)}\omega_m^{(4)}\Big] ,\end{aligned} \tag{2.9b}$$

$$\begin{aligned}\gamma_m = &-a_m^{(1)}a_m^{(2)}a_m^{(3)} - a_m^{(1)}a_m^{(2)}a_m^{(4)} - a_m^{(1)}a_m^{(3)}a_m^{(4)} - a_m^{(2)}a_m^{(3)}a_m^{(4)} \\ &+a_m^{(1)}\omega_m^{(2)}\omega_m^{(3)} + a_m^{(1)}\omega_m^{(2)}\omega_m^{(4)} + a_m^{(1)}\omega_m^{(3)}\omega_m^{(4)} + a_m^{(2)}\omega_m^{(1)}\omega_m^{(3)} \\ &+a_m^{(2)}\omega_m^{(1)}\omega_m^{(4)} + a_m^{(2)}\omega_m^{(3)}\omega_m^{(4)} + a_m^{(3)}\omega_m^{(1)}\omega_m^{(2)} + a_m^{(3)}\omega_m^{(1)}\omega_m^{(4)} \\ &+a_m^{(3)}\omega_m^{(2)}\omega_m^{(4)} + a_m^{(4)}\omega_m^{(1)}\omega_m^{(2)} + a_m^{(4)}\omega_m^{(1)}\omega_m^{(3)} + a_m^{(4)}\omega_m^{(2)}\omega_m^{(3)} \\ &+\mathbf{i}\Big[a_m^{(2)}a_m^{(3)}\omega_m^{(1)} + a_m^{(2)}a_m^{(4)}\omega_m^{(1)} + a_m^{(3)}a_m^{(4)}\omega_m^{(1)} + a_m^{(1)}a_m^{(3)}\omega_m^{(2)} \\ &+a_m^{(1)}a_m^{(4)}\omega_m^{(2)} + a_m^{(3)}a_m^{(4)}\omega_m^{(2)} + a_m^{(1)}a_m^{(2)}\omega_m^{(3)} + a_m^{(1)}a_m^{(4)}\omega_m^{(3)} \\ &+a_m^{(2)}a_m^{(4)}\omega_m^{(3)} + a_m^{(1)}a_m^{(2)}\omega_m^{(4)} + a_m^{(1)}a_m^{(3)}\omega_m^{(4)} + a_m^{(2)}a_m^{(3)}\omega_m^{(4)} \\ &-\omega_m^{(1)}\omega_m^{(2)}\omega_m^{(3)} - \omega_m^{(1)}\omega_m^{(2)}\omega_m^{(4)} - \omega_m^{(1)}\omega_m^{(3)}\omega_m^{(4)} - \omega_m^{(2)}\omega_m^{(3)}\omega_m^{(4)}\Big] ,\end{aligned} \tag{2.9c}$$

$$\begin{aligned}\delta_m = &-a_m^{(1)}a_m^{(2)}a_m^{(3)}a_m^{(4)} + a_m^{(3)}a_m^{(4)}\omega_m^{(1)}\omega_m^{(2)} + a_m^{(2)}a_m^{(4)}\omega_m^{(1)}\omega_m^{(3)} + a_m^{(2)}a_m^{(3)}\omega_m^{(1)}\omega_m^{(4)} \\ &+a_m^{(1)}a_m^{(4)}\omega_m^{(2)}\omega_m^{(3)} + a_m^{(1)}a_m^{(3)}\omega_m^{(2)}\omega_m^{(4)} + a_m^{(1)}a_m^{(2)}\omega_m^{(3)}\omega_m^{(4)} - \omega_m^{(1)}\omega_m^{(2)}\omega_m^{(3)}\omega_m^{(4)} \\ &+\mathbf{i}\Big[a_m^{(2)}a_m^{(3)}a_m^{(4)}\omega_m^{(1)} + a_m^{(1)}a_m^{(3)}a_m^{(4)}\omega_m^{(2)} + a_m^{(1)}a_m^{(2)}a_m^{(4)}\omega_m^{(3)} \\ &+a_m^{(1)}a_m^{(2)}a_m^{(3)}\omega_m^{(4)} - a_m^{(4)}\omega_m^{(1)}\omega_m^{(2)}\omega_m^{(3)} - a_m^{(3)}\omega_m^{(1)}\omega_m^{(2)}\omega_m^{(4)} \\ &-a_m^{(2)}\omega_m^{(1)}\omega_m^{(3)}\omega_m^{(4)} - a_m^{(1)}\omega_m^{(2)}\omega_m^{(3)}\omega_m^{(4)}\Big] .\end{aligned} \tag{2.9d}$$

As explained in the following Section 3, it is also possible to invert these equations, i. e. to write formulas expressing the $8N$ *real* parameters $a_m^{(1)}, a_m^{(2)}, a_m^{(3)}, a_m^{(4)}, \omega_m^{(1)}, \omega_m^{(2)}, \omega_m^{(3)}, \omega_m^{(4)}$ in terms of the $4N$ *complex* parameters $\alpha_m,\ \beta_m,\ \gamma_m,\ \delta_m$, but in view of the complicated nature of these expressions—essentially based on the solution of algebraic equations of fourth degree—this does not seem useful (see below **Remark 3.2**).

As shown in the following Section 3, the *general solution* of this $(2N)$-body problem is provided by the following prescription: the values of the coordinates $w_n(t)$ are of course provided by (2.8a), while the values of the N coordinates $z_n(t)$ are the N zeros of the monic polynomial (1.2b) where the N coefficients $c_m(t)$—being the solutions of the *solvable* dynamical system (3.1)—are provided by the following formulas:

$$c_m(t) = \sum_{k=1}^{4} \left\{ b_m^{(k)} \exp\left[\left(-a_m^{(k)} + \mathbf{i}\, \omega_m^{(k)} \right) t \right] \right\} . \tag{2.10}$$

Here the coefficients $b_m^{(k)}$ are $4N$ *a priori* arbitrary *complex* parameters. And the solution of the *initial value problem* for this $(2N)$-body problem, (2.8), is obtained by determining the $4N$ coefficients $b_m^{(k)}$ as solutions, for every value of the parameter m, of the system of 4 *linear* algebraic equations

$$\sum_{k=1}^{4} \left[b_m^{(k)} \left(-a_m^{(k)} + \mathbf{i}\, \omega_m^{(k)} \right)^s \right] = \left. \frac{d^s c_m(t)}{dt^s} \right|_{t=0} , \quad s = 0,1,2,3 , \tag{2.11}$$

with, in the right-hand side, $c_m(0)$, $\dot{c}_m(0)$ expressed in terms of the initial data $z_n(0)$, $\dot{z}_n(0)$ by (1.4) and (1.7a) (at $t=0$), and $\ddot{c}_m(0)$, $\dddot{c}_m(0)$ expressed in terms of the initial data $z_n(0)$, $\dot{z}_n(0)$ and $w_n(0)$, $\dot{w}_n(0)$ by (2.8c) and (2.8d) (at $t=0$).

Remark 2.1. Above and hereafter we assume for simplicity that the $4N$ *complex* numbers $\lambda_{m,k} = -a_m^{(k)} + \mathbf{i}\, \omega_m^{(k)}$ (see (2.10)) are *all different among themselves*; otherwise some appropriate limit should be taken in (2.10) and some of the statements made in the following **Remark 2.2** would require additional restrictions. ■

Remark 2.2. The following properties of various subcases of the many-body problem characterized by the Newtonian equations of motion (2.8) are obviously implied by its *general solution*, as detailed above.

(i) If the $4N$ *real* parameters $a_m^{(k)}$ are *all nonnegative*, $a_m^{(k)} \geq 0$, then *all* solutions of this many-body problem are, for *all future time*, confined to a *finite* region—the dimensions of which depend on the initial data—of the complex z and w planes; and in particular if the $4N$ *real* parameters $a_m^{(k)}$ are *all positive*, $a_m^{(k)} > 0$, *all* solutions of this many-body problem converge to the origin,

$$\lim_{t\to\infty} [z_n(t)] = 0 , \quad \lim_{t\to\infty} [w_n(t)] = 0 ; \tag{2.12}$$

while if only *some* of the $4N$ *real* parameters $a_m^{(k)}$ are *positive* and *all* others *vanish*, then this many-body problem is *asymptotically multiply periodic*; and if in addition the *real* parameters $\omega_m^{(k)}$, such that the corresponding parameter $a_m^{(k)}$ vanishes, are *all integer multiples* of a common (nonvanishing) *real* factor $\omega \neq 0$, i. e. if for *some* values of the indices m and k the parameters $a_m^{(k)}$ are positive, $a_m^{(k)} > 0$, while for *all other* values of the indices m and k

$$a_m^{(k)} = 0 , \quad \omega_m^{(k)} = p_{mk}\, \omega \tag{2.13}$$

with these parameters p_{mk} being *all integers* (positive, negative or vanishing, but all different among themselves), then this many-body system is *asymptotically isochronous.* [5]

(ii) If the $4N$ *real* parameters $a_m^{(k)}$ *all* vanish, $a_m^{(k)} = 0$, and the $4N$ *real* parameters $\omega_m^{(k)}$ are *all integer multiples* of a common (nonvanishing) *real* factor $\omega \neq 0$, $\omega_m^{(k)} = p_{mk}\, \omega$ with the $4N$ parameters p_{mk} *all integers* (positive, negative or vanishing, and of course all different among themselves), then this many-body system is *isochronous* [6].

(iii) If some or even all of the $4N$ *real* parameters $a_m^{(k)}$ are *negative*, then the solutions of this many-body problem need not be confined to a bounded region of the complex plane, indeed they generally describe *scattering* phenomena: for a detailed analysis of such behaviors see Appendix G ("Asymptotic behavior of the zeros of a polynomial whose coefficients diverge exponentially") of the book [7]. ■

Example 1. If $N = 2$, system (2.8) reduces to

$$\ddot{z}_1 = w_1, \;\; \ddot{z}_2 = w_2,$$
$$\ddot{w}_1 = G(\vec{z},\dot{\vec{z}},\vec{w},\dot{\vec{w}}) + \frac{1}{z_1 - z_2}\left[z_1 F_1(\vec{z},\dot{\vec{z}},\vec{w},\dot{\vec{w}}) - F_2(\vec{z},\dot{\vec{z}},\vec{w},\dot{\vec{w}})\right],$$
$$\ddot{w}_2 = -G(\vec{z},\dot{\vec{z}},\vec{w},\dot{\vec{w}}) + \frac{1}{z_1 - z_2}\left[-z_2 F_1(\vec{z},\dot{\vec{z}},\vec{w},\dot{\vec{w}}) + F_2(\vec{z},\dot{\vec{z}},\vec{w},\dot{\vec{w}})\right], \quad (2.14a)$$

where

$$G(\vec{z},\dot{\vec{z}},\vec{w},\dot{\vec{w}}) = \frac{4\dot{w}_1\dot{z}_2 + 4\dot{w}_2\dot{z}_1 + 6w_1w_2}{z_1 - z_2}, \quad (2.14b)$$
$$F_1(\vec{z},\dot{\vec{z}},\vec{w},\dot{\vec{w}}) = \alpha_1(\dot{w}_1 + \dot{w}_2) + \beta_1(w_1 + w_2)$$
$$+\gamma_1(\dot{z}_1 + \dot{z}_2) + \delta_1(z_1 + z_2), \quad (2.14c)$$
$$F_2(\vec{z},\dot{\vec{z}},\vec{w},\dot{\vec{w}}) = \alpha_2(\dot{w}_1 z_2 + 3w_1\dot{z}_2 + 3\dot{z}_1 w_2 + z_1\dot{w}_2)$$
$$+\beta_2(w_1 z_2 + 2\dot{z}_1\dot{z}_2 + z_1 w_2) + \gamma_2(\dot{z}_1 z_2 + z_1\dot{z}_2) + \delta_2 z_1 z_2. \quad (2.14d)$$

In Appendix B we provide plots of the solutions of this system (2.14) with the following values of the parameters $\alpha_m, \beta_m, \gamma_m, \delta_m$, $m = 1,2$,

$$\alpha_m = 5\mathbf{i}, \beta_m = 5, \gamma_m = 5\mathbf{i}, \delta_m = 6, \text{ for } m = 1,2, \quad (2.15a)$$

and the initial conditions

$$z_1(0) = 1 + \mathbf{i}, \dot{z}_1(0) = 1, z_2(0) = 5 + \mathbf{i}, \dot{z}_2(0) = 1,$$
$$w_1(0) = 1, \dot{w}_1(0) = \mathbf{i}, w_2(0) = -\mathbf{i}, \dot{w}_2(0) = 1. \quad (2.15b)$$

For system (2.14), (2.15a), each characteristic equation (3.2) has the four roots $-\mathbf{i}, \mathbf{i}, 2\mathbf{i}, 3\mathbf{i}$. Therefore, by (ii) of **Remark 2.2,** system (2.14), (2.15a) is *isochronous*, see Figures 3, 4, 5, 6, 7, 8, 9, 10 in Appendix B.

Next,we provide plots of the solutions of system (2.14) with the following values of the parameters $\alpha_m, \beta_m, \gamma_m, \delta_m$, $m = 1,2$,

$$\alpha_m = -3, \beta_m = -3, \gamma_m = -3, \delta_m = -2, m = 1,2, \quad (2.16a)$$

and the initial conditions

$$z_1(0) = -2 - \mathbf{i}, \dot{z}_1(0) = 1, z_2(0) = 2 + \mathbf{i}, \dot{z}_2(0) = 1,$$
$$w_1(0) = 1, \dot{w}_1(0) = \mathbf{i}, w_2(0) = -\mathbf{i}, \dot{w}_2(0) = 1. \quad (2.16b)$$

For the initial value problem (2.14), (2.16a), each characteristic equation (3.2) has the four roots $-\mathbf{i}, \mathbf{i}, -1, -2$. Therefore, by (i) of **Remark 2.2,** system (2.14), (2.16a) is *asymptotically isochronous*, see Figures 11, 12, 13, 14, 15, 16, 17, 18 in Appendix B.

Next,we provide plots of the solutions of system (2.14) with the following values of the parameters $\alpha_m, \beta_m, \gamma_m, \delta_m$, $m = 1,2$,

$$\alpha_m = -3+(\pi-1)\mathbf{i}, \beta_m = -(\pi+2)+3(\pi-1)\mathbf{i},$$
$$\gamma_m = -3\pi+2(\pi-1)\mathbf{i}, \delta_m = -2\pi, m = 1,2, \tag{2.17a}$$

and the initial conditions

$$z_1(0) = -2-\mathbf{i}, \dot{z}_1(0) = 1, z_2(0) = 2+\mathbf{i}, \dot{z}_2(0) = -1,$$
$$w_1(0) = \mathbf{i}, \dot{w}_1(0) = 1, w_2(0) = -\mathbf{i}, \dot{w}_2(0) = -1. \tag{2.17b}$$

For system (2.14), (2.17a), each characteristic equation (3.2) has the four roots $-\mathbf{i}, \pi\mathbf{i}, -1, -2$. Therefore, by (i) of **Remark 2.2,** system (2.14), (2.17a) is *asymptotically multiply periodic*, see Figures 19, 20, 21, 22, 23, 24, 25, 26 in Appendix B.

Next,we provide plots of the solutions of system (2.14) with the following values of the parameters $\alpha_m, \beta_m, \gamma_m, \delta_m$, $m = 1,2$,

$$\alpha_1 = 0.222+1.4\mathbf{i}, \beta_1 = 0.41208-0.2208\mathbf{i},$$
$$\gamma_1 = -0.038436-0.018968\mathbf{i}, \delta_1 = 0.000866464+0.0010224\mathbf{i},$$
$$\alpha_2 = 0.172+1.1\mathbf{i}, \beta_2 = 0.06952-0.1512\mathbf{i},$$
$$\gamma_2 = -0.006696+0.026376\mathbf{i}, \delta_2 = 0.000104896-0.00047584\mathbf{i}, \tag{2.18a}$$

and the initial conditions

$$z_1(0) = -2+3\mathbf{i}, \dot{z}_1(0) = 7, z_2(0) = 3+2\mathbf{i}, \dot{z}_2(0) = -5,$$
$$w_1(0) = 2+4.2\mathbf{i}, \dot{w}_1(0) = 4.5, w_2(0) = 3.1\mathbf{i}, \dot{w}_2(0) = 2.4. \tag{2.18b}$$

For system (2.14), (2.18a), the characteristic equation (3.2) for $m = 1$ has the four roots $0.04, 0.062+\mathbf{i}, 0.08+0.3\mathbf{i}, 0.04+0.1\mathbf{i}$ and the characteristic equation (3.2) for $m = 2$ has the four roots $0.02, 0.032+\mathbf{i}, 0.06-0.1\mathbf{i}, 0.06+0.2\mathbf{i}$. In agreement with (iii) of **Remark 2.2,** the components z_1 and w_1 of the solution of system (2.14), (2.18a) exhibit *scattering* phenomena, see Figures 27, 28, 29, 30, 31, 32, 33, 34 in Appendix B. From these figures, it is clear that $z_1(t)$ diverges exponentially as $t \to \infty$ (and of course $w_1(t)$ features the same behavior), while $z_2(t)$ and $w_2(t)$ converge to zero as $t \to \infty$, which is consistent with the behavior of the zeros of polynomials whose coefficients depend on t exponentially, as reported in Appendix G of [7].

Example 2. If $N = 3$, system (2.8) reduces to

$$\ddot{z}_1 = w_1, \ddot{z}_2 = w_2, \ddot{z}_3 = w_3,$$
$$\ddot{w}_1 = \frac{4\dot{w}_1\dot{z}_2 + 4\dot{w}_2\dot{z}_1 + 6w_1w_2}{z_1 - z_2} + \frac{4\dot{w}_1\dot{z}_3 + 4\dot{w}_3\dot{z}_1 + 6w_1w_3}{z_1 - z_3}$$
$$-\frac{1}{(z_1 - z_2)(z_1 - z_3)}\left\{12\left[w_1\dot{z}_2\dot{z}_3 + \dot{z}_1(w_2\dot{z}_3 + w_3\dot{z}_2)\right] + z_1^2 K_1(\vec{z}, \dot{\vec{z}}, \vec{w}, \dot{\vec{w}})\right.$$
$$\left. + z_1 K_2(\vec{z}, \dot{\vec{z}}, \vec{w}, \dot{\vec{w}}) + K_3(\vec{z}, \dot{\vec{z}}, \vec{w}, \dot{\vec{w}})\right\},$$
$$\ddot{w}_2 = \frac{4\dot{w}_2\dot{z}_1 + 4\dot{w}_1\dot{z}_2 + 6w_1w_2}{z_2 - z_1} + \frac{4\dot{w}_2\dot{z}_3 + 4\dot{w}_3\dot{z}_2 + 6w_2w_3}{z_2 - z_3}$$
$$-\frac{1}{(z_2 - z_1)(z_2 - z_3)}\left\{12\left[w_2\dot{z}_1\dot{z}_3 + \dot{z}_2(w_1\dot{z}_3 + w_3\dot{z}_1)\right] + z_2^2 K_1(\vec{z}, \dot{\vec{z}}, \vec{w}, \dot{\vec{w}})\right.$$
$$\left. + z_2 K_2(\vec{z}, \dot{\vec{z}}, \vec{w}, \dot{\vec{w}}) + K_3(\vec{z}, \dot{\vec{z}}, \vec{w}, \dot{\vec{w}})\right\},$$
$$\ddot{w}_3 = \frac{4\dot{w}_3\dot{z}_1 + 4\dot{w}_1\dot{z}_3 + 6w_1w_3}{z_3 - z_1} + \frac{4\dot{w}_3\dot{z}_2 + 4\dot{w}_2\dot{z}_3 + 6w_2w_3}{z_3 - z_2}$$
$$-\frac{1}{(z_3 - z_1)(z_3 - z_2)}\left\{12\left[w_3\dot{z}_1\dot{z}_2 + \dot{z}_3(w_1\dot{z}_2 + w_2\dot{z}_1)\right] + z_3^2 K_1(\vec{z}, \dot{\vec{z}}, \vec{w}, \dot{\vec{w}})\right.$$
$$\left. + z_3 K_2(\vec{z}, \dot{\vec{z}}, \vec{w}, \dot{\vec{w}}) + K_3(\vec{z}, \dot{\vec{z}}, \vec{w}, \dot{\vec{w}})\right\}, \tag{2.19a}$$

where

$$K_1(\vec{z}, \dot{\vec{z}}, \vec{w}, \dot{\vec{w}}) = -\Big[\alpha_1(\dot{w}_1 + \dot{w}_2 + \dot{w}_3) + \beta_1(w_1 + w_2 + w_3)$$
$$+\gamma_1(\dot{z}_1 + \dot{z}_2 + \dot{z}_3) + \delta_1(z_1 + z_2 + z_3)\Big], \tag{2.19b}$$

$$K_2(\vec{z}, \dot{\vec{z}}, \vec{w}, \dot{\vec{w}}) = \alpha_2\Big[\dot{w}_1(z_2 + z_3) + \dot{w}_2(z_1 + z_3) + \dot{w}_3(z_1 + z_2) + 2w_1(\dot{z}_2 + \dot{z}_3)$$
$$+2w_2(\dot{z}_1 + \dot{z}_3) + 2w_3(\dot{z}_1 + \dot{z}_2) + \dot{z}_1(w_2 + w_3) + \dot{z}_2(w_1 + w_3) + \dot{z}_3(w_1 + w_2)\Big]$$
$$+\beta_2\Big[w_1(z_2 + z_3) + w_2(z_1 + z_3) + w_3(z_1 + z_2) + \dot{z}_1(\dot{z}_2 + \dot{z}_3) + \dot{z}_2(\dot{z}_1 + \dot{z}_3)$$
$$+\dot{z}_3(\dot{z}_1 + \dot{z}_2)\Big] + \gamma_2\Big[\dot{z}_1(z_2 + z_3) + \dot{z}_2(z_1 + z_3) + \dot{z}_3(z_1 + z_2)\Big]$$
$$+\delta_2(z_1z_2 + z_1z_3 + z_2z_3), \tag{2.19c}$$

$$K_3(\vec{z},\dot{\vec{z}},\vec{w},\dot{\vec{w}}) = -\Big\{\alpha_3\Big[\dot{w}_1 z_2 z_3 + \dot{w}_2 z_1 z_3 + \dot{w}_3 z_1 z_2 + 2w_1(\dot{z}_2 z_3 + z_2\dot{z}_3)$$
$$+2w_2(\dot{z}_1 z_3 + z_1\dot{z}_3) + 2w_3(\dot{z}_1 z_2 + z_1\dot{z}_2) + \dot{z}_1(w_2 z_3 + 2\dot{z}_2\dot{z}_3 + z_2 w_3)$$
$$+\dot{z}_2(w_1 z_3 + 2\dot{z}_1\dot{z}_3 + z_1 w_3) + \dot{z}_3(w_1 z_2 + 2\dot{z}_1\dot{z}_2 + z_1 w_2)\Big]$$
$$+\beta_3\Big[w_1 z_2 z_3 + w_2 z_1 z_3 + w_3 z_1 z_2 + \dot{z}_1(\dot{z}_2 z_3 + z_2\dot{z}_3) + \dot{z}_2(\dot{z}_1 z_3 + z_1\dot{z}_3)$$
$$+\dot{z}_3(\dot{z}_1 z_2 + z_1\dot{z}_2)\Big] + \gamma_3\Big[\dot{z}_1 z_2 z_3 + z_1\dot{z}_2 z_3 + z_1 z_2\dot{z}_3\Big] + \delta_3 z_1 z_2 z_3\Big\}. \quad (2.19d)$$

In Appendix B we provide plots of the solutions of system (2.19) with the following values of the parameters $\alpha_m, \beta_m, \gamma_m, \delta_m$, $m = 1,2,3$,

$$\alpha_1 = 5\mathbf{i}, \beta_1 = 5, \gamma_1 = 5\mathbf{i}, \delta_1 = 6,$$
$$\alpha_2 = 4\mathbf{i}, \beta_2 = -1, \gamma_2 = 16\mathbf{i}, \delta_2 = 12,$$
$$\alpha_3 = 0, \beta_3 = -5, \gamma_3 = 0, \delta_3 = -4, \quad (2.20a)$$

and the initial conditions

$$z_1(0) = -1.45 + 1.1\mathbf{i}, \dot{z}_1(0) = 0.9,$$
$$z_2(0) = 5.1 + 0.8\mathbf{i}, \dot{z}_2(0) = 1.2,$$
$$z_3(0) = 2.5 - 0.2\mathbf{i}, \dot{z}_3(0) = -1.04,$$
$$w_1(0) = 1.23, \dot{w}_1(0) = 0.84\mathbf{i},$$
$$w_2(0) = -2.26\mathbf{i}, \dot{w}_2(0) = 2.16,$$
$$w_3(0) = 1.32\mathbf{i}, \dot{w}_3(0) = -1.12. \quad (2.20b)$$

For system (2.19), (2.20a), the characteristic equation (3.2) for $m = 1$ has the four roots $-\mathbf{i}, \mathbf{i}, 2\mathbf{i}, 3\mathbf{i}$, the characteristic equation (3.2) for $m = 2$ has the four roots $-2\mathbf{i}, \mathbf{i}, 2\mathbf{i}, 3\mathbf{i}$ and the characteristic equation (3.2) for $m = 3$ has the four roots $-2\mathbf{i}, -\mathbf{i}, \mathbf{i}, 2\mathbf{i},$. Therefore, by (ii) of **Remark 2.2,** system (2.19), (2.20a) is *isochronous*, see Figures 35, 36, 37, 38, 39, 40, 41, 42, 43, 44 in Appendix B.

3. New *solvable* dynamical systems and their solutions

In this section we indicate how to identify new *solvable* dynamical systems describing the motion in the complex z-plane of point-particles interacting among themselves with certain forces depending on their positions and velocities. Let us reiterate that a many-body model is considered *solvable* if the configuration of the system at any arbitrary time t can be obtained—from any given *initial* data: the *initial* positions and velocities of the N particles in the complex z-plane—by *algebraic* operations, such as finding the zeros of an *explicitly* known time-dependent polynomial.

Remark 3.1. Note however that knowledge of the configuration of the many-body system at time t, with the (generally complex) values of its coordinates given as the *unordered* set of the zeros of a known polynomial, does *not* allow to identify the specific coordinate that has evolved over time from the assignment of its specific initial position and velocity; this additional information can only be gained by following over time the evolution of the system, either by integrating numerically the equations of motion, or by identifying the configurations of the system at a sequence of time intervals sufficiently close to each other so as to guarantee the identification by *contiguity* of the trajectory of each particle (or at least of the specific particle under consideration). But these additional operations need not be performed with great accuracy, even when one wishes the final configuration—including the identity of each particle—to be known with much greater accuracy.

Likewise—in the case of systems which have been identified as *isochronous* because their solution is provided by the zeros of a time-dependent polynomial which is itself periodic in time with period, say, T—an analogous procedure must be followed to ascertain whether the period of the time evolution of a specific particle is T, or pT (with p a *positive integer*), due to the possibility of a T-periodic exchange of the correspondence between the zeros of the polynomial and the particle identities (for a general discussion of this possibility in a specific context see [8]). ■

The key formulas for the following developments are the identities (1.3), relating the time evolution of the *zeros* $z_n(t)$ of a time-dependent (monic) polynomial to that of the *coefficients* $c_m(t)$ of the same polynomial, as well as the relations (1.4) respectively (1.7) expressing the coefficients $c_m(t)$ of a monic polynomial respectively their time derivatives in terms of the *zeros* of the same polynomial and their time derivatives.

In this paper we restrict for simplicity attention to the case of a *linear decoupled* evolution of the coefficients $c_m(t)$, namely we assume that these N coefficients of the time-dependent polynomial (1.2) evolve in time according to the following system of ODEs,

$$\ddot{\ddot{c}}_m = \alpha_m \dddot{c}_m + \beta_m \ddot{c}_m + \gamma_m \dot{c}_m + \delta_m \; , \tag{3.1}$$

where the parameters $\alpha_m,\ \beta_m,\ \gamma_m,\ \delta_m$ are $4N$ *generic* complex numbers such that for each m, the characteristic equation

$$(\lambda_m)^4 = \alpha_m\ (\lambda_m)^3 + \beta_m\ (\lambda_m)^2 + \gamma_m\ \lambda_m + \delta_m \; , \tag{3.2}$$

has four *distinct* roots $\lambda_{m,k}, k = 1,2,3,4$ (see below). It is then plain that the *general solution* of this system reads as follows :

$$c_m(t) = \sum_{k=1}^{4} \left[b_m^{(k)}\ \exp(\lambda_{m,k}\, t) \right] \; . \tag{3.3a}$$

The $4N$ numbers $\lambda_{m,k}$, labeled by the 4 values of the index k, are denoted as follows:

$$\lambda_{m,k} = -a_m^{(k)} + \mathbf{i}\ \omega_m^{(k)} \; , \quad k = 1,2,3,4 \; , \tag{3.3b}$$

introducing thereby the $8N$ *real* parameters $a_m^{(1)}$, $a_m^{(2)}$, $a_m^{(3)}$, $a_m^{(4)}$, $\omega_m^{(1)}$, $\omega_m^{(2)}$, $\omega_m^{(3)}$, $\omega_m^{(4)}$, implying that the general solution (3.3a) can be equivalently written as (2.10).

Remark 3.2. The fourth-degree algebraic equations (3.2) could be *explicitly solved*, but the formulas expressing, for every value of the index m, the 4 exponents $\lambda_{m,k}$ in terms of the 4 parameters α_m, β_m, γ_m, δ_m are too complicated to be of much use. The converse formulas, expressing, for every value of the index m, the 4 parameters α_m, β_m, γ_m, δ_m in terms of the 4 exponents $\lambda_{m,k}$, or rather their real and imaginary parts, see (3.3b), are instead rather neat, see (2.9). ■

As for the $4N$ numbers $b_m^{(k)}$ in (3.3a), they are *a priori* arbitrary; but can of course be determined in terms of the initial data (thereby solving the *initial value problem* of the dynamical system (3.1)) by solving, for each of the N values of the index m, the following system of 4 *linear algebraic* equations,

$$\sum_{k=1}^{4} \left[b_m^{(k)} \ (\lambda_{m,k})^s \right] = \left. \frac{d^s c_m(t)}{dt^s} \right|_{t=0} , \quad s = 0,1,2,3 \ . \tag{3.3c}$$

Remark 3.3. It is plain that one could have considered, instead of the system of N *linear decoupled* ODEs (3.1), the more general system of N *linear coupled* ODEs

$$\ddddot{c}_m = \sum_{n=1}^{N} \left(A_{mn} \dddot{c}_n + B_{mn} \ddot{c}_n + C_{mn} \dot{c}_n + D_{mn} c_n \right) , \tag{3.4}$$

which is of course also *solvable* by algebraic operations, while featuring more arbitrary constants ($4N^2$ instead than $4N$). ■

The *solvable* character of the dynamical system characterized by the following N *coupled non-linear* ODEs to be satisfied by the N dependent variables $z_n \equiv z_n(t)$ is then clearly implied by the formula (1.3d):

$$\begin{aligned}
\ddddot{z}_n = & \sum_{\ell=1}^{N}{}' \left(\frac{4 \dddot{z}_n \dot{z}_\ell + 4 \dddot{z}_\ell \dot{z}_n + 6 \ddot{z}_n \ddot{z}_\ell}{z_n - z_\ell} \right) \\
& -6 \sum_{\ell_1,\ell_2=1}^{N}{}' \left[\frac{\ddot{z}_n \dot{z}_{\ell_1} \dot{z}_{\ell_2} + 2 \ddot{z}_{\ell_1} \dot{z}_n \dot{z}_{\ell_2}}{(z_n - z_{\ell_1}) \ (z_n - z_{\ell_2})} \right] \\
& +4 \sum_{\ell_1,\ell_2,\ell_3=1}^{N}{}' \left[\frac{\dot{z}_n \dot{z}_{\ell_1} \dot{z}_{\ell_2} \dot{z}_{\ell_3}}{(z_n - z_{\ell_1}) \ (z_n - z_{\ell_2}) \ (z_n - z_{\ell_3})} \right] - \left[\prod_{\ell=1, \ \ell \neq n}^{N} (z_n - z_\ell)^{-1} \right] \cdot \\
& \cdot \sum_{m=1}^{N} \left[\left(\alpha_m \dddot{c}_m + \beta_m \ddot{c}_m + \gamma_m \dot{c}_m + \delta_m c_m \right) \ (z_n)^{N-m} \right] .
\end{aligned} \tag{3.5}$$

In the last term the 4 quantities $\dddot{c}_m$, $\ddot{c}_m$, $\dot{c}_m$ and c_m must of course be expressed in terms of the dependent variables z_n and their time-derivatives by the formulas (1.7) and (1.4) (of course with (1.1)). Indeed the solution of this dynamical system—(3.5) with (1.7) and (1.4)—is clearly provided by the N zeros of the monic polynomial (1.2b) where the coefficients $c_m(t)$ are given by the formulas (2.10)—with the coefficients $b_m^{(k)}$ appearing there expressed, as indicated above before eq. (3.3c), in terms of the initial data $c_m(0)$, $\dot{c}_m(0)$, $\ddot{c}_m(0)$, $\dddot{c}_m(0)$, themselves expressed in terms of the initial data $z_n(0)$, $\dot{z}_n(0)$, $\ddot{z}_n(0)$, $\dddot{z}_n(0)$ via the formulas (1.4) and (1.7) at $t = 0$.

Remark 3.4. If in the (last term in the) right-hand side of (3.5) any one of the parameters α_m, β_m, γ_m, δ_m is independent of the index m, say $\beta_m = \beta$, then the corresponding term can be replaced by a simpler expression via the appropriate identity (1.3), implying, say,

$$\left[\prod_{\ell=1,\ \ell\neq n}^{N}(z_n-z_\ell)^{-1}\right]\ \sum_{m=1}^{N}\left[\beta\ \ddot{c}_m\ (z_n)^{N-m}\right]=\beta\ \left[\ddot{z}_n-\sum_{\ell=1}^{N}{}'\left(\frac{2\,\dot{z}_n\,\dot{z}_\ell}{z_n-z_\ell}\right)\right]\ , \qquad (3.6)$$

see (1.3b). ■

4. Outlook

The findings reported in this paper suggest further developments, which ourselves or others might pursue and report in future publications.

One direction of future research is the exploration of the *solvable* dynamical systems and many-body problems of Newtonian type that are obtained by *iterating* the type of approach described above, along the lines discussed in [4].

It would also be of interest to obtain generalizations of the identities (1.3) to derivatives of order higher than 4, indeed hopefully of arbitrary order. [**Note added in proof:** this problem has now been solved [9].]

And of course further explorations are appealing of the detailed behaviors of the solutions of the dynamical systems obtained via this approach, as well as—in the case of *solvable* many-body problems of Newtonian type allowing a Hamiltonian formulation—the exploration of their quantal versions.

Appendix A. Relations among the time derivatives of the zeros and the coefficients of a time-dependent polynomial

In this Appendix A we tersely outline for the convenience of the reader the proof of the 4 identities (1.3) relating the time evolution of the N zeros $z_n(t)$ of a time-dependent monic polynomial of degree N in the independent variable z to the time-evolution of its N coefficients $c_m(t)$, see (1.2). A proof of the first 2 of these 4 identities was already provided in [1], hence the first part of the following treatment reports almost *verbatim* that presentation.

The starting point to prove the relation (1.3a) are the two relations

$$\psi_t(z;t)=\sum_{m=1}^{N}\left[\dot{c}_m\ z^{N-m}\right]\ , \qquad (A.1a)$$

$$\psi_t(z;t)=-\sum_{m=1}^{N}\left[\dot{z}_m\prod_{\ell=1,\ \ell\neq m}^{N}(z-z_\ell)\right]\ , \qquad (A.1b)$$

which are obtained by time-differentiating (1.2b) respectively (1.2a). They imply the relation

$$\sum_{m=1}^{N}\left[\dot{z}_m\prod_{\ell=1,\ \ell\neq m}^{N}(z-z_\ell)\right]=-\sum_{m=1}^{N}\left[\dot{c}_m\ z^{N-m}\right]\ , \qquad (A.1c)$$

and it is plain that, for $z=z_n$, this formula yields (1.3a).

Likewise, an additional time-differentiation of (A.1a) yields

$$\psi_{tt}(z;t) = \sum_{m=1}^{N} \left(\ddot{c}_m \, z^{N-m} \right) \, , \tag{A.2a}$$

while an additional time-differentiation of (A.1b) yields

$$\begin{aligned} \psi_{tt}(z;t) = & - \sum_{m=1}^{N} \left\{ \ddot{z}_m \left[\prod_{\ell=1,\, \ell\neq m}^{N} (z - z_\ell) \right] \right\} \\ & + \sum_{\ell_1,\ell_2=1,\, \ell_1\neq\ell_2}^{N} \left\{ \dot{z}_{\ell_1} \, \dot{z}_{\ell_2} \left[\prod_{\ell'=1,\, \ell'\neq\ell_1,\ell_2}^{N} (z - z_{\ell'}) \right] \right\} \\ = & \sum_{m=1}^{N} \left(\ddot{c}_m \, z^{N-m} \right) \, , \end{aligned} \tag{A.2b}$$

where the second equality is implied by (A.2a). It is then again plain that, for $z = z_n$, one obtains (1.3b).

To obtain (1.3c) and (1.3d) we proceeded in an analogous manner: the calculations involved in the additional time-differentiations of (1.2b) are clearly trivial, while the successive time-differentiations of (1.2a) become progressively more complicated; but the two yielding (1.3c) and (1.3d) are still quite manageable by hand, so that their detailed treatment can be left to the diligent reader.

Appendix B. Plots of the Solutions of the Initial Value Problems in Examples 1 and 2 of Section 2

The Mathematica code used to obtain the plots in this section is included in the arXiv version of this paper available at http://arxiv.org/pdf/1601.04793v1.pdf.

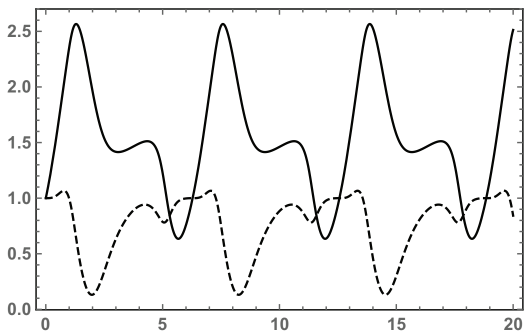

Fig. 1. Initial value problem (2.14), (2.15). Graphs of the real (bold curve) and imaginary (dashed curve) parts of the coordinate $z_1(t)$; period 2π.

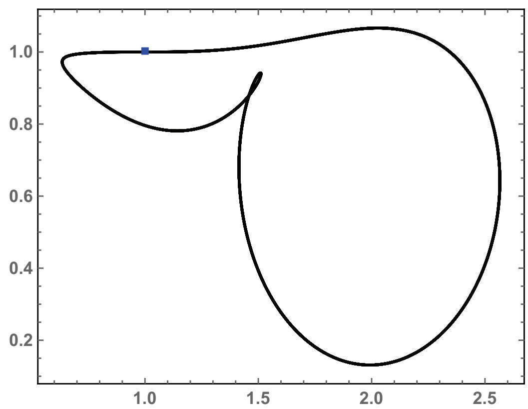

Fig. 2. Initial value problem (2.14), (2.15). Trajectory, in the complex z-plane, of $z_1(t)$; period 2π. The square indicates the initial condition $z_1(0) = 1 + \mathbf{i}$.

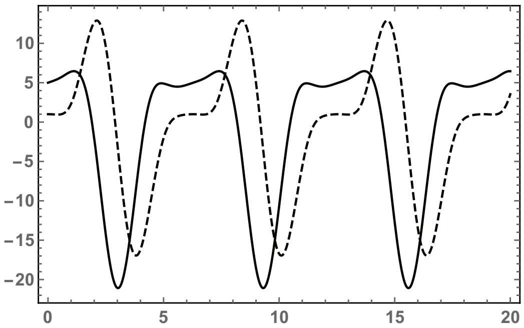

Fig. 3. Initial value problem (2.14), (2.15). Graphs of the real (bold curve) and imaginary (dashed curve) parts of the coordinate $z_2(t)$; period 2π.

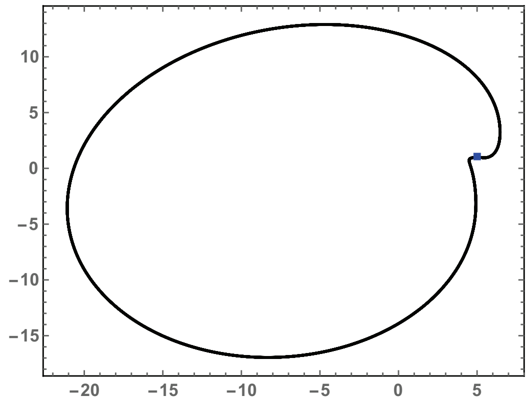

Fig. 4. Initial value problem (2.14), (2.15). Trajectory, in the complex z-plane, of $z_2(t)$; period 2π. The square indicates the initial condition $z_2(0) = 5 + \mathbf{i}$.

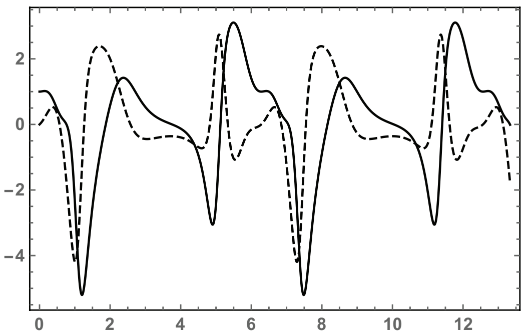

Fig. 5. Initial value problem (2.14), (2.15). Graphs of the real (bold curve) and imaginary (dashed curve) parts of the coordinate $w_1(t)$; period 2π.

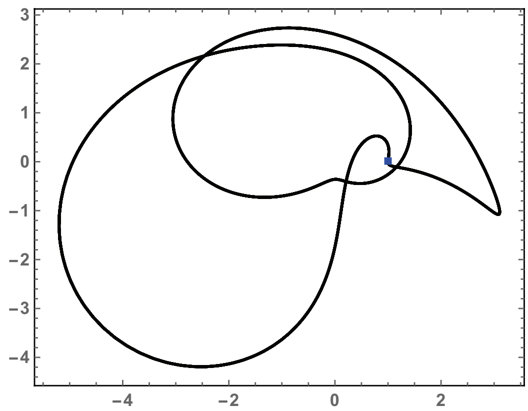

Fig. 6. Initial value problem (2.14), (2.15). Trajectory, in the complex z-plane, of $w_1(t)$; period 2π. The square indicates the initial condition $w_1(0) = 1$.

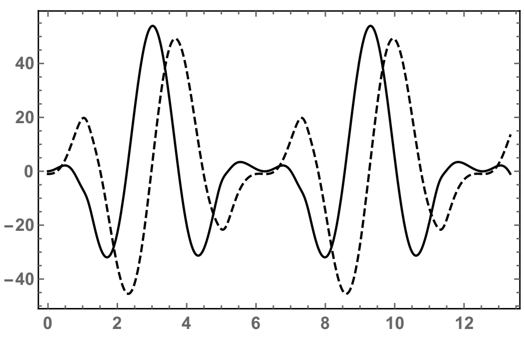

Fig. 7. Initial value problem (2.14), (2.15). Graphs of the real (bold curve) and imaginary (dashed curve) parts of the coordinate $w_2(t)$; period 2π.

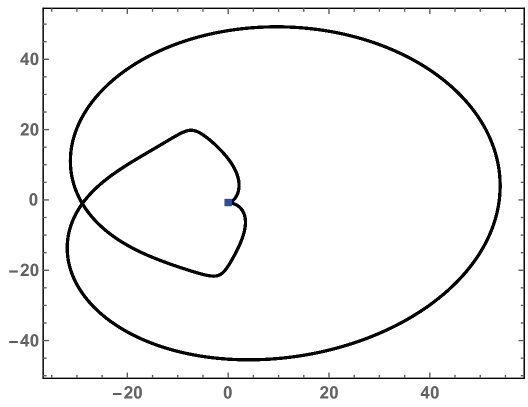

Fig. 8. Initial value problem (2.14), (2.15). Trajectory, in the complex z-plane, of $w_2(t)$; period 2π. The square indicates the initial condition $w_2(0) = -\mathbf{i}$.

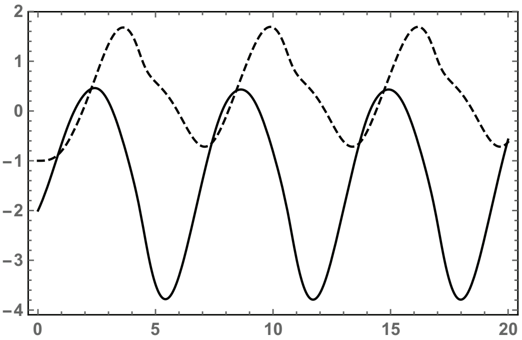

Fig. 9. Initial value problem (2.14), (2.16). Graphs of the real (bold curve) and imaginary (dashed curve) parts of the coordinate $z_1(t)$.

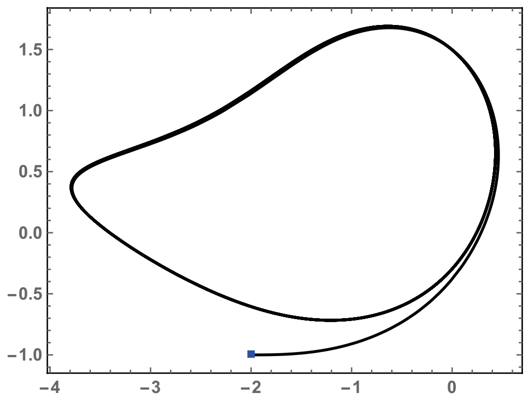

Fig. 10. Initial value problem (2.14), (2.16). Trajectory, in the complex z-plane, of $z_1(t)$. The square indicates the initial condition $z_1(0) = -2 - \mathbf{i}$.

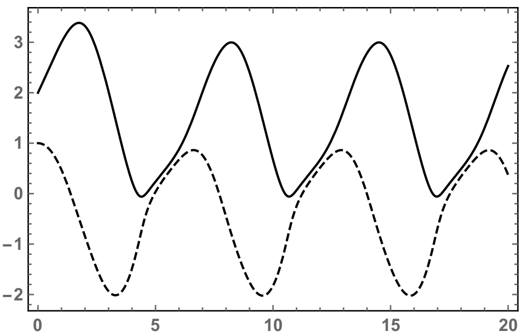

Fig. 11. Initial value problem (2.14), (2.16). Graphs of the real (bold curve) and imaginary (dashed curve) parts of the coordinate $z_2(t)$.

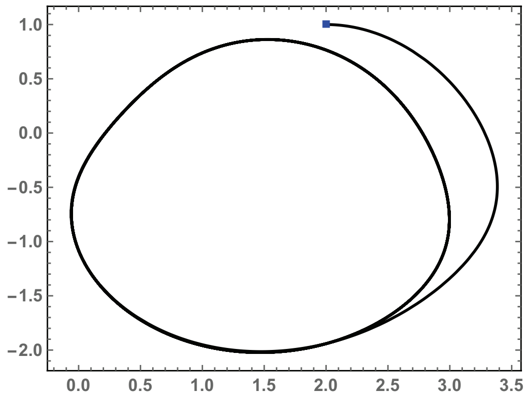

Fig. 12. Initial value problem (2.14), (2.16). Trajectory, in the complex z-plane, of $z_2(t)$. The square indicates the initial condition $z_2(0) = 2 + \mathbf{i}$.

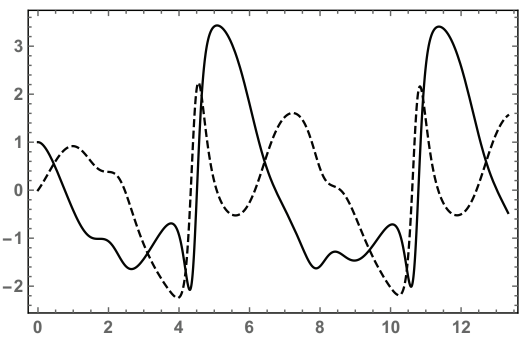

Fig. 13. Initial value problem (2.14), (2.16). Graphs of the real (bold curve) and imaginary (dashed curve) parts of the coordinate $w_1(t)$.

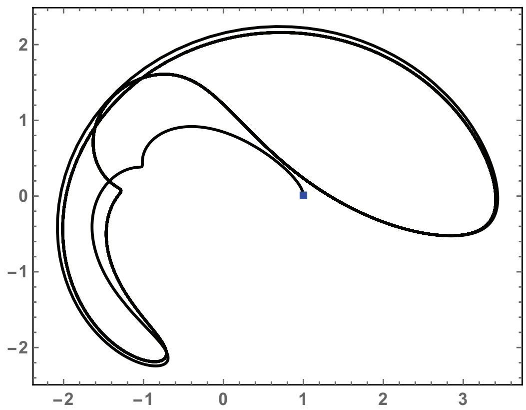

Fig. 14. Initial value problem (2.14), (2.16). Trajectory, in the complex z-plane, of $w_1(t)$. The square indicates the initial condition $w_1(0) = 1$.

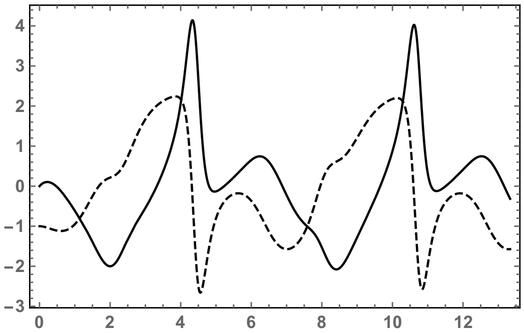

Fig. 15. Initial value problem (2.14), (2.16). Graphs of the real (bold curve) and imaginary (dashed curve) parts of the coordinate $w_2(t)$.

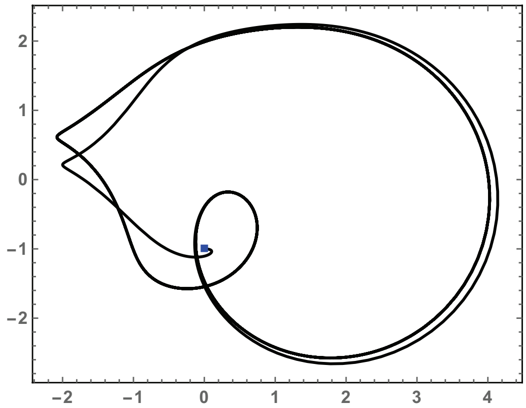

Fig. 16. Initial value problem (2.14), (2.16). Trajectory, in the complex z-plane, of $w_2(t)$. The square indicates the initial condition $w_2(0) = -\mathbf{i}$.

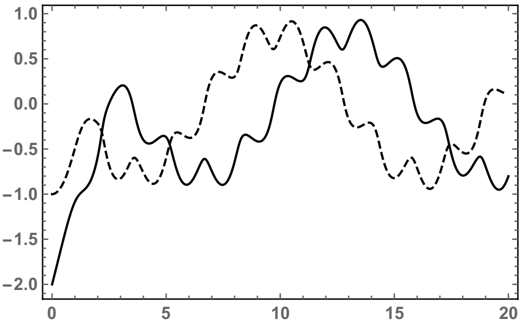

Fig. 17. Initial value problem (2.14), (2.17). Graphs of the real (bold curve) and imaginary (dashed curve) parts of the coordinate $z_1(t)$.

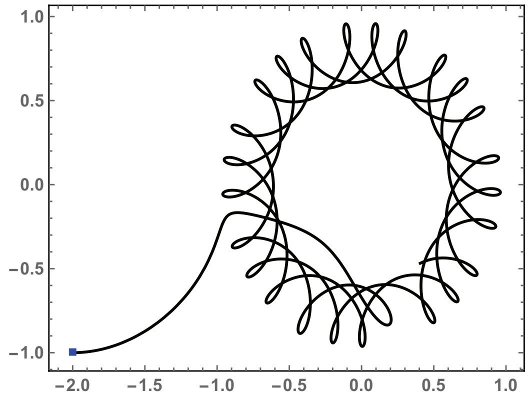

Fig. 18. Initial value problem (2.14), (2.17). Trajectory, in the complex z-plane, of $z_1(t)$. The square indicates the initial condition $z_1(0) = -2 - \mathbf{i}$.

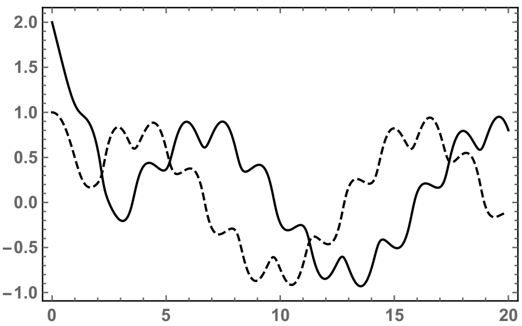

Fig. 19. Initial value problem (2.14), (2.17). Graphs of the real (bold curve) and imaginary (dashed curve) parts of the coordinate $z_2(t)$.

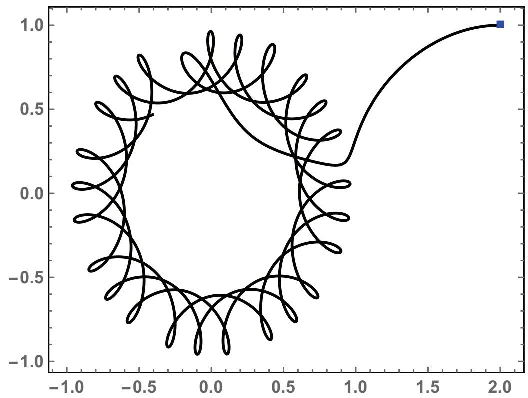

Fig. 20. Initial value problem (2.14), (2.17). Trajectory, in the complex z-plane, of $z_2(t)$. The square indicates the initial condition $z_2(0) = 2 + \mathbf{i}$.

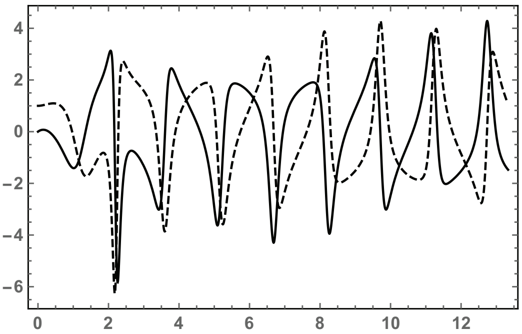

Fig. 21. Initial value problem (2.14), (2.17). Graphs of the real (bold curve) and imaginary (dashed curve) parts of the coordinate $w_1(t)$.

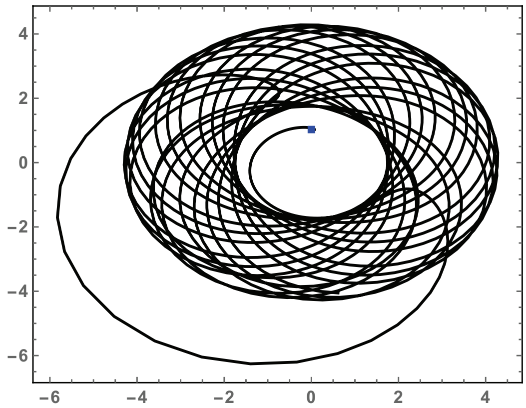

Fig. 22. Initial value problem (2.14), (2.17). Trajectory, in the complex z-plane, of $w_1(t)$. The square indicates the initial condition $w_1(0) = \mathbf{i}$.

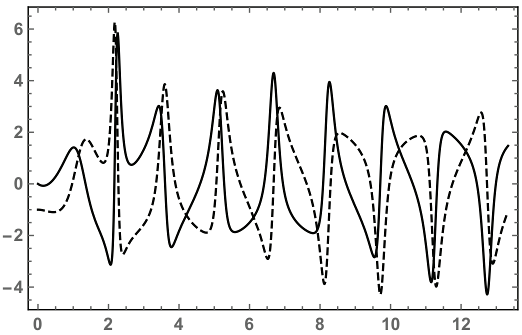

Fig. 23. Initial value problem (2.14), (2.17). Graphs of the real (bold curve) and imaginary (dashed curve) parts of the coordinate $w_2(t)$.

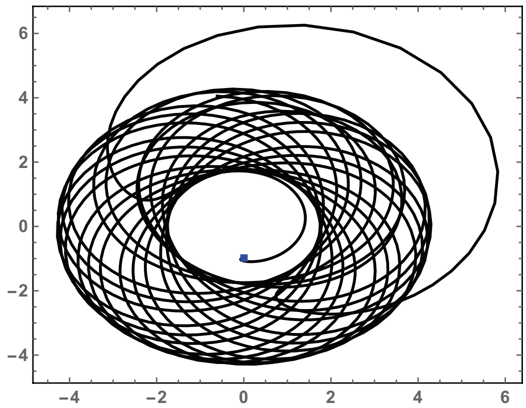

Fig. 24. Initial value problem (2.14), (2.17). Trajectory, in the complex z-plane, of $w_2(t)$. The square indicates the initial condition $w_2(0) = -\mathbf{i}$.

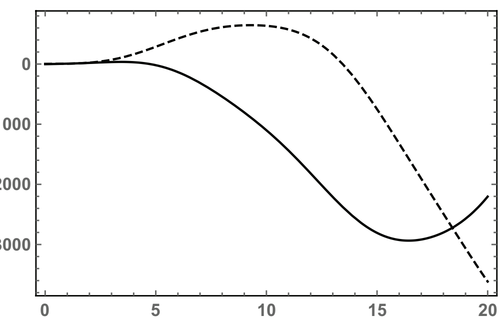

Fig. 25. Initial value problem (2.14), (2.18). Graphs of the real (bold curve) and imaginary (dashed curve) parts of the coordinate $z_1(t)$.

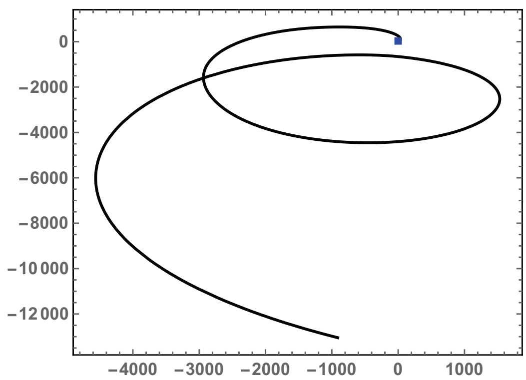

Fig. 26. Initial value problem (2.14), (2.18). Trajectory, in the complex z-plane, of $z_1(t)$. The square indicates the initial condition $z_1(0) = -2 + 3\mathbf{i}$.

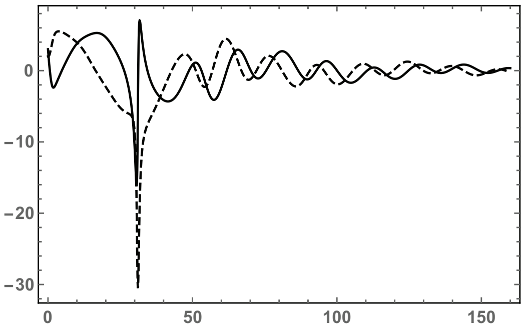

Fig. 27. Initial value problem (2.14), (2.18). Graphs of the real (bold curve) and imaginary (dashed curve) parts of the coordinate $z_2(t)$.

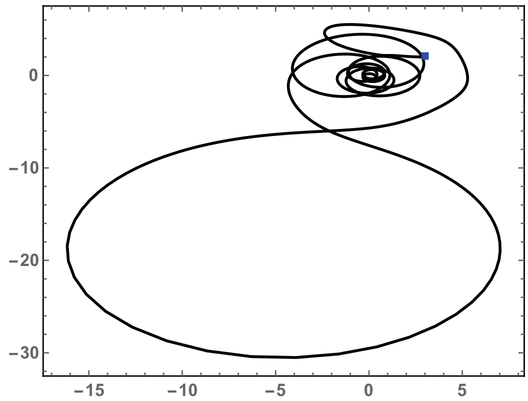

Fig. 28. Initial value problem (2.14), (2.18). Trajectory, in the complex z-plane, of $z_2(t)$. The square indicates the initial condition $z_2(0) = 3 + 2\mathbf{i}$.

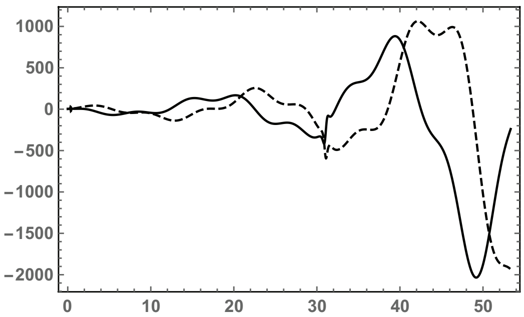

Fig. 29. Initial value problem (2.14), (2.18). Graphs of the real (bold curve) and imaginary (dashed curve) parts of the coordinate $w_1(t)$.

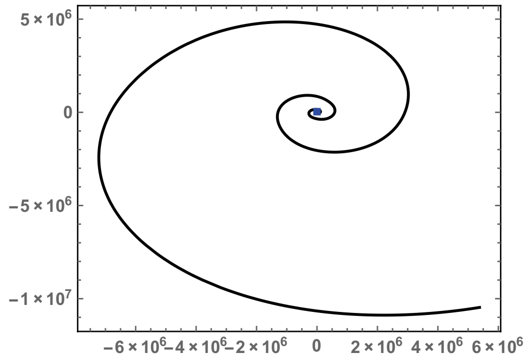

Fig. 30. Initial value problem (2.14), (2.18). Trajectory, in the complex z-plane, of $w_1(t)$. The square indicates the initial condition $w_1(0) = 2 + 4.2\mathbf{i}$.

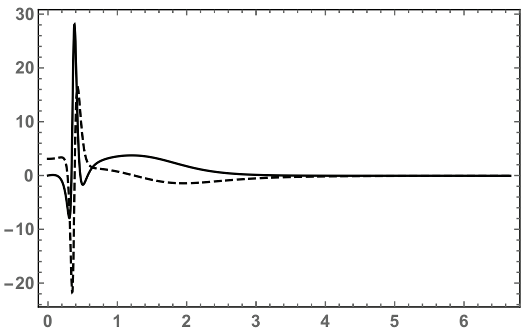

Fig. 31. Initial value problem (2.14), (2.18). Graphs of the real (bold curve) and imaginary (dashed curve) parts of the coordinate $w_2(t)$.

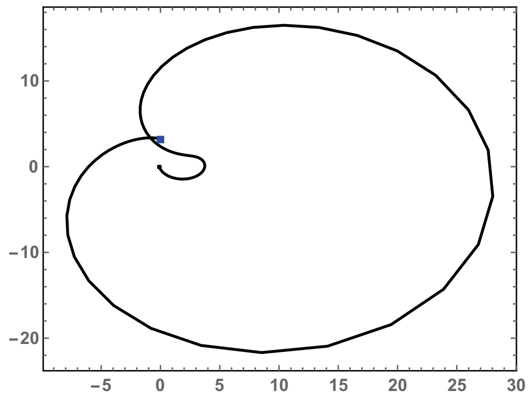

Fig. 32. Initial value problem (2.14), (2.18). Trajectory, in the complex z-plane, of $w_2(t)$. The square indicates the initial condition $w_2(0) = 3.1\mathbf{i}$.

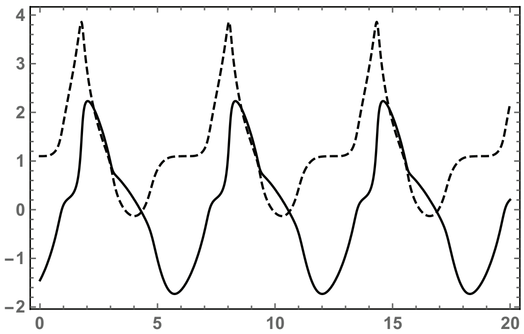

Fig. 33. Initial value problem (2.19), (2.20). Graphs of the real (bold curve) and imaginary (dashed curve) parts of the coordinate $z_1(t)$; period 2π.

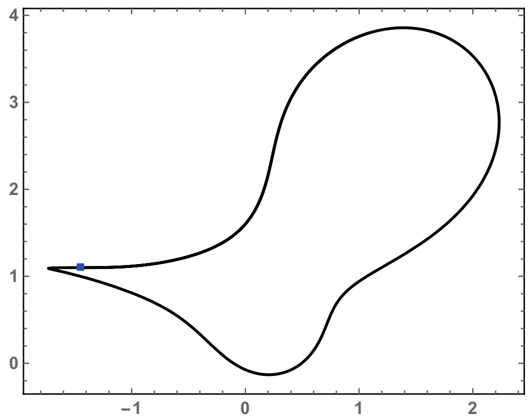

Fig. 34. Initial value problem (2.19), (2.20). Trajectory, in the complex z-plane, of $z_1(t)$; period 2π. The square indicates the initial condition $z_1(0) = -1.45 + 1.1\mathbf{i}$.

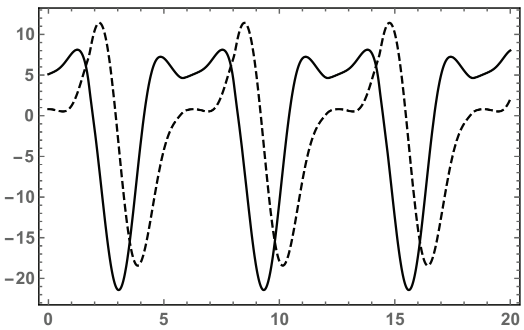

Fig. 35. Initial value problem (2.19), (2.20). Graphs of the real (bold curve) and imaginary (dashed curve) parts of the coordinate $z_2(t)$; period 4π.

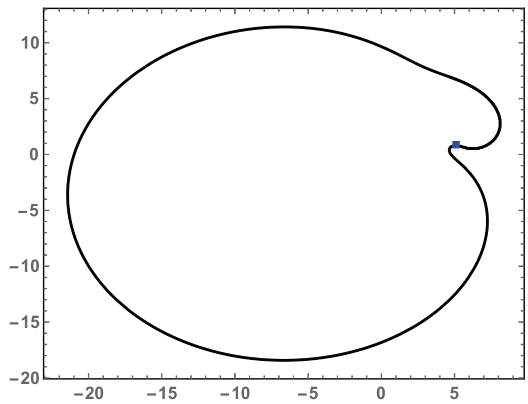

Fig. 36. Initial value problem (2.19), (2.20). Trajectory, in the complex z-plane, of $z_2(t)$; period 4π. The square indicates the initial condition $z_2(0) = 5.1 + 0.8\mathbf{i}$.

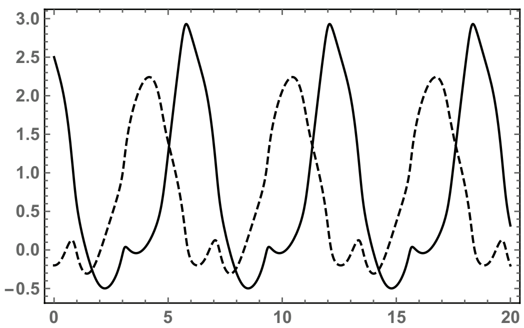

Fig. 37. Initial value problem (2.19), (2.20). Graphs of the real (bold curve) and imaginary (dashed curve) parts of the coordinate $z_3(t)$; period 4π.

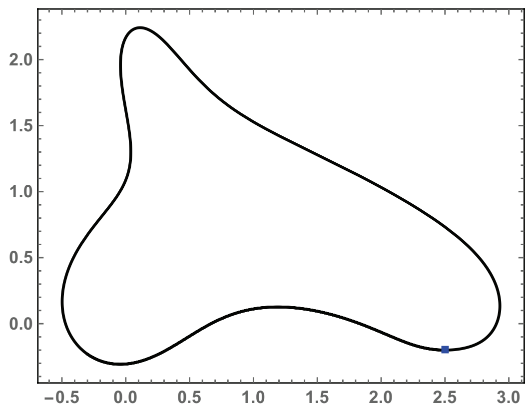

Fig. 38. Initial value problem (2.19), (2.20). Trajectory, in the complex z-plane, of $z_3(t)$; period 4π. The square indicates the initial condition $z_3(0) = 2.5 - 0.2\mathbf{i}$.

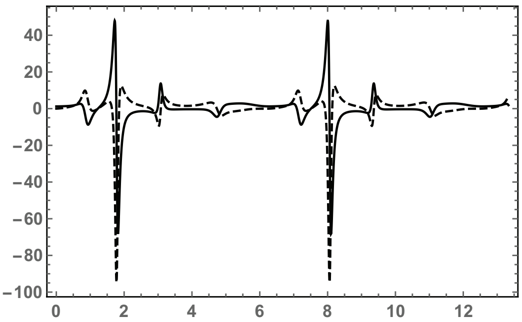

Fig. 39. Initial value problem (2.19), (2.20). Graphs of the real (bold curve) and imaginary (dashed curve) parts of the coordinate $w_1(t)$; period 2π.

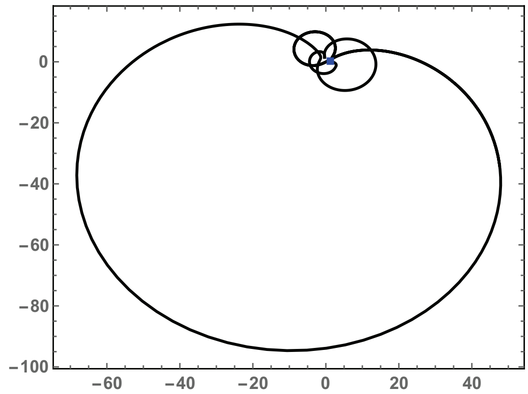

Fig. 40. Initial value problem (2.19), (2.20). Trajectory, in the complex z-plane, of $w_1(t)$; period 2π. . The square indicates the initial condition $w_1(0) = 1.23$.

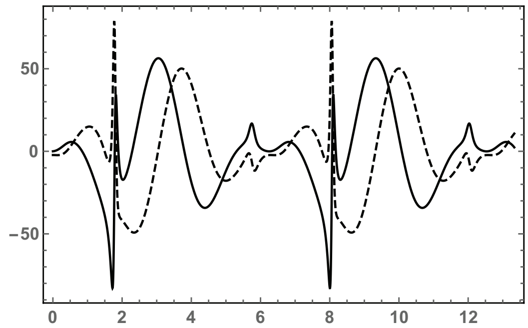

Fig. 41. Initial value problem (2.19), (2.20). Graphs of the real (bold curve) and imaginary (dashed curve) parts of the coordinate $w_2(t)$; period 4π.

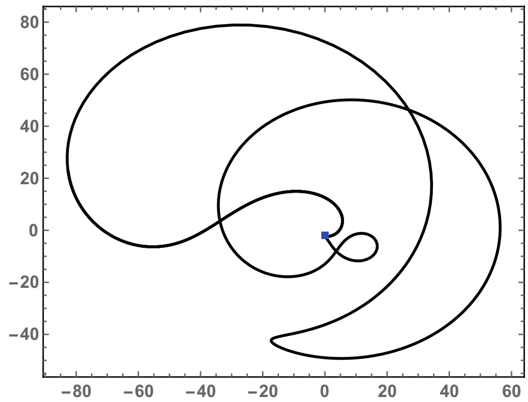

Fig. 42. Initial value problem (2.19), (2.20). Trajectory, in the complex z-plane, of $w_2(t)$; period 4π. The square indicates the initial condition $w_2(0) = -2.26\mathbf{i}$.

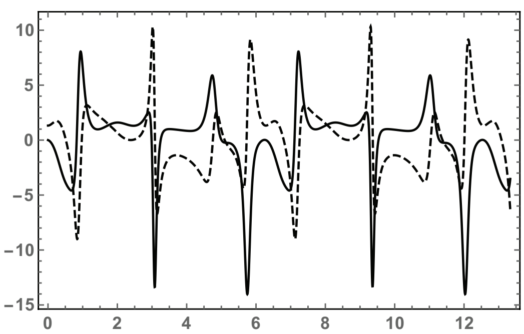

Fig. 43. Initial value problem (2.19), (2.20). Graphs of the real (bold curve) and imaginary (dashed curve) parts of the coordinate $w_3(t)$; period 4π.

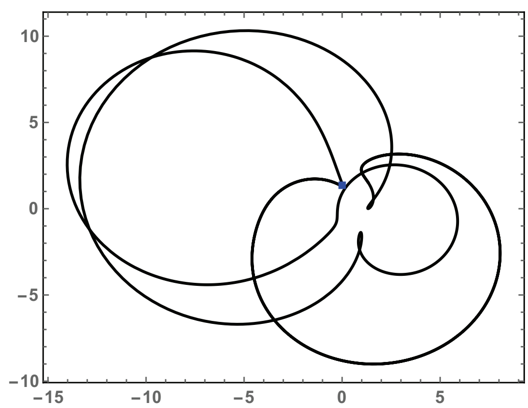

Fig. 44. Initial value problem (2.19), (2.20). Trajectory, in the complex z-plane, of $w_3(t)$; period 4π. The square indicates the initial condition $w_3(0) = 1.32\mathbf{i}$.

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