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Oksana Bihun, Francesco Calogero

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A new solvable many-body problem of goldfish type

Oksana Bihun

*Department of Mathematics, University of Colorado, Colorado Springs,
1420 Austin Bluffs Pkwy,
Colorado Springs, CO 80918, USA
obihun@uccs.edu*

Francesco Calogero

*Physics Department, University of Rome “La Sapienza”
p. Aldo Moro, I-00185 Roma, Italy
Istituto Nazionale di Fisica Nucleare, Sezione di Roma, Italy
francesco.calogero@roma1.infn.it, francesco.calogero@uniroma1.it*

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A new solvable many-body problem of goldfish type is introduced and the behavior of its solutions is tersely discussed.

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2000 Mathematics Subject Classification: 70F10, 70K42.

1. Introduction

Notation 1.1. Hereafter N is (unless otherwise indicated) an *arbitrary integer*, $N \geq 2$, the (generally *complex*) numbers $z_n \equiv z_n(t)$ are the dependent variables, t (“time”) is the independent variable, superimposed dots denote time-differentiations, and indices such as n, m, ℓ run over the integers from 1 to N unless otherwise indicated (see for instance below in (1.1) the limitation $\ell \neq n$ on the N values of ℓ). Below we often omit to indicate explicitly the t -dependence of various quantities, when this can be done without causing misunderstandings. Hereafter $N \times N$ matrices are denoted by upper-case **boldface** letters (so that, for instance, the matrix \mathbf{C} has the N^2 elements C_{nm}). Lower-case **boldface** letters stand for N -vectors (so that, for instance, the N -vector \mathbf{z} has the N components z_n); and the *imaginary unit* is denoted by \mathbf{i} (so that $\mathbf{i}^2 = -1$, and \mathbf{i} is of course *not* a N -vector!). We occasionally use the Kronecker symbol, with its standard definition: $\delta_{mn} = 1$ for $m = n$, $\delta_{mn} = 0$ for $m \neq n$. And let us mention the standard convention according to which an empty sum vanishes and an empty product equals unity, i. e. $\sum_{j=J}^K (f_j) = 0$, $\prod_{j=J}^K (f_j) = 1$ if $K < J$. ■

The prototypical “goldfish” many-body model [2] is characterized by the *translation-invariant* equations of motion

$$\ddot{z}_n = \mathbf{i} \omega \dot{z}_n + \sum_{\ell=1, \ell \neq n}^N \left(\frac{2 \dot{z}_n \dot{z}_\ell}{z_n - z_\ell} \right). \quad (1.1a)$$

A Hamiltonian producing these equations of motion reads as follows:

$$H(\zeta; \mathbf{z}) = \sum_{n=1}^N \left[\mathbf{i} \omega z_n + \exp(\zeta_n) \prod_{\ell=1, \ell \neq n}^N (z_n - z_\ell)^{-1} \right], \quad (1.1b)$$

where of course the N coordinates $\zeta_n \equiv \zeta_n(t)$ are the canonical momenta corresponding to the canonical particle coordinates $z_n \equiv z_n(t)$. The solution of the corresponding initial-values problem is provided by the N roots $z_n \equiv z_n(t)$ of the following, rather neat, *algebraic* equation in the variable z :

$$\sum_{\ell=1, \ell \neq n}^N \left[\frac{\dot{z}_\ell(0) + \mathbf{i} \omega z_\ell(0)}{z - z_\ell(0)} \right] = \frac{\mathbf{i} \omega}{\exp(\mathbf{i} \omega t) - 1}. \quad (1.1c)$$

(Note that this is actually a *polynomial* equation of degree N in z , as seen by multiplying it by $\prod_{m=1}^N [z - z_m(0)]$). Hence this model is *isochronous* (whenever the parameter ω is *positive*, as we generally assume hereafter; the special case $\omega = 0$ is “the” prototypical, *nonisochronous*, case...): all its solutions are *completely periodic*, with the period $T = 2\pi/\omega$ or, possibly, due to an exchange of the particle positions through the motion, with a period that is a (generally small with respect to its possible maximal value $N!$, see [6]) *integer multiple* of T .

Several *solvable* generalizations of the goldfish model, characterized by Newtonian equations of motion featuring additional forces besides those appearing in the right-hand side of (1.1a), are known: see for instance [1, 3–5] and references therein.

Remark 1.1. Above and hereafter we call a many-body model *solvable* if its initial value problem can be solved by *algebraic* operations, such as finding the N zeros of a known t -dependent *polynomial* of degree N (of course such an algebraic equation can be *explicitly* solved only for $N \leq 4$). ■

Recently a simple technique has been introduced [7], which allows to identify and investigate additional *solvable* models of goldfish type; and a few examples of such models yielded by this new approach have been identified and tersely discussed [7]. The model treated in this paper is another such new model, which is perhaps itself *interesting* (as all solvable models tend to be), and moreover is related to the properties of the *zeros* of (monic) polynomials of degree N the *coefficients* of which are the *zeros* of Hermite polynomials (see, for instance, [8]) of degree N .

2. The Model and its Solutions

The Newtonian equations of motion of the new many-body problem of goldfish type read as follows:

$$\ddot{z}_n = \sum_{\ell=1, \ell \neq n}^N \left(\frac{2 \dot{z}_n \dot{z}_\ell}{z_n - z_\ell} \right) - \left[\prod_{\ell=1, \ell \neq n}^N (z_n - z_\ell) \right]^{-1} \cdot \sum_{m=1}^N \left\{ (z_n)^{N-m} \left[-\omega^2 c_m + 2 \sum_{\ell=1, \ell \neq m}^N (c_m - c_\ell)^{-3} \right] \right\}, \quad (2.1a)$$

with

$$c_m = (-1)^m \sum_{1 \leq s_1 < s_2 < \dots < s_m \leq N} \left[\prod_{r=1}^m (z_{s_r}) \right]. \quad (2.1b)$$

Here and hereafter the symbol $\sum_{1 \leq s_1 < s_2 \dots < s_m \leq N}$ signifies the sum from 1 to N over the m (integer) indices s_j with $j = 1, \dots, m$ and the restriction $s_1 < s_2 < \dots < s_m$; of course this sum vanishes if $m > N$, consistently with **Notation 1.1**.

The solutions $z_n \equiv z_n(t)$ of this N -body problem are provided—consistently with the expressions (2.1b)—by the N zeros of the following t -dependent (monic) polynomial $\psi_N(z; t)$ of degree N in z :

$$\psi_N(z; t) = z^N + \sum_{m=1}^N [c_m(t) z^{N-m}] , \quad (2.2)$$

where the coefficients $c_m(t)$ are themselves the solutions of the system of N ODEs

$$\ddot{c}_m = -\omega^2 c_m + 2 \sum_{\ell=1, \ell \neq m}^N (c_m - c_\ell)^{-3} . \quad (2.3)$$

Because this is a well-known *solvable* model, the time-dependence of these N quantities $c_m(t)$ can be obtained by solving an *algebraic* (in fact *polynomial*) problem, indeed the solution of the initial-value problem of this dynamical system, (2.3), is provided by the following prescription (see for instance section 4.2.2 in [5] or [4, 9]): the N quantities $c_m \equiv c_m(t)$ are the N *eigenvalues* of the $N \times N$ (t -dependent) matrix

$$\mathbf{C}(t) = \mathbf{C}(0) \cos(\omega t) + \dot{\mathbf{C}}(0) \frac{\sin(\omega t)}{\omega} , \quad (2.4a)$$

with

$$\mathbf{C}(0) = \text{diag}[c_m(0)] , \quad (2.4b)$$

$$\dot{\mathbf{C}}(0) = \text{diag}[\dot{c}_m(0)] + \mathbf{i} [\mathbf{M}(0), \mathbf{C}(0)] , \quad (2.4c)$$

where of course (see (2.1b))

$$c_m(0) = (-1)^m \sum_{1 \leq s_1 < s_2 \dots < s_m \leq N} \left\{ \prod_{r=1}^m [z_{s_r}(0)] \right\} , \quad (2.4d)$$

$$\dot{c}_m(0) = (-1)^m \sum_{1 \leq s_1 < s_2 \dots < s_m \leq N} \sum_{q=1}^m \left\{ \dot{z}_{s_q}(0) \prod_{r=1, r \neq q}^m [z_{s_r}(0)] \right\} , \quad (2.4e)$$

and in the right-hand side of (2.4c)

$$[\mathbf{M}(0), \mathbf{C}(0)] \equiv \mathbf{M}(0) \mathbf{C}(0) - \mathbf{C}(0) \mathbf{M}(0) \quad (2.4f)$$

with the matrix $\mathbf{M}(0)$ defined componentwise in terms of the initial data $z_n(0)$ as follows:

$$\begin{aligned} M_{nm}(0) &= -[z_n(0) - z_m(0)]^{-2} , \quad n \neq m , \\ M_{nn}(0) &= - \sum_{\ell=1, \ell \neq n}^N M_{n\ell}(0) = \sum_{\ell=1, \ell \neq n}^N [z_n(0) - z_\ell(0)]^{-2} . \end{aligned} \quad (2.4g)$$

Note that these formulas provide an explicit definition of the N time-dependent coefficients $c_m(t)$ in terms of the initial data $z_n(0)$, $\dot{z}_n(0)$ of the N -body problem of goldfish type characterized

by the Newtonian equations of motion (2.1), via algebraic operations, amounting essentially to the solution of polynomial equations of degree N ; and that the values $z_n(t)$ at time t of the particle coordinates z_n are then provided by the N zeros of the polynomial $\psi_N(z; t)$, explicitly known (see (2.2)) in terms of its N coefficients $c_m(t)$. It is thereby demonstrated that the N -body problem of goldfish type characterized by the Newtonian equations of motion (2.1) is *solvable* (see **Remark 1.1**).

Remark 2.1. Let us call attention to a (well known) tricky point associated with the solution—as described above—of the N -body problem of goldfish type characterized by the Newtonian equations of motion (2.1). The identification of the N eigenvalues of a given matrix is only *unique up to permutations*, and likewise the identification of the zeros of a polynomial is only *unique up to permutations*. Therefore the N coordinates $z_n = z_n(t)$ yielded by the solution detailed above are only identified *up to permutations of their N labels n* . The (only) way to identify a *specific* coordinate—say, the coordinate $z_1(t)$ that corresponds to the initial data $z_1(0)$, $\dot{z}_1(0)$ —is by following the (*continuous*) time evolution of the coordinate $z_1(t)$ from its (assigned) initial value $z_1(0)$ to its value $z_1(t)$ at time t . In this manner one arrives at the *uniquely* defined value of the coordinate $z_1(t)$ corresponding to the initial value $z_1(0)$, which coincides of course with that *uniquely* yielded by the time evolution of the N -body problem (2.1). An analogous phenomenology is also relevant to the solution of model (1.1a). ■

Because of the way system (2.1) is constructed, its equilibria can be obtained by finding the *zeros* of the polynomials whose *coefficients* are equilibria of system (2.3). On the other hand, it is known that the *zeros* of the N -th degree Hermite polynomial are equilibria of system (2.3) (see for instance [4]). This relationship allows to study the properties of the polynomials whose *coefficients* are the *zeros* of Hermite polynomials.

The findings reported above imply the possibility to display in *completely explicit* form the solution of the N -body problem of goldfish type characterized by the Newtonian equations of motion (2.1) for $N = 2, 3, 4$; but we doubt this would be very illuminating, and we therefore leave this task to the eager reader. We rather like to emphasize that these findings imply that, for arbitrary N —and arbitrary positive ω —this N -body problem is *isochronous*, all its solutions satisfying the periodicity property

$$z_n(t + T) = z_n(t) , \quad (2.5)$$

with $T = 2\pi/\omega$ (or, possibly, the period T might be replaced by an integer multiple of $2\pi/\omega$, generally small with respect to its possible maximal value $N!$, which is of course a quite large number if N is large: see [6]). We end this paper by displaying a few examples of this phenomenology.

Example 2.1. For $N = 2$ and $\omega = 1$, taking into account that $c_1 = -z_1 - z_2$ and $c_2 = z_1 z_2$, we see that system (2.1) reduces to

$$\begin{aligned} \ddot{z}_1 &= \frac{2\dot{z}_1\dot{z}_2}{z_1 - z_2} - \frac{1}{(z_1 - z_2)} \left[z_1^2 - \frac{2(z_1 - 1)}{(z_1 + z_2 + z_1 z_2)^3} \right], \\ \ddot{z}_2 &= -\frac{2\dot{z}_1\dot{z}_2}{z_1 - z_2} + \frac{1}{(z_1 - z_2)} \left[z_2^2 - \frac{2(z_2 - 1)}{(z_1 + z_2 + z_1 z_2)^3} \right]. \end{aligned} \quad (2.6)$$

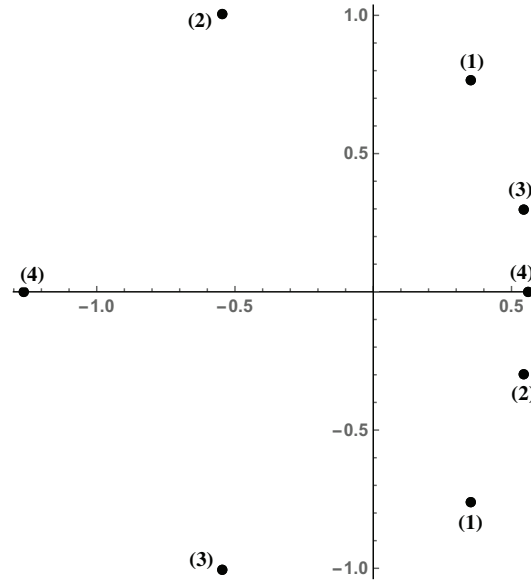


Fig. 1. Equilibria of system (2.6) in the complex z -plane. Each equilibrium is represented by the two points $\hat{z}_1^{(j)}$ and $\hat{z}_2^{(j)}$ labeled by the index $j = 1, 2, 3, 4$, see the display (2.6 Equilibria).

The assignment $\omega = 1$ is motivated by the fact that this quantity, ω , can be eliminated from system (1.1a) by a *constant* rescaling of the dependent variables $z_n(t)$ and of the independent variable t .

System (2.6) has 4 equilibrium configurations $(\hat{z}_1^{(j)}, \hat{z}_2^{(j)})$, $j = 1, 2, 3, 4$, up to the exchange of $\hat{z}_1^{(j)}$ with $\hat{z}_2^{(j)}$, whose approximate numerical values are given below:

$$\begin{aligned} (\hat{z}_1^{(1)}, \hat{z}_2^{(1)}) &= (0.353553 - 0.762959 \mathbf{i}, 0.353553 + 0.762959 \mathbf{i}), \\ (\hat{z}_1^{(2)}, \hat{z}_2^{(2)}) &= (-0.54455 + 1.00281 \mathbf{i}, 0.54455 - 0.295704 \mathbf{i}), \\ (\hat{z}_1^{(3)}, \hat{z}_2^{(3)}) &= (-0.54455 - 1.00281 \mathbf{i}, 0.54455 + 0.295704 \mathbf{i}), \\ (\hat{z}_1^{(4)}, \hat{z}_2^{(4)}) &= (-1.26575, 0.558645). \end{aligned} \quad (2.6 \text{ Equilibria})$$

These equilibria can be obtained either by the substitution $z_n(t) = \hat{z}_n$, $n = 1, 2$ into system (2.6) and the subsequent solution of the resulting system of algebraic equations for \hat{z}_1, \hat{z}_2 , or by finding the zeros of the polynomials whose coefficients are the equilibria of system (2.3) for $N = 2$. We note that, in this case where $N = 2$ and $\omega = 1$, system (2.3) has the four equilibria $(\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}})$, $(-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}})$, $(\frac{\mathbf{i}}{\sqrt{2}}, -\frac{\mathbf{i}}{\sqrt{2}})$, and $(-\frac{\mathbf{i}}{\sqrt{2}}, \frac{\mathbf{i}}{\sqrt{2}})$. Two of them are the zeros $\pm \frac{1}{\sqrt{2}}$ of the Hermite polynomial $H_2(c) = 4c^2 - 2$, which is consistent with the known fact that the zeros of the N -th order Hermite polynomial are equilibria of system (2.3).

In Fig. 1 each equilibrium of system (2.6) is represented by the two points $\hat{z}_1^{(j)}$ and $\hat{z}_2^{(j)}$, labeled by (j) , where $j = 1, 2, 3, 4$.

Below we provide *graphs* of the real and imaginary parts of the components of the solution $(z_1(t), z_2(t))$ of system (2.6) as functions of time, and some *trajectories* of the particles z_1 and z_2 in the complex z -plane. These graphs have been obtained by numerical integration of system (2.6), using Mathematica 10. We employed the command NDSolve with the automatic choice of the method and the accuracy. The program script is provided in the Appendix.

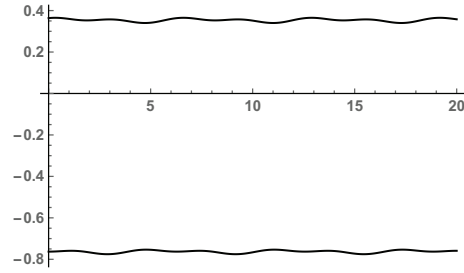


Fig. 2. System (2.6), initial conditions (2.6a). Graphs of the real (upper curve) and imaginary (lower curve) parts of the coordinate $z_1(t)$; period 2π .

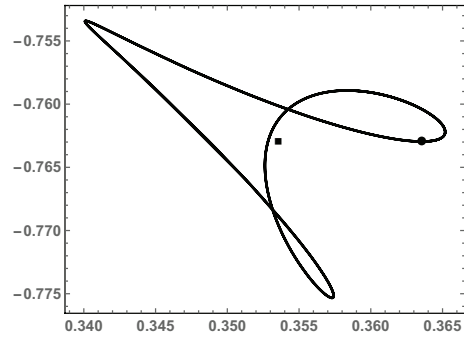


Fig. 3. System (2.6), initial conditions (2.6a). Trajectory, in the complex z -plane, of $z_1(t)$. The square respectively the dot indicate the positions of $\hat{z}_1^{(1)}$ respectively $z_1(0)$, i.e. of a nearby equilibrium point respectively the initial value of $z_1(t)$; period 2π .

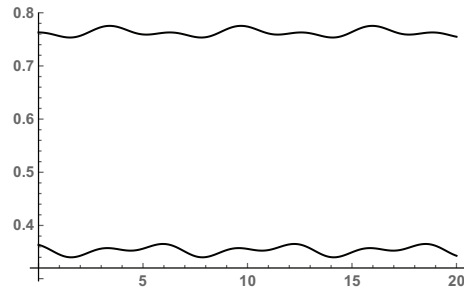


Fig. 4. System (2.6), initial conditions (2.6a). Graphs of the real (lower curve) and the imaginary (upper curve) parts of the coordinate $z_2(t)$; period 2π .

(2.6a) Initial conditions $z_1(0) = \hat{z}_1^{(1)} + 0.01$, $z_2(0) = \hat{z}_2^{(1)} + 0.01$,
 $\dot{z}_1(0) = 0.01$, $\dot{z}_2(0) = -0.01$. See Fig. 2, 3, 4, 5.

(2.6b) Initial conditions $z_1(0) = \hat{z}_1^{(1)} + 0.01 + 0.01 \mathbf{i}$, $z_2(0) = \hat{z}_2^{(1)} + 0.01 + 0.01 \mathbf{i}$,
 $\dot{z}_1(0) = -0.01$, $\dot{z}_2(0) = 0.01$. See Fig. 6, 7.

(2.6c) Initial conditions $z_1(0) = \hat{z}_1^{(2)} + 0.01 + 0.01 \mathbf{i}$, $z_2(0) = \hat{z}_2^{(2)} + 0.01 + 0.01 \mathbf{i}$,
 $\dot{z}_1(0) = -0.01$, $\dot{z}_2(0) = 0.01$. See Fig. 8, 9.

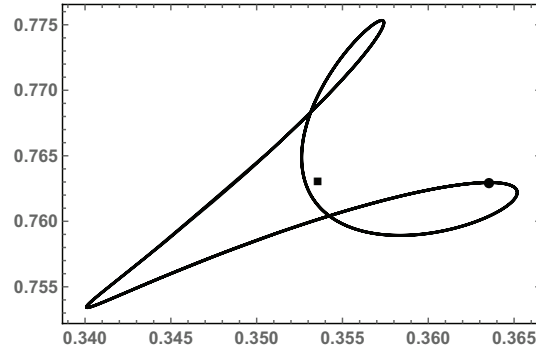


Fig. 5. System (2.6), initial conditions (2.6a). Trajectory, in the complex z -plane, of $z_2(t)$. The square respectively the dot indicate the positions of $\hat{z}_2^{(1)}$ respectively $z_2(0)$, i.e. of a nearby equilibrium point respectively the initial value of $z_2(t)$; period 2π .

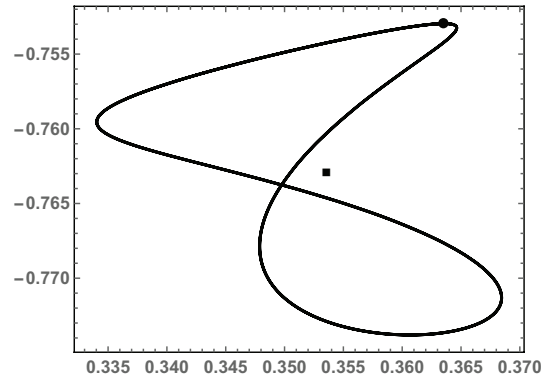


Fig. 6. System (2.6), initial conditions (2.6b). Trajectory, in the complex z -plane, of $z_1(t)$. The square respectively the dot indicate the positions of $\hat{z}_1^{(1)}$ respectively $z_1(0)$, i.e. of a nearby equilibrium point respectively the initial value of $z_1(t)$; period 2π .

(2.6d) Initial conditions $z_1(0) = \hat{z}_1^{(3)} + 0.01 + 0.01 \mathbf{i}$, $z_2(0) = \hat{z}_2^{(3)} + 0.01 + 0.01 \mathbf{i}$, $\dot{z}_1(0) = -0.005$, $\dot{z}_2(0) = 0.005$. See Fig. 10 and 11.

(2.6e) Initial conditions $z_1(0) = \hat{z}_1^{(4)} + 0.01 + 0.01 \mathbf{i}$, $z_2(0) = \hat{z}_2^{(4)} + 0.01 + 0.01 \mathbf{i}$, $\dot{z}_1(0) = -0.01$, $\dot{z}_2(0) = 0.01$. See Fig. 10 and 11.

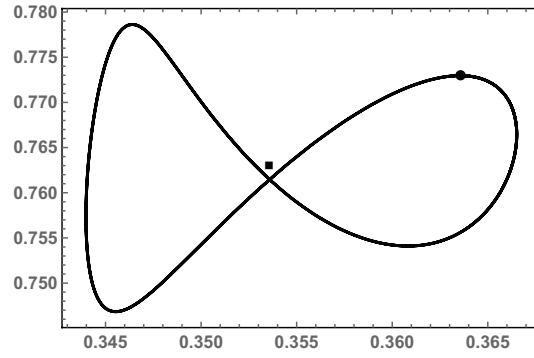


Fig. 7. System (2.6), initial conditions (2.6b). Trajectory, in the complex z -plane, of $z_2(t)$. The square respectively the dot indicate the positions of $\hat{z}_2^{(1)}$ respectively $z_2(0)$, i.e. of a nearby equilibrium point respectively the initial value of $z_2(t)$; period 2π .

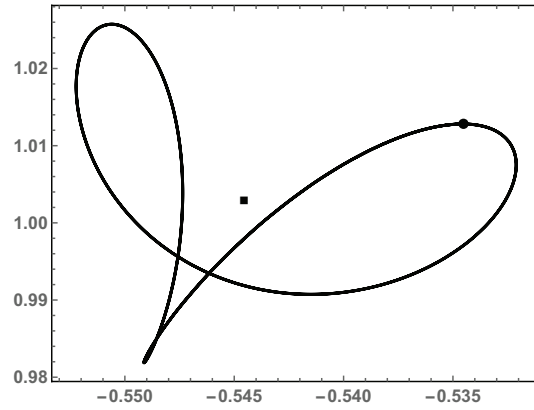


Fig. 8. System (2.6), initial conditions (2.6c). Trajectory, in the complex z -plane, of $z_1(t)$. The square respectively the dot indicate the positions of $\hat{z}_1^{(2)}$ respectively $z_1(0)$, i.e. of a nearby equilibrium point respectively the initial value of $z_1(t)$; period 2π .

Example 2.2. For $N = 3$ and $\omega = 1$, taking into account that $c_1 = -z_1 - z_2 - z_3$, $c_2 = z_1 z_2 + z_1 z_3 + z_2 z_3$, and $c_3 = -z_1 z_2 z_3$, we see that system (2.1) reduces to

$$\begin{aligned} \ddot{z}_1 &= \frac{2\dot{z}_1\dot{z}_2}{z_1 - z_2} + \frac{2\dot{z}_1\dot{z}_3}{z_1 - z_3} \\ &\quad - \frac{1}{(z_1 - z_2)(z_1 - z_3)} [z_1^2 F_1(z_1, z_2, z_3) + z_1 F_2(z_1, z_2, z_3) + F_3(z_1, z_2, z_3)], \\ \ddot{z}_2 &= -\frac{2\dot{z}_1\dot{z}_2}{z_1 - z_2} + \frac{2\dot{z}_2\dot{z}_3}{z_2 - z_3} \\ &\quad + \frac{1}{(z_1 - z_2)(z_2 - z_3)} [z_2^2 F_1(z_1, z_2, z_3) + z_2 F_2(z_1, z_2, z_3) + F_3(z_1, z_2, z_3)], \\ \ddot{z}_3 &= -\frac{2\dot{z}_1\dot{z}_3}{z_1 - z_3} - \frac{2\dot{z}_2\dot{z}_3}{z_2 - z_3} \\ &\quad - \frac{1}{(z_1 - z_3)(z_2 - z_3)} [z_3^2 F_1(z_1, z_2, z_3) + z_3 F_2(z_1, z_2, z_3) + F_3(z_1, z_2, z_3)], \end{aligned} \quad (2.7a)$$

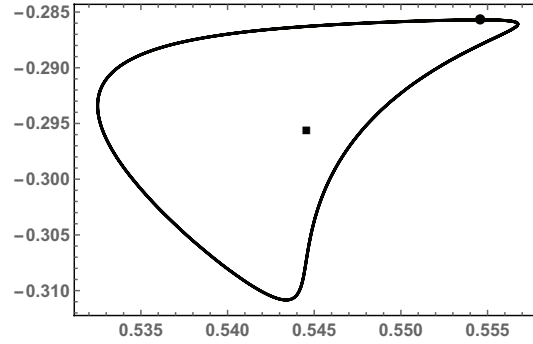


Fig. 9. System (2.6), initial conditions (2.6c). Trajectory, in the complex z -plane, of $z_2(t)$. The square respectively the dot indicate the positions of $\hat{z}_2^{(2)}$ respectively $z_2(0)$, i.e. of a nearby equilibrium point respectively the initial value of $z_2(t)$; period 2π .

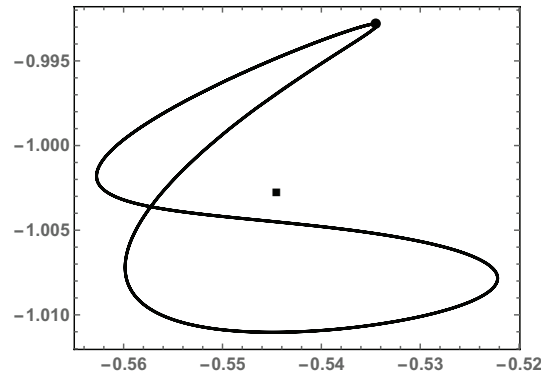


Fig. 10. System (2.6), initial conditions (2.6d). Trajectory, in the complex z -plane, of $z_1(t)$. The square respectively the dot indicate the positions of $\hat{z}_1^{(3)}$ respectively $z_1(0)$, i.e. of a nearby equilibrium point respectively the initial value of $z_1(t)$; period 2π .

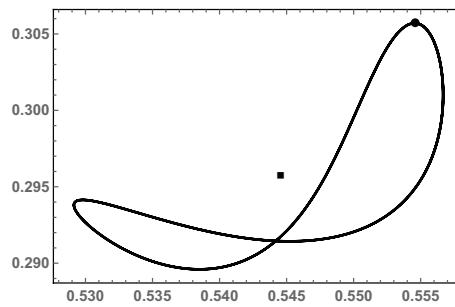


Fig. 11. System (2.6), initial conditions (2.6d). Trajectory, in the complex z -plane, of $z_2(t)$. The square respectively the dot indicate the positions of $\hat{z}_2^{(3)}$ respectively $z_2(0)$ of a nearby equilibrium point respectively the initial value of $z_2(t)$; period 2π .

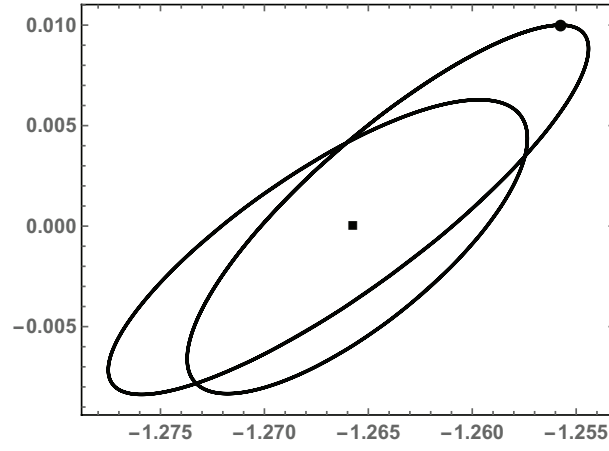


Fig. 12. System (2.6), initial conditions (2.6e). Trajectory, in the complex z -plane, of $z_1(t)$. The square respectively the dot indicate the positions of $\hat{z}_1^{(4)}$ respectively $z_1(0)$, i.e. of a nearby equilibrium point respectively the initial value of $z_1(t)$; period 2π .

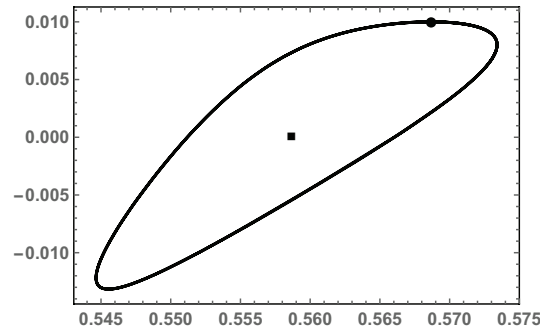


Fig. 13. System (2.6), initial conditions (2.6e). Trajectory, in the complex z -plane, of $z_2(t)$. The square respectively the dot indicate the positions of $\hat{z}_2^{(4)}$ respectively $z_2(0)$, i.e. of a nearby equilibrium point respectively the initial value of $z_2(t)$; period 2π .

where

$$\begin{aligned}
 F_1(z_1, z_2, z_3) &= z_1 + z_2 + z_3 - \frac{2}{(z_1 + z_2 + z_3 - z_1 z_2 z_3)^3} \\
 &\quad - \frac{2}{(z_1 + z_2 + z_3 + z_1 z_2 + z_1 z_3 + z_2 z_3)^3}, \\
 F_2(z_1, z_2, z_3) &= -z_1 z_2 - z_1 z_3 - z_2 z_3 \\
 &\quad + \frac{2}{(z_1 + z_2 + z_3 + z_1 z_2 + z_1 z_3 + z_2 z_3)^3} + \frac{2}{(z_1 z_2 + z_1 z_3 + z_2 z_3 + z_1 z_2 z_3)^3}, \\
 F_3(z_1, z_2, z_3) &= z_1 z_2 z_3 \\
 &\quad + \frac{2}{(z_1 + z_2 + z_3 - z_1 z_2 z_3)^3} - \frac{2}{(z_1 z_2 + z_1 z_3 + z_2 z_3 + z_1 z_2 z_3)^3}.
 \end{aligned} \tag{2.7b}$$

We obtained equilibria of system (2.7a), (2.7b) as follows. First, we found the equilibria of system (2.3) for $N = 3$; they are given by $(0, \sqrt{\frac{3}{2}}, -\sqrt{\frac{3}{2}})$ and $(0, \mathbf{i}\sqrt{\frac{3}{2}}, -\mathbf{i}\sqrt{\frac{3}{2}})$, up to the permutations of the three coordinates. Second, we found the zeros of the monic polynomials whose coefficients are the equilibria of system (2.3) for $N = 3$. These zeros are equilibrium solutions of system (2.1) because of how this system is constructed. Therefore, system (2.7a), (2.7b) has at least 12 equilibrium configurations $(\hat{z}_1^{(j)}, \hat{z}_2^{(j)}, \hat{z}_3^{(j)})$, $j = 1, 2, \dots, 12$, up to the permutations of $\hat{z}_1^{(j)}$, $\hat{z}_2^{(j)}$ and $\hat{z}_3^{(j)}$, whose approximate numerical values are given below:

$$\begin{aligned}
 (\hat{z}_1^{(1)}, \hat{z}_2^{(1)}, \hat{z}_3^{(1)}) &= (0.720239 - 0.575751 \mathbf{i}, 0.720239 + 0.575751 \mathbf{i}, -1.44048), \\
 (\hat{z}_1^{(2)}, \hat{z}_2^{(2)}, \hat{z}_3^{(2)}) &= (0.397225 + 1.07661 \mathbf{i}, -1.12106 - 0.854451 \mathbf{i}, 0.72384 - 0.222154 \mathbf{i}), \\
 (\hat{z}_1^{(3)}, \hat{z}_2^{(3)}, \hat{z}_3^{(3)}) &= (0.397225 - 1.07661 \mathbf{i}, 0.72384 + 0.222154 \mathbf{i}, -1.12106 + 0.854451 \mathbf{i}), \\
 (\hat{z}_1^{(4)}, \hat{z}_2^{(4)}, \hat{z}_3^{(4)}) &= (0.709, -0.3545 - 1.2656 \mathbf{i}, -0.3545 + 1.2656 \mathbf{i}), \\
 (\hat{z}_1^{(5)}, \hat{z}_2^{(5)}, \hat{z}_3^{(5)}) &= (-0.781352, 1.00305 - 0.749241 \mathbf{i}, 1.00305 + 0.749241 \mathbf{i}), \\
 (\hat{z}_1^{(6)}, \hat{z}_2^{(6)}, \hat{z}_3^{(6)}) &= (0, 0.612372 - 0.921816 \mathbf{i}, 0.612372 + 0.921816 \mathbf{i}), \\
 (\hat{z}_1^{(7)}, \hat{z}_2^{(7)}, \hat{z}_3^{(7)}) &= (-0.82853 - 0.22063 \mathbf{i}, 0.82853 - 0.22063 \mathbf{i}, 1.666 \mathbf{i}), \\
 (\hat{z}_1^{(8)}, \hat{z}_2^{(8)}, \hat{z}_3^{(8)}) &= (0, -0.673004 + 1.52228 \mathbf{i}, 0.673004 - 0.297537 \mathbf{i}), \\
 (\hat{z}_1^{(9)}, \hat{z}_2^{(9)}, \hat{z}_3^{(9)}) &= (0.82853 + 0.22063 \mathbf{i}, -1.666 \mathbf{i}, -0.82853 + 0.22063 \mathbf{i}), \\
 (\hat{z}_1^{(10)}, \hat{z}_2^{(10)}, \hat{z}_3^{(10)}) &= (0, -0.673004 - 1.52228 \mathbf{i}, 0.673004 + 0.297537 \mathbf{i}), \\
 (\hat{z}_1^{(11)}, \hat{z}_2^{(11)}, \hat{z}_3^{(11)}) &= (-1.00305 + 0.749241 \mathbf{i}, -1.00305 - 0.749241 \mathbf{i}, 0.781352), \\
 (\hat{z}_1^{(12)}, \hat{z}_2^{(12)}, \hat{z}_3^{(12)}) &= (0, -1.87718, 0.652438).
 \end{aligned} \tag{2.7 Equilibria}$$

It is possible that the system of algebraic equations characterizing the equilibria of (2.7a), (2.7b) has additional solutions besides those listed above. A direct attempt to solve this system of algebraic equations using Mathematica 10 was unsuccessful, and we did not deem the matter sufficiently relevant to justify further investigations.

In Fig. 14 each equilibrium of system (2.7a), (2.7b) is represented by the three points $\hat{z}_1^{(j)}$, $\hat{z}_2^{(j)}$ and $\hat{z}_3^{(j)}$, labeled by the index j , where $j = 1, 2, \dots, 12$.

Below we provide some *trajectories*, in the complex plane, of the particles z_1 , z_2 and z_3 whose evolution is described by system (2.7a), (2.7b) with the initial conditions

$$\begin{aligned}
 z_1(0) &= \hat{z}_1^{(3)} + 0.3 + 0.3\mathbf{i}, \quad \dot{z}_1(0) = 0.1 \\
 z_2(0) &= \hat{z}_2^{(3)} + 0.1, \quad \dot{z}_2(0) = 0.1 + 0.1\mathbf{i}, \\
 z_3(0) &= \hat{z}_3^{(3)} + 0.1, \quad \dot{z}_3(0) = 0.1,
 \end{aligned} \tag{2.7c}$$

see Fig. 15, 16 and 17. These graphs have been obtained by numerical integration of system (2.7a), (2.7b) with the initial conditions (2.7c), using Mathematica 10. We employed the command NDSolve with the automatic choice of the method and the accuracy. The program script is provided in the Appendix.

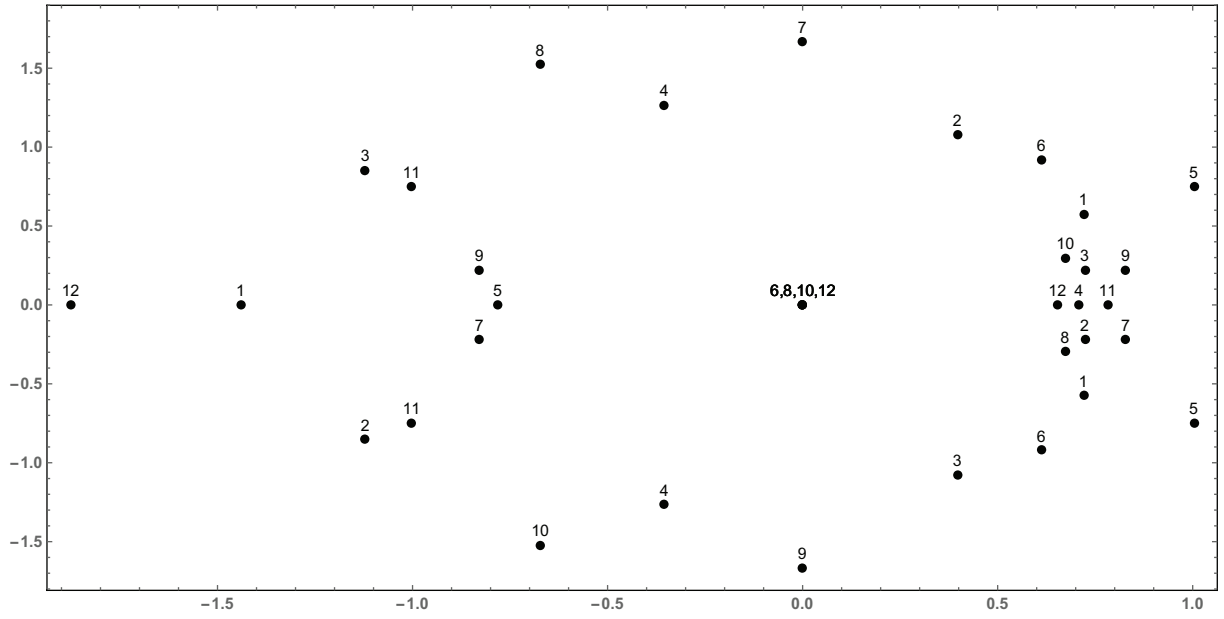


Fig. 14. Equilibria of system (2.7a), (2.7b) in the complex z -plane. Each equilibrium is represented by the three points $\hat{z}_1^{(j)}$, $\hat{z}_2^{(j)}$ and $\hat{z}_3^{(j)}$ indexed and labeled by $j = 1, 2, \dots, 12$, see display (2.7 Equilibria).

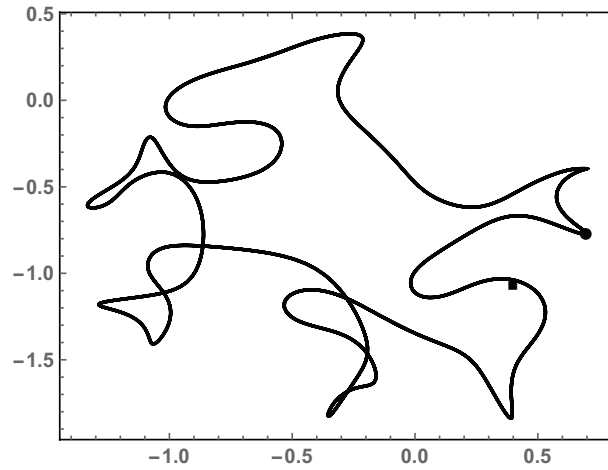


Fig. 15. System (2.7a), (2.7b), initial conditions (2.7c). Trajectory, in the complex z -plane, of $z_1(t)$. The square respectively the dot indicate the positions of $\hat{z}_1^{(3)}$ respectively $z_1(0)$, i.e. of a nearby equilibrium point respectively the initial value of $z_1(t)$; period 6π .

3. Outlook

The interest of the many-body model introduced and discussed above is demonstrated by its *solvable* and *isochronous* character as well as by the remarkable trajectories it features – already in the simple $N = 2$ and $N = 3$ cases, as shown by the graphs reported above. It is also amusing to observe—as the cognoscenti will have noted—that this solvable model has been obtained by appropriately

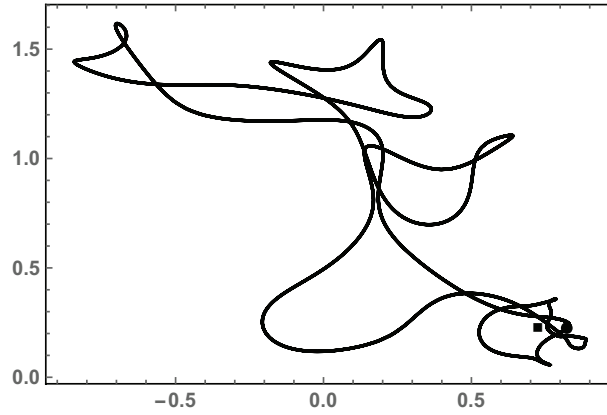


Fig. 16. System (2.7a), (2.7b), initial conditions (2.7c). Trajectory, in the complex z -plane, of $z_2(t)$. The square respectively the dot indicate the positions of $\hat{z}_2^{(3)}$ respectively $z_2(0)$, i.e. of a nearby equilibrium point respectively the initial value of $z_2(t)$; period 6π .

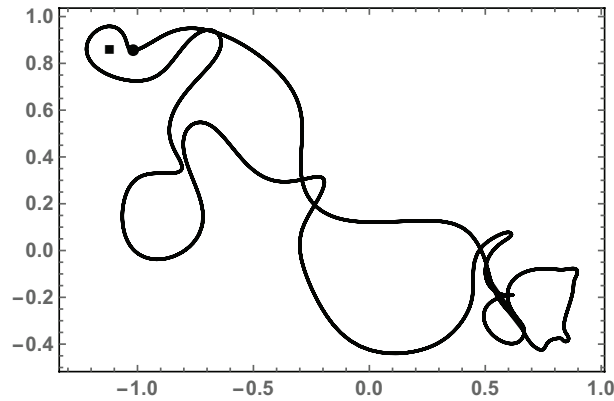


Fig. 17. System (2.7a), (2.7b), initial conditions (2.7c). Trajectory, in the complex z -plane, of $z_3(t)$. The square respectively the dot indicate the positions of $\hat{z}_3^{(3)}$ respectively $z_3(0)$, i.e. of a nearby equilibrium point respectively the initial value of $z_3(t)$; period 6π .

combining the two solvable equations—see (1.1a) and (2.3)—which are the prototypes of two, quite different, basic families of solvable many-body problems of Newtonian type. Moreover, the findings reported above provide the point of departure to obtain *Diophantine* properties of the N zeros of each of the $N!$ (monic) polynomials the *coefficients* of which are the N *zeros* of the Hermite polynomial $H_N(z)$ of degree N (these $N!$ polynomials correspond of course to the $N!$ permutations of the N *zeros* of $H_N(z)$). And a rather ample vista of further developments is provided by these findings.

4. Acknowledgements

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5. Appendix. Scripts to Plot Approximate Solutions of Systems (2.6) and (2.7a), (2.7b)

5.1. A Mathematica Script to Plot Approximate Solutions of System (2.6)

```
(* This is a script to plot approximate solutions of system (2.6)
with initial conditions (2.6a) through (2.6e). Please feel free to
modify and to try other initial conditions *)

(* To find the equilibria of (2.6), we first find the equilibria of
system (2.3) with omega=1 *)

omega=1;
LHSEqnEquilibriaC1[cc1_,cc2_]=Simplify[-omega^2 cc1+2/(cc1-cc2)^3];
LHSEqnEquilibriaC2[cc1_,cc2_]=Simplify[-omega^2 cc2+2/(cc2-cc1)^3];
EquilibriaCoatedC=Simplify[Solve[{LHSEqnEquilibriaC1[zz1,zz2]==0,
LHSEqnEquilibriaC2[zz1,zz2]==0},{zz1,zz2}]];
EquilibriaRepeatedC=Simplify[{zz1,zz2}/.EquilibriaCoatedC]
CC=EquilibriaRepeatedC;

(* P[[n,1]] and P[[n,2]] are the zeros of the polynomial
z^2+CC[[n,1]] z +CC[[n,2]],
so (P[[n,1]], P[[n,2]]) are equilibria of (2.6) *)

P=Table[0,{n,1,4},{m,1,2}];
Do[{SolCoated=Solve[zz^2+CC[[n,1]] zz +CC[[n,2]]==0,zz];
Sol=Simplify[zz/.SolCoated];
P[[n,1]]=Simplify[Sol[[1]]];
P[[n,2]]=Simplify[Sol[[2]]];},{n,1,4}
P=N[P]

(* Equilibria of (2.6), compare with (2.7) in the paper *)
MatrixForm[P]

(* List of the initial conditions (2.6a) through (2.6e)

(2.6a) EquilibriumZ1=P[[1,1]]; EquilibriumZ2=P[[1,2]];
z10=EquilibriumZ1+0.01;z1p0=0.01; z20=EquilibriumZ2+0.01;z2p0=-0.01;

(2.6b) EquilibriumZ1=P[[1,1]]; EquilibriumZ2=P[[1,2]];
z10=EquilibriumZ1+0.01+0.01I;z1p0=-0.01; z20=EquilibriumZ2+0.01+0.01 I;z2p0=0.01;

(2.6c) EquilibriumZ1=P[[2,1]]; EquilibriumZ2=P[[2,2]];
z10=EquilibriumZ1+0.01+0.01I;z1p0=-0.01; z20=EquilibriumZ2+0.01+0.01 I;z2p0=0.01;

(2.6d) EquilibriumZ1=P[[3,1]]; EquilibriumZ2=P[[3,2]];
z10=EquilibriumZ1+0.01+0.01I;z1p0=-0.005; z20=EquilibriumZ2+0.01+0.01 I;z2p0=0.005;

(2.6e) EquilibriumZ1=P[[4,1]]; EquilibriumZ2=P[[4,2]];
z10=EquilibriumZ1+0.01+0.01I;z1p0=-0.01; z20=EquilibriumZ2+0.01+0.01 I;z2p0=0.01;
*)

(* Choose the initial conditions for system (2.6),
either from the list above or your own*)

EquilibriumZ1=P[[4,1]]; EquilibriumZ2=P[[4,2]];
z10=EquilibriumZ1+0.02+0.02 I;z1p0=-0.02; z20=EquilibriumZ2+0.02;z2p0=0.02;

(* Solve system (2.6) with the last initial conditions, numerically*)
```

```

s=NDSolve[{z1'[t]==(2 z1'[t] z2'[t])/(z1[t]-z2[t])-1/(z1[t]-z2[t])
(omega^2 z1[t]^2-(2(z1[t]-1))/(z1[t]+z2[t]+z1[t] z2[t])^3),
z2'[t]==-((2 z1'[t] z2'[t])/(z1[t]-z2[t]))+1/(z1[t]-z2[t])
(omega^2 z2[t]^2-(2(z2[t]-1))/(z1[t]+z2[t]+z1[t] z2[t])^3),z1[0]==z10,
z1'[0]==z1p0,z2[0]==z20,z2'[0]==z2p0},{z1,z2},{t,40}];

(* Plot the real and the imaginary parts of z1[t], the first component
of the solution of system (2.6) with the assigned initial conditions *)

Plot[{Re[Evaluate[{z1[t]}/.s]][[1]][[1]],PlotStyle->{Black,Thick}},
Im[Evaluate[{z1[t]}/.s]][[1]][[1]]},{t,0,20},PlotRange->All,
Axes->True, AxesStyle->Directive[Bold,12], PlotStyle->{Black,Thick}]

(* From the last plot, guess the period of z1[t], verify if the guess is
correct by plotting the real and the imaginary parts
of z1[t+Period Guess]-z1[t] *)

PeriodGuess=N[2 Pi];Plot[{Re[Evaluate[{z1[t+PeriodGuess]}/.s]][[1]][[1]]
-Re[Evaluate[{z1[t]}/.s]][[1]][[1]],PlotStyle->{Black,Thick}},
Im[Evaluate[{z1[t+PeriodGuess]}/.s]][[1]][[1]]
-Im[Evaluate[{z1[t]}/.s]][[1]][[1]]},{t,0,20},
PlotRange->All, Axes->True, AxesStyle->Directive[Bold,12],
PlotStyle->{Black,Thick}]

(* Plot the real and the imaginary parts of z2[t], the first
component of the solution of system (2.6) with the assigned
initial conditions *)

Plot[{Re[Evaluate[{z2[t]}/.s]][[1]][[1]],
PlotStyle->{Black,Thick}},
{Im[Evaluate[{z2[t]}/.s]][[1]][[1]],
PlotStyle->{Black,Thick}}},{t,0,20},
PlotRange->All, Axes->True,
AxesStyle->Directive[Bold,12],PlotStyle->{Black,Thick}]

(* From the last plot, guess the period of z2[t], verify if the
guess is correct by plotting the real and the imaginary parts
of z2[t+Period Guess]-z2[t] *)

PeriodGuess=N[2 Pi];
Plot[{Re[Evaluate[{z2[t+PeriodGuess]}/.s]][[1]][[1]]
-Re[Evaluate[{z2[t]}/.s]][[1]][[1]],PlotStyle->{Black,Thick}},
Im[Evaluate[{z2[t+PeriodGuess]}/.s]][[1]][[1]]
-Im[Evaluate[{z2[t]}/.s]][[1]][[1]]},
{t,0,20},PlotRange->All, Axes->True,
AxesStyle->Directive[Bold,12],
PlotStyle->{Black,Thick}]

(* Plot the trajectory, in the complex z-plane, of the solution z1[t] of
system (2.6) with the assigned initial conditions *)

Graph2=ParametricPlot[{Re[Evaluate[{z1[t]}/.s]][[1]][[1]],
Im[Evaluate[{z1[t]}/.s]][[1]][[1]]},{t,0,20},PlotRange->All,
Axes->True, AxesStyle->Directive[Bold,12],
PlotStyle->{Black,Thick},
AspectRatio->1/1.3];

```



```
Show[Graph2,Graphics[
{PointSize[Large],
Point[{Re[EquilibriumZ1],Im[EquilibriumZ1]}],
Point[{Re[z10],Im[z10]}]
}]]

(* Plot the trajectory, in the complex z-plane, of the solution z2[t] of
system (2.6) with the assigned initial conditions *)

Graph3=ParametricPlot[{Re[Evaluate[{z2[t]}/.s]][[1]][[1]],
Im[Evaluate[{z2[t]}/.s]][[1]][[1]]},{t,0,20},PlotRange->All,
Axes->True,
AxesStyle->Directive[Bold,12],PlotStyle->{Black,Thick},
AspectRatio->1/1.5];
Show[Graph3,Graphics[
{PointSize[Large],Point[{Re[EquilibriumZ2],Im[EquilibriumZ2]}],
Point[{Re[z20],Im[z20]}]
}]]
```

5.2. A Mathematica Script to Plot Approximate Solutions of System (2.7a), (2.7b)

(* This is a script to plot approximate solutions of system (2.7ab) with initial conditions (2.7c). Please feel free to modify and to try other initial conditions *)

```
c1=(-z1-z2-z3);
c2=z1 z2+z1 z3+z2 z3;
c3=-z1 z2 z3;

(* To find (some of the) equilibria of system (2.7ab),
we first find the equilibria of system (2.3) for
N=3 and omega=1 and then find the zeros of the
monic polynomials whose coefficients are equilibria of (2.3)*)

FF1[z1_,z2_,z3_]=Simplify[-c1+2/(c1-c2)^3+2/(c1-c3)^3];
FF2[z1_,z2_,z3_]=Simplify[-c2+2/(c2-c1)^3+2/(c2-c3)^3];
FF3[z1_,z2_,z3_]=Simplify[-c3+2/(c3-c1)^3+2/(c3-c2)^3];

omega=1;

LHSEqnEquilibriaC1[cc1_,cc2_,cc3_]=Simplify[-cc1+2/(cc1-cc2)^3+2/(cc1-cc3)^3];
LHSEqnEquilibriaC2[cc1_,cc2_,cc3_]=Simplify[-cc2+2/(cc2-cc1)^3+2/(cc2-cc3)^3];
LHSEqnEquilibriaC3[cc1_,cc2_,cc3_]=Simplify[-cc3+2/(cc3-cc1)^3+2/(cc3-cc2)^3];
EquilibriaCoatedC=Simplify[Solve[{LHSEqnEquilibriaC1[zz1,zz2,zz3]==0,
LHSEqnEquilibriaC2[zz1,zz2,zz3]==0,
LHSEqnEquilibriaC3[zz1,zz2,zz3]==0},{zz1,zz2,zz3}]]];

EquilibriaRepeatedC=Simplify[{zz1,zz2,zz3}/.EquilibriaCoatedC]
CC=EquilibriaRepeatedC;

(* P[[n,1]], P[[n,2]] and P[[n,3]] are the zeros of the polynomial
z^3+CC[[n,1]] z^2 +CC[[n,2]] z+CC[[n,3]],
so (P[[n,1]], P[[n,2]], P[[n,3]])
are equilibria of (2.7ab) *)
```

```

P=Table[0,{n,1,12}, {m,1,3}];
Do[{SolCoated=Solve[zz^3+CC[[n,1]] zz^2 +CC[[n,2]]zz+CC[[n,3]]==0,zz];
  Sol=Simplify[zz/.SolCoated];
  P[[n,1]]=Simplify[Sol[[1]]];
  P[[n,2]]=Simplify[Sol[[2]]];
  P[[n,3]]=Simplify[Sol[[3]]];},{n,1,12}]

(* (Some) equilibria of (2.7ab), compare with the list (2.7 Equilibria) in this paper *)

P=N[P];
MatrixForm[P]

(* Initial Conditions (2.7c)*)
EquilibriumZ1 = P[[3, 1]];
EquilibriumZ2 = P[[3, 2]];
EquilibriumZ3 = P[[3, 3]];
z10 = EquilibriumZ1 + 0.3 + 0.3 I; z1p0 = 0.1;
z20 = EquilibriumZ2 + 0.1 ; z2p0 = 0.1 + 0.1 I;
z30 = EquilibriumZ3 + 0.1; z3p0 = 0.1;

(* Solve system (2.7ab) with the last initial conditions, numerically*)

LHSEquilibriaEqn1[z1_,z2_,z3_]=1/((z1-z2)(z1-z3))
Simplify[ (z1^2FF1[z1,z2,z3]+z1 FF2[z1,z2,z3]+FF3[z1,z2,z3])];

LHSEquilibriaEqn2[z1_,z2_,z3_]=1/((z2-z1)(z2-z3))
Simplify[ (z2^2FF1[z1,z2,z3]+z2 FF2[z1,z2,z3]+FF3[z1,z2,z3])];

LHSEquilibriaEqn3[z1_,z2_,z3_]=1/((z3-z1)(z3-z2))
Simplify[ (z3^2FF1[z1,z2,z3]+z3 FF2[z1,z2,z3]+FF3[z1,z2,z3])];

Tmax = 76;
s = NDSolve[{z1'[t] == (2 z1'[t] z2'[t])/(z1[t] - z2[t]) + (
  2 z1'[t] z3'[t])/(z1[t] - z3[t]) -
  LHSEquilibriaEqn1[z1[t], z2[t], z3[t]],
  z2'[t] == (2 z2'[t] z1'[t])/(z2[t] - z1[t]) + (2 z2'[t] z3'[t])/(
  z2[t] - z3[t]) - LHSEquilibriaEqn2[z1[t], z2[t], z3[t]],
  z3'[t] == (2 z3'[t] z1'[t])/(z3[t] - z1[t]) + (2 z3'[t] z2'[t])/(
  z3[t] - z2[t]) - LHSEquilibriaEqn3[z1[t], z2[t], z3[t]],
  z1[0] == z10, z1'[0] == z1p0, z2[0] == z20, z2'[0] == z2p0,
  z3[0] == z30, z3'[0] == z3p0}, {z1, z2, z3}, {t, Tmax},
  AccuracyGoal -> Automatic, Method -> Automatic];

(* Plot the real and the imaginary parts of z1[t], the first component
of the solution of system (2.7ab) with the assigned initial conditions *)

Plot[{Re[Evaluate[{z1[t]} /. s]][[1]][[1]],
  PlotStyle -> {Black, Thick}},
  Im[Evaluate[{z1[t]} /. s]][[1]][[1]], {t, 0, Tmax},
  PlotRange -> All, Axes -> True, AxesStyle -> Directive[Bold, 12],
  PlotStyle -> {Black, Thick}]

(* From the last plot, guess the period of z1[t], verify if the guess
is correct by plotting the real and the imaginary parts of
z1[t+Period Guess]-z1[t] *)

```

```

PeriodGuess =
N[6 Pi]; Plot[{Re[Evaluate[{z1[t + PeriodGuess]} /. s]][[1]][[1]] -
  Re[Evaluate[{z1[t]} /. s]][[1]][[1]],
  PlotStyle -> {Black, Thick}},
  Im[Evaluate[{z1[t + PeriodGuess]} /. s]][[1]][[1]] -
  Im[Evaluate[{z1[t]} /. s]][[1]][[1]]], {t, 0, Tmax/2},
PlotRange -> All, Axes -> True, AxesStyle -> Directive[Bold, 12],
PlotStyle -> {Black, Thick}]

(* Plot the real and the imaginary parts of z2[t], the second
component of the solution of system (2.7ab) with the assigned i
nitial conditions *)

Plot[{Re[Evaluate[{z2[t]} /. s]][[1]][[1]],
  PlotStyle -> {Black, Thick}}, {Im[Evaluate[{z2[t]} /. s]][[1]][[
  1]], PlotStyle -> {Black, Thick}}, {t, 0, Tmax},
PlotRange -> All, Axes -> True, AxesStyle -> Directive[Bold, 12],
PlotStyle -> {Black, Thick}]

(* From the last plot, guess the period of z2[t], verify if the
guess is correct by plotting the real and the imaginary
parts of z2[t+Period Guess]-z2[t] *)

PeriodGuess =
N[6 Pi]; Plot[{Re[Evaluate[{z2[t + PeriodGuess]} /. s]][[1]][[1]] -
  Re[Evaluate[{z2[t]} /. s]][[1]][[1]],
  PlotStyle -> {Black, Thick}},
  Im[Evaluate[{z2[t + PeriodGuess]} /. s]][[1]][[1]] -
  Im[Evaluate[{z2[t]} /. s]][[1]][[1]]], {t, 0, Tmax/2},
PlotRange -> All, Axes -> True, AxesStyle -> Directive[Bold, 12],
PlotStyle -> {Black, Thick}]

(* Plot the real and the imaginary parts of z3[t], the third component
of the solution of system (2.7ab) with the assigned initial conditions *)

Plot[{Re[Evaluate[{z3[t]} /. s]][[1]][[1]],
  PlotStyle -> {Black, Thick}}, {Im[Evaluate[{z3[t]} /. s]][[1]][[
  1]], PlotStyle -> {Black, Thick}}, {t, 0, Tmax},
PlotRange -> All, Axes -> True, AxesStyle -> Directive[Bold, 12],
PlotStyle -> {Black, Thick}]

(* From the last plot, guess the period of z3[t], verify if the guess
is correct by plotting the real and the imaginary parts of
z3[t+Period Guess]-z3[t] *)

PeriodGuess =
N[6 Pi]; Plot[{Re[Evaluate[{z3[t + PeriodGuess]} /. s]][[1]][[1]] -
  Re[Evaluate[{z3[t]} /. s]][[1]][[1]],
  PlotStyle -> {Black, Thick}},
  Im[Evaluate[{z3[t + PeriodGuess]} /. s]][[1]][[1]] -
  Im[Evaluate[{z3[t]} /. s]][[1]][[1]]], {t, 0, Tmax/2},
PlotRange -> All, Axes -> True, AxesStyle -> Directive[Bold, 12],
PlotStyle -> {Black, Thick}]

(* Plot the trajectory, in the complex z-plane, of the solution z1[t] of
system (2.7ab) with the assigned initial conditions *)

```

```

Graph2 = ParametricPlot[{Re[Evaluate[{z1[t]} /. s]][[1]][[1]],
  Im[Evaluate[{z1[t]} /. s]][[1]][[1]]}, {t, 0, Tmax},
  PlotRange -> All, Axes -> False, Frame -> True,
  FrameStyle -> Directive[Bold, 12], PlotStyle -> {Black, Thick},
  AspectRatio -> 1/1.3];
Show[Graph2, Graphics[
  {PointSize[Large],
  Text[Style[\[FilledSquare], Medium], {Re[EquilibriumZ1],
  Im[EquilibriumZ1]}], Point[{Re[z10], Im[z10]}] }]]

(* Plot the trajectory, in the complex z-plane, of the
solution z2[t] of system (2.7ab) with the assigned initial conditions *)

Graph3 = ParametricPlot[{Re[Evaluate[{z2[t]} /. s]][[1]][[1]],
  Im[Evaluate[{z2[t]} /. s]][[1]][[1]]}, {t, 0, Tmax},
  PlotRange -> All, Axes -> False, Frame -> True,
  FrameStyle -> Directive[Bold, 12], PlotStyle -> {Black, Thick},
  AspectRatio -> 1/1.5];
Show[Graph3, Graphics[
  {PointSize[Large],
  Text[Style[\[FilledSquare], Medium], {Re[EquilibriumZ2],
  Im[EquilibriumZ2]}], Point[{Re[z20], Im[z20]}] }]]

(* Plot the trajectory, in the complex z-plane, of the
solution z3[t] of system (2.7ab) with the assigned initial conditions *)

Graph4 = ParametricPlot[{Re[Evaluate[{z3[t]} /. s]][[1]][[1]],
  Im[Evaluate[{z3[t]} /. s]][[1]][[1]]}, {t, 0, Tmax},
  PlotRange -> All, Axes -> False, Frame -> True,
  FrameStyle -> Directive[Bold, 12], PlotStyle -> {Black, Thick},
  AspectRatio -> 1/1.5];
Show[Graph4, Graphics[
  {PointSize[Large],
  Text[Style[\[FilledSquare], Medium], {Re[EquilibriumZ3],
  Im[EquilibriumZ3]}], Point[{Re[z30], Im[z30]}] }]]

```

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