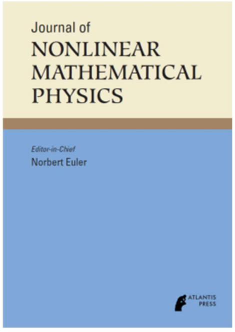


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# Darboux integrability of generalized Yang–Mills Hamiltonian system

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We show that the generalized Yang–Mills system with Hamiltonian $H = (p_1^2 + p_2^2)/2 + V(q_1,q_2)$ where $V = 1/2(aq_1^2 + bq_2^2) + (cq_1^4 + 2eq_1^2q_2^2 + dq_2^4)/4$ is not completely integrable with Darboux first integrals.

*Keywords*: Hamiltonian systems; weight-homogenous differential systems; polynomial integrability.

2000 Mathematics Subject Classification: 37J35, 37K10

## 1. Introduction and statement of the main results

We consider $\mathbb{R}^4$ as a symplectic linear space with canonical variables $q = (q_1,q_2)$ and $p = (p_1, p_2)$, with $q_i$ called the *coordinates* and $p_i$ called the *momenta*. We want to study the Hamiltonian systems with Hamilton's function of the form

$$H = \frac{1}{2}\sum_{i=1}^{2} p_i^2 + V(q_1,q_2)$$

where

$$V(q_1,q_2) = \frac{1}{2}\sum_{i=1}^{2} p_i^2 + \frac{1}{2}(aq_1^2 + bq_2^2) + \frac{c}{4}q_1^4 + \frac{d}{2}q_1^2q_2^2 + \frac{e}{4}q_2^4 \tag{1.1}$$

being $a,b,c,d,e$ constants. We will study the Hamiltonian systems

$$\dot{q}_i = p_i, \quad \dot{p}_i = -\frac{\partial V}{\partial q_i}, \quad i = 1,2. \tag{1.2}$$

These systems are the generalized form of the classical Yang–Mills Hamiltonian systems and can be reduced from them in scalar field theory [25]. They appear in a variety of problems in scalar field theory [10, 25], celestial mechanics, cosmological models [2, 5, 14] and quantum mechanics [1, 20].

We start by recalling some definitions. Let $A = A(p,q)$ and $B = B(p,q)$ be two functions. Their *Poisson bracket* $\{A,B\}$ is defined as

$$\{A,B\} = \sum_{i=1}^{2} \left( \frac{\partial A}{\partial q_i}\frac{\partial B}{\partial p_i} - \frac{\partial A}{\partial p_i}\frac{\partial B}{\partial q_i} \right).$$

We say that two functions $A$ and $B$ are *in involution* if $\{A,B\} = 0$, and that a non-constant function $F = F(q,p)$ is a *first integral* for the Hamiltonian system (1.2) if it commutes with the Hamiltonian function $H$, i.e., $\{H,F\} = 0$. Moreover, we say that the Hamiltonian system (1.2) is *completely integrable* if it has 2 functionally independent first integrals which are in involution. One first integral will always be the Hamiltonian $H$. We say that two functions $H$ and $F$ are *functionally independent* if their gradients are independent at all points of $\mathbb{R}^4$ except perhaps in a zero Lebesgue measure set.

Many papers have been published regarding the integrability and non-integrability of the Yang–Mills Hamiltonian systems (1.2) by using different methods such as the Painlevé method, direct calculations, algebraic geometry tools,..., see for instance [1, 2, 9, 13, 22, 24] and the references therein.

During the last century Hamiltonian systems with potential $V(q_1,q_2)$ of degree at most 5 have been intensively studied in the view point of integrability looking for a second independent first integral (see for instance [3, 11, 12, 21] and the references therein).

A relevant result providing a method that gives a necessary condition for the existence of an additional meromorphic first integral for Hamiltonian systems with homogeneous potential was given by Morales and Ramis (see page 100 of [18] and the references therein). However system (1.2) has no a homogeneous polynomial $V$. Nevertheless in [23] the authors were able to use Morales–Ramis theory [17, 18] and its generalization to higher order variational equations [19] to characterize the meromorphic integrability of the Yang–Mills system (1.2). More precisely, due to the symmetry of the parameters $c$ and $d$ in (1.1) and that if $c \neq 0$ with the transformation $q_i = x_i/\sqrt{c}$, $p_i = y_i/\sqrt{c}$ for $i = 1,2$ and setting $a_1 = a$, $a_2 = b$, $a_3 = d/c$, $a_4 = e/c$, the Hamiltonian with potential given in (1.1) is equivalent to the Hamiltonian

$$H = \frac{1}{2}(p_1^2 + p_2^2) + \frac{1}{2}(a_1 q_1^2 + a_2 q_2^2) + \frac{1}{4}q_1^4 + \frac{1}{2}a_4 q_1^2 q_2^2 + \frac{1}{4}a_3 q_2^4,$$

the authors in [23] prove the following result.

**Theorem 1.1.** *System* (1.2) *is meromorphically integrable if and only if one of the following conditions hold:*

- *(i)* $d = 0$. *In this case the additional first integral is* $G = \frac{1}{2}p_1^2 + \frac{1}{2}aq_1^2 + cq_1^4/4$*;*
- *(ii)* $c = d = e$. *In this case the additional first integral is* $q_1p_1 - q_2p_2$*;*
- *(iii)* $a = b$, $d = 3c = 3e$. *In this case the additional first integral first integral is* $G = aq_1q_2 + cq_1q_2(q_1^2 + q_2^2) + p_1p_2$*;*
- *(iv)* $b = 4a$, $d = 3c$, $e = 8c$, *or* $b = 4a$, $d = 3e$, $c = 8e$. *In these cases the additional first integral is* $G = p_1^4 + a^2q_1^4 + c(q_1^4 + 6q_1^2q_2^2 + 2aq_1^2)p_1^2 - 4cq_1^3q_2p_1p_2 + cq_1^4p_2^2 + ca^2q_1^4 + caq_1^6 + 2acq_1^4q_2^2 + \frac{1}{4}c^2q_1^8 + c^2q_1^6q_2^2 + c^2q_1^4q_2^4$*, or* $G = p_1^4 + a^2q_1^4 + 8c(q_1^4 + 6q_1^2q_2^2 + 2aq_1^2)p_1^2 - 36cq_1^3q_2p_1p_2 + 8cq_1^4p_2^2 + 8ca^2q_1^4 + 8caq_1^6 + 16acq_1^4q_2^2 + 16c^2q_1^8 + 64c^2q_1^6q_2^2 + 64c^2q_1^4q_2^4$*;*
- *(v)* $b = 4a$, $d = 6c$, $e = 16c$, *or* $b = 4a$, $d = 6e$, $c = 16e$. *In these cases the additional first integral is* $G = aq_1^2q_2 - q_2p_1^2 + q_1p_1p_2 + \sqrt{c}a_1^4q_2 + 2\sqrt{c}q_1^2q_2^3$*, or* $G = aq_1^2q_2 - q_2p_1^2 + q_1p_1p_2 + 4\sqrt{c}a_1^4q_2 + 8\sqrt{c}q_1^2q_2^3$.

Note that Theorem 1.1 can be reformulated as follows:

**Corollary 1.1.** *System* (1.2) *is meromorphically integrable if and only if it is polynomially integrable and one of the conditions (i)–(v) in Theorem 1.1 hold.*

Our main objective is to study when system (1.2) is completely integrable with Darboux functions (see below for a definition). We recall that, as it will be clear below from its definition, not all meromorphic functions are Darboux functions, and that not all Darboux functions are meromorphic functions. This shows that our results given in Theorem 1.2 and the results in [23] given in Theorem 1.1 are independent.

To study the existence of Darboux first integrals we will use the well-known Darboux theory of integrability. The Darboux theory of integrability in dimension 4 is based on the existence of invariant algebraic hypersurfaces (or Darboux polynomials). For more details see [6, 7] and [15]. This theory is one of the best theories for studying the existence of first integrals for polynomial differential systems.

A *Darboux polynomial* of system (1.2) is a polynomial $f \in \mathbb{C}[p_1, p_2, q_1, q_2] \setminus \mathbb{C}$ such that

$$\sum_{i=1}^{2} \left( p_i \frac{\partial f}{\partial q_i} - \frac{\partial V}{\partial q_i} \frac{\partial f}{\partial p_i} \right) = Kf$$

for some polynomial $K$ called the *cofactor* of $f$ and with degree at most two.

Since on the points of the algebraic hypersurface $f = 0$, the vector field of system (1.2) is tangent to it, it is thus formed by orbits of the vector field and so the hypersurface $f = 0$ is invariant under the flow of system (1.2). It is called *an invariant algebraic hypersurface*. On the other hand, a *polynomial first integral* (a first integral which is a polynomial) is a Darboux polynomial with zero cofactor. We recall that if $f \notin \mathbb{R}[p_1, p_2, q_1, q_2] \setminus \mathbb{R}$ is a Darboux polynomial then there exists another Darboux polynomial $\bar{f}$ (the conjugate of $f$) with cofactor $\bar{K}$ (the conjugate of $K$).

An *exponential factor* $F = F(p_1, p_2, q_1, q_2)$ of system (1.2) is a function of the form $F = \exp(g_0/g_1) \notin \mathbb{C}$ with $g_0, g_1 \in \mathbb{C}[p_1, p_2, q_1, q_2]$ coprime satisfying that

$$\sum_{i=1}^{2} \left( p_i \frac{\partial F}{\partial q_i} - \frac{\partial V}{\partial q_i} \frac{\partial F}{\partial p_i} \right) = LF,$$

for some polynomial $L = L(p_1, p_2, q_1, q_2)$ called the *cofactor* of $F$ and with degree at most two. We recall that if $F \in \mathbb{C}[p_1, p_2, q_1, q_2] \setminus \mathbb{C}$ is an exponential factor that is not real, then there exists another exponential factor $\bar{F}$ (the conjugate of $F$) with cofactor $\bar{L}$ (the conjugate of $L$).

A *Darboux first integral* $G$ of system (1.2) is a first integral of the form

$$G = f_1^{\lambda_1} \cdots f_p^{\lambda_p} F_1^{\mu_1} \cdots F_q^{\mu_q}, \tag{1.3}$$

where $f_1, \ldots, f_p$ are Darboux polynomials and $F_1, \ldots, F_q$ are exponential factors and $\lambda_j, \mu_k \in \mathbb{C}$ for $j = 1, \ldots, p$ and $k = 1, \ldots, q$. Note that a Darboux first integral is always a real function due to the fact that if there are complex Darboux polynomials or complex exponential factors always also appear their conjugates.

Our main result is the following one.

**Theorem 1.2.** *The Hamiltonian system* (1.2) *is completely integrable with Darboux first integrals if and only if it is polynomially integrable and one of the conditions (i)–(v) in Theorem 1.1 hold.*

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As explained above, Theorems 1.1 and 1.2 are independent.

The proof of Theorem 1.2 is given in section 3. For this, we will restrict to the potentials that do not satisfy any of the conditions (i)–(v) in Theorem 1.1, otherwise they are polynomially integrable. First we will prove the non–existence of Darboux polynomial with non–zero cofactor for the potentials not satisfying any of the conditions (i)–(v) in Theorem 1.1.

## 2. The Darboux polynomials with nonzero cofactor

It follows from Theorem 1.1 that the unique polynomial first integrals are the ones satisfying any of the conditions (i)–(v) in Theorem 1.1. Therefore, we will only study the Hamiltonian system (1.2) with potential (1.1) that do not satisfy any of the conditions (i)–(v) in Theorem 1.1. In particular, we consider that $d \neq 0$.

Our main result in this section is the following.

**Theorem 2.1.** *The Hamiltonian system* (1.2) *with potential* (1.1) *satisfying* $d \neq 0$ *has no irreducible Darboux polynomials with non–zero cofactor.*

To prove Theorem 2.1 we recall the following auxiliary result that was proved in [4].

**Lemma 2.1.** *Let $f$ be a polynomial and $f = \prod_{j=1}^{s} f_j^{\alpha_j}$ its decomposition into irreducible factors in $\mathbb{C}[x,y,z]$. Then $f$ is a Darboux polynomial if and only if all the $f_j$ are Darboux polynomials. Moreover, if $K$ and $K_j$ are the cofactors of $f$ and $f_j$, then $K = \sum_{j=1}^{s} \alpha_j K_j$.*

Now we recall some properties of our Hamiltonian system (1.2) with homogenous potential (1.1).

Let $\tau\colon \mathbb{C}[p_1,p_2,q_1,q_2] \to \mathbb{C}[p_1,p_2,q_1,q_2]$ be the automorphism

$$\tau(p_1,p_2,q_1,q_2) = (-p_1,-p_2,-q_1,-q_2).$$

**Proposition 2.1.** *If $g$ is an irreducible Darboux polynomial for system* (1.2) *with cofactor*

$$\begin{aligned} K = {} & \alpha_0 + \alpha_1 q_1 + \alpha_2 q_2 + \alpha_3 p_1 + \alpha_4 p_2 + \alpha_5 q_1^2 + \alpha_6 q_1 q_2 + \alpha_7 q_1 p_1 + \alpha_8 q_1 p_2 \\ & + \alpha_9 q_2^2 + \alpha_{10} q_2 p_1 + \alpha_{11} q_2 p_2 + \alpha_{12} p_1^2 + \alpha_{13} p_1 p_2 + \alpha_{14} p_2^2, \end{aligned} \tag{2.1}$$

*then $f = g \cdot \tau g$ is a Darboux polynomial invariant by $\tau$ with a cofactor of the form*

$$\begin{aligned} K_\tau = {} & 2\alpha_0 + 2\alpha_5 q_1^2 + 2\alpha_6 q_1 q_2 + 2\alpha_7 q_1 p_1 + 2\alpha_8 q_1 p_2 + 2\alpha_9 q_2^2 + 2\alpha_{10} q_2 p_1 \\ & + 2\alpha_{11} q_2 p_2 \quad + 2\alpha_{12} p_1^2 + 2\alpha_{13} p_1 p_2 + 2\alpha_{14} p_2^2. \end{aligned} \tag{2.2}$$

**Proof.** Since system (1.2) with $V$ as in (1.1) is invariant under $\tau$, $\tau g$ is a Darboux polynomial of system (1.2) with $V$ as in (1.1) and with cofactor $\tau(K)$. Moreover, by Lemma 2.1, $g \cdot \tau g$ is also a Darboux polynomial oof system (1.2) with $V$ as in (1.1) and with cofactor $K + \tau(K)$. Therefore again by Lemma 2.1, the cofactor of $f$ is the $K_\tau$ which is given in (2.2). □

We first study the Darboux polynomials of the Hamiltonian system (1.2) with potential (1.1) not satisfying any of the conditions (i)–(v) in Theorem 1.1 and that are invariant by $\tau$.

**Proposition 2.2.** *The Hamiltonian system* (1.2) *with potential* (1.1) *not satisfying any of the conditions (i)–(v) in Theorem 1.1 and that are invariant by $\tau$ have no Darboux polynomials with non-zero cofactor $K_\tau$ as in* (2.2).

**Proof.** Let $f$ be a Darboux polynomial of the Hamiltonian system (1.2) with potential (1.1) not satisfying any of the conditions (i)–(v) in Theorem 1.1, that is invariant by $\tau$ and has non-zero cofactor $K_\tau$. We write it as $f = \sum_{j=0}^{n} f_j(p_1, p_2, q_1, q_2)$ where each $f_j$ is a homogeneous polynomial of degree $j$ in the variables $p_1, p_2, q_1, q_2$ and is invariant by $\tau$. Without loss of generality we can assume that $f_n \neq 0$ with $n > 0$. We have

$$p_1\frac{\partial f}{\partial q_1} + p_2\frac{\partial f}{\partial q_2} - (aq_1 + cq_1^3 + dq_1q_2^2)\frac{\partial f}{\partial p_1} - \Big(bq_2 + dq_1^2q_2 + eq_2^3\Big)\frac{\partial f}{\partial p_2} = K_\tau f, \tag{2.3}$$

where $K_\tau$ is as in (2.2) with $\alpha_i \in \mathbb{C}$ not all zero.

The terms of degree $n+1$ in (2.3) satisfy

$$\begin{aligned} &-(cq_1^3 + dq_1q_2^2)\frac{\partial f_n}{\partial p_1} - \Big(dq_1^2q_2 + eq_2^3\Big)\frac{\partial f_n}{\partial p_2} = \big(2\alpha_5q_1^2 + 2\alpha_6q_1q_2 + 2\alpha_7q_1p_1 \\ &+ 2\alpha_8q_1p_2 + 2\alpha_9q_2^2 + 2\alpha_{10}q_2p_1 + 2\alpha_{11}q_2p_2 + 2\alpha_{12}p_1^2 + 2\alpha_{13}p_1p_2 \\ &+ 2\alpha_{14}p_2^2\big)f_n. \end{aligned} \tag{2.4}$$

Solving the differential equation in (2.4) we have

$$f_n = F_n\left(q_1, q_2, p_2 - \frac{p_1(3dq_1^2q_2 + 3eq_2^3)}{3q_1(cq_1^2 + dq_2^2)}\right)\exp\left(\frac{p_1T_1}{27q_1^3(cq_1^2 + dq_2^2)^3}\right) \tag{2.5}$$

where $K_n$ is an arbitrary function and

$$\begin{aligned} T_1 = &-9c^2q_1^6(2\alpha_{12}p_1^2 + 3\alpha_{13}p_1p_2 + 6\alpha_{14}{p_2}^2 + 3\alpha_7p_1q_1 + 6\alpha_8p_2q_1 + 6\alpha_5q_1^2) \\ &- 3q_1^5q_2\big(9c^2q_1(\alpha_{10}p_1 + 2\alpha_{11}p_2 + 2\alpha_6q_1) - 3cdp_1(\alpha_{13}p_1 + 6\alpha_{14}p_2 \\ &+ 3\alpha_8q_1)\big) - 3q_1^4q_2^2(-4a_2dp_1(\alpha_{13}p_1 + 6\alpha_{14}p_2) + 6d\big((2c\alpha_{12} + d\alpha_{14})p_1^2 \\ &+ 3c\alpha_{13}p_1p_2 + 6c\alpha_{14}{p_2}^2\big) - 9cd(\alpha_{11}p_1 - 2\alpha_7p_1 - 4\alpha_8p_2)q_1 \\ &+ 18c(2d\alpha_5 + c\alpha_9)q_1^2) - 3q_1^3q_2^3\big(18cdq_1(\alpha_{10}p_1 + 2\alpha_{11}p_2 + 2\alpha_6q_1) \\ &- 3d^2p_1(\alpha_{13}p_1 + 6\alpha_{14}p_2 + 3\alpha_8q_1) - 3cep_1(\alpha_{13}p_1 + 6\alpha_{14}p_2 + 3\alpha_8q_1)\big) \\ &+ 9q_1^2q_2^4\big(-3ce\alpha_{11}p_1q_1 + d^2(2\alpha_{12}p_1^2 + 3\alpha_{13}p_1p_2 + 6\alpha_{14}{p_2}^2 - 3\alpha_{11}p_1q_1 \\ &+ 3\alpha_7p_1q_1 + 6\alpha_8p_2q_1 + 6\alpha_5q_1^2) + 2d(2e\alpha_{14}p_1^2 + 6c\alpha_9q_1^2)\big) \\ &+ 9q_1q_2^5\big(-3d^2q_1(\alpha_{10}p_1 + 2\alpha_{11}p_2 + 2\alpha_6q_1) + dep_1(\alpha_{13}p_1 + 6\alpha_{14}p_2 \\ &+ 3\alpha_8q_1)\big) - 9q_2^6(2e^2\alpha_{14}p_1^2 - 3de\alpha_{11}p_1q_1 + 6d^2\alpha_9q_1^2). \end{aligned} \tag{2.6}$$

Let

$$Y = p_2 - \frac{p_1(3dq_1^2q_2 + 3eq_2^3)}{3q_1(cq_1^2 + dq_2^2)} \quad \text{then} \quad p_2 = Y + \frac{p_1(3dq_1^2q_2 + 3eq_2^3)}{3q_1(cq_1^2 + dq_2^2)}.$$

Then we can rewrite (2.5) and (2.6) as

$$f_n = K_n(q_1, q_2, Y)\exp\left(\frac{p_1\tilde{T}_1}{27q_1^3(cq_1^2 + dq_2^2)^3}\right),$$

where $\tilde{T}_1 = \tilde{T}_1(q_1,q_2,p_1,Y) = T_1(q_1,q_2,p_1,p_2)$. Since $f_n$ must be a polynomial and the function $K_n$ in the variables $q_1,q_2,p_1$ and $Y$ does not depend on $p_1$, we must have $\tilde{T}_1 = T_1 = 0$. Computing the coefficient in $T_1$ of $q_1^2q_2^6$ we get $54d^2\alpha_9 = 0$. Since $d \neq 0$ (otherwise the potential $V$ in (1.1) would satisfy condition (i) in Theorem 1.1) we get $\alpha_9 = 0$. Then the coefficients in $T_1$ of $p_2^2q_1^4q_2^3$, $q_1^4q_2^4$, $q_1^4p_1^2q_2^2$ and $q_1^2q_2^4p_2^2$ are, respectively, $-54d^2\alpha_{11}$, $-54d^2\alpha_9$, $-18d^2\alpha_{12}$ and $36d^2\alpha_{14}$. Setting them equal to zero we obtain $\alpha_9 = \alpha_{11} = \alpha_{12} = \alpha_{14} = 0$. Now computing in $T_1$ the coefficients of $q_1^5q_2^2$, $q_1^5q_2^3$ and $p_1^2q_1^5q_2$ we get $-54d^2\alpha_8$, $-54d^2\alpha_6$ and $3d^2\alpha_{13}$. Setting them equal to zero we obtain $\alpha_6 = \alpha_8 = \alpha_{13} = 0$. Finally, the coefficients of $p_1q_1^5q_2^2$, $q_1^6q_2^2$ and $p_1q_1^4q_2^3$ are, respectively, $-27d^2\alpha_7$, $-54d^2\alpha_5$ and $-27d^2\alpha_{10}$. Setting them equal to zero we get $\alpha_6 = \alpha_7 = \alpha_{10} = 0$. In short, we have proved that $K_\tau = 2\alpha_0$.

Now we show that $\alpha_0 = 0$. If $\alpha_0 \neq 0$, from the terms of degree 0 in (2.3) we have $\alpha_0 f_0 = 0$, and so $f_0 = 0$. We consider four different cases: assume first that $a = b = 0$. In this case the terms of degree 1 in (2.3) become

$$p_1\frac{\partial f_1}{\partial q_1} + p_2\frac{\partial f_1}{\partial q_2} = 2\alpha_0 f_1.$$

Solving it we get

$$f_1 = C_1\left(p_1,p_2,\frac{p_1q_2-p_2q_1}{p_1}\right)\exp\left(\frac{2\alpha_0q_1}{p_1}\right).$$

Since $f_1$ must be a polynomial and $\alpha_0 \neq 0$ we must have $C_1 = 0$ and thus $f_1 = 0$. Proceeding inductively we get that $f_j = 0$ for $j = 2,3,\ldots,n$ in contradiction with the fact that $f_n \neq 0$. This implies that $\alpha_0 = 0$, i.e., $K_\tau = 0$ and concludes the proof of the proposition in this case.

Assume now $a = 0$ and $b \neq 0$. Then the terms of degree 1 in (2.3) become

$$p_1\frac{\partial f_1}{\partial q_1} + p_2\frac{\partial f_1}{\partial q_2} - aq_1\frac{\partial f_1}{\partial p_1} = 2\alpha_0 f_1.$$

Solving it we get

$$f_1 = C_1\left(p_1,p_2,\frac{q_1}{p_1} - \frac{\arctan\left(\frac{\sqrt{b}q_2}{p_2}\right)}{\sqrt{b}}\right)\exp\left(\frac{2\alpha_0q_1}{p_1}\right).$$

Proceeding as above, since $f_1$ must be a polynomial and $\alpha_0 \neq 0$ we must have that $C_1 = 0$ and thus $f_1 = 0$. Proceeding inductively we get that $f_j = 0$ for $j = 2,3,\ldots,n$ in contradiction with the fact that $f_n \neq 0$. This implies that $\alpha_0 = 0$, i.e., $K_\tau = 0$ and concludes the proof of the proposition in this case.

Now consider the case $a \neq 0$ and $b = 0$. Then the terms of degree 1 in (2.3) become

$$p_1\frac{\partial f_1}{\partial q_1} + p_2\frac{\partial f_1}{\partial q_2} - bq_2\frac{\partial f_1}{\partial p_2} = 2\alpha_0 f_1.$$

Solving it we get

$$f_1 = C_1\left(p_1,p_2,\frac{q_2}{p_2} - \frac{\arctan\left(\frac{\sqrt{a}q_1}{p_1}\right)}{\sqrt{a}}\right)\exp\left(\frac{2\alpha_0q_2}{p_2}\right).$$

Proceeding as above, since $f_1$ must be a polynomial and $\alpha_0 \neq 0$ we must have that $C_1 = 0$ and thus $f_1 = 0$. Proceeding inductively we get that $f_j = 0$ for $j = 2,3,\ldots,n$ in contradiction with the fact that $f_n \neq 0$. This implies that $\alpha_0 = 0$, i.e., $K_\tau = 0$ and concludes the proof of the proposition in this case.

Finally, assume $ab \neq 0$. Then the terms of degree 1 in (2.3) become

$$p_1 \frac{\partial f_1}{\partial q_1} + p_2 \frac{\partial f_1}{\partial q_2} - aq_1 \frac{\partial f_1}{\partial p_1} - bq_2 \frac{\partial f_1}{\partial p_2} = 2\alpha_0 f_1.$$

Solving it we get

$$f_1 = C_1 \left( p_1, p_2, \frac{\arctan\left(\frac{\sqrt{a}q_1}{p_1}\right)}{\sqrt{a}} + \frac{\arctan\left(\frac{\sqrt{b}q_2}{p_2}\right)}{\sqrt{b}} \right) \exp\left( \frac{2\alpha_0}{\sqrt{a}} \arctan\left( \frac{\sqrt{a}q_1}{p_1} \right) \right) \Bigg).$$

Proceeding as above, since $f_1$ must be a polynomial and $\alpha_0 \neq 0$ we must have that $C_1 = 0$ and thus $f_1 = 0$. Proceeding inductively we get that $f_j = 0$ for $j = 2, 3, \ldots, n$ in contradiction with the fact that $f_n \neq 0$. This implies that $\alpha_0 = 0$, i.e., $K_\tau = 0$ and concludes the proof of the proposition. □

**Proof of Theorem 2.1.** Let $g$ be an irreducible Darboux polynomial of the Hamiltonian system (1.2) with potential (1.1) not satisfying any of the conditions (i)–(v) in Theorem 1.1 and with non-zero cofactor $K$ of the form in (2.1). Then, from Proposition 2.1, we can assume that $f = g \cdot \tau g$ is a Darboux polynomial of the Hamiltonian system (1.2) with potential (1.1) not satisfying any of the conditions (i)–(v) in Theorem 1.1 and invariant by $\tau$, with degree $2n$ and non–zero cofactor $K_\tau$ of the form in (2.2). From Proposition 2.2, we get that $K_\tau = 0$, otherwise we get a contradiction. Hence, $f$ must be a polynomial first integral of the Hamiltonian system (1.2)with potential (1.1) not satisfying any of the conditions (i)–(v) in Theorem 1.1. Hence, $f$ is of the form $f = H^{2n}$ Then, from the definition of $f$ and since $H$ and $g$ are irreducible, invariant by $\tau$, and $g$, $\tau g$ have the same dimension, it follows that $g = H^n$, in contradiction with the fact that the cofactor of $g$ is not zero. □

## 3. Proof of Theorem 1.2

In order to proof Theorem 1.2 we will introduce some well-known results. The first one whose whose proof and geometrical meaning is given in [4, 16] is the following.

**Proposition 3.1.** *The following statements hold.*

- *(a) If $E = \exp(g_0/g)$ is an exponential factor for the polynomial system* (1.2) *and g is not a constant polynomial, then $g = 0$ is an invariant algebraic hypersurface.*
- *(b) Eventually $e^{g_0}$ can be an exponential factor, coming from the multiplicity of the infinite invariant hyperplane.*

The proof of the following results is given in [8].

**Theorem 3.1.** *If system* (1.2) *has a rational first integral then either it has a polynomial first integral or two Darboux polynomials with the same nonzero cofactor.*

**Theorem 3.2.** *Suppose that system* (1.2) *admits p Darboux polynomials and with cofactors $K_i$ and q exponential factors $F_j$ with cofactors $L_j$. Then there exists $\lambda_j, \mu_j \in \mathbb{C}$ not all zero such that*

$$\sum_{i=1}^{q} \lambda_k K_i + \sum_{i=1}^{q} \mu_i L_i = 0$$

*if and only if the function G given in* (1.3) *(called of Darboux type) is a first integral of system* (1.2).

In view of Theorems 2.1 and 3.2 if $G$ is a Darboux first integral then it must be of the form $G = F_1^{\mu_1} \cdots F_q^{\mu_q}$ with $\sum_{i=1}^q \mu_i L_i = 0$ and such that $F_i = \exp(g_i/H^j)$ with $j \in \mathbb{N}$, cofactor $\mu_i L_i$ being $g_i$ a polynomial. Take

$$h = \sum_{i=1}^{q} \frac{\mu_i g_i}{H^j}.$$

Note that $G = \exp(h)$ being $h$ a rational function, that has cofactor $L = \sum_{i=1}^q \mu_i L_i = 0$ and that satisfies, after simplifying by $G$

$$p_1 \frac{\partial g}{\partial q_1} + p_2 \frac{\partial g}{\partial q_2} - (cq_1^3 + dq_1 q_2^2)\frac{\partial g}{\partial p_1} - (dq_1^2 q_2 + eq_2^3)\frac{\partial g}{\partial p_2} = \sum_{i=1}^{q} \mu_i L_i = 0.$$

In particular $h$ must be a rational first integral. However, in view of Theorems 2.1 and 3.1, if system (1.2) has a potential (1.1) not satisfying any of the conditions (i)–(v) in Theorem 1.1, then it cannot have rational first integrals. This completes the proof.

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