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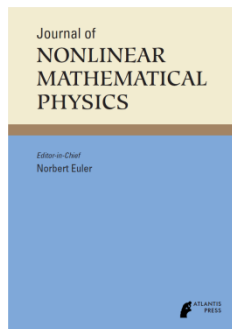
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**A convenient expression of the time-derivative $z_n^{(k)}(t)$, of arbitrary order k ,
of the zero $z_n(t)$ of a time-dependent polynomial $p_N(z;t)$ of arbitrary degree N in z ,
and solvable dynamical systems**

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Let $p_N(z;t)$ be a (monic) time-dependent polynomial of arbitrary degree N in z , and let $z_n \equiv z_n(t)$ be its N zeros: $p_N(z;t) = \prod_{n=1}^N [z - z_n(t)]$. In this paper we report a convenient expression of the k -th time-derivative $z_n^{(k)}(t)$ of the zero $z_n(t)$. This formula plays a key role in the identification of classes of *solvable* dynamical systems describing the motion of point-particles moving in the *complex* z -plane while nonlinearly interacting among themselves; one such example, featuring *many arbitrary* parameters, is reported, including its variation describing the motion of many particles moving in the *real* Cartesian xy -plane and interacting among themselves via *rotation-invariant* Newtonian equations of motion (“accelerations equal forces”).

Keywords: time-derivatives of time-dependent polynomials, time-derivatives of the zeros of time-dependent polynomials, solvable dynamical systems, solvable many-body problems, isochronous many-body problems

1. Introduction

Let $p_N(z;t) \equiv p_N(z; \vec{c}(t); \underline{z}(t))$ be a (monic) time-dependent (t = time) polynomial of degree N in z , where $\vec{c} \equiv \vec{c}(t)$ is the N -vector of its *coefficients* $c_m \equiv c_m(t)$ and $\underline{z} \equiv \underline{z}(t)$ is the *unordered* set of its N *zeros* $z_n \equiv z_n(t)$:

$$p_N(z; \vec{c}(t); \underline{z}(t)) \equiv p_N(z; t) = z^N + \sum_{m=1}^N [c_m(t) z^{N-m}] = \prod_{n=1}^N [z - z_n(t)] . \quad (1.1a)$$

Note that the notation $p_N(z; \vec{c}(t); \underline{z}(t))$ is somewhat redundant, because clearly this time-dependent monic polynomial is uniquely identified by assigning either its N *coefficients* $c_m(t)$ or its N *zeros* $z_n(t)$. Indeed the N *coefficients* $c_m(t)$ can themselves be *explicitly* expressed in terms of the N *zeros* $z_n(t)$ as follows:

$$c_m(t) = \frac{(-1)^m}{m!} \sum_{n_1, n_2, \dots, n_m=1}^N * \left\{ \prod_{j=1}^m [z_{n_j}(t)] \right\} , \quad m = 1, 2, \dots, N . \quad (1.1b)$$

Notation 1.1. Above and hereafter N is an arbitrary positive integer ($N \geq 2$), indices such as n, n_j, m, ℓ run over the integers from 1 to N (unless otherwise indicated) and the “*-modified” summation symbol $\sum_{n_1, n_2, \dots, n_m=1}^N *$ indicates that this “*-modified” sum runs over each of the N integers n_1, n_2, \dots, n_m from 1 to N with the restriction that these m indices be *all different among themselves*; while the “*n*-modified” summation symbol $\sum_{n_1, n_2, \dots, n_m=1}^N *n*$ (see below) indicates that the sum

runs over each of the N integers n_1, n_2, \dots, n_m from 1 to N with the restriction that these m indices be *all different among themselves* and moreover be *all different from the (external) index n* . Note that this implies that the “*-modified” sum vanishes identically (whatever its argument) if $m > N$ (which is not the case in (1.1b)) and likewise the “* n *-modified” sum vanishes identically if $m \geq N$, as implied by the standard convention according to which *a void sum vanishes*; and note that we also assume hereafter the standard convention according to which *a void product equals unity* (see below). Let us emphasize that all variables and parameters—except of course indices—are generally considered to be *complex* numbers (unless otherwise indicated: for instance the time t is *real*, see above). And we denote below by \mathbf{i} the *imaginary unit*, so that $\mathbf{i}^2 = -1$.

Clearly the time-derivative of arbitrary order k of this polynomial, $p_N^{(k)}(z;t) \equiv (d/dt)^k p_N(z;t)$, is expressed in terms of the k -th time-derivative $c_m^{(k)}(t) \equiv (d/dt)^k c_m(t)$ of its N coefficients $c_m(t)$ by the following formula:

$$p_N^{(k)}(z;t) \equiv (d/dt)^k p_N(z;t) = \sum_{m=1}^N \left[c_m^{(k)}(t) z^{N-m} \right], \quad k = 1, 2, 3, \dots \quad (1.2)$$

The first goal of this paper is to report a formula which instead expresses the time-derivative $p_N^{(k)}(z;t)$ in terms of the time-derivatives (of order from 1 to k) of the N zeros $z_n(t)$ of the polynomial $p_N(z;t)$. A version of this formula is rather directly obtained by inserting the k -th time-derivative $c_m^{(k)}(t)$ of the right-hand side of (1.1b),

$$c_m^{(k)}(t) = \frac{(-1)^m}{m!} \left(\frac{d}{dt} \right)^k \left\{ \sum_{n_1, n_2, \dots, n_m=1}^N * \prod_{\ell=1}^m [z_{n_\ell}(t)] \right\}, \quad m = 1, 2, \dots, N, \quad (1.3)$$

in the right-hand side of (1.2). But this version is *not* convenient for our main goal, see below. Indeed the motivation for obtaining another version of this formula, (1.2), is to evince from it an expression of the k -th time-derivative of any one of the zeros $z_n(t)$ in terms of the k -th time-derivative $c_m^{(k)}(t)$ of the N coefficients $c_m(t)$ and of the N zeros $z_m(t)$ and their time-derivatives of order less than k , because such a formula plays a key role in the identification of classes of *solvable* dynamical systems describing the motion of N point-particles moving in the complex z -plane while nonlinearly interacting among themselves. Indeed a version of this formula—but only for $k = 1, 2$ —was already obtained in [1] and used for this purpose in [1–4], and—but only for $k = 3, 4$ —in [5].

These formulas are reported in the following Section 2. In Section 3 we indicate how *solvable* dynamical systems—more general than those treated in the papers quoted above—are identified by using, for *arbitrary (positive integer) values of k* , the key formula reported in Section 2; and we describe such an example featuring *many arbitrary* parameters, including its variation describing the motion of *many* particles moving in the *real* Cartesian xy -plane and interacting among themselves via *rotation-invariant* Newtonian equations of motion (“accelerations equal forces”). Explicit versions of the key formula expressing $z_n^{(k)}(t)$ are displayed in the Appendix for k from 1 to 6 and N arbitrary ($N \geq k$), and for $N = 2$ and k arbitrary ($k \geq 2$).

2. A convenient formula

A compact way to express the k -th time-derivative $p_N^{(k)}(z;t)$ of the polynomial $p_N(z;t)$, see (1.1a), reads

$$\frac{d^k p_N(z;t)}{dt^k} \equiv p_N^{(k)}(z;t) = p_N(z;t) b_{N,k}(z;t), \quad (2.1a)$$

with the following definition of the function $b_{N,k}(z;t)$: let

$$q_{N,k}(\underline{x}) = \left[\sum_{j=1}^N (x_j) \right]^k, \quad (2.1b)$$

where of course the N quantities x_j are the N components of the *unordered* set \underline{x} ; then expand this expression of $q_{N,k}(\underline{x})$ in powers of x_j and perform, for every *positive* value of the exponent r of $(x_j)^r$, the replacement

$$(x_j)^r \implies -[z - z_j(t)]^{-1} z_j^{(r)}(t), \quad r = 1, 2, \dots, k, \quad (2.1c)$$

where of course, above and hereafter,

$$z_j^{(r)}(t) \equiv \frac{d^r z_j(t)}{dt^r}, \quad r = 1, 2, 3, \dots \quad (2.1d)$$

This yields $b_{N,k}(z;t)$.

In a previous version of this paper a proof of this result was provided, because we had been unable to find this formula in the literature; but a competent referee convinced us that this result is not sufficiently new to justify a report of its proof.

To apply this rule it is convenient for our purposes to reformulate the standard multinomial theorem according to which (see, for instance, [6])

$$\left[\sum_{j=1}^N (x_j) \right]^k = \sum_{\substack{k_1, k_2, \dots, k_N=0; \\ k_1+k_2+\dots+k_N=k}}^k \left[\left(\frac{k!}{k_1! k_2! \dots k_N!} \right) \prod_{j=1}^N (x_j)^{k_j} \right], \quad (2.2)$$

by treating *separately* the terms with *vanishing* exponents k_j , so that

$$\left[\sum_{j=1}^N (x_j) \right]^k = \sum_{s=1}^{\min(N,k)} [b_{N,k,s}(\underline{x})], \quad (2.3a)$$

with (see **Notation 1.1**)

$$b_{N,k,s}(\underline{x}) = \left(\frac{k!}{s!} \right) \sum_{n_1, n_2, \dots, n_s=1}^N * \left\{ \sum_{\substack{k_1, k_2, \dots, k_N=1; \\ k_1+k_2+\dots+k_s=k}}^k \left[\prod_{r=1}^s \frac{(x_{n_r})^{k_r}}{k_r!} \right] \right\}. \quad (2.3b)$$

In these formulas, (2.3), s identifies of course the number of terms in the right hand side of (2.2) with a *nonvanishing* exponent k_j .

Then, via the rule described above (see, in particular, (2.1c)) one gets

$$b_{N,k}(z;t) = \sum_{s=1}^{\min(N,k)} \left[(-)^s \left(\frac{k!}{s!} \right) \sum_{n_1, n_2, \dots, n_s=1}^N * \cdot \right. \\ \cdot \left. \left\{ \sum_{\substack{k_1, k_2, \dots, k_s=1; \\ k_1+k_2+\dots+k_s=k}}^k \left[\prod_{r=1}^s \left(\frac{z_{n_r}^{(k_r)}(t)}{k_r! [z - z_{n_r}(t)]} \right) \right] \right\} \right], \quad (2.4)$$

which, via (2.1a) and the last of the formulas (1.1a), yields the identity

$$p_N^{(k)}(z;t) = k! \sum_{s=1}^{\min(N,k)} \left[\frac{(-)^s}{s!} \sum_{n_1, n_2, \dots, n_s=1}^N * \cdot \right. \\ \cdot \left. \left\{ \left(\prod_{\ell=1; \ell \neq n_1, n_2, \dots, n_s}^N [z - z_\ell(t)] \right) \sum_{\substack{k_1, k_2, \dots, k_s=1; \\ k_1+k_2+\dots+k_s=k}}^k \left[\prod_{r=1}^s \left(\frac{z_{n_r}^{(k_r)}(t)}{k_r!} \right) \right] \right\} \right]. \quad (2.5)$$

It is now convenient to rewrite this formula by separating out the term with $s = 1$, which is the only term containing in its right-hand side the highest derivative (of order k) of the zeros:

$$p_N^{(k)}(z;t) = - \sum_{n_1=1}^N \left[\left(\prod_{\ell=1; \ell \neq n_1}^N [z - z_\ell(t)] \right) z_{n_1}^{(k)}(t) \right] \\ + k! \sum_{s=2}^{\min(N,k)} \left[\frac{(-)^s}{s!} \sum_{n_1, n_2, \dots, n_s=1}^N * \left\{ \left(\prod_{\ell=1; \ell \neq n_1, n_2, \dots, n_s}^N [z - z_\ell(t)] \right) \cdot \right. \right. \\ \cdot \left. \left. \sum_{\substack{k_1, k_2, \dots, k_s=1; \\ k_1+k_2+\dots+k_s=k}}^k \left[\prod_{r=1}^s \left(\frac{z_{n_r}^{(k_r)}(t)}{k_r!} \right) \right] \right\} \right]. \quad (2.6)$$

Setting $z = z_n(t)$ in this identity one then clearly gets (see again **Notation 1.1**)

$$p_N^{(k)}(z_n(t);t) = - \left(\prod_{\ell=1; \ell \neq n}^N [z_n - z_\ell(t)] \right) z_n^{(k)}(t) + \sum_{s=2}^{\min(N,k)} \tilde{b}_{N,k,n,s}(t) \quad (2.7a)$$

with

$$\tilde{b}_{N,k,n,s}(t) = \frac{(-)^s k!}{s!} \sum_{n_1, n_2, \dots, n_s=1}^N * \left\{ \left(\prod_{\ell=1; \ell \neq n_1, n_2, \dots, n_s}^N [z_n - z_\ell(t)] \right) \cdot \right. \\ \cdot \left. \sum_{\substack{k_1, k_2, \dots, k_s=1; \\ k_1+k_2+\dots+k_s=k}}^k \left[\prod_{r=1}^s \left(\frac{z_{n_r}^{(k_r)}(t)}{k_r!} \right) \right] \right\}. \quad (2.7b)$$

It is on the other hand plain (see (1.2)) that

$$p_N^{(k)}[z_n(t);t] = \sum_{m=1}^N \left\{ c_m^{(k)}(t) [z_n(t)]^{N-m} \right\}, \quad k = 1, 2, 3, \dots \quad (2.8)$$

Hence a comparison of this last formula with (2.7a) yields, for $k = 1, 2, 3, \dots$, the formula

$$z_n^{(k)}(t) = \left\{ \prod_{\ell=1, \ell \neq n}^N [z_n(t) - z_\ell(t)] \right\}^{-1} \cdot \left(\sum_{s=2}^{\min(N,k)} \tilde{b}_{N,k,n,s}(t) - \sum_{m=1}^N \left\{ c_m^{(k)}(t) [z_n(t)]^{N-m} \right\} \right), \quad (2.9)$$

This formula, with (2.7b), provides a convenient starting point for the identification of *solvable* dynamical systems, as explained in the following section; hence we display it more explicitly in the Appendix for k from 1 to 6 (with $N \geq k$) and for $N = 2$ with $k \geq 2$.

3. Solvable dynamical systems

Let us recall the standard definition of *solvable* dynamical systems: they are systems of an arbitrary number N of Ordinary Differential Equations (ODEs) involving N time-dependent variables $z_n(t)$, the *general* solution of which—and as well the solution of their *initial-values* problem—can be achieved by *algebraic* operations, essentially by finding the N roots of a known time-dependent polynomial $p_N(z;t)$, see (1.1a). These dynamical systems—in contrast to a *generic* autonomous *nonlinear* dynamical system with $N \geq 3$ —generally do *not* yield a *chaotic* time evolution, indeed in some cases their general behavior can be remarkably neat, for instance *isochronous* or *multiply periodic*—i. e., *all* their solutions are *completely periodic* with a *fixed* period independent of the initial data or feature a, possibly *nonlinear*, superposition of a *finite* number of different fixed periods—or can display such behaviors *asymptotically*, in the remote future—i. e. be *asymptotically isochronous* or *asymptotically multiply periodic* [7, 8]. Note, however, that when the solution of the *initial-values* problem is provided by the N roots $z_n(t)$ of a known time-dependent polynomial $p_N(z;t)$, this solution identifies the overall configuration of the system at any time t , but generally it does *not* allow to identify one by one the particles, namely to recognize which one of the N coordinates $z_n(t)$ corresponds to the specific *initial data* characterizing that specific coordinate; although such an identification can be obtained by evaluating—possibly even with lower accuracy—a sequence of solutions over sufficiently short time subintervals by an argument of *contiguity* applied to the positions of the particles at the beginning and the end of each such time subinterval. Alternatively, one can follow the time evolution of each zero on the Riemann surface associated to the polynomial $p_N(z;t)$, but this is generally a nontrivial task, see for instance [9–13].

The manufacture/identification of *solvable* dynamical systems has been an important sector of mathematical physics research in the last few decades. A natural approach in this context is to consider a (monic) time-dependent polynomial $p_N(z;t)$, see (1.1a), and to assume that the time-evolution of its N *coefficients* $c_m(t)$ is characterized by a *solvable* dynamical system; it is then clear that the time evolution of its N *zeros* $z_n(t)$ is—as it were, by definition—also *solvable*. This procedure allows to infer from *trivially solvable* systems—including those characterized by *linear* equations of motion—other dynamical systems which are still *solvable* while featuring less trivial, generally highly *nonlinear*, equations of motion. Moreover, this procedure can be repeated, by letting the time-dependent solutions—yielded by the *zeros* of a known time-dependent polynomial—play the role of *coefficients* of new polynomials, and by then considering the *zeros* of the new polynomials as the coordinates of new dynamical systems, which of course shall also be—again, as it were, by definition—*solvable*. Note that this procedure yields, from any polynomial $p_N(z;t)$ —in

the *generic* case with its N zeros *all different among themselves*— $N!$ new polynomials, since $N!$ different assignments of the *ordered* set of N coefficients of the *new* polynomials are obtained by identifying them with the $N!$ different elements of the *unordered* set of the N zeros $z_n(t)$ of $p_N(z;t)$. And an iteration of this procedure—leading to the notion of *generations* of polynomials [4]—allows the identification of endless sequences of *solvable* dynamical systems.

In order to exhibit the equations of motion of these *solvable* dynamical systems, one must be able to relate explicitly the time evolution of the *zeros* of a time-dependent polynomial to the time evolution of its *coefficients*. A convenient way to do so is provided by the key formula (2.9) with (2.7b). We now illustrate this notion by outlining a simple example. The analogous derivation of vast classes of novel *solvable* dynamical systems, and the detailed investigation of their time evolutions—going beyond the results already reported in [1–5, 14–16]—is a task for the future, to be pursued by ourselves and/or by others.

So, let us assume that the coefficients $c_m(t)$ evolve in time according to the following system of N *decoupled autonomous* and *linear* ODEs:

$$c_m^{(k)}(t) = \sum_{s=1}^k \left[a_{ms} c_m^{(s-1)}(t) \right], \quad m = 1, 2, \dots, N, \quad (3.1a)$$

where of course $c_m^{(r)}(t)$ is, for all *nonnegative integer* values of r , the r -th time-derivative of $c_m(t)$. This system, featuring kN arbitrary constants a_{ms} , is of course *solvable*, indeed its *general* solution reads

$$c_m(t) = \sum_{s=1}^k [\gamma_{ms} \exp(\lambda_{ms} t)], \quad (3.1b)$$

where, for every value of m , the k numbers λ_{ms} with $s = 1, 2, \dots, k$ are the k roots of the following polynomial equations of degree k in the variable λ_m ,

$$(\lambda_m)^k = \sum_{s=1}^k \left[a_{ms} (\lambda_m)^{s-1} \right], \quad m = 1, 2, \dots, N; \quad (3.1c)$$

while the kN constants γ_{ms} can be arbitrarily assigned, or determined as the solutions of the system of kN linear equations

$$c_m^{(s-1)}(0) = \sum_{s=1}^k \left[\gamma_{ms} (\lambda_m)^{s-1} \right], \quad s = 1, \dots, k, \quad m = 1, \dots, N \quad (3.1d)$$

in order to solve the *initial-values* problem, with the kN initial values $c_m^{(s-1)}(0)$ assigned.

Remark 3.1. Note that here, for simplicity, we restricted consideration to the *solvable* dynamical system (3.1a), and we moreover assumed that, for every value of m , the k roots of the algebraic equation (3.1c) are *all different among themselves*. While of course the treatment—above and below—could be easily generalized to the more general *solvable* dynamical system characterized

by the (now *coupled* but still *linear*) equations of motion

$$c_m^{(k)}(t) = \sum_{n=1}^N \sum_{s=1}^k \left[a_{mns} c_n^{(s-1)}(t) \right], \quad m = 1, 2, \dots, N, \quad (3.2)$$

featuring the kN^2 arbitrary constants a_{mns} .

Remark 3.2. It is plain that the time evolution of the dynamical system (3.1) is essentially characterized by the values of the kN exponents λ_{ms} . For instance clearly if these exponents are *all purely imaginary* (and *all different among themselves*), then *all* solutions of the dynamical system (3.1) will, for *all* time, remain confined to a *finite* region of the complex c -plane, and if moreover all the kN exponents λ_{ms} are (different) *rational* multiples of the same *imaginary* number $\mathbf{i}\omega$, $\lambda_m = \mathbf{i}(q_m/p_m)\omega$ with q_m and p_m *coprime real integers* and $p_m > 0$, then the system (3.1) is *isochronous*, *all* its solutions being periodic with period $T = 2\pi p/\omega$,

$$c_m(t+T) = c_m(t), \quad (3.3)$$

where of course the integer p is the Minimum Common Multiple of the N integers p_m . Likewise, if some of the exponents λ_{ms} have the features we just described and the remaining ones are *complex* numbers all featuring a *negative real part*, then the corresponding dynamical system is clearly *asymptotically isochronous*.

It is then plain that also *solvable* is the, also *autonomous* but *nonlinearly coupled*, dynamical system the solutions of which are identified as the N zeros of the polynomial $p_N(z;t)$, see (1.1a), with the coefficients $c_m(t)$ satisfying the system (3.1). And—on the basis of the above treatment—it is clear that the equations of motions of this dynamical system can be *explicitly* written as follows:

$$z_n^{(k)}(t) = \left\{ \prod_{\ell=1, \ell \neq n}^N [z_n(t) - z_\ell(t)] \right\}^{-1} \cdot \left(\sum_{s=2}^{\min(N,k)} \tilde{b}_{N,k,n,s}(t) - \sum_{m=1}^N \left\{ [z_n(t)]^{N-m} \sum_{s=1}^k [a_{ms} c_m^{(s-1)}(t)] \right\} \right), \quad (3.4a)$$

with the quantities $\tilde{b}_{N,k,n,s}(t)$ expressed in terms of the N quantities $z_\ell(t)$ and their time-derivatives of order less than k by (2.7b), and with the quantities $c_m^{(s-1)}(t)$ also expressed in terms of the N quantities $z_\ell(t)$ and their time-derivatives of order less than k by (1.1b) and by the time-derivatives of this formula of order less than k ,

$$c_m^{(s-1)}(t) = \frac{(-1)^m}{m!} \left(\frac{d}{dt} \right)^{s-1} \left\{ \sum_{n_1, n_2, \dots, n_m=1}^N * \left[\prod_{\ell=1}^m [z_{n_\ell}(t)] \right] \right\}, \quad m = 1, 2, \dots, N, \quad s = 1, 2, \dots, k. \quad (3.4b)$$

And it is moreover plain that this nonlinear dynamical system, featuring the kN arbitrary constants a_{ms} —the *solvable* character of which we believe to be a *novel* finding—also inherits properties of the system (3.1), for instance those described in **Remark 3.2**; except for the fact that, in the *isochronous* cases—and likewise in the *asymptotically isochronous* cases—the period might turn

out to be an *integer multiple* of T , due to the fact that the N zeros $z_n(t)$ of a time-dependent polynomial $p_N(z; t)$ which is *periodic* with period T are certainly also *all periodic*, but possibly with a larger period νT , with ν a *positive integer*, $\nu \leq N!$ (for a discussion in an analogous context of the relevance of this phenomenon see [17]).

To illustrate this finding—and to display a remarkable *isochronous* system featuring many *arbitrary* (but *rational*) parameters—let us report the following:

Remark 3.3. The dynamical system (3.4) with (2.7b) is *isochronous* provided the Nk constants a_{ms} are given by the following formulas:

$$a_{ms} = -\frac{(-i\omega)^s}{s!} \sum_{k_1, k_2, \dots, k_s=1}^k {}^*(r_{mk_1} r_{mk_2} \cdots r_{mk_s}), \quad (3.5a)$$

in terms of the arbitrary (*nonvanishing*) *real* parameter ω and of the Nk *real rational* numbers r_{ms} —which are also *arbitrary*, except for the restriction to be *all different among themselves*,

$$r_{ms} = \frac{q_{ms}}{p_{ms}}, \quad m = 1, 2, \dots, N, \quad s = 1, 2, \dots, k; \quad r_{ms} \neq r_{m's'} \text{ unless } m = m', s = s', \quad (3.5b)$$

with q_{ms} and p_{ms} *arbitrary coprime real integers* and, for definiteness, $p_{ms} > 0$. Indeed then *all* its *generic* solutions—except for a set of *nongeneric* solutions which are singular due to “particle collisions”—are then periodic, $z_n(t+T) = z_n(t)$, with the overall period

$$T = (N!) p \left(\frac{2\pi}{\omega} \right), \quad p = \text{MinCommMult}[p_{ms}]_{m=1, \dots, N; s=1, \dots, k}; \quad (3.5c)$$

but of course there also are sets of *generic* initial data yielding solutions with periods which are *integer submultiples* of T .

The formulation of the analogous but more general result encompassing *asymptotically isochronous* systems is left to the interested reader.

Let us conclude this paper with the following two rather elementary remarks, which underscore the interest of the *solvable* dynamical system described above.

Remark 3.4. For any *even integer* $k = 2S$ with S any *positive integer*, the *solvable* dynamical system (3.4) can be rephrased as a system of equations of motion of *Newtonian type* (“accelerations equal forces”) describing the evolution of SN unit-mass particles the coordinates of which can be denoted as $w_{ns} \equiv w_{ns}(t)$, by rephrasing the equations of motion (3.4)—which themselves read (in self-evident notation: identify the right-hand side of the following equation with the right-hand side of (3.4))

$$z_n^{(k)} = f_{nk} \left(\vec{z}, \vec{z}^{(2)}, \dots, \vec{z}^{(2S-2)}; \dot{\vec{z}}, \dot{\vec{z}}^{(2)}, \dots, \dot{\vec{z}}^{(2S-2)} \right) \quad (3.6a)$$

—by setting

$$z_n^{(2s-2)} = w_{n,s}, \quad s = 1, 2, \dots, S, \quad (3.6b)$$

so that the equations of motion in terms of the new dependent variables $w_{ns} \equiv w_{ns}(t)$ then read as follows:

$$\begin{aligned} \ddot{w}_{n,s} &= w_{n,s-1}, \quad s = 1, 2, \dots, S-1, \\ \ddot{w}_{n,S} &= f_{nk} \left(\vec{w}_1, \vec{w}_2, \dots, \vec{w}_S; \dot{\vec{w}}_1, \dot{\vec{w}}_2, \dots, \dot{\vec{w}}_S \right). \end{aligned} \quad (3.6c)$$

Here of course the superimposed arrows denote S -vectors.

Remark 3.5. The equations of motion (3.4), as well as the Newtonian equations of motion presented in the preceding **Remark 3.4**, are clearly *invariant* under a constant rescaling of the dependent variables, $z_n \Rightarrow C z_n$ and likewise $w_{ns} \Rightarrow C w_{ns}$ with C an *arbitrary nonvanishing complex* constant, $\dot{C} = 0$; hence in particular they are *invariant under rotations* around the origin of the *complex* z -plane, i. e. under the transformations $z_n \Rightarrow \exp(i\theta) z_n$ and likewise $w_{ns} \Rightarrow \exp(i\theta) w_{ns}$ with θ an *arbitrary constant* angle. This indicates the possible interest of an additional reformulation of these *solvable* many-body problems in terms of unit-mass particles moving in the *real* Cartesian xy -plane—their positions being identified by the *real* 2-vectors $\vec{r}_n = (x_n, y_n)$ where $z_n = x_n + i y_n$ or $\vec{r}_{ns} = (x_{ns}, y_{ns})$ where $w_{ns} = x_{ns} + i y_{ns}$.

Appendix: explicit display of the formula (2.9) in special cases

In this Appendix we display the formula (2.9) for $k = 1, 2, 3, 4, 5, 6$ and $N \geq k$; and for $N = 2$ and $k \geq N$. The diligent reader will verify that, for $k = 1, 2, 3, 4$, the formulas (3.7a)-(3.7d) displayed below are consistent with the findings obtained previously [1, 5].

Notation A. As above, $z^{(k)} \equiv z^{(k)}(t) \equiv d^k z(t)/dt^k$ and $c^{(k)} \equiv c^{(k)}(t) \equiv d^k c(t)/dt^k$; and recall **Notation 1.1** for the significance of the symbol \sum^{*n*} .

$$z_n^{(1)} = - \left[\prod_{\ell=1, \ell \neq n}^N (z_n - z_\ell) \right]^{-1} \sum_{m=1}^N \left[c_m^{(1)} (z_n)^{N-m} \right], \quad (3.7a)$$

$$\begin{aligned} z_n^{(2)} &= \sum_{\ell=1; \ell \neq n}^N \left(\frac{2 z_n^{(1)} z_\ell^{(1)}}{z_n - z_\ell} \right) \\ &- \left[\prod_{\ell=1, \ell \neq n}^N (z_n - z_\ell) \right]^{-1} \sum_{m=1}^N \left[c_m^{(2)} (z_n)^{N-m} \right], \end{aligned} \quad (3.7b)$$

$$\begin{aligned}
 z_n^{(3)} = & 3 \sum_{\ell=1; \ell \neq n}^N \left(\frac{z_n^{(2)} z_\ell^{(1)} + z_\ell^{(2)} z_n^{(1)}}{z_n - z_\ell} \right) \\
 & - 3 \sum_{\ell_1, \ell_2=1}^N {}^{*n*} \left[\frac{z_n^{(1)} z_{\ell_1}^{(1)} z_{\ell_2}^{(1)}}{(z_n - z_{\ell_1}) (z_n - z_{\ell_2})} \right] \\
 & - \left[\prod_{\ell=1, \ell \neq n}^N (z_n - z_\ell) \right]^{-1} \sum_{m=1}^N \left[c_m^{(3)} (z_n)^{N-m} \right], \quad (3.7c)
 \end{aligned}$$

$$\begin{aligned}
 z_n^{(4)} = & \sum_{\ell=1; \ell \neq n}^N \left(\frac{4 z_n^{(3)} z_\ell^{(1)} + 4 z_\ell^{(3)} z_n^{(1)} + 6 z_n^{(2)} z_\ell^{(2)}}{z_n - z_\ell} \right) \\
 & - 6 \sum_{\ell_1, \ell_2=1}^N {}^{*n*} \left[\frac{z_n^{(2)} z_{\ell_1}^{(1)} z_{\ell_2}^{(1)} + 2 z_{\ell_1}^{(2)} z_{\ell_2}^{(1)} z_n^{(1)}}{(z_n - z_{\ell_1}) (z_n - z_{\ell_2})} \right] \\
 & + 4 \sum_{\ell_1, \ell_2, \ell_3=1}^N {}^{*n*} \left[\frac{z_n^{(1)} z_{\ell_1}^{(1)} z_{\ell_2}^{(1)} z_{\ell_3}^{(1)}}{(z_n - z_{\ell_1}) (z_n - z_{\ell_2}) (z_n - z_{\ell_3})} \right] \\
 & - \left[\prod_{\ell=1, \ell \neq n}^N (z_n - z_\ell)^{-1} \right] \sum_{m=1}^N \left[c_m^{(4)} (z_n)^{N-m} \right], \quad (3.7d)
 \end{aligned}$$

$$\begin{aligned}
 z_n^{(5)} = & 5 \sum_{\ell=1; \ell \neq n}^N \frac{z_n^{(4)} z_\ell^{(1)} + z_\ell^{(4)} z_n^{(1)} + 2 \left(z_n^{(3)} z_\ell^{(2)} + z_\ell^{(3)} z_n^{(2)} \right)}{z_n - z_\ell} \\
 & - 5 \sum_{\ell_1, \ell_2=1}^N {}^{*n*} \left[\frac{\left(2 z_n^{(3)} z_{\ell_1}^{(1)} z_{\ell_2}^{(1)} + 4 z_n^{(1)} z_{\ell_1}^{(3)} z_{\ell_2}^{(1)} \right) + 6 z_n^{(2)} z_{\ell_1}^{(2)} z_{\ell_2}^{(1)} + 3 z_n^{(1)} z_{\ell_1}^{(2)} z_{\ell_2}^{(2)}}{(z_n - z_{\ell_1}) (z_n - z_{\ell_2})} \right] \\
 & + 10 \sum_{\ell_1, \ell_2, \ell_3=1}^N {}^{*n*} \left[\frac{z_n^{(2)} z_{\ell_1}^{(1)} z_{\ell_2}^{(1)} z_{\ell_3}^{(1)} + 3 z_n^{(1)} z_{\ell_1}^{(2)} z_{\ell_2}^{(1)} z_{\ell_3}^{(1)}}{(z_n - z_{\ell_1}) (z_n - z_{\ell_2}) (z_n - z_{\ell_3})} \right] \\
 & - 5 \sum_{\ell_1, \ell_2, \ell_3, \ell_4=1}^N {}^{*n*} \left[\frac{z_n^{(2)} z_{\ell_1}^{(1)} z_{\ell_2}^{(1)} z_{\ell_3}^{(1)} z_{\ell_4}^{(1)}}{(z_n - z_{\ell_1}) (z_n - z_{\ell_2}) (z_n - z_{\ell_3}) (z_n - z_{\ell_4})} \right] \\
 & - \left[\prod_{\ell=1, \ell \neq n}^N (z_n - z_\ell) \right]^{-1} \sum_{m=1}^N \left[c_m^{(5)} (z_n)^{N-m} \right], \quad (3.7e)
 \end{aligned}$$

$$\begin{aligned}
 z_n^{(6)} = & \sum_{\ell=1; \ell \neq n}^N \frac{6 \left(z_n^{(5)} z_\ell^{(1)} + z_\ell^{(5)} z_n^{(1)} \right) + 15 \left(z_n^{(4)} z_\ell^{(2)} + z_\ell^{(4)} z_n^{(2)} \right) + 20 z_n^{(3)} z_\ell^{(3)}}{z_n - z_\ell} \\
 & - \sum_{\ell_1, \ell_2=1}^N *n* \left[\frac{15 z_n^{(4)} z_{\ell_1}^{(1)} z_{\ell_2}^{(1)} + 30 z_n^{(1)} z_{\ell_1}^{(4)} z_{\ell_2}^{(1)} + 60 z_n^{(3)} z_{\ell_1}^{(2)} z_{\ell_2}^{(1)} + 60 z_n^{(2)} z_{\ell_1}^{(3)} z_{\ell_2}^{(1)} + 60 z_n^{(1)} z_{\ell_1}^{(3)} z_{\ell_2}^{(2)} + 45 z_n^{(2)} z_{\ell_1}^{(2)} z_{\ell_2}^{(2)}}{(z_n - z_{\ell_1}) (z_n - z_{\ell_2})} \right] \\
 & + 10 \sum_{\ell_1, \ell_2, \ell_3=1}^N *n* \left[\frac{\left(2 z_n^{(3)} z_{\ell_1}^{(1)} z_{\ell_2}^{(1)} z_{\ell_3}^{(1)} + 6 z_n^{(1)} z_{\ell_1}^{(3)} z_{\ell_2}^{(1)} z_{\ell_3}^{(1)} \right) + 9 z_n^{(2)} z_{\ell_1}^{(2)} z_{\ell_2}^{(1)} z_{\ell_3}^{(1)} + 9 z_n^{(1)} z_{\ell_1}^{(2)} z_{\ell_2}^{(2)} z_{\ell_3}^{(1)}}{(z_n - z_{\ell_1}) (z_n - z_{\ell_2}) (z_n - z_{\ell_3})} \right] \\
 & - 15 \sum_{\ell_1, \ell_2, \ell_3, \ell_4=1}^N *n* \left[\frac{z_n^{(2)} z_{\ell_1}^{(1)} z_{\ell_2}^{(1)} z_{\ell_3}^{(1)} z_{\ell_4}^{(1)} + 4 z_n^{(1)} z_{\ell_1}^{(2)} z_{\ell_2}^{(1)} z_{\ell_3}^{(1)} z_{\ell_4}^{(1)}}{(z_n - z_{\ell_1}) (z_n - z_{\ell_2}) (z_n - z_{\ell_3}) (z_n - z_{\ell_4})} \right] \\
 & + 6 \sum_{\ell_1, \ell_2, \ell_3, \ell_4, \ell_5=1}^N *n* \left[\frac{z_n^{(1)} z_{\ell_1}^{(1)} z_{\ell_2}^{(1)} z_{\ell_3}^{(1)} z_{\ell_4}^{(1)} z_{\ell_5}^{(1)}}{(z_n - z_{\ell_1}) (z_n - z_{\ell_2}) (z_n - z_{\ell_3}) (z_n - z_{\ell_4}) (z_n - z_{\ell_5})} \right] \\
 & - \left[\prod_{\ell=1, \ell \neq n}^N (z_n - z_\ell) \right]^{-1} \sum_{m=1}^N \left[c_m^{(6)} (z_n)^{N-m} \right]. \quad (3.7f)
 \end{aligned}$$

For $N = 2$ and $k \geq 2$,

$$\begin{aligned}
 z_n^{(k)} = & (z_n - z_{\tilde{n}(n)})^{-1} \left\{ \sum_{k_1=1}^{k-1} \left[\left(\frac{k!}{(k-k_1)! k_1!} \right) z_n^{(k-k_1)} z_{\tilde{n}(n)}^{(k_1)} \right] \right. \\
 & \left. - \sum_{m=1}^2 \left[c_m^{(k)} (z_n)^{2-m} \right] \right\}, \quad n = 1, 2, \quad \tilde{n}(1) = 2, \quad \tilde{n}(2) = 1. \quad (3.8)
 \end{aligned}$$

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