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Orbital stability of the smooth solitary wave with nonzero asymptotic value for the mCH equation

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This paper is concerned with orbital stability of the smooth solitary wave with nonzero asymptotic value for the mCH equation

$$u_t - u_{xxt} + 2ku_x + au^2u_x = 2u_xu_{xx} + uu_{xxx}.$$

Under the parametric conditions $a > 0$ and $k < \frac{1}{8a}$, an interesting phenomenon is discovered, that is, for the stability there exist three bifurcation wave speeds

$$c_1 = \frac{1 + \sqrt{1 - 8ak}}{2a}, \quad c_2 = \frac{9}{2a} \quad \text{and} \quad c_3 = \frac{9 + 3\sqrt{9 - 8ak}}{2a}$$

such that the following conclusions hold.

(i) When wave speed belongs to the interval (c_1, c_2) for $-\frac{63}{8a} < k < \frac{1}{8a}$, the smooth solitary wave is orbitally stable.

(ii) When wave speed belongs to the interval (c_2, c_3) for $-\frac{63}{8a} < k < \frac{1}{8a}$, the smooth solitary wave is orbitally unstable.

(iii) When wave speed belongs to the interval (c_1, c_3) for $k \leq -\frac{63}{8a}$, the smooth solitary wave is orbitally unstable.

Keywords: Orbital stability; Solitary wave; The mCH equation.

2000 Mathematics Subject Classification: 35B35, 35G25, 35D30

1. Introduction

The celebrated Camassa-Holm(CH) equation is of the form

$$u_t - u_{xxt} + 2ku_x + 3uu_x = 2u_xu_{xx} + uu_{xxx}, \quad (1.1)$$

for the function $u(x, t)$ of a single spatial variable x and time t . Eq. (1.1) was introduced as a new integrable system by Fuchssteiner and Fokas [13] in 1981.

Later, Camassa and Holm [3] recovered Eq. (1.1) as a shallow water wave model and showed that when $k = 0$, it has a non-smooth solitary wave with expression

$$u_0(x, t) = ce^{-|x-ct|}. \quad (1.2)$$

They discovered that the non-smooth solitary wave is a soliton [4]. The soliton is not a classical solution of the shallow water equation, it has to be understood (see [8]) as a weak solution.

When $k = 0$, Constantin, Strauss and Molinet [9,10] demonstrated that $u_0(x, t)$ is orbitally stable. That is, a wave starting close to $u_0(x, t)$ remains close to some translate of it at all later times. Thus the shape of the wave remains approximately the same for all times.

When $k \neq 0$, Liu et. al [29] proved that Eq. (1.1) has a non-smooth solitary wave with expression

$$u_k(x, t) = (k + c)e^{-|x-ct|} - k. \quad (1.3)$$

Ouyang et. al [30] confirmed that $u_k(x, t)$ is orbitally stable too. Many authors, for instance [7, 11, 17–19, 22, 37, 39], studied others properties of the CH equation.

Further, some generalized forms of the CH equation have been considered successively. Li and Olver [23] established local well-posedness in the Sobolev space H^s with any $s > \frac{3}{2}$ for the equation

$$u_t - u_{xxt} + 2ku_x + auu_x = 2u_xu_{xx} + uu_{xxx}. \quad (1.4)$$

Qian and Tang [31] studied some peakons and periodic cusp waves for Eq. (1.4).

Based on Eq. (1.4), Liu and Qian [28] suggested the equation

$$u_t - u_{xxt} + 2ku_x + au^2u_x = 2u_xu_{xx} + uu_{xxx}, \quad (1.5)$$

which is called the mCH equation. When $a = 3$, Eq. (1.5) becomes

$$u_t - u_{xxt} + 2ku_x + 3u^2u_x = 2u_xu_{xx} + uu_{xxx}, \quad (1.6)$$

which has been investigated by many authors. For example, Tian and Song [34] gave some physical explanation for Eq. (1.5). He et. al [16] used the integral bifurcation method to construct some exact traveling wave solutions of Eq. (1.5). Khuri [20] obtained some periodic wave solutions for Eq. (1.5). Shen and Xu [33] gave the condition under which compactons and cusp waves appear. When $k = 0$, Wazwaz [35] showed that there is a bell-shaped solitary wave solution

$$u(x, t) = -2\text{sech}^2\frac{1}{2}(x - 2t). \quad (1.7)$$

Liu and Liang [27] showed that there are several bifurcation wave speeds such that peakon and anti-peakon appear in Eq. (1.5). Specially, Yin et. al [38] showed that the solitary waves of Eq. (1.5) are stable for arbitrary wave speed. For stability of solitary waves to the other CH-types, many interesting results have been obtained, see [21, 22, 24, 32] for example.

Just as mentioned above, on the study of stability of solitary waves for the CH equation and its generalized forms, pioneers only considered the case of zero asymptotic value. It was showed that the solitary waves with zero asymptotic value are orbital stability for any wave speed [38]. Orbital stability of solitary waves for the KdV type was investigated by Yin [36] and Zhang [40]. Yin [36] showed that the smooth solitary waves of the generalized Korteweg-de-Vries equation are stable for any speed of wave propagation. Zhang [40] proved that the stability of the solitary waves with nonzero asymptotic value is different from that of solitary waves with zero asymptotic value.

The solitary waves with zero asymptotic value are orbitally stable. However, for one of the solitary waves with nonzero asymptotic value, it is orbitally unstable just in part of the range of wave speed which makes the solution meaningful. Therefore, we cannot derive orbital stability of solitary waves with nonzero asymptotic value from that of solitary waves with zero asymptotic value.

In this paper, attention will be directed to orbital stability of the smooth solitary wave with nonzero asymptotic value for Eq. (1.5). Firstly, we give the explicit expression of the smooth solitary wave of Eq. (1.5). Secondly, we derive its orbital stability by applying the theory presented by Grillakis-Shatah-Strauss [14, 15]. Finally, via decaying estimates on the semigroup and the methods used in [2, 25, 26], we obtain its orbital instability. From the results of the stability analysis it follows that, unlike the CH equation (1.1), not all the allowed solitary waves are stable.

2. Explicit expression of the smooth solitary wave

In this section, we give the explicit expression of solitary wave solution with nonzero asymptotic value for Eq. (1.5).

We substitute $u = \varphi(\xi)$ with $\xi = x - ct$ into Eq. (1.5). Then we get

$$-c\varphi' + c\varphi''' + 2k\varphi' + a\varphi^2\varphi' = 2\varphi'\varphi'' + \varphi\varphi''', \quad (2.1)$$

where c is a constant wave speed.

Integrating (2.1) once, we have

$$\varphi''(\varphi - c) = \frac{a}{3}\varphi^3 + (2k - c)\varphi - \frac{1}{2}(\varphi')^2 + g, \quad (2.2)$$

where g is a constant of integration.

Letting $y = \varphi'$, we get the following planar system

$$\begin{cases} \frac{d\varphi}{d\xi} = y, \\ \frac{dy}{d\xi} = \frac{1}{\varphi - c} \left(\frac{a}{3}\varphi^3 + (2k - c)\varphi - \frac{1}{2}y^2 + g \right). \end{cases} \quad (2.3)$$

Since in system (2.3) there is a singular line $l : \varphi = c$ which is inconvenient for our study, we multiply both sides of system (2.3) by $\varphi - c$. Then we have

$$\begin{cases} (\varphi - c)\frac{d\varphi}{d\xi} = (\varphi - c)y, \\ (\varphi - c)\frac{dy}{d\xi} = \frac{a}{3}\varphi^3 + (2k - c)\varphi - \frac{1}{2}y^2 + g. \end{cases} \quad (2.4)$$

Using the transformation $d\xi = (\varphi - c)d\tau$, (2.4) is carried into the Hamiltonian system

$$\begin{cases} \frac{d\varphi}{d\tau} = (\varphi - c)y, \\ \frac{dy}{d\tau} = \frac{a}{3}\varphi^3 + (2k - c)\varphi - \frac{1}{2}y^2 + g. \end{cases} \quad (2.5)$$

Systems (2.4) and (2.5) have the same first integral

$$H(\varphi, y) = h,$$

where

$$H(\varphi, y) = y^2(\varphi - c) - \frac{a}{6}\varphi^4 - (2k - c)\varphi^2 - 2g\varphi. \quad (2.6)$$

Let

$$f(\varphi) = \frac{a}{6}\varphi^4 + (2k - c)\varphi^2 + 2g\varphi + h. \quad (2.7)$$

In order to get the solitary wave solution, we suppose that

$$f(\varphi) = \frac{a}{6}(\varphi - c)(\varphi - \alpha)^2(\varphi - \beta), \quad (2.8)$$

where α and β are the roots of $f(\varphi) = 0$.

From (2.7) and (2.8), we get

$$\alpha = \frac{-ac + \Delta}{3a},$$

$$\beta = \frac{-ac - 2\Delta}{3a},$$

where $\Delta = \sqrt{-2a^2c^2 - 18a(2k - c)}$. Then (2.6) can be rewritten as

$$y^2 = \frac{a}{6}(\varphi - \alpha)^2(\varphi - \beta), \quad (2.9)$$

or

$$y = \pm \sqrt{\frac{a}{6}} \sqrt{\varphi - \beta}(\varphi - \alpha), \quad (a > 0). \quad (2.10)$$

Substituting the expression (2.10) into $\frac{d\varphi}{d\xi} = y$ and integrating it, we have

$$\int_{\beta}^{\varphi} \frac{ds}{(s - \alpha)\sqrt{s - \beta}} = \sqrt{\frac{a}{6}} \int_0^{\xi} d\xi. \quad (2.11)$$

In (2.11) completing the integration and solving the equation for φ , it follows that

$$\varphi(\xi) = \alpha + (\beta - \alpha)\operatorname{sech}^2 \frac{\eta}{2}, \quad (2.12)$$

where $\eta = \sqrt{\frac{a(\alpha - \beta)}{6}}\xi$.

Note that $u(x, t) = \varphi(\xi)$, we obtain the smooth solitary wave solution

$$u(x, t) = \alpha + (\beta - \alpha)\operatorname{sech}^2 \frac{\eta}{2} \quad (2.13)$$

with nonzero asymptotic value $D = \alpha$ for Eq. (1.5).

Next, we transform Eq. (1.5) into a new nonlinear equation by a translation transformation. Therefore, we only need to study the stability of the solitary wave with zero asymptotic value for the new equation.

3. Spectral analysis

In this section, we prove that the equation which derives from Eq. (1.5) by using the translation transformation is a Hamiltonian system, and satisfies the conditions of the orbital stability theory presented by Grillakis-Shatah-Strauss [14, 15].

By using the translation transformation $u = \varphi + D$ to Eq. (1.5), we have

$$\varphi_t - \varphi_{xxt} + 2k\varphi_x + a(\varphi + D)^2\varphi_x = 2\varphi_x\varphi_{xx} + \varphi\varphi_{xxx} + D\varphi_{xxx}. \quad (3.1)$$

For convenience, Eq. (3.1) can be rewritten as

$$u_t - u_{xxt} + 2ku_x + a(u + D)^2u_x = 2u_xu_{xx} + uu_{xxx} + Du_{xxx}. \quad (3.2)$$

Analogously to the case [3] for Eq. (1.1), Eq. (3.2) can be written in Hamiltonian form and has the invariants

$$\begin{aligned} E(u) &= \frac{1}{2} \int_R (u^2 + u_x^2) dx, \\ F(u) &= \int_R \left[\frac{a}{12} (u^4 + 4u^3D + 6u^2D^2 + 4uD^3) + \frac{1}{2}(u + D)u_x^2 + ku^2 \right] dx. \end{aligned} \quad (3.3)$$

In fact, if u is a classical solution of Eq. (3.2), it is straightforward to check that $E(u)$ is conserved. To see that $F(u)$ is an invariant, a different approach is needed.

Let $v := u - u_{xx}$. Furthermore, Eq. (3.2) can be written in a Hamiltonian form

$$v_t = -\partial_x(1 - \partial_x^2) \frac{\partial F}{\partial v}, \quad (3.4)$$

where $\frac{\partial F}{\partial v}$ denotes the variational derivative of the functional F , defined by

$$\left\langle \frac{\partial F}{\partial v}, f \right\rangle_{L^2} = \frac{d}{d\varepsilon} F(v + \varepsilon f)|_{\varepsilon=0}. \quad (3.5)$$

Indeed, since $v = u - u_{xx}$, we have (see [5], page 70)

$$\frac{\partial F}{\partial u} = (1 - \partial_x^2) \frac{\partial F}{\partial v}. \quad (3.6)$$

Furthermore, an easy computation yields

$$\frac{\partial F}{\partial u} = \frac{a}{3}u^3 + aDu^2 + aD^2u - \frac{1}{2}u_x^2 - (u + D)u_{xx} + 2ku. \quad (3.7)$$

Therefore, Eq. (3.2) takes the form

$$v_t = -\partial_x \frac{\partial F}{\partial u} = -\partial_x(1 - \partial_x^2) \frac{\partial F}{\partial v}, \quad (3.8)$$

where $u \in X$, $X = H^2(R)$ whose dual space is denoted by $X^* = H^{-2}(R)$. (3.8) ensures that F is an invariant.

Let the inner product of X be

$$(f, g) = \int_R fg + f'g' + f''g'' dx, \quad \forall f, g \in X. \quad (3.9)$$

There exists a natural isomorphism $I : X \rightarrow X^*$ defined by

$$\langle If, g \rangle = (f, g), \quad (3.10)$$

where

$$\langle f, g \rangle = \int_R fg dx. \quad (3.11)$$

From (3.9)-(3.11), it is clearly that $I = 1 - \frac{\partial^2}{\partial x^2} + \frac{\partial^4}{\partial x^4}$.

Let T be a one-parameter group of unitary operator on X defined by

$$T(s)u(\cdot) = u(\cdot - s), \quad \forall s \in R, u(\cdot) \in X. \quad (3.12)$$

From (3.12), we get $T'(0) = -\frac{\partial}{\partial x}$.

Now we consider the orbital stability of solitary waves $\varphi_c = T(ct)\varphi(x)$ with zero asymptotic value for Eq. (3.2), where

$$\varphi(x) = (\beta - \alpha) \operatorname{sech}^2 \sqrt{\frac{a(\alpha - \beta)}{24}} x.$$

Next, we give the existence of solutions to the initial value problem of Eq. (3.2) from Theorem 2.2 in [39].

Lemma 3.1. [39] Assume that $s > \frac{3}{2}$. For any fixed $u_0 \in H^s$, there exists a unique solution $u \in C([0, T^*), H^s(R))$ for some $0 < T^* < \infty$, such that $u(0) = u_0$ for Eq. (3.2).

Since $E(u)$ and $F(u)$ are invariants, it is easy to know that $E(u)$ and $F(u)$ satisfy

$$E(u(t)) = E(u(0)) = E(u_0),$$

$$F(u(t)) = F(u(0)) = F(u_0),$$

respectively. From (2.12), we know that $\varphi_c(\xi) = \varphi(\xi) - D$ is a bounded state solution.

Lemma 3.2. φ_c satisfies $cE'(\varphi_c) - F'(\varphi_c) = 0$.

Proof. Notice that φ_c satisfies Eq. (3.2), we have

$$-c\varphi_{cx} + c\varphi_{cxx} + 2k\varphi_{cx} + a\varphi_c^2\varphi_{cx} + 2aD\varphi_c\varphi_{cx} + aD^2\varphi_{cx} = 2\varphi_{cx}\varphi_{cxx} + \varphi_c\varphi_{cxxx} + D\varphi_{cxxx}. \quad (3.13)$$

Integrating (3.13) once leads to

$$-c\varphi_c + c\varphi_{cxx} + 2k\varphi_c + \frac{a}{3}\varphi_c^3 + aD\varphi_c^2 + aD^2\varphi_c = \frac{1}{2}(\varphi_{cx})^2 + \varphi_c\varphi_{cxx} + D\varphi_{cxx} + g_1,$$

where g_1 is a constant of integration. Since $\varphi_c, \varphi_{cx}, \varphi_{cxx} \rightarrow 0$ as $x \rightarrow \infty$, $g_1 = 0$, i.e.

$$-c\varphi_c + c\varphi_{cxx} + 2k\varphi_c + \frac{a}{3}\varphi_c^3 + aD\varphi_c^2 + aD^2\varphi_c = \frac{1}{2}(\varphi_{cx})^2 + \varphi_c\varphi_{cxx} + D\varphi_{cxx}. \quad (3.14)$$

Furthermore,

$$cE'(\varphi_c) - F'(\varphi_c) = c\varphi_c - c\varphi_{cxx} - 2k\varphi_c - \frac{a}{3}\varphi_c^3 - aD\varphi_c^2 - aD^2\varphi_c + \frac{1}{2}(\varphi_{cx})^2 + \varphi_c\varphi_{cxx} + D\varphi_{cxx}. \quad (3.15)$$

Therefore, from (3.14) and (3.15), we obtain that φ_c satisfies

$$cE'(\varphi_c) - F'(\varphi_c) = 0. \quad (3.16)$$

□

Now we define the operator $H_c : X \rightarrow X^*$,

$$H_c = cE''(\varphi_c) - F''(\varphi_c), \quad (3.17)$$

where

$$E''(\varphi_c) = 1 - \partial_{xx}, \quad (3.18)$$

$$F''(\varphi_c) = a\varphi_c^2 + 2aD\varphi_c + aD^2 - \varphi_{cx}\partial_x - \varphi_c\partial_{xx} - \varphi_{cxx} - D\partial_{xx} + 2k. \quad (3.19)$$

Therefore,

$$H_c = -a\varphi_c^2 - 2aD\varphi_c + (\varphi_c + D - c)\partial_{xx} + \varphi_{cx}\partial_x + \varphi_{cxx} - (aD^2 + 2k - c). \quad (3.20)$$

For any $\varphi_1, \varphi_2 \in H^2(\mathbb{R})$, we have

$$\langle H_c\varphi_1, \varphi_2 \rangle = \langle \varphi_1, H_c\varphi_2 \rangle, \quad (3.21)$$

which suggests that H_c is a self-conjugate operator in the sense that $H_c = H_c^*$.

In fact,

$$\begin{aligned} \langle H_c\varphi_1, \varphi_2 \rangle &= \int_{\mathbb{R}} [-a\varphi_c^2\varphi_1 - 2aD\varphi_c\varphi_1 + (\varphi_c + D - c)\varphi_{1xx} + \varphi_{cx}\varphi_{1x} + \varphi_{cxx}\varphi_1 - (aD^2 + 2k - c)\varphi_1] \varphi_2 dx, \end{aligned} \quad (3.22)$$

where

$$\begin{aligned} & \int_{\mathbb{R}} [(\varphi_c + D - c)\varphi_{1xx}\varphi_2 + \varphi_{cx}\varphi_{1x}\varphi_2] dx \\ &= \int_{\mathbb{R}} [(\varphi_c + D - c)\varphi_2] d\varphi_{1x} + \int_{\mathbb{R}} \varphi_{cx}\varphi_2 d\varphi_1 \\ &= - \int_{\mathbb{R}} [\varphi_{cx}\varphi_2 + (\varphi_c + D - c)\varphi_{2x}] \varphi_{1x} dx - \int_{\mathbb{R}} (\varphi_{cxx}\varphi_2\varphi_1 + \varphi_{cx}\varphi_{2x}\varphi_1) dx \\ &= \int_{\mathbb{R}} [\varphi_{cxx}\varphi_2 + 2\varphi_{cx}\varphi_{2x} + (\varphi_c + D - c)\varphi_{2xx}] \varphi_1 dx - \int_{\mathbb{R}} (\varphi_{cxx}\varphi_2\varphi_1 + \varphi_{cx}\varphi_{2x}\varphi_1) dx \\ &= \int_{\mathbb{R}} [\varphi_{cx}\varphi_1\varphi_{2x} + (\varphi_c + D - c)\varphi_1\varphi_{2xx}] dx. \end{aligned}$$

Therefore,

$$\begin{aligned} & \langle H_c\varphi_1, \varphi_2 \rangle \\ &= \int_{\mathbb{R}} \varphi_1 \left[-a\varphi_c^2\varphi_2 - 2aD\varphi_c\varphi_2 + (\varphi_c + D - c)\varphi_{2xx} + \varphi_{cx}\varphi_{2x} + \varphi_{cxx}\varphi_2 - (aD^2 + 2k - c)\varphi_2 \right] dx \\ &= \langle \varphi_1, H_c\varphi_2 \rangle. \end{aligned}$$

This means that $I^{-1}H_c$ is a bounded self-conjugate operator on X . The eigenvalues of H_c consist of the real numbers λ which ensure that $H_c - \lambda I$ has a non-trivial kernel.

We claim that $\lambda = 0$ belongs to the eigenvalues of H_c . From (3.13) and (3.20), we have $H_c T'(0)\varphi_c(x) = -H_c \varphi_{cx} = 0$. Letting $Z' = \{r\varphi_{cx} | r \in R\}$, then Z' is contained in the kernel of H_c denoted by $Z = \{u \in X | H_c u = 0\}$.

Obviously, $x = 0$ is a unique zero point of φ_{cx} , by using Sturm-Liouville theorem, we know that zero is the second eigenvalue of H_c . Furthermore, H_c only has a negative eigenvalue $-\sigma^2$, whose corresponding eigenfunction is denoted by χ . Namely, $H\chi = -\sigma^2\chi$, where $\langle \chi, \chi \rangle = 1$.

Since $\varphi, \varphi_{cx}, \varphi_{cxx} \rightarrow 0$ exponentially fast as $|x| \rightarrow \infty$, we obtain that the essential spectrum of H_c is $essH_c = [c - 2k - aD^2, +\infty)$ by Weyl's essential spectrum theorem, where $c - 2k - aD^2 > 0$.

According to the above analysis, we make spectrum decomposition for H_c . Let

$$\begin{aligned} Z &= \{r_1 \varphi_{cx} | r_1 \in R\}, \\ N &= \{r_2 \chi | r_2 \in R\}, \\ P &= \{p \in X | (p, \chi) = (p, \varphi_{cx}) = 0\}. \end{aligned}$$

Due to $\langle H_c r_2 \chi, r_2 \chi \rangle = r_2^2 \langle H_2 \chi, \chi \rangle = -r_2^2 \sigma^2 \langle \chi, \chi \rangle = -r_2^2 \sigma^2 < 0$, we get $\langle H_c u, u \rangle < 0$ for any $u \in N$. Due to $\langle H_c r_1 \varphi_{cx}, r_1 \varphi_{cx} \rangle = 0$, we get $\langle H_c z, z \rangle = 0$ for any $z \in Z$.

For any $p \in P$, along the lines of proof in Appendix of [32], we find that for any real function $p \in H^2(R)$ with $\langle p, \chi \rangle = \langle p, \varphi_{cx} \rangle = 0$, there exists $\delta > 0$ independent of p such that $\langle H_c p, p \rangle \geq \delta \|p\|_X^2$. Therefore, $\langle H_c p, p \rangle > 0$.

So the space X can be decomposed as a direct sum $X = N + Z + P$, where Z is the kernel space of H_c , N is a finite-dimensional subspace and P is a closed subspace. Furthermore, it is known [14, 15] that the stability would be ensured by the convexity of the scalar function [6]

$$d(c) = cE(\varphi_c) - F(\varphi_c), \quad (3.23)$$

and $d''(c)$ as the Hessian matrix of function d .

4. Orbital stability of the solitary wave

Definition 4.1. Assume that $\varphi_c = T(ct)\varphi(x)$ is a solitary wave of Eq. (3.2). The solitary wave φ_c is called orbitally stable if for any $\varepsilon > 0$, there exists $\delta > 0$ such that if $u \in C([0, T^*), H^2(R))$ for some $0 < T^* < \infty$ is a solution of Eq. (3.2) with $u(0) = u_0$ and $\|u_0 - \varphi_c\|_{H^2} < \delta$, then for every $t \in [0, T^*)$,

$$\sup_{0 < t < +\infty} \inf_{s \in R} \|u(t) - T(s)\varphi_c\|_{H^2} < \varepsilon.$$

Otherwise, $T(ct)\varphi(x)$ is called orbitally unstable.

According to the general theory of Bona, Grillakis, Shatah, Souganidis, and Strauss in [2, 14, 15], the stability of the solitary wave depends on the convexity of the function $d(c)$. We have the following theorem:

Theorem 4.1. *The solitary wave φ_c is stable if the function d is strictly convex, i.e. $d''(c) > 0$. The solitary wave φ_c is unstable if the function d is strictly concave, i.e. $d''(c) < 0$.*

In view of (3.21) and taking (3.16) into account, we have

$$d'(c) = \langle cE'(\varphi_c) - F'(\varphi_c), \frac{\partial \varphi_c}{\partial c} \rangle + E(\varphi) = E(\varphi_c) = \frac{1}{2} \int_R (\varphi_c^2 + \varphi_{cx}^2) dx. \quad (4.1)$$

From (2.12) we have

$$\varphi_c = \frac{-4\Delta}{a} \frac{1}{e^{\sqrt{\frac{\Delta}{6}}x} + e^{-\sqrt{\frac{\Delta}{6}}x} + 2}, \quad (4.2)$$

and

$$\varphi_{cx} = \frac{4\Delta^{\frac{3}{2}} e^{\sqrt{\frac{\Delta}{6}}x} (e^{\sqrt{\frac{\Delta}{6}}x} - 1)}{\sqrt{6}a (1 + e^{\sqrt{\frac{\Delta}{6}}x})^3}. \quad (4.3)$$

Then, we have

$$d'(c) = \frac{1}{2} \int_R (\varphi_c^2 + \varphi_{cx}^2) dx = \frac{4\Delta^{\frac{3}{2}}(30 + \Delta)}{15\sqrt{6}a^2}, \quad (4.4)$$

and

$$d''(c) = \frac{\sqrt{2}(9 - 2ac)(18 + \Delta)}{3\sqrt{3}a\Delta^{\frac{1}{2}}}. \quad (4.5)$$

According to the orbital stability theorem, we only need to observe the sign of $d''(c)$. We note that $c - 2k - aD^2 > 0$ in the essential spectrum of H_c . Then we get $c_1 < c < c_2$. From (4.5) we know that the wave speed c has a critical point $c = c_2$, where $c_1 = \frac{1 - \sqrt{1 - 8ak}}{2a}$, $c_2 = \frac{9}{2a}$ and $c_3 = \frac{9 + 3\sqrt{9 - 8ak}}{2a}$. Then we have the following conclusions.

- (1) When $c_1 < c < c_2$ for $-\frac{63}{8a} < k < \frac{1}{8a}$, we have $d''(c) > 0$.
- (2) When $c_2 < c < c_3$ for $-\frac{63}{8a} < k < \frac{1}{8a}$, we have $d''(c) < 0$.
- (3) When $c_1 < c < c_3$ for $k \leq -\frac{63}{8a}$, we have $d''(c) < 0$.

Applying the abstract result on orbital stability of solitary wave, the solitary wave obtained in the Section 2 are orbitally stable when $d''(c) > 0$. Next, we will give the proof of orbital instability of the solitary wave when $d''(c) < 0$.

5. Orbital instability of the solitary wave

Due to the operator $\partial_x(1 - \partial_x^2)$ in (3.8) is not a one-to-one mapping, the abstract theory of orbital instability in [14, 15] cannot be applied directly. Bona et. al [2] proved that the solitary waves are indeed orbitally unstable when $d''(c) < 0$. The instability proof in [2] requires the use of two special ingredients which are the invariant $I(u)$ and an estimate of the primitives of the solution. In this section, we obtain the orbital instability of solitary wave when $d''(c) < 0$ by detailed decaying estimates on the semigroup and the methods of proof in [2].

For the instability, we first state Lemma 5.1 and Theorem 5.1.

Lemma 5.1. *Let $I(u) = \int_R u dx$. If $\int_R u(x, 0) dx$ converges, then for $0 \leq t \leq T^*$, $I(u)$ converges to constant, where T^* denotes the maximum existence time for $u(x, t)$.*

Proof. Integrating (3.2) over the domain $\{(x, t) : a \leq x \leq b, 0 \leq t \leq T^*\}$, we have

$$\begin{aligned} & \int_a^b [u(x, t) - u(x, 0)] dx \\ &= \int_a^b \int_0^t [2u_x u_{xx} + (u + D)u_{xxx} - a(u + D)^2 u_x - 2ku_x - u_{xxt}] dt dx \\ &= \int_a^b \int_0^t \left[\frac{1}{2}(u_x^2)_x + (uu_{xx})_x + Du_{xxx} - \frac{a}{3}((u + D)^3)_x - k(u^2)_x - (u_{xt})_x \right] dt dx. \end{aligned}$$

Since $u \in H^2$, the above formula tends to zero as $a \rightarrow -\infty$ and $b \rightarrow +\infty$. Hence if $I(u_0) = \int_R u_0(x) dx$ exists as an improper integral, then $I(u)$ exists and $I(u) = I(u_0)$. This completes the proof of Lemma 5.1. \square

The next theorem is the principal result of this section and also a key step in the proof of orbital instability.

Theorem 5.1. *Assume that $u(x, t)$ satisfies Eq. (3.2) with initial data $u_0(x) = u_0$. Then*

$$\sup_{-\infty < z < +\infty} \left| \int_{-\infty}^z u(x, t) dx \right| \leq C_0(1 + t^{\frac{7}{8}}), \quad (5.1)$$

where the constant C_0 depends only on u_0 .

To prove Theorem 5.1, we need a series of lemmas given below. The first one is the well-known Van der Corput Lemma which we state without proof.

Lemma 5.2. *Let h be either convex or concave on $[a, b]$ with $-\infty \leq a < b \leq +\infty$. Then*

$$\left| \int_a^b e^{ih(\omega)} d\omega \right| \leq 4 \{ \min_{[a, b]} |h''(\omega)| \}^{-\frac{1}{2}}, \quad \text{if } h'' \neq 0 \text{ in } [a, b].$$

Lemma 5.3. *For $1 \leq t < +\infty$, we have*

$$\sup_{-\infty < \gamma < +\infty} \left| \int_{-n}^n e^{ih(\omega, \gamma)} d\omega \right| \leq C_0(t^{-\frac{1}{3}} + t^{-\frac{1}{2}} n^{\frac{3}{2}}), \quad (5.2)$$

where $h(\omega, \gamma) = \frac{(k_1 - D)\omega}{1 + \omega^2} + \gamma\omega$, $k_1 = 2k + aD^2$, D is bounded and $h'(\omega, \gamma)$ denotes the derivative with respect to ω .

Proof. Through simple calculation, we have

$$h'(\omega, \gamma) = \frac{(k_1 - D)(1 - \omega^2)}{(1 + \omega^2)^2} + \gamma,$$

and

$$h''(\omega, \gamma) = \frac{2(k_1 - D)\omega(\omega^2 - 1)}{(1 + \omega^2)^3}.$$

Define $A = \{\omega \in \mathbb{R} \mid |\omega| < t^{-\theta}\}$, $B = \{\omega \in \mathbb{R} \mid 1 - t^{-\theta} < |\omega| < 1 + t^{-\theta}\}$ and $\Omega = [-n, n] \setminus (A \cup B)$, where $t > 1$ and $0 < \theta < 1$. By Lemma 5.2, we have

$$\left| \int_{\Omega} e^{ith(\omega, \gamma)} d\omega \right| \leq 4 \{t \min_{\Omega} |h''(\omega, \gamma)|\}^{-\frac{1}{2}} \leq C_0 \left(t^{-\frac{1}{2}} n^{\frac{3}{2}} + t^{-\frac{1}{2} + \frac{\theta}{2}} \right).$$

On the other hand,

$$\left| \int_{A \cup B} e^{ith(\omega, \gamma)} d\omega \right| \leq 6t^{-\theta}.$$

Hence,

$$\sup_{-\infty < \gamma < +\infty} \left| \int_{-n}^n e^{ith(\omega, \gamma)} d\omega \right| \leq C_0 \left(t^{-\frac{1}{2}} n^{\frac{3}{2}} + t^{-\frac{1}{2} + \frac{\theta}{2}} + t^{-\theta} \right).$$

Choosing $\theta = \frac{1}{3}$, we complete the proof of Lemma 5.5. □

Lemma 5.4. $(1 - \partial_x^2)^{-1}$ is a convolution by a function in L^1 , that is, $(1 - \partial_x^2)^{-1}u = K * u$, where $K(x) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} e^{ix\omega} (1 + \omega^2)^{-1} d\omega$, $K(x) \in L^1(\mathbb{R})$ and $\|(1 - \partial_x^2)^{-1}u\|_{L^1(\mathbb{R})} \leq 2\|u\|_{L^1(\mathbb{R})}$.

Proof. Since

$$\frac{1}{1 + \xi^2} = \frac{1}{4\pi} \int_0^{+\infty} e^{-\frac{1+\xi^2}{4\pi}y} dy,$$

using the Fubini's theorem, we obtain

$$K(x) = \frac{1}{2\pi} \int_0^{+\infty} e^{-\frac{\pi|x|^2}{y}} \cdot e^{-\frac{y}{4\pi}} \cdot y^{-\frac{1}{2}} dy.$$

Notice that

$$\int_{-\infty}^{+\infty} e^{-\frac{\pi|x|^2}{y}} dx = y^{\frac{1}{2}}.$$

Applying the Fubini's theorem again, we get

$$\int_{-\infty}^{+\infty} |K(x)| dx = \frac{1}{2\pi} \int_0^{+\infty} e^{-\frac{y}{4\pi}} dy = 2.$$

Therefore using Young's inequality, we obtain

$$\|(1 - \Delta)^{-1}u\|_{L^1(\mathbb{R})} \leq \|K(x)\|_{L^1(\mathbb{R})} \cdot \|u\|_{L^1(\mathbb{R})} \leq 2\|u\|_{L^1(\mathbb{R})}.$$

□

Lemma 5.5. Let $S(t)$ be the evolution operator for the linear equation

$$(1 - \partial_x^2)u_t + k_1 u_x - Du_{xxx} = 0, \quad (5.3)$$

that is $u(t) = S(t)u(0)$. Then there exists a constant $C_0 > 0$ such that

$$|u(t)| \leq C_0 t^{-\frac{1}{8}} \left(\|u(0)\|_{L^1(R)} + \|u(0)\|_{H^1(R)} \right).$$

Proof. The solution of the linear equation (5.3) is

$$u(t) = S(t)u(0) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} \left[e^{ix\omega} e^{i\frac{k_1\omega}{1+\omega^2}t + i\frac{D\omega^3}{1+\omega^2}t} \hat{u}_0(\omega) \right] d\omega,$$

$\hat{u}_0(\omega)$ is the Fourier transform of $u_0(\omega)$.

In virtue of Lemma 5.3 and Lemma 5.4, we have

$$\begin{aligned} |u(t)| &= |S(t)u(0)| \\ &= \left| \frac{1}{2\pi} \int_{-\infty}^{+\infty} \left[e^{it\left(\frac{(k_1-D)\omega}{1+\omega^2} + (D+\frac{x}{t})\omega\right)} \hat{u}_0(\omega) \right] d\omega \right| \\ &\leq \frac{1}{2\pi} \left[\int_{|\omega|>n} |\hat{u}_0(\omega)| d\omega + \left| \int_{-n}^n e^{it\left(\frac{(k_1-D)\omega}{1+\omega^2} + (D+\frac{x}{t})\omega\right)} d\omega \right| |\hat{u}_0(\omega)|_{L^\infty(R)} \right] \\ &\leq \frac{1}{2\pi} \left[\|u_0\|_{H^1(R)} \left(\int_{|\omega|>n} (1+|\omega|)^{-2} d\omega \right)^{\frac{1}{2}} + C_0(t^{-\frac{1}{3}} + t^{-\frac{1}{2}}n^{\frac{3}{2}}) \|u_0\|_{L^1(R)} \right] \\ &\leq C_0 \left(n^{-\frac{1}{2}} + t^{-\frac{1}{3}} + t^{-\frac{1}{2}}n^{\frac{3}{2}} \right) \left(\|u(0)\|_{L^1(R)} + \|u(0)\|_{H^1(R)} \right). \end{aligned}$$

Choosing $n = t^{\frac{1}{4}} (t \geq 1)$, we obtain

$$|u(t)| \leq C_0 t^{-\frac{1}{8}} \left(\|u(0)\|_{L^1(R)} + \|u(0)\|_{H^1(R)} \right).$$

This completes the proof of Lemma 5.5. □

Proof of Theorem 5.1. Let $z(t) = S(t)u(0)$, that is

$$(1 - \partial_x^2)z_t + k_1 z_x - Dz_{xxx} = 0, \quad z(0) = u(0) = u_0.$$

Then

$$\begin{aligned} u(t) &= z(t) - \int_0^t \left[S(t-\tau)(1 - \partial_x^2)^{-1} \partial_x \left(\frac{a}{3}u^3 + aDu^2 - \frac{1}{2}u_x^2 - uu_{xx} \right) \right] d\tau \\ &= z(t) - \partial_x \int_0^t \left[S(t-\tau)(1 - \partial_x^2)^{-1} \left(\frac{a}{3}u^3 + c_1(1 - \partial_x^2)u^2 + c_2u^2 - \frac{3}{2}u_x^2 \right) \right] d\tau, \end{aligned}$$

where $c_1 = -\frac{1}{2}$, $c_2 = aD + \frac{1}{2}$. Let

$$U(x, t) = \int_{-\infty}^x u(y, t) dy, \quad Z(x, t) = \int_{-\infty}^x z(y, t) dy.$$

Then

$$U(t) = Z(t) - \int_0^t \left[S(t-\tau) \left((1 - \partial_x^2)^{-1} \frac{a}{3}u^3 + c_1u^2 + c_2(1 - \partial_x^2)^{-1}u^2 - \frac{3}{2}(1 - \partial_x^2)^{-1}u_x^2 \right) \right] d\tau.$$

We estimate the two terms on the right-hand side of the above equation separately. From the equation for $z(x, t)$ we write

$$\begin{aligned} z(x, t) &= u_0 - \int_0^t (1 - \partial_x^2)^{-1} \partial_x (k_1 z(\tau) - D \partial_{xx} z(\tau)) d\tau \\ &= u_0 - \partial_x \int_0^t (1 - \partial_x^2)^{-1} (k_1 z(\tau) - D \partial_{xx} z(\tau)) d\tau, \end{aligned}$$

so that the first term is

$$\begin{aligned} Z(t) &= U_0 - \partial_x \int_0^t (1 - \partial_x^2)^{-1} (k_1 z(\tau) - D \partial_{xx} z(\tau)) d\tau \\ &= U_0 - \int_0^t S(\tau) [(k_1 - D)(1 - \partial_x^2)^{-1} u_0 + D u_0] d\tau, \end{aligned}$$

where $U_0(x) = \int_{-\infty}^x u_0(y) dy$. Using Lemma 5.5, we have

$$|Z(x, t)| \leq C_0(1 + t^{\frac{7}{8}}) \|u_0\|_{H^1 \cap L^1}.$$

If $P(x, t)$ denotes the second term, then from Lemma 5.5, we have

$$\begin{aligned} |P(x, t)| &\leq \int_0^t |c_1 S(t - \tau) u^2| d\tau + \int_0^t \left| \frac{a}{3} S(t - \tau) (1 - \partial_x^2)^{-1} u^3 \right| d\tau \\ &\quad + \int_0^t |c_2 S(t - \tau) (1 - \partial_x^2)^{-1} u^2| d\tau + \int_0^t \left| \frac{3}{2} S(t - \tau) (1 - \partial_x^2)^{-1} u_x^2 \right| d\tau \\ &\leq \int_0^t C_1 (1 + t + \tau)^{-\frac{1}{8}} \|u^2\|_{H^1 \cap L^1} d\tau + \int_0^t C_2 (1 + t + \tau)^{-\frac{1}{8}} \|(1 - \partial_x^2)^{-1} u^3\|_{H^1 \cap L^1} d\tau \\ &\quad + \int_0^t C_3 (1 + t + \tau)^{-\frac{1}{8}} \|(1 - \partial_x^2)^{-1} u^2\|_{H^1 \cap L^1} d\tau + \int_0^t C_4 (1 + t + \tau)^{-\frac{1}{8}} \|(1 - \partial_x^2)^{-1} u_x^2\|_{H^1 \cap L^1} d\tau. \end{aligned}$$

Since $u \in H^1$, then $u \in L^\infty$. So $\|u^2\|_{L^1} \leq C_0$, $\|u^3\|_{L^1} \leq C_0$, $\|(1 - \partial_x^2)^{-1} u^3\|_{L^1} \leq C_0$, $\|(1 - \partial_x^2)^{-1} u^2\|_{L^1} \leq C_0$. Since H^1 is an algebra,

$$\|u^p\|_{H^1} \leq \|u\|_{H^1}^p \leq \|u_0\|_{H^1}^p,$$

and

$$\|(1 - \partial_x^2)^{-1} u^p\|_{H^1} \leq \|u^p\|_{H^1} \leq \|u_0\|_{H^1}^p.$$

Next, we observe that

$$\|(1 - \partial_x^2)^{-1} u_x u_x\|_{H^1} \leq \|u_x u_x\|_{H^1} \leq \|u_x\|_{H^0} \|u_x\|_{H^0} \leq C_0 \|u\|_{H^1}^2,$$

and

$$\|(1 - \partial_x^2)^{-1} u_x u_x\|_{L^1} \leq C_0 \|u_x u_x\|_{L^1} = C_0 \int_{\mathbb{R}} |u_x|^2 dx \leq C_0 \|u\|_{H^1}.$$

Thus

$$|P(x, t)| \leq C_0 \int_0^t (1 + t + \tau)^{-\frac{1}{8}} d\tau \leq C_0 (1 + t)^{\frac{7}{8}}.$$

This completes the proof of Theorem 5.1. □

Next, we will give the outline of the proof of instability. Let $\varepsilon > 0$ be sufficiently small. To prove the instability of φ_c , it suffice to show that there exists $u_0 \in X$ sufficiently close to φ_c such that the solution u subject to the initial data u_0 exists outside the “tube” $U_\varepsilon \triangleq \{u \in X : \inf_{s \in \mathbb{R}} \|u - T(s)\varphi_c\|_X < \varepsilon\}$ in finite time. Let $[0, t_1)$ denote the maximal interval for which u lies continuously in U_ε . Next we will show that $t_1 < +\infty$.

We define the functional

$$A(t) = \int_{-\infty}^{+\infty} Y(x - \beta(t))u(x, t)dx, \text{ for } 0 \leq t < t_1,$$

where $Y(x) = \int_{-\infty}^x y(p)dp - \partial_x y(x)$, $y = \frac{d}{ds}(\varphi_{c(s)} + s\chi)|_{s=0}$. Here χ is defined in Section 3. By Lemma 5.1 and Theorem 5.1, using Minkowski’s inequality, we can obtain the following estimate

$$|A(t)| \leq C_0(1 + t^{\frac{7}{8}}), \quad 0 \leq t < t_1. \quad (5.4)$$

On the other hand, using the similar ideas in [2, 12, 26, 41], we can prove that there exists some $\delta_0 > 0$ such that

$$\frac{dA}{dt} < -\delta_0. \quad (5.5)$$

From (5.4) and (5.5), we have $t_1 \neq +\infty$, which implies $t_1 < +\infty$. Thus the solitary wave obtained in Section 2 is orbitally unstable when $d''(c) < 0$.

6. Conclusions

In this paper, we have investigated the stability of the smooth solitary wave with nonzero asymptotic value for the mCH equation. We have not only showed that there exists a smooth solitary wave with explicit expression, but also derived its orbital stability. From the results of the stability analysis it follows that, unlike the CH equation, not all the allowed solitary waves are stable.

Note that in this paper there are two problems waiting to solve. The first one is that the wave speed c has a critical point $c = c_2$. At this critical point we do not know whether the solitary wave is orbitally stable. The second one is that we have investigated the orbital stability of the smooth solitary wave. But the stability of the non-smooth solitary wave awaits further study.

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