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A Solvable N -body Problem of Goldfish Type Featuring N^2 Arbitrary Coupling Constants

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A N -body problem “of goldfish type” is introduced, the Newtonian (“acceleration equal force”) equations of motion of which describe the motion of N pointlike unit-mass particles moving in the complex z -plane. The model—for arbitrary N —is *solvable*, namely its configuration (positions and velocities of the N “particles”) at *any later* time t can be obtained from its configuration at the *initial* time by *algebraic* operations. It features specific nonlinear velocity-dependent many-body forces depending on N^2 arbitrary (complex) coupling constants. Sufficient conditions on these constants are identified which cause the model to be *isochronous*—so that *all* its solutions are then periodic with a fixed period independent of the initial data. A variant with *twice* as many arbitrary coupling constants, or even more, is also identified.

Keywords: Solvable N -body problems; integrable N -body problems; integrable dynamical systems.

2000 Mathematics Subject Classification: 34C15, 70K75, 70K90

1. Introduction

Recently a technique has been introduced [1] which allows a rather straightforward identification of solvable N -body problems “of goldfish type” (for the identification of this class of *solvable* N -body problems, and a selection of previous developments concerning this class, see for instance [2–18]). The usefulness of this technique to identify *new* many-body problems characterized by certain Newtonian (“acceleration equal force”) equations of motion describing the motion of N nonlinearly interacting pointlike unit-mass particles moving in the complex z -plane has been demonstrated by several examples [1, 19–21]. In this short communication we report one more such example with the *novel* property to feature as many as N^2 *arbitrary* coupling constants.

In the following Section 2 the equations of motion of this *solvable* model are displayed, and their solution is reported and tersely discussed. The following Section 3 explains how this model, and its solution, were obtained. A final Section 4 entitled “Outlook” completes the paper: in this section a variant of the N -body model treated in the preceding sections is also identified, which is as well *solvable*, while featuring *twice* as many arbitrary coupling constants, or even more.

2. Results

Notation 2.1. In this paper N is an arbitrary integer ($N \geq 2$), indices such as n, m, ℓ, j run over the integers from 1 to N (unless otherwise indicated, see for instance the restriction $\ell \neq n$ in (2.1a) below), all quantities other than indices are generally *complex* numbers (unless otherwise indicated, see for instance below “time”), **boldface** lower-case Latin letters are N -vectors (for instance

the N -vector \mathbf{z} features the N components z_n , **boldface** upper-case Latin letters are $N \times N$ matrices (for instance the $N \times N$ matrix \mathbf{A} features the N^2 elements A_{nm}), the *real* variable t indicates “time” and differentiations with respects to this variable are denoted by superimposed dots (so that $\dot{z}_n(t) \equiv d z_n(t) / d t$, $\ddot{z}_n(t) \equiv d^2 z_n(t) / d t^2$). In the following the time-dependence of quantities will not be explicitly displayed whenever the context allows to do so with a negligible risk to cause misunderstandings. There is one exception to the above definition of N -vectors: we hereafter denote (for typographic convenience) as \mathbf{i} the imaginary unit, so that $\mathbf{i}^2 = -1$. We also adopt the standard convention according to which an empty sum has a *vanishing* value and an empty product has *unit* value, for instance $\sum_{j=K}^J (x_j) = 0$, $\prod_{j=K}^J (x_j) = 1$ if $K > J$. Finally we denote as usual with δ_{nm} the Kronecker symbol, so that $\delta_{nm} = 1$ if $n = m$, $\delta_{nm} = 0$ if $n \neq m$. \square

Proposition 2.1. The *solvable* N -body problem is characterized by the following N Newtonian equations of motion:

$$\ddot{z}_n = \sum_{\ell=1, \ell \neq n}^N \left(\frac{2 \dot{z}_n \dot{z}_\ell}{z_n - z_\ell} \right) + \left[\prod_{\ell=1, \ell \neq n}^N (z_n - z_\ell) \right]^{-1} \sum_{m,j=1}^N \left[A_{mj} c_j (z_n)^{N-m} \right], \quad (2.1a)$$

where of course $z_n \equiv z_n(t)$ are the coordinates of the moving particles, A_{mj} are N^2 arbitrary (coupling) constants, and the N quantities $c_j \equiv c_j(t)$ are—up to a sign, see below—the symmetrical sums of j coordinates $z_n(t)$, being defined in terms of them as follows:

$$c_m = (-1)^m \sum_{1 \leq s_1 < s_2 < \dots < s_m \leq N} (z_{s_1} z_{s_2} \dots z_{s_m}), \quad (2.1b)$$

where of course (above and hereafter) the symbol $\sum_{1 \leq s_1 < s_2 < \dots < s_m \leq N}$ denotes the sum from 1 to N over the m indices s_1, s_2, \dots, s_m with the restriction $s_1 < s_2 < \dots < s_m$. Note that the first sum in the right-hand side of (2.1a) is the characterizing mark of N -body problems “of goldfish type”. Also note (see below) that the right-hand side of the equations of motion (2.1a) blows up when two different coordinates coincide, corresponding to a “collision” of two “particles”; but the occurrence of such events is *not generic* for motions taking place in the complex z -plane.

The configuration at time t of this system is provided by the following prescription: the N coordinates $z_n(t)$ are the N zeros of the t -dependent (monic) polynomial $p_N(z;t)$, of degree N in z , given by the following formulas in terms of the initial data $z_n(0)$, $\dot{z}_m(0)$:

$$p_N(z;t) = z^N + \sum_{m=1}^N [c_m(t) z^{N-m}], \quad (2.2a)$$

where

$$c_m(t) = \sum_{n=1}^N \left\{ \left[\gamma_n^{(+)} \exp(\mathbf{i} \alpha_n t) + \gamma_n^{(-)} \exp(-\mathbf{i} \alpha_n t) \right] u_m^{(n)} \right\}. \quad (2.2b)$$

Here the quantities α_n respectively $u_m^{(n)}$ are defined via the N eigenvalues α_n^2 respectively the corresponding N eigenvectors $\mathbf{u}^{(n)}$ (with components $u_m^{(n)}$) of the (time-independent) eigenvalue problem

$$\mathbf{A} \mathbf{u}^{(n)} = \alpha_n^2 \mathbf{u}^{(n)}, \quad (2.3)$$

where the $N \times N$ matrix \mathbf{A} features as its elements the N^2 coupling constants A_{nm} , see (2.1a); while the $2N$ time-independent quantities $\gamma_n^{(\pm)}$ are the solutions of the following linear system of $2N$

algebraic equations (see (2.2b)):

$$\sum_{n=1}^N \left\{ \left[\gamma_n^{(+)} + \gamma_n^{(-)} \right] u_m^{(n)} \right\} = c_m(0) , \quad (2.4a)$$

$$\mathbf{i} \sum_{n=1}^N \left\{ \alpha_n \left[\gamma_n^{(+)} - \gamma_n^{(-)} \right] u_m^{(n)} \right\} = \dot{c}_m(0) , \quad (2.4b)$$

where of course the $2N$ quantities $c_m(0)$ respectively $\dot{c}_m(0)$ are expressed as follows in terms of the initial data $z_n(0)$, $\dot{z}_n(0)$ (see (2.1b)):

$$c_m(0) = (-1)^m \sum_{1 \leq s_1 < s_2 < \dots < s_m \leq N} [z_{s_1}(0) z_{s_2}(0) \dots z_{s_m}(0)] , \quad (2.4c)$$

$$\begin{aligned} \dot{c}_m(0) = (-1)^m \sum_{n=1}^N \left(\dot{z}_n(0) \left\{ \delta_{m1} \right. \right. \\ \left. \left. + \sum_{1 \leq s_1 < s_2 < \dots < s_{m-1} \leq N; s_j \neq n, j=1, \dots, m-1} [z_{s_1}(0) z_{s_2}(0) \dots z_{s_{m-1}}(0)] \right\} \right) , \quad (2.4d) \end{aligned}$$

where the symbol $\sum_{1 \leq s_1 < s_2 < \dots < s_{m-1} \leq N; s_j \neq n, j=1, \dots, m-1}$ denotes the sum from 1 to N over the $m-1$ indices s_1, s_2, \dots, s_{m-1} with the restriction $s_1 < s_2 < \dots < s_{m-1}$ and the additional restriction that all these indices be different from n (note that, for $m=1$, this sum vanishes, see the above **Notation 2.1**).

Note that in writing this solution we are implicitly assuming that the N eigenvalues α_n^2 are *all different among themselves*; otherwise the standard limit must be taken, yielding terms in the right-hand side of (2.2b) featuring powers of t (see below, after eq. (3.9)). \square

Remark 2.1. It is plain that a rescaling respectively a shifting of the N coordinates $z_n(t)$,

$$z_n(t) \Rightarrow \beta_n z_n(t) + \gamma_n \quad (2.5)$$

with β_n respectively γ_n arbitrary (constant) parameters, changes only rather trivially the equations of motions (2.1): indeed if $\beta_n = \beta$ and $\gamma_n = \gamma$ it amounts almost only to a redefinition of the N^2 coupling constants A_{nm} . \square

Remark 2.2. Let us emphasize that, while the above technique of solution, as formulated in **Proposition 2.1**, provides the *configuration* of the system at time t as an *unordered* set of N coordinates $z_n(t)$, it does of course also allow to identify, say, which is the value at time t of the specific coordinate $z_1(t)$ which has evolved by *continuity* in time from the initial data $z_1(0)$, $\dot{z}_1(0)$. To do so one must investigate the time evolution of each coordinate over the *Riemann surface* associated with the configuration of the zeros of the polynomial (2.2). This is not a trivial endeavour, as demonstrated by various papers where this phenomenology has been studied in considerable detail [8, 12, 22–25]. In practice to get such information it may be easier to integrate *numerically* the equations of motion from the initial data (possibly only with rather poor precision); or to chop up the time interval from 0 to t into several (say, s) subintervals (from 0 to t_1 , from t_1 to t_2 , ..., from t_{s-1} to $t_s = t$), to solve in every subinterval by the technique described in this paper, and to make sure that each subinterval is

sufficiently short to allow the identification of each moving particles by an argument of *contiguity* (approximating *continuity*) of their positions over their time evolution. On the other hand let us also emphasize that the technique of solution described in **Proposition 2.1** also yields some important *general properties* of the solutions of the system (2.1), see for instance the following **Remark 2.3**. \square

Remark 2.3. It is plain from **Proposition 2.1** that the following *general properties* of the N -body problem characterized by the Newtonian equations of motion (2.1) hold.

(i) For *generic* (*complex*) initial data $z_n(0)$, $\dot{z}_n(0)$ the time evolution of this N -body system is, for *all* (finite) time, *nonsingular*.

(ii) If the N eigenvalues α_n^2 of the $N \times N$ matrix \mathbf{A} are *all positive* and *distinct*, for *generic* (*complex*) initial data the time evolution of this N -body system is *multiply periodic*, hence *confined* to a *finite* region of its phase space.

(iii) If and only if some (say, ν) of the N eigenvalues α_n^2 of the $N \times N$ matrix \mathbf{A} are—up to a common *positive* parameter η^2 —the *squares of distinct integers*, say

$$\alpha_p^2 = \eta^2 q_p^2, \quad p = 1, 2, \dots, \nu < N, \quad (2.6a)$$

with *all* q_p 's *positive integers* and $q_p \neq q_{p'}$ if $p \neq p'$, then the system (2.1) features ν completely periodic solutions,

$$z_n(t + T_p) = z_n(t), \quad (2.6b)$$

with periods $T_p = 2\pi/|\eta q_p|$; or possibly their periods might be a *finite integer multiple* of T_p —generally *small* with respect to its possible *maximal* value $N!$ —due to the possibility that the *zeros* of a polynomial $p_N(z;t)$ of order N in z which itself evolves *periodically* in time with period T_p exchange their roles when the polynomial becomes equal to itself after one period: see [8].

(iv) If and only if *all* the N eigenvalues α_n^2 of the $N \times N$ matrix \mathbf{A} are—up to a common *positive* parameter η^2 —the *squares of N distinct integers*,

$$\alpha_p^2 = \eta^2 q_p^2, \quad p = 1, 2, \dots, N, \quad (2.7a)$$

again with q_p *integer* and $q_p \neq q_{p'}$ if $p \neq p'$, then the system (2.1) is *isochronous*, i. e. *all* its nonsingular solutions are completely periodic with a period $\tilde{T} = 2\pi/|\eta\tilde{q}|$ where \tilde{q} is the minimum common multiple of the N integers $|q_p|$ (or, again, the period might be a *finite integer multiple* of \tilde{T} , generally *small* with respect to its possible *maximal* value $N!$: see [8, 12, 22–25]). \square

3. Proofs

The starting point of our proof are the identities

$$\ddot{z}_n = \sum_{\ell=1, \ell \neq n}^N \left(\frac{2 \dot{z}_n \dot{z}_\ell}{z_n - z_\ell} \right) - \left[\prod_{\ell=1, \ell \neq n}^N (z_n - z_\ell) \right]^{-1} \sum_{m=1}^N \left[\ddot{c}_m (z_n)^{N-m} \right], \quad (3.8)$$

which—as proven in [1]—relate the (time-evolutions of the) N zeros $z_n \equiv z_n(t)$ and the N coefficients $c_m = c_m(t)$ of *any* time-dependent monic polynomial of degree N in z , see (2.2a).

Let us now assume that the coefficients $c_m(t)$ of such a polynomial evolve in time according to the following system of N linearly coupled Ordinary Differential Equations (ODEs):

$$\ddot{c}_m(t) = - \sum_{j=1}^N [A_{mj} c_j(t)] . \quad (3.9)$$

It is then plain that the time evolution of these quantities is provided by (2.2b) with (2.3) and (2.4) (here the assumption that the N eigenvalues α_n^2 be *all different among themselves* plays a crucial role; otherwise the standard treatment of systems of *autonomous linear* ODEs implies that the solution of (3.9) features powers of t besides exponentials); while clearly the insertion of (3.9) in the right-hand side of (3.8) yields (2.1a), and (2.1b) is an identical consequence of (2.2a). The validity of **Proposition 2.1** is an immediate consequence.

4. Outlook

The results reported in this paper are rather immediate consequences of the findings reported in [1]; yet the identification of a *solvable* N -body problem “of goldfish type” featuring as many as N^2 arbitrary coupling constants is, to the best of my knowledge, a *novel* fact: hence deserving to be reported.

It is also possible to consider the more general *solvable* N -body model of goldfish type, the equations of motion of which obtain by inserting, in the right-hand side of (3.8), instead of (3.9), the *linear* set of N ODEs

$$\ddot{c}_m(t) = - \sum_{j=1}^N [B_{mj} \dot{c}_j(t) + A_{mj} c_j(t)] , \quad (4.1)$$

which feature the additional set of N^2 arbitrary constants B_{mj} (and of course *additional* arbitrary parameters can be introduced by the trivial trick mentioned in **Remark 2.1**). But we leave the details of this extension as a task for the interested reader.

And let us end by mentioning the possibility to, as it were, *iterate* these findings, i. e. to introduce the *sequel* of *solvable* N -body problems of goldfish type which obtain by inserting in the right-hand side of (3.8)—to characterize the evolution of the quantities $c_m(t)$ —the *solvable* model obtained at the previous level of the iteration. [20]

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