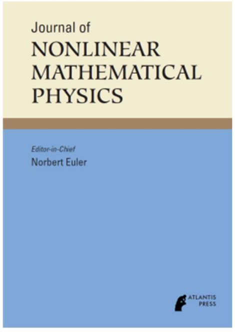

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A reciprocal transformation is introduced for a three-component Camassa-Holm type equation and it is showed that the transformed system is a reduction of the first negative flow in a generalized MKdV hierarchy.

Keywords: Camassa-Holm type equation; Reciprocal transformation; Hamiltonian structure

2000 Mathematics Subject Classification: 37K10, 35Q51, 35Q58

1. Introduction

The Camassa-Holm (CH) equation

$$u_t - u_{xxt} + 3uu_x + 2ku_x = 2u_x u_{xx} + uu_{xxx} \tag{1.1}$$

was derived by Camassa and Holm as a model for unidirectional dispersive shallow-water motion using an asymptotic expansion directly in the Hamiltonian for Euler's equations [1, 2]. It is completely integrable with a Lax pair, bi-Hamiltonian structure and infinitely many conserved quantities. The equation can be solved by the inverse scattering transformation [4], and is found to admit multi-soliton solutions and algebro-geometric solutions [10, 21, 23]. In particular, the discovery of peakons [1, 2] (if $k = 0$) makes the CH equation a subject of extensive research in recent years. The peakons are weak solutions in certain Sobolov space and of interest for water wave theory as well as from the point of view of general analysis for PDEs. Besides, the CH equation is linked to the first negative flow in the KdV hierarchy [7, 17] via a reciprocal transformation. It is remarked that reciprocal transformations, originated from the study of invariance theory in gasdynamics and magneto-gasdynamics, have diverse physical applications and been used extensively to reveal connections between integrable hierarchies (see e.g., [16, 26] and references therein).

Subsequently, various other CH type equations (possessing peakon solutions) are proposed and studied, to mention just a few we cite the Degasperis-Procesi (DP) equation, the Novikov equation and the Geng-Xue equation [5, 6, 9, 14, 15, 20, 24]. While those systems are completely integrable, they have some nonstandard features such as the weak Painlevé property [5, 11, 13]. Reciprocal transformations are a useful tool to connect the hierarchies displaying weak Painlevé behaviour with the hierarchies displaying the standard (strong) Painlevé test of Weiss, Tabor and Carnevale (WTC) [12]. Just as the CH equation is reciprocally linked to the negative KdV equation, the DP equation, the Novikov equation and the Geng-Xue equation are reciprocally connected to the negative flows in the Kaup-Kupershmidt hierarchy, the Sawada-Kotera hierarchy and the modified Boussinesq hierarchy respectively [15, 20].

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Recently, a three-component CH type system admitting the linear problem

$$\psi_x = \begin{pmatrix} 0 & 1 & 0 \\ 1+\lambda v & 0 & u \\ \lambda w & 0 & 0 \end{pmatrix} \psi \tag{1.2}$$

and N-peakon solutions was proposed by Geng and Xue [8], that is

$$\begin{aligned} u_t &= -v p_x + u_x q + \frac{3}{2} u q_x - \frac{3}{2} u (p_x r_x - p r), \\ v_t &= 2 v q_x + v_x q, \\ w_t &= v r_x + w_x q + \frac{3}{2} w q_x + \frac{3}{2} w (p_x r_x - p r), \end{aligned} \tag{1.3}$$

where

$$\begin{aligned} u &= p - p_{xx}, \\ v &= \frac{1}{2}(q_{xx} - 4q + p_{xx} r_x - r_{xx} p_x + 3 p_x r - 3 p r_x), \\ w &= r_{xx} - r. \end{aligned} \tag{1.4}$$

The bi-Hamiltonian structure as well as infinitely many conserved quantities and the dynamical system for the N-peakon solutions of the system are obtained [8, 18].

Very recently, based on the following spectral problem

$$\phi_x = \begin{pmatrix} 0 & 0 & 1 \\ \lambda m_1 & 0 & \lambda m_3 \\ 1 & \lambda m_2 & 0 \end{pmatrix} \phi, \tag{1.5}$$

Liu, Popowicz and the present author constructed another three-component CH type system [19]

$$\begin{aligned} & m_{1t} + u_2 g m_{1x} - m_3(u_{2x} f - u_2 g) - m_1(3 u_2 f - m_3 u_2) = 0, \\ & m_{2t} + u_2 g m_{2x} + m_2(3 u_{2x} g + m_3 u_2) = 0, \\ & m_{3t} + u_2 g m_{3x} - m_3(2 u_2 f + u_{2x} g - m_3 u_2) = 0, \\ & m_i = u_i - u_{ixx}, \ i = 1,2,3, \quad f = u_3 - u_{1x}, \quad g = u_1 - u_{3x}. \end{aligned} \tag{1.6}$$

We showed that the CH type system (1.6) admits a bi-Hamiltonian structure as well as infinitely many conserved quantities. It was pointed out that the spectral problem (1.5) may be transformed to the spectral problem (1.2) by a gauge transformation [19], and we will only consider to construct a reciprocal transformation for the system (1.6).

The purpose of this paper is to construct a reciprocal transformation for the three-component CH type equation (1.6), and establish a relationship between the transformed system and a generalized MKdV hierarchy. These will be done in Sec. 2 and Sec. 3.

2. A reciprocal transformation

In this section, we will construct a reciprocal transformation for the CH type system (1.6). As pointed out in [19], the three-component system (1.6) is exactly the compatibility condition of the spectral problem (1.5) and associated auxiliary problem

$$\phi_t = \begin{pmatrix} -u_2 f & \lambda^{-1}u_2 & -u_2 g \\ \lambda^{-1}f - \lambda m_1 u_2 g & u_2 f + u_{2x} g - \lambda^{-2} & \lambda^{-1}g - \lambda m_3 u_2 g \\ -u_{2x} f & \lambda^{-1}u_{2x} - \lambda m_2 u_2 g & -u_{2x} g \end{pmatrix} \phi. \tag{2.1}$$

Based on the above Lax pair, an infinite sequence of conservation laws may be constructed. In particular, we have

$$((m_2 m_3)^{\frac{1}{2}})_t = -((m_2 m_3)^{\frac{1}{2}} u_2 g)_x$$

which introduces a reciprocal transformation

$$dy = adx - au_2 g dt, \quad d\tau = dt, \tag{2.2}$$

where $a = (m_2 m_3)^{\frac{1}{2}}$.

On the one hand, writing the column vector ϕ in components as $\phi = (\phi_1, \phi_2, \phi_3)^T$ and elimating ϕ_3 from the spectral problem (1.5), we get

$$\phi_{1xx} - \phi_1 = \lambda m_2 \phi_2, \quad \phi_{2x} = \lambda m_1 \phi_1 + \lambda m_3 \phi_{1x}$$

which in the new variable can be written as

$$\phi_{1yy} + \frac{a_y}{a}\phi_{1y} - \frac{1}{a^2}\phi_1 = \lambda \frac{1}{m_3}\phi_2, \tag{2.3}$$

$$\phi_{2y} = \lambda \frac{m_1}{a}\phi_1 + \lambda m_3 \phi_{1y}. \tag{2.4}$$

Setting $b = e^{-\partial_y^{-1} \frac{m_1}{am_3}}$ and after a gauge transformation

$$\phi_1 = b\varphi_1, \quad \phi_2 = m_3 b \varphi_2,$$

the spectral problem (2.3)-(2.4) is transformed to

$$\varphi_{1yy} - Q_2 \varphi_{1y} - Q_1 \varphi_1 = \lambda \varphi_2, \tag{2.5}$$

$$\varphi_{2y} = \lambda \varphi_{1y} + Q_3 \varphi_2, \tag{2.6}$$

where

$$Q_1 = -\frac{b_{yy}}{b} - \frac{a_y b_y}{ab} + \frac{1}{a^2}, \quad Q_2 = -2\frac{b_y}{b} - \frac{a_y}{a}, \quad Q_3 = \frac{m_1}{am_3} - \frac{m_{3y}}{m_3}. \tag{2.7}$$

Eliminating φ_2 between (2.5) and (2.6), we find

$$(\partial_y^2 + u\partial_y + v + \partial_y^{-1} w)\varphi_1 = \lambda^2 \varphi_1. \tag{2.8}$$

The hierarchy associated with this spectral problem is an extended MKdV hierarchy which is related to an extended KdV hierarchy (the Yajima-Oikawa hierarchy [3, 27]) via a Miura transformation

[22, 25]. In fact the relationship between (Q_1, Q_2, Q_3) and (u, v, w) is established by the following Miura transformation

$$u = -(Q_2 + Q_3), \quad v = Q_2Q_3 - Q_1 + Q_{3y}, \quad w = Q_1Q_3 - (Q_2Q_3)_y - Q_{3yy} \tag{2.9}$$

which may be obtained from factorizing the Lax operator in (2.8), that is

$$\partial_y^2 + u\partial_y + v + \partial_y^{-1}w = \partial_y^{-1}(\partial_y - Q_3)(\partial_y^2 - Q_2\partial_y - Q_1).$$

Furthermore, the spectral problem (2.5)-(2.6) may be reduced to the spectral problem of the MKdV hierarchy as $Q_1 = Q_2 = 0$, thus we call the hierarchy associated with the spectral problem (2.5)-(2.6) a generalized MKdV hierarchy.

Similarly, the auxiliary problem (2.1) may be changed to the following form

$$\begin{aligned} \varphi_{1\tau} &= \lambda^{-1}q_3\varphi_2, \\ \varphi_{2\tau} &= (-\lambda^{-2} + q_3)\varphi_2 + \lambda^{-1}q_2\varphi_{1y} + \lambda^{-1}q_1\varphi_1, \end{aligned}$$

where

$$q_1 = \frac{u_3 - u_{1y}a}{m_3} - \frac{m_1}{m_3^2}(u_1 - u_{3y}a), \quad q_2 = \frac{a}{m_3}(u_1 - u_{3y}a), \quad q_3 = m_3u_2. \tag{2.10}$$

Then the Lax pair (1.5)-(2.1) for the three-component CH type system (1.6) is transformed to

$$\varphi_y = \begin{pmatrix} 0 & 1 & 0 \\ Q_1 & Q_2 & \lambda \\ 0 & \lambda & Q_3 \end{pmatrix} \varphi, \tag{2.11}$$

and

$$\varphi_\tau = \begin{pmatrix} 0 & 0 & \lambda^{-1}q_3 \\ 0 & q_3 & \lambda^{-1}(q_{3y} + Q_3q_3) \\ \lambda^{-1}q_1 & \lambda^{-1}q_2 & -\lambda^{-2} + q_3 \end{pmatrix} \varphi. \tag{2.12}$$

On the other hand, under the transformation (2.2), the CH type system (1.6) is transformed to

$$\begin{aligned} m_{1\tau} &= m_1(3u_2f - m_3u_2) + m_3(u_{2y}af - u_2g), & m_1 &= u_1 - a(u_{1y}a)_y, \\ m_{2\tau} &= -m_2(3u_{2y}ag + m_3u_2), & m_2 &= u_2 - a(u_{2y}a)_y, \\ m_{3\tau} &= m_3(2u_2f + u_{2y}ag - m_3u_2), & m_3 &= u_3 - a(u_{3y}a)_y, \end{aligned}$$

where

$$f = u_3 - au_{1y}, \quad g = u_1 - au_{3y}.$$

With the help of the Liouville transformation

$$\begin{pmatrix} Q_1 \\ Q_2 \\ Q_3 \end{pmatrix} = \begin{pmatrix} (m_2m_3)^{-1}[(\frac{m_1}{m_3})_x + 1 - (\frac{m_1}{m_3})^2] \\ (m_2m_3)^{-\frac{3}{2}}[2m_1m_2 - \frac{1}{2}(m_2m_3)_x] \\ m_2^{-\frac{1}{2}}m_3^{-\frac{3}{2}}(m_1 - m_{3x}) \end{pmatrix}, \quad y = \int_{-\infty}^{x} (m_2m_3)^{\frac{1}{2}}dx,$$

and through tedious but direct calculations, we obtain the system for Q_1, Q_2, Q_3 as follows

$$\begin{aligned} &Q_{1\tau} = Q_1 q_3 - q_1, && S_1 = -1,\\ &Q_{2\tau} = 2q_{3y} + Q_3 q_3 - q_2, && S_2 = 0,\\ &Q_{3\tau} = -Q_3 q_3 + q_2, && S_3 = -1, \end{aligned} \tag{2.13}$$

where

$$\begin{aligned} S_1 &= [(\partial_y + Q_3 - Q_2)(\partial_y + Q_3) - Q_1]q_3,\\ S_2 &= (-\partial_y + Q_3)q_1 - Q_1 q_2,\\ S_3 &= (-\partial_y + Q_3 - Q_2)q_2 - q_1. \end{aligned}$$

Direct calculation shows that the compatibility condition of the Lax pair (2.11)-(2.12) yields just the transformed system (2.13). Therefore we conclude that, under the reciprocal transformation (2.2), the three-component CH type system (1.6) with a Lax pair (1.5)-(2.1) are changed to (2.13) and its Lax pair (2.11)-(2.12), accordingly.

3. The link between the transformed CH type system and the generalized MKdV hierarchy

According to [25], the extended MKdV hierarchy associated with the spectral problem (2.8) may be formulated as a bi-Hamiltonian system admitting the Hamiltonian pair

$$\mathscr{J}_1 = \begin{pmatrix} 0 & 0 & 2\partial_y \\ 0 & 2\partial_y & \partial_y^2 + u\partial_y \\ 2\partial_y & -\partial_y^2 + \partial_y u & 0 \end{pmatrix},$$

$$\mathscr{J}_2 = \begin{pmatrix} 6\partial_y & * & * \\ 4u\partial_y & 2\partial_y^3 + 2u\partial_y u + \partial_y v + v\partial_y & * \\ 2\partial_y^3 - 2\partial_y u\partial_y + 2v\partial_y & \theta_1 & \theta_2 \end{pmatrix},$$

where

$$\begin{aligned} \theta_1 &= -\partial_y^4 + \partial_y^3 u + \partial_y u\partial_y^2 - \partial_y u\partial_y u - v\partial_y^2 + v\partial_y u + 2w\partial_y + \partial_y w,\\ \theta_2 &= \partial_y uw + uw\partial_y + w\partial_y^2 - \partial_y^2 w \end{aligned}$$

and the omitted terms in the operator $\mathscr{J}_2$ are determined by the skew-symmetry. Taken account of the Miura transformation (2.9), the compatible Hamiltonian operators for the generalized MKdV hierarchy associated with the spectral problem (2.11) (equivalent to the spectral problem (2.5)-(2.6)) may be obtained, which are

$$\tilde{\mathscr{J}}_i = \mathscr{F}^{-1}\mathscr{J}_i\mathscr{F}^{-1*},\ i = 1,2, \qquad \mathscr{F} = \begin{pmatrix} 0 & -1 & -1 \\ -1 & Q_3 & \partial_y + Q_2 \\ Q_3 & -\partial_y Q_3 & Q_1 - \partial_y Q_2 - \partial_y^2 \end{pmatrix}.$$

Now we may construct the first negative flow in this generalized MKdV hierarchy, which reads as

$$\begin{pmatrix} Q_1 \\ Q_2 \\ Q_3 \end{pmatrix}_\tau = \mathscr{F}^{-1}\mathscr{J}_1 \begin{pmatrix} A \\ B \\ C \end{pmatrix}, \qquad \mathscr{F}^{-1}\mathscr{J}_2 \begin{pmatrix} A \\ B \\ C \end{pmatrix} = 0. \tag{3.1}$$

In order to establish the link between the transformed system (2.13) and the first negative flow (3.1) in the generalized MKdV hierarchy, we introduce

$$\begin{aligned} A &= \frac{1}{4}(Q_2+Q_3)\partial_y^{-1}(S_1-S_3)-\frac{1}{2}\partial_y^{-1}S_2+\frac{1}{4}(S_1+S_3)-Q_3q_{3y}-Q_3^2q_3, \\ B &= \frac{1}{2}\partial_y^{-1}(S_1-S_3)-Q_3q_3, \\ C &= -q_3, \end{aligned} \tag{3.2}$$

(here all integration constants are assumed to be zero). Then substituting (3.2) into (3.1), the first negative flow is reformulated as

$$\begin{pmatrix} Q_1 \\ Q_2 \\ Q_3 \end{pmatrix}_\tau = \begin{pmatrix} Q_1q_3-q_1 \\ 2q_{3y}+Q_3q_3-q_2 \\ -Q_3q_3+q_2 \end{pmatrix}, \quad \mathscr{F}^{-1}\mathscr{K}\begin{pmatrix} S_1 \\ S_2 \\ S_3 \end{pmatrix}=0, \tag{3.3}$$

where

$$\mathscr{K} = \begin{pmatrix} \frac{1}{2}\partial_y u\partial_y^{-1}-\frac{1}{2}\partial_y & -3 & \frac{3}{2}\partial_y-\frac{1}{2}\partial_y u\partial_y^{-1} \\ Q_3\partial_y+v+\frac{1}{2}v_y\partial_y^{-1} & -2u & u\partial_y-\partial_y^2-v-\frac{1}{2}v_y\partial_y^{-1} \\ \chi_1 & \partial_y u-\partial_y^2-v & \chi_2 \end{pmatrix},$$

herein

$$\begin{aligned} \chi_1 &= -Q_3\partial_y^2-(2Q_{3y}+Q_2Q_3)\partial_y+\frac{1}{2}(3w+w_y\partial_y^{-1}), \\ \chi_2 &= \partial_y^3+v\partial_y-\partial_y u\partial_y-\frac{1}{2}(3w+w_y\partial_y^{-1}), \end{aligned}$$

with (u,v,w) given by (2.9).

From the definitions of A,B,C, it is easy to see that $S_1=S_3=-1$ and $S_2=0$ imply $\mathscr{K}(S_1,S_2,S_3)^T=0$. Therefore the transformed system (2.13) may be regarded as a reduction of the first negative flow (3.3) in the generalized MKdV hierarchy.

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