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We present a 3×3 Riemann-Hilbert problem formalism for the initial-boundary value problem of the two-component nonlinear Schrödinger (2-NLS) equation on the half-line. And we also get the Dirichlet to Neuemann map through analysising the global relation in this paper.

Keywords: Riemann-Hilbert problem; Initial-boundary value problem; two-component nonlinear Schrödinger equation.

2000 Mathematics Subject Classification: 37K40, 35Q53

1. Introduction

Integrable PDEs can be written as the compatibility condition of two linear eigenvalue equations, which are called Lax pairs [9]. In 1967, Gardner, Greene, Kruskal, Miura [7] used the Inverse Scattering Transform (IST) formalism to analyze the initial value problem for KdV equation. From then on, the IST method became a powerful tool to analyze the other integrable PDEs, such as NLS equation, Sine-Gordon equation and so on. Until the 1990s the IST methodology was pursued almost entirely for pure initial value problems. However, in many situations, we need to consider an initial-boundary value (IBV) problem instead of a pure initial value problem.

In 1997, Fokas announced a new unified approach for the analysis of IBV problems for linear and nonlinear integrable PDEs [2, 3](see also [4]). The Fokas method provides a generalization of the IST formalism from initial value to IBV problems. As the IST on the line, the unified method yields an expression for the solution of an IBV problem in terms of the solution of a Riemann-Hilbert problem. But there has a big difference between the unified method and IST method. The IST method just need analyze the x-part of Lax pairs to obtain the scattering data, the other t-part of

Lax pairs is used to determine the time-dependent of the scattering data. But the unified method need analyze simultaneously the x-part and t-part of Lax pairs. This difference leads to some difficulties in analyzing the IBV problem. Usually, the derivative order involved in the t-part of Lax pairs is less than the original integrable PDEs one. Hence, we need know all these boundary data to the spectral analysis. However, it needs "more" boundary data than a well-posed IBV problem. We need determine these "more" boundary data. This can be done by analyzing the global relation, which expresses the initial value data and all boundary data are not independent.

The unified method was usually used to analyze the IBV problem for integrable PDES with 2×2 Lax pairs [1, 5] (see also a series of paper [6, 12, 13]). But many important integrable PDEs involve 3×3 Lax pairs, such as Boussnissq equation, DP equation, manakov systems and so on. It naturally leads to extend unified method from 2×2 Lax pairs to 3×3 Lax pairs. In 2012, Lenells developed an extended methodology for analyzing IBV problems for integrable evolution equations with Lax pairs involving 3×3 matrices [10], Degasperis-Procesi equation [11]. And some other 3×3 integrable equations, such as Sasa-Satsuma equation [15], the three-wave equation [16] are considered by the authors.

In this paper, we will consider IBV problem for the manakov systems. It is well known that the nonlinear Schrödinger(NLS) equation

$$iq_t + q_{xx} + 2\sigma |q|^2 q = 0, \quad \sigma = \pm 1, \tag{1.1}$$

describes slowly varying wave envelopes in dispersive media and arises in various physical systems such as water waves, plasma physics, solid-state physics and nonlinear optics. One of the most successful among them is the description of optical solitons in fibers. But, many complex systems, such as nonlinear optics, etal, involves more than one-component. It is nature to extend the studies to the two-component case, see [14].

The two-component nonlinear Schrödinger equation or Manokov equation is

$$\begin{cases} iq_{1t} + q_{1xx} - 2\sigma(|q_1|^2 + |q_2|^2)q_1 = 0, \\ iq_{2t} + q_{2xx} - 2\sigma(|q_1|^2 + |q_2|^2)q_2 = 0. \end{cases} \quad \sigma = \pm 1. \tag{1.2}$$

Here, $\sigma = 1$ means defocusing case and $\sigma = -1$ means focusing case. In this paper, we consider the defocusing case[a] on the half-line $x \geq 0$. That is to say, we consider the following initial-boundary

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[a]After we finish this paper, we are told that the focusing case is considered in a very recently paper [8]. So here we just consider the defocusing case. And we compare this paper with the paper [8], the Lax pairs of equation (1.2) are chosen different form and we find that although we all use the same unified method, but we use block matrix to analyze the global relation, as we did in [15]. And in the results of Dirichlet to Neumann map, we also consider the possible finite simple poles, which wasn't considered in [8].

value problem of the 2-NLS equation,

$$
\begin{aligned}
q_{10}(x) &= q_1(x, t = 0), \quad q_{20}(x) = q_2(x, t = 0), \\
g_{01}(t) &= q_1(x = 0, t), \quad g_{02}(t) = q_2(x = 0, t), \\
g_{11}(t) &= q_{1x}(x = 0, t), \quad g_{12}(t) = q_{2x}(x = 0, t).
\end{aligned}
\tag{1.3}
$$

where $q_{10}(x)$ and $q_{20}(x)$ lie in Schwartz space.

In fact, the most important motivation for us to consider this IBV problem for 2-component NLS equation or manakov systems is that we try to extend the IBV problem on the half-line to the IBV problem on the finite interval for integrable PDEs with 3×3 Lax pairs. And we indeed do this extension in the paper [17], where we consider both the defocusing and focusing case. And the results about the Dirichlet to Neumann map obtained in this paper can be viewed as the results of the length of the finite interval goes to infity, in [17].

Organization of the paper: In section 2 we perform the spectral analysis of the associated Lax pair. And we formulate the main Riemann-Hilbert problem in section 3. We also get the map between the Dirichlet and the Neumann boundary problem through analysising the global relation in section 4.

2. Spectral analysis

The 2-NLS equation admits a 3×3 Lax pair,

$$
\Psi_x = U\Psi, \quad \Psi = \begin{pmatrix} \Psi_1 \\ \Psi_2 \\ \Psi_3 \end{pmatrix}.
\tag{2.1a}
$$

$$
\Psi_t = V\Psi.
\tag{2.1b}
$$

where

$$
U = ik\Lambda + V_1.
\tag{2.2}
$$

and

$$
V = 2ik^2\Lambda + V_2
\tag{2.3}
$$

here

$$
\Lambda = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, V_1 = \begin{pmatrix} 0 & q_1 & q_2 \\ \bar{q}_1 & 0 & 0 \\ \bar{q}_2 & 0 & 0 \end{pmatrix}, V_2 = kV_2^{(1)} + V_2^{(0)}.
\tag{2.4}
$$

where

$$V_2^{(1)} = 2V_1, \qquad V_2^{(0)} = i\Lambda(V_1^2 - V_{1x}). \tag{2.5}$$

2.1. *The closed one-form*

Suppose that $u(x,t)$ is sufficiently smooth function of (x,t) in the half-line domain $\Omega = \{0 < x < \infty, 0 < t < T\}$ which decay as $x \to \infty$. Introducing a new eigenfunction $\mu(x,t,k)$ by

$$\Psi = \mu e^{i\Lambda kx + 2i\Lambda k^2 t} \tag{2.6}$$

then we find the Lax pair equations

$$\begin{cases} \mu_x - [ik\Lambda, \mu] = V_1\mu, \\ \mu_t - [2ik^2\Lambda, \mu] = V_2\mu. \end{cases} \tag{2.7}$$

Letting $\hat{A}$ denotes the operators which acts on a 3×3 matrix X by $\hat{A}X = [A, X]$, then the equations in (2.7) can be written in differential form as

$$d(e^{-(ikx+2ik^2t)\hat{\Lambda}}\mu) = W, \tag{2.8}$$

where $W(x,t,k)$ is the closed one-form defined by

$$W = e^{-(ikx+2ik^2t)\hat{\Lambda}}(V_1dx + V_2dt)\mu. \tag{2.9}$$

2.2. *The μ_j's*

We define three eigenfunctions $\{\mu_j\}_1^3$ of (2.7) by the Volterra integral equations

$$\mu_j(x,t,k) = \mathbb{I} + \int_{\gamma_j} e^{(ikx+2ik^2t)\hat{\Lambda}} W_j(x',t',k). \qquad j = 1,2,3. \tag{2.10}$$

where W_j is given by (2.9) with μ replaced with μ_j, and the contours $\{\gamma_j\}_1^3$ are showed in Figure 1. The first, second and third column of the matrix equation (2.10) involves the exponentials
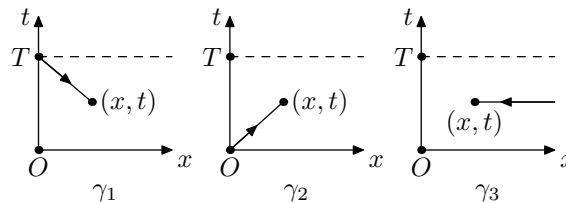

Fig. 1. The three contours γ_1, γ_2 and γ_3 in the (x,t)−domain.

$$[\mu_j]_1: e^{2ik(x-x')+4ik^2(t-t')}, e^{2ik(x-x')+4ik^2(t-t')}$$
$$[\mu_j]_2: e^{-2ik(x-x')-4ik^2(t-t')},$$
$$[\mu_j]_3: e^{-2ik(x-x')-4ik^2(t-t')}.$$

(2.11)

And we have the following inequalities on the contours:

$$\gamma_1 : x-x' \geq 0, t-t' \leq 0,$$
$$\gamma_2 : x-x' \geq 0, t-t' \geq 0,$$
$$\gamma_3 : x-x' \leq 0.$$

(2.12)

So, these inequalities imply that the functions $\{\mu_j\}_1^3$ are bounded and analytic for $k \in \mathbb{C}$ such that k belongs to

$$\mu_1 : (D_2, D_3, D_3),$$
$$\mu_2 : (D_1, D_4, D_4),$$
$$\mu_3 : (D_3 \cup D_4, D_1 \cup D_2, D_1 \cup D_2).$$

(2.13)

where $\{D_n\}_1^4$ denote four open, pairwisely disjoint subsets of the complex $k-$sphere showed in Figure 2. And the sets $\{D_n\}_1^4$ has the following properties:
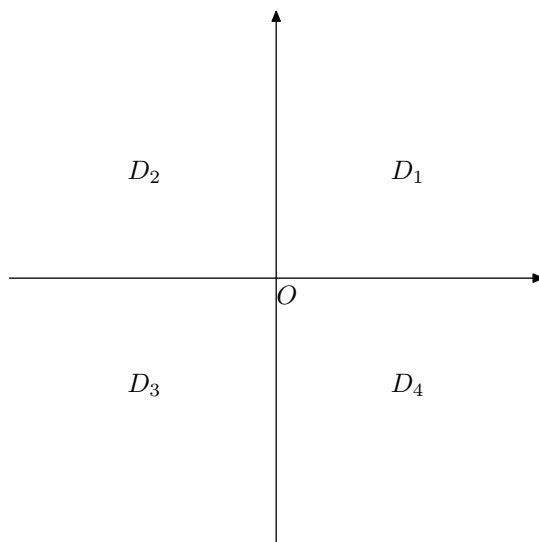

Fig. 2. The sets D_n, $n = 1, \ldots, 4$, which decompose the complex $k-$plane.

$$D_1 = \{k \in \mathbb{C} | \mathrm{Re}l_1 > \mathrm{Re}l_2 = \mathrm{Re}l_3, \mathrm{Re}z_1 > \mathrm{Re}z_2 = \mathrm{Re}z_3\},$$
$$D_2 = \{k \in \mathbb{C} | \mathrm{Re}l_1 > \mathrm{Re}l_2 = \mathrm{Re}l_3, \mathrm{Re}z_1 < \mathrm{Re}z_2 = \mathrm{Re}z_3\},$$
$$D_3 = \{k \in \mathbb{C} | \mathrm{Re}l_1 < \mathrm{Re}l_2 = \mathrm{Re}l_3, \mathrm{Re}z_1 > \mathrm{Re}z_2 = \mathrm{Re}z_3\},$$
$$D_4 = \{k \in \mathbb{C} | \mathrm{Re}l_1 < \mathrm{Re}l_2 = \mathrm{Re}l_3, \mathrm{Re}z_1 < \mathrm{Re}z_2 = \mathrm{Re}z_3\},$$

where $l_i(k)$ and $z_i(k)$ are the diagonal entries of matrices $ik\Lambda$ and $2ik^2\Lambda$, respectively.

In fact, for $x = 0$, $\mu_1(0,t,k)$ has enlarged domain of boundedness: $(D_2 \cup D_4, D_1 \cup D_3, D_1 \cup D_3)$, and $\mu_2(0,t,k)$ has enlarged domain of boundedness: $(D_1 \cup D_3, D_2 \cup D_4, D_2 \cup D_4)$.

2.3. *The M_n's*

For each $n = 1, \ldots, 4$, define a solution $M_n(x,t,k)$ of (2.7) by the following system of integral equations:

$$(M_n)_{ij}(x,t,k) = \delta_{ij} + \int_{\gamma_{ij}^n} (e^{(ikx+2ik^2t)\hat{\Lambda}} W_n(x',t',k))_{ij}, \quad k \in D_n, \quad i,j = 1,2,3. \tag{2.14}$$

where W_n is given by (2.9) with μ replaced with M_n, and the contours γ_{ij}^n, $n = 1, \ldots, 4$, $i,j = 1,2,3$ are defined by

$$\gamma_{ij}^n = \begin{cases} \gamma_1 \ \text{if} \ \mathrm{Re}\,l_i(k) < \mathrm{Re}\,l_j(k) \ \text{and} \ \mathrm{Re}\,z_i(k) \geq \mathrm{Re}\,z_j(k), \\ \gamma_2 \ \text{if} \ \mathrm{Re}\,l_i(k) < \mathrm{Re}\,l_j(k) \ \text{and} \ \mathrm{Re}\,z_i(k) < \mathrm{Re}\,z_j(k), \quad \text{for} \quad k \in D_n. \\ \gamma_3 \ \text{if} \ \mathrm{Re}\,l_i(k) \geq \mathrm{Re}\,l_j(k) \qquad . \end{cases} \tag{2.15}$$

The following proposition ascertains that the M_n's defined in this way have the properties required for the formulation of a Riemann-Hilbert problem.

Proposition 2.1. *For each $n = 1, \ldots, 4$, the function $M_n(x,t,k)$ is well-defined by equation (2.14) for $k \in \bar{D}_n$ and $(x,t) \in \Omega$. Moreover, M_n admits a bounded and contious extension to $\bar{D}_n$ and*

$$M_n(x,t,k) = \mathbb{I} + O(\frac{1}{k}), \qquad k \to \infty, \quad k \in D_n. \tag{2.16}$$

Proof. Substituting the expansion

$$M = M_0 + \frac{M^{(1)}}{k} + \frac{M^{(2)}}{k^2} + \cdots, \qquad k \to \infty.$$

into the Lax pair (2.7) and comparing the terms of the same order of k yield the equation (2.16). $\square$

Remark 2.1. Of course, for any fixed point (x,t), M_n is bounded and analytic as a function of $k \in D_n$ away from a possible discrete set of singularities $\{k_j\}$ at which the Fredholm determinant vanishes. The bounedness and analyticity properties are established in appendix B in [10].

2.4. *The jump matrices*

We define spectral functions $S_n(k)$, $n = 1, \ldots, 4$, and

$$S_n(k) = M_n(0,0,k), \qquad k \in D_n, \quad n = 1, \ldots, 4. \tag{2.17}$$

Let M denote the sectionally analytic function on the Riemann $k-$sphere which equals M_n for $k \in D_n$. Then M satisfies the jump conditions

$$M_n = M_m J_{m,n}, \qquad k \in \bar{D}_n \cap \bar{D}_m, \qquad n,m = 1,\ldots,4, \quad n \neq m, \tag{2.18}$$

where the jump matrices $J_{m,n}(x,t,k)$ are defined by

$$J_{m,n} = e^{(ikx+2ik^2t)\hat{\Lambda}}(S_m^{-1}S_n). \tag{2.19}$$

According to the definition of the γ^n, we find that

$$\gamma^1 = \begin{pmatrix} \gamma_3 & \gamma_3 & \gamma_3 \\ \gamma_2 & \gamma_3 & \gamma_3 \\ \gamma_2 & \gamma_3 & \gamma_3 \end{pmatrix} \quad \gamma^2 = \begin{pmatrix} \gamma_3 & \gamma_3 & \gamma_3 \\ \gamma_1 & \gamma_3 & \gamma_3 \\ \gamma_1 & \gamma_3 & \gamma_3 \end{pmatrix}$$

$$\gamma^3 = \begin{pmatrix} \gamma_3 & \gamma_1 & \gamma_1 \\ \gamma_3 & \gamma_3 & \gamma_3 \\ \gamma_3 & \gamma_3 & \gamma_3 \end{pmatrix} \quad \gamma^4 = \begin{pmatrix} \gamma_3 & \gamma_2 & \gamma_2 \\ \gamma_3 & \gamma_3 & \gamma_3 \\ \gamma_3 & \gamma_3 & \gamma_3 \end{pmatrix}. \tag{2.20}$$

2.5. *The adjugated eigenfunctions*

As the expressions of $S_n(k)$ will involve the adjugate matrix of $\{s(k), S(k)\}$ defined in the next subsection. We will also need the analyticity and boundedness properties of the minors of the matrices $\{\mu_j(x,t,k)\}_1^3$. We recall that the adjugate matrix X^A of a 3×3 matrix X is defined by

$$X^A = \begin{pmatrix} m_{11}(X) & -m_{12}(X) & m_{13}(X) \\ -m_{21}(X) & m_{22}(X) & -m_{23}(X) \\ m_{31}(X) & -m_{32}(X) & m_{33}(X) \end{pmatrix},$$

where $m_{ij}(X)$ denote the (ij)th minor of X.

It follows from (2.7) that the adjugated eigenfunction μ^A satisfies the Lax pair

$$\begin{cases} \mu_x^A + [ik\Lambda, \mu^A] = -V_1^T \mu^A, \\ \mu_t^A + [2ik^2\Lambda, \mu^A] = -V_2^T \mu^A. \end{cases} \tag{2.21}$$

where V^T denote the transform of a matrix V. Thus, the eigenfunctions $\{\mu_j^A\}_1^3$ are solutions of the integral equations

$$\mu_j^A(x,t,k) = \mathbb{I} - \int_{\gamma_j} e^{-ik(x-x')-2ik^2(t-t')\hat{\Lambda}}(V_1^T dx + V_2^T)\mu^A, \quad j = 1,2,3. \tag{2.22}$$

Then we can get the following analyticity and boundedness properties:

$$\begin{aligned} &\mu_1^A : (D_3, D_2, D_2), \\ &\mu_2^A : (D_4, D_1, D_1), \\ &\mu_3^A : (D_1 \cup D_2, D_3 \cup D_4, D_3 \cup D_4). \end{aligned} \tag{2.23}$$

In fact, for $x = 0$, $\mu_1^A(0,t,k)$ has enlarged domain of boundedness: $(D_1 \cup D_3, D_2 \cup D_4, D_2 \cup D_4)$, and $\mu_2^A(0,t,k)$ has enlarged domain of boundedness: $(D_2 \cup D_4, D_1 \cup D_3, D_1 \cup D_3)$.

2.6. *The $J_{m,n}$'s computation*

Let us define the $3 \times 3-$matrix value spectral functions $s(k)$ and $S(k)$ by

$$\mu_3(x,t,k) = \mu_2(x,t,k)e^{(ikx+2ik^2t)\hat{\Lambda}}s(k), \tag{2.24a}$$

$$\mu_1(x,t,k) = \mu_2(x,t,k)e^{(ikx+2ik^2t)\hat{\Lambda}}S(k), \tag{2.24b}$$

Thus,

$$s(k) = \mu_3(0,0,k), \qquad S(k) = \mu_1(0,0,k). \tag{2.25}$$

And we deduce from the properties of μ_j and μ_j^A that $s(k)$ and $S(k)$ have the following boundedness properties:

$$
\begin{aligned}
s(k): &\quad (D_3 \cup D_4, D_1 \cup D_2, D_1 \cup D_2), \\
S(k): &\quad (D_2 \cup D_4, D_1 \cup D_3, D_1 \cup D_3), \\
s^A(k): &\quad (D_1 \cup D_2, D_3 \cup D_4, D_3 \cup D_4), \\
S^A(k): &\quad (D_1 \cup D_3, D_2 \cup D_4, D_2 \cup D_4).
\end{aligned}
$$

Moreover,

$$M_n(x,t,k) = \mu_2(x,t,k)e^{(ikx+2ik^2t)\hat{\Lambda}}S_n(k), \quad k \in D_n. \tag{2.26}$$

Proposition 2.2. *The S_n can be expressed in terms of the entries of $s(k)$ and $S(k)$ as follows:*

$$S_1 = \begin{pmatrix} \frac{1}{m_{11}(s)} & s_{12} & s_{13} \\ 0 & s_{22} & s_{23} \\ 0 & s_{32} & s_{33} \end{pmatrix}, \; S_2 = \begin{pmatrix} \frac{S_{11}}{(S^T s^A)_{11}} & s_{12} & s_{13} \\ \frac{S_{21}}{(S^T s^A)_{11}} & s_{22} & s_{23} \\ \frac{S_{31}}{(S^T s^A)_{11}} & s_{32} & s_{33} \end{pmatrix}, \tag{2.27a}$$

$$S_3 = \begin{pmatrix} s_{11} & \frac{m_{33}(s)M_{21}(S)-m_{23}(s)M_{31}(S)}{(s^T S^A)_{11}} & \frac{m_{32}(s)M_{21}(S)-m_{22}(s)M_{31}(S)}{(s^T S^A)_{11}} \\ s_{21} & \frac{m_{33}(s)M_{11}(S)-m_{13}(s)M_{31}(S)}{(s^T S^A)_{11}} & \frac{m_{32}(s)M_{11}(S)-m_{12}(s)M_{31}(S)}{(s^T S^A)_{11}} \\ s_{31} & \frac{m_{23}(s)M_{11}(S)-m_{13}(s)M_{21}(S)}{(s^T S^A)_{11}} & \frac{m_{22}(s)M_{11}(S)-m_{12}(s)M_{21}(S)}{(s^T S^A)_{11}} \end{pmatrix},$$

$$S_4 = \begin{pmatrix} s_{11} & 0 & 0 \\ s_{21} & \frac{m_{33}(s)}{s_{11}} & \frac{m_{32}(s)}{s_{11}} \\ s_{31} & \frac{m_{23}(s)}{s_{11}} & \frac{m_{22}(s)}{s_{11}} \end{pmatrix}. \tag{2.27b}$$

Proof. Let $\gamma_3^{X_0}$ denote the contour $(X_0, 0) \to (x, t)$ in the (x, t)−plane, here $X_0 > 0$ is a constant. We introduce $\mu_3(x, t, k; X_0)$ as the solution of (2.10) with $j = 3$ and with the contour γ_3 replaced by $\gamma_3^{X_0}$. Similarly, we define $M_n(x, t, k; X_0)$ as the solution of (2.14) with γ_3 replaced by $\gamma_3^{X_0}$. We will first derive expression for $S_n(k; X_0) = M_n(0, 0, k; X_0)$ in terms of $S(k)$ and $s(k; X_0) = \mu_3(0, 0, k; X_0)$. Then (2.27) will follow by taking the limit $X_0 \to \infty$.

First, We have the following relations:

$$\begin{cases} M_n(x, t, k; X_0) = \mu_1(x, t, k)e^{(ikx + 2ik^2 t)\hat{\Lambda}}R_n(k; X_0), \\ M_n(x, t, k; X_0) = \mu_2(x, t, k)e^{(ikx + 2ik^2 t)\hat{\Lambda}}S_n(k; X_0), \\ M_n(x, t, k; X_0) = \mu_3(x, t, k)e^{(ikx + 2ik^2 t)\hat{\Lambda}}T_n(k; X_0). \end{cases} \tag{2.28}$$

Then we get $R_n(k; X_0)$ and $T_n(k; X_0)$ are fedined as follows:

$$R_n(k; X_0) = e^{-2ik^2 T\hat{\Lambda}}M_n(0, T, k; X_0), \tag{2.29a}$$

$$T_n(k; X_0) = e^{-ikx\hat{\Lambda}}M_n(X_0, 0, k; X_0). \tag{2.29b}$$

The relations (2.28) imply that

$$s(k; X_0) = S_n(k; X_0)T_n^{-1}(k; X_0), \qquad S(k) = S_n(k; X_0)R_n^{-1}(k; X_0). \tag{2.30}$$

These equations constitute a matrix factorization problem which, given $\{s, S\}$ can be solved for the $\{R_n, S_n, T_n\}$. Indeed, the integral equations (2.14) together with the definitions of $\{R_n, S_n, T_n\}$ imply that

$$\begin{cases} (R_n(k; X_0))_{ij} = 0 & \text{if } \gamma_{ij}^n = \gamma_1, \\ (S_n(k; X_0))_{ij} = 0 & \text{if } \gamma_{ij}^n = \gamma_2, \\ (T_n(k; X_0))_{ij} = \delta_{ij} & \text{if } \gamma_{ij}^n = \gamma_3. \end{cases} \tag{2.31}$$

It follows that (2.30) are 18 scalar equations for 18 unknowns. By computing the explicit solution of this algebraic system, we find that $\{S_n(k; X_0)\}_1^4$ are given by the equation obtained from (2.27) by replacing $\{S_n(k), s(k)\}$ with $\{S_n(k; X_0), s(k; X_0)\}$. taking $X_0 \to \infty$ in this equation, we arrive at (2.27). $\square$

2.7. *The residue conditions*

Since μ_2 is an entire function, it follows from (2.26) that M can only have sigularities at the points where the $S'_n s$ have singularities. We denote the possible zeros by $\{k_j\}_1^N$ and assume they satisfy the following assumption.

Assumption 2.1. We assume that

- $m_{11}(s)(k)$ has n_0 possible simple zeros in D_1 denoted by $\{k_j\}_1^{n_0}$;
- $(S^T s^A)_{11}(k)$ has $n_1 - n_0$ possible simple zeros in D_2 denoted by $\{k_j\}_{n_0+1}^{n_1}$;
- $(s^T S^A)_{11}(k)$ has $n_2 - n_1$ possible simple zeros in D_2 denoted by $\{k_j\}_{n_1+1}^{n_2}$;
- $s_{11}(k)$ has $N - n_2$ possible simple zeros in D_4 denoted by $\{k_j\}_{n_2+1}^{N}$;

and that none of these zeros coincide. Moreover, we assume that none of these functions have zeros on the boundaries of the D_n's.

We determine the residue conditions at these zeros in the following:

Proposition 2.3. *Let $\{M_n\}_1^4$ be the eigenfunctions defined by (2.14) and assume that the set $\{k_j\}_1^N$ of singularities are as the above assumption. Then the following residue conditions hold:*

$$Res_{k=k_j}[M]_1 = \frac{s_{33}(k_j)[M(k_j)]_2 - s_{23}(k_j)[M(k_j)]_3}{\dot{s}_{33}(k_j)m_{22}(s)(k_j)}e^{2\theta(k_j)}, \quad 1 \leq j \leq n_0, k_j \in D_1 \qquad (2.32a)$$

$$Res_{k=k_j}[M]_1 = \frac{S_{21}(k_j)s_{33}(k_j) - S_{31}(k_j)s_{23}(k_j)}{(S^T s^A)_{33}(k_j)m_{11}(k_j)}e^{2\theta(k_j)}[M(k_j)]_2$$
$$+ \frac{S_{31}(k_j)s_{22}(k_j) - S_{21}(k_j)s_{32}(k_j)}{(S^T s^A)_{33}(k_j)m_{11}(k_j)}e^{2\theta(k_j)}[M(k_j)]_3 \qquad (2.32b)$$
$$n_0 + 1 \leq j \leq n_1, k_j \in D_2,$$

$$Res_{k=k_j}[M]_2 = \frac{m_{33}(s)(k_j)M_{21}(S)(k_j) - m_{23}(s)(k_j)M_{31}(S)(k_j)}{(s^T S^A)_{11}(k_j)\dot{s}_{11}(k_j)}e^{-2\theta(k_j)}[M(k_j)]_1$$
$$n_1 + 1 \leq j \leq n_2, k_j \in D_3, \qquad (2.32c)$$

$$Res_{k=k_j}[M]_3 = \frac{m_{32}(s)(k_j)M_{21}(S)(k_j) - m_{22}(s)(k_j)M_{31}(S)(k_j)}{(s^T S^A)_{11}(k_j)\dot{s}_{11}(k_j)}e^{-2\theta(k_j)}[M(k_j)]_1$$
$$n_1 + 1 \leq j \leq n_2, k_j \in D_3. \qquad (2.32d)$$

$$Res_{k=k_j}[M]_2 = \frac{m_{33}(s)(k_j)}{\dot{s}_{11}(k_j)s_{21}(k_j)}e^{-2\theta(k_j)}[M(k_j)]_1, \quad n_2 + 1 \leq j \leq N, k_j \in D_4. \qquad (2.32e)$$

$$Res_{k=k_j}[M]_3 = \frac{m_{32}(s)(k_j)}{\dot{s}_{11}(k_j)s_{21}(k_j)}e^{-2\theta(k_j)}[M(k_j)]_1, \quad n_2 + 1 \leq j \leq N, k_j \in D_4. \qquad (2.32f)$$

where $\dot{f} = \frac{df}{dk}$, and θ is defined by

$$\theta(x,t,k) = ikx + 2ik^2 t. \qquad (2.33)$$

Proof. We will prove (2.32a), (2.32c), the other conditions follow by similar arguments. Equation (2.26) implies the relation

$$M_1 = \mu_2 e^{(ikx+2ik^2 t)\hat{\Lambda}} S_1, \tag{2.34a}$$

$$M_3 = \mu_2 e^{(ikx+2ik^2 t)\hat{\Lambda}} S_3, \tag{2.34b}$$

In view of the expressions for S_1 and S_3 given in (2.27), the three columns of (2.34a) read:

$$[M_1]_1 = [\mu_2]_1 \frac{1}{m_{11}(s)}, \tag{2.35a}$$

$$[M_1]_2 = [\mu_2]_1 e^{-2\theta} s_{12} + [\mu_2]_2 s_{22} + [\mu_2]_3 s_{32}, \tag{2.35b}$$

$$[M_1]_3 = [\mu_2]_1 e^{-2\theta} s_{13} + [\mu_2]_2 s_{23} + [\mu_2]_3 s_{33}. \tag{2.35c}$$

while the three columns of (2.34b) read:

$$[M_3]_1 = [\mu_2]_1 s_{11} + [\mu_2]_2 s_{21} e^{2\theta} + [\mu_2]_3 s_{31} e^{2\theta} \tag{2.36a}$$

$$[M_3]_2 = [\mu_2]_1 \frac{m_{33}(s)M_{21}(S) - m_{23}(s)M_{31}(S)}{(s^T S^A)_{11}} e^{-2\theta} \\ + [\mu_2]_2 \frac{m_{33}(s)M_{11}(S) - m_{13}(s)M_{31}(S)}{(s^T S^A)_{11}} \\ + [\mu_2]_3 \frac{m_{23}(s)M_{11}(S) - m_{13}(s)M_{21}(S)}{(s^T S^A)_{11}} \tag{2.36b}$$

$$[M_3]_3 = [\mu_2]_1 \frac{m_{32}(s)M_{21}(S) - m_{22}(s)M_{31}(S)}{(s^T S^A)_{11}} e^{-2\theta} \\ + [\mu_2]_2 \frac{m_{32}(s)M_{11}(S) - m_{12}(s)M_{31}(S)}{(s^T S^A)_{11}} \\ + [\mu_2]_3 \frac{m_{22}(s)M_{11}(S) - m_{12}(s)M_{21}(S)}{(s^T S^A)_{11}}. \tag{2.36c}$$

We first suppose that $k_j \in D_1$ is a simple zero of $m_{11}(s)(k)$. Solving (2.35b) and (2.35c) for $[\mu_2]_1$ and substituting the result in to (2.35a), we find

$$[M_1]_1 = \frac{s_{33}[M_1]_2 - s_{23}[M_1]_3}{m_{11}(s)m_{22}(s)} e^{2\theta} - \frac{[\mu_2]_2}{m_{21}(s)} e^{2\theta}.$$

Taking the residue of this equation at k_j, we find the condition (2.32a) in the case when $k_j \in D_1$.

In order to prove (2.32c), we solve (2.36a) for $[\mu_2]_1$, then substituting the result into (2.36b) and (2.36c), we find

$$[M_3]_2 = \frac{m_{33}(s)}{s_{11}} [\mu_2]_2 + \frac{m_{23}(s)}{s_{11}} [\mu_2]_3 + \frac{m_{33}(s)M_{21}(S) - m_{23}(s)M_{31}(S)}{(s^T \dot{S}^A)_{11} s_{11}} e^{-2\theta} [M_3]_1, \tag{2.37a}$$

$$[M_3]_3 = \frac{m_{32}(s)}{s_{11}} [\mu_2]_2 + \frac{m_{22}(s)}{s_{11}} [\mu_2]_3 + \frac{m_{32}(s)M_{21}(S) - m_{22}(s)M_{31}(S)}{(s^T \dot{S}^A)_{11} s_{11}} e^{-2\theta} [M_3]_1. \tag{2.37b}$$

Taking the residue of this equation at k_j, we find the condition (2.32c) in the case when $k_j \in D_3$. $\square$

2.8. *The global relation*

The spectral functions $S(k)$ and $s(k)$ are not independent but satisfy an important relation. Indeed, it follows from (2.24) that

$$\mu_1(x,t,k)e^{(ikx+2ik^2t)\hat{\Lambda}}S^{-1}(k)s(k) = \mu_3(x,t,k), \quad k \in (D_3 \cup D_4, D_3 \cup D_4, D_1 \cup D_2). \tag{2.38}$$

Since $\mu_1(0,T,k) = \mathbb{I}$, evaluation at $(0,T)$ yields the following global relation:

$$S^{-1}(k)s(k) = e^{-2ik^2T\hat{\Lambda}}c(T,k), \quad k \in (D_3 \cup D_4, D_3 \cup D_4, D_1 \cup D_2). \tag{2.39}$$

where $c(T,k) = \mu_3(0,T,k)$.

3. The Riemann-Hilbert problem

In this section, we state our main result that the solutions $q_1(x,t)$ and $q_2(x,t)$ of (1.2) can be recovered from a 3×3 Riemann-Hilbert problem.

Theorem 3.1. *Suppose that $q_1(x,t)$ and $q_2(x,t)$ are a pair of solutions of (1.2) in the half-line domain Ω with sufficient smoothness and decays as $x \to \infty$. Then $q_1(x,t)$ and $q_2(x,t)$ can be reconstructed from the initial value $\{q_{10}(x), q_{20}(x)\}$ and boundary values $\{g_{01}(t), g_{02}(t), g_{11}(t), g_{12}(t)\}$ defined as follows,*

$$
\begin{aligned}
&q_{10}(x) = q_1(x,0), \; q_{20}(x) = q_2(x,0), \; g_{01}(t) = q_1(0,t), \\
&g_{02}(t) = q_2(0,t), \; g_{11}(t) = q_{1x}(0,t), \; g_{12}(t) = q_{2x}(0,t).
\end{aligned} \tag{3.1}
$$

Use the initial and boundary data to define the jump matrices $J_{m,n}(x,t,k)$ as well as the spectral $s(k)$ and $S(k)$ by equation (2.24). Assume that the possible zeros $\{k_j\}_1^N$ of the functions $m_{11}(s)(k), (S^T s^A)_{11}(k), (s^T S^A)_{11}(k)$ and $s_{11}(k)$ are as in assumption 2.3.

Then the solution $\{q_1(x,t), q_2(x,t)\}$ is given by

$$q_1(x,t) = 2i\lim_{k\to\infty}(kM(x,t,k))_{12}, \quad q_2(x,t) = 2i\lim_{k\to\infty}(kM(x,t,k))_{13}. \tag{3.2}$$

where $M(x,t,k)$ satisfies the following 3×3 matrix Riemann-Hilbert problem:

- *M is sectionally meromorphic on the Riemann $k-$sphere with jumps across the contours $\bar{D}_n \cap \bar{D}_m, n, m = 1, \cdots, 4$, see Figure 2.*
- *Across the contours $\bar{D}_n \cap \bar{D}_m$, M satisfies the jump condition*

$$M_n(x,t,k) = M_m(x,t,k)J_{m,n}(x,t,k), \quad k \in \bar{D}_n \cap \bar{D}_m, n, m = 1, 2, 3, 4. \tag{3.3}$$

- $M(x,t,k) = \mathbb{I} + O(\frac{1}{k}), \quad k \to \infty.$
- *The residue condition of M is showed in Proposition 2.3.*

Proof. It only remains to prove (3.2) and this equation follows from the large k asymptotics of the eigenfunctions. $\qquad\square$

4. Non-linearizable Boundary Conditions

As we state in the introduction, the most difficulty of initial-boundary value problems is that some of the boundary values are unkown for a well-posed problem. All boundary values are needed for the definition of $S(k)$, and hence for the formulation of the Riemann-Hilbert problem. Our main result in this section expresses the spectral function $S(k)$ in terms of the prescribed boundary data and the initial data via the solution of a system of nonlinear integral equations.

4.1. *Asymptotics*

An analysis of (2.7) shows that the eigenfunctions $\{\mu_j\}_1^3$ have the following asymptotics as $k \to \infty$:

$$
\mu_j(x,t,k) = \mathbb{I} + \frac{1}{k}\begin{pmatrix} \mu_{11}^{(1)} & \mu_{12}^{(1)} & \mu_{13}^{(1)} \\ \mu_{21}^{(1)} & \mu_{22}^{(1)} & \mu_{23}^{(1)} \\ \mu_{31}^{(1)} & \mu_{32}^{(1)} & \mu_{33}^{(1)} \end{pmatrix} + \frac{1}{k^2}\begin{pmatrix} \mu_{11}^{(2)} & \mu_{12}^{(2)} & \mu_{13}^{(2)} \\ \mu_{21}^{(2)} & \mu_{22}^{(2)} & \mu_{23}^{(2)} \\ \mu_{31}^{(2)} & \mu_{32}^{(2)} & \mu_{33}^{(2)} \end{pmatrix} + O(\tfrac{1}{k^3})
$$

$$
= \mathbb{I} + \frac{1}{k}\begin{pmatrix} \int_{(x_j,t_j)}^{(x,t)} \Delta_{11} & \frac{q_1}{2i} & \frac{q_2}{2i} \\ -\frac{\bar{q}_1}{2i} & \int_{(x_j,t_j)}^{(x,t)} \Delta_{22}^{(1)} & \int_{(x_j,t_j)}^{(x,t)} \Delta_{23}^{(1)} \\ -\frac{\bar{q}_2}{2i} & \int_{(x_j,t_j)}^{(x,t)} \Delta_{32}^{(1)} & \int_{(x_j,t_j)}^{(x,t)} \Delta_{33}^{(1)} \end{pmatrix} \tag{4.1a}
$$

$$
+ \frac{1}{k^2}\begin{pmatrix} \int_{(x_j,t_j)}^{(x,t)} \eta_j & \frac{q_1\mu_{22}^{(1)}-q_2\mu_{32}^{(1)}-\mu_{12x}^{(1)}}{2i} & \frac{q_1\mu_{23}^{(1)}-q_2\mu_{33}^{(1)}-\mu_{13x}^{(1)}}{2i} \\ \frac{\mu_{21x}^{(1)}-\bar{q}_1\mu_{11}^{(1)}}{2i} & \int_{(x_j,t_j)}^{(x,t)} \Delta_{22}^{(2)} & \int_{(x_j,t_j)}^{(x,t)} \Delta_{23}^{(2)} \\ \frac{\mu_{31x}^{(1)}-\bar{q}_2\mu_{11}^{(1)}}{2i} & \int_{(x_j,t_j)}^{(x,t)} \Delta_{32}^{(2)} & \int_{(x_j,t_j)}^{(x,t)} \Delta_{33}^{(2)} \end{pmatrix} + O(\tfrac{1}{k^3}).
$$

where

$$
\begin{aligned}
\Delta_{11} &= \tfrac{i}{2}(|q_1|^2 + |q_2|^2)dx + \tfrac{1}{2}(q_1\bar{q}_{1x} - \bar{q}_1 q_{1x} + q_2\bar{q}_{2x} - \bar{q}_2 q_{2x})dt \\
\Delta_{22}^{(1)} &= -\tfrac{i}{2}|q_1|^2 dx - \tfrac{1}{2}(\bar{q}_{1x}q_1 - \bar{q}_1 q_{1x})dt \\
\Delta_{23}^{(1)} &= -\tfrac{i}{2}\bar{q}_1 q_2 dx - \tfrac{1}{2}(\bar{q}_{1x}q_2 - \bar{q}_1 q_{2x})dt \\
\Delta_{32}^{(1)} &= -\tfrac{i}{2}\bar{q}_2 q_1 dx - \tfrac{1}{2}(\bar{q}_{2x}q_1 - \bar{q}_2 q_{1x})dt \\
\Delta_{33}^{(1)} &= -\tfrac{i}{2}|q_2|^2 dx - \tfrac{1}{2}(\bar{q}_{2x}q_2 - \bar{q}_2 q_{2x})dt
\end{aligned} \tag{4.2a}
$$

$$
\begin{aligned}
\eta_j &= d\left(\tfrac{1}{2}\left(\int_{(x_j,t_j)}^{(x,t)} \Delta_{11}\right)^2\right) + \tfrac{1}{4}(q_1\bar{q}_{1x} + q_2\bar{q}_{2x})dx \\
&\quad + [\tfrac{i}{4}(|q_1|^2 + |q_2|^2)^2 + \tfrac{i}{4}(q_{1x}\bar{q}_{1x} - q_1\bar{q}_{1xx} + q_{2x}\bar{q}_{2x} - q_2\bar{q}_{2xx})]dt
\end{aligned} \tag{4.2b}
$$

$$\Delta_{22}^{(2)} = (-\tfrac{i}{2}|q_1|^2\mu_{22}^{(1)} - \tfrac{i}{2}\bar{q}_1 q_2\mu_{32}^{(1)} + \tfrac{1}{4}\bar{q}_1 q_{1x})dx$$
$$+[-\tfrac{1}{2}(\bar{q}_{1x}q_1 - \bar{q}_1 q_{1x})\mu_{22}^{(1)} - \tfrac{1}{2}(\bar{q}_{1x}q_2 - \bar{q}_1 q_{2x})\mu_{32}^{(1)} - \tfrac{i}{4}(|q_1|^4 + |q_1|^2|q_2|^2) + \tfrac{i}{4}(\bar{q}_1 q_{1xx} - \bar{q}_{1x}q_{1x})]dt$$
$$\Delta_{23}^{(2)} = (-\tfrac{i}{2}|q_1|^2\mu_{23}^{(1)} - \tfrac{i}{2}\bar{q}_1 q_2\mu_{33}^{(1)} + \tfrac{1}{4}\bar{q}_1 q_{2x})dx$$
$$+[-\tfrac{1}{2}(\bar{q}_{1x}q_1 - \bar{q}_1 q_{1x})\mu_{23}^{(1)} - \tfrac{1}{2}(\bar{q}_{1x}q_2 - \bar{q}_1 q_{2x})\mu_{33}^{(1)} - \tfrac{i}{4}(|q_1|^2\bar{q}_1 q_2 + \bar{q}_1 q_2|q_2|^2) + \tfrac{i}{4}(\bar{q}_1 q_{2xx} - \bar{q}_{1x}q_{2x})]dt$$
$$\Delta_{32}^{(2)} = (-\tfrac{i}{2}\bar{q}_2 q_1\mu_{22}^{(1)} - \tfrac{i}{2}|q_2|^2\mu_{32}^{(1)} + \tfrac{1}{4}\bar{q}_2 q_{1x})dx$$
$$+[-\tfrac{1}{2}(\bar{q}_{2x}q_1 - \bar{q}_2 q_{1x})\mu_{22}^{(1)} - \tfrac{1}{2}(\bar{q}_{2x}q_2 - \bar{q}_2 q_{2x})\mu_{32}^{(1)} - \tfrac{i}{4}(|q_1|^2\bar{q}_2 q_1 + |q_2|^2\bar{q}_2 q_1) + \tfrac{i}{4}(\bar{q}_2 q_{1xx} - \bar{q}_{2x}q_{1x})]dt$$
$$\Delta_{33}^{(2)} = (-\tfrac{i}{2}\bar{q}_2 q_1\mu_{23}^{(1)} - \tfrac{i}{2}|q_2|^2\mu_{33}^{(1)} + \tfrac{1}{4}\bar{q}_2 q_{2x})dx$$
$$+[-\tfrac{1}{2}(\bar{q}_{2x}q_1 - \bar{q}_2 q_{1x})\mu_{23}^{(1)} - \tfrac{1}{2}(\bar{q}_{2x}q_2 - \bar{q}_2 q_{2x})\mu_{33}^{(1)} - \tfrac{i}{4}(|q_1|^2|q_2|^2 + |q_2|^4) + \tfrac{i}{4}(\bar{q}_2 q_{2xx} - \bar{q}_{2x}q_{2x})]dt.$$

$$(4.2c)$$

We define functions $\{\Phi_{ij}(t,k)\}_{i,j=1}^3$ by:

$$\mu_2(0,t,k) = \begin{pmatrix} \Phi_{11}(t,k) & \Phi_{12}(t,k) & \Phi_{13}(t,k) \\ \Phi_{21}(t,k) & \Phi_{22}(t,k) & \Phi_{23}(t,k) \\ \Phi_{31}(t,k) & \Phi_{32}(t,k) & \Phi_{33}(t,k) \end{pmatrix}. \tag{4.3}$$

From the global relation (2.39)and replacing T by t, we find

$$\mu_2(0,t,k)e^{2ik^2t\hat{\Lambda}}s(k) = c(t,k), \quad k \in (D_3 \cup D_4, D_1 \cup D_2, D_1 \cup D_2). \tag{4.4}$$

We partition matrix as following,

$$\mu_2(0,t,k) = \begin{pmatrix} \Phi_{11} & \Phi_{1j} \\ \Phi_{j1} & \Phi_{2\times 2} \end{pmatrix}, \quad j = 2,3, \tag{4.5}$$

where $\Phi_{2\times 2}$ denotes a 2×2 matrix, Φ_{1j} denotes a 1×2 vector, Φ_{j1} denotes a 2×1 vector. Then, we can write the second column of the global relation, undering the matrix partitioned as (4.5), as

$$\Phi_{11}(t,k)s_{1j}(k)s_{2\times 2}^{-1}(k)e^{-4ik^2t} + \Phi_{1j}(t,k) = c_{1j}(t,k), \quad k \in D_1 \cup D_2, \tag{4.6a}$$

$$\Phi_{j1}(t,k)s_{1j}(k)s_{2\times 2}^{-1}(k)e^{-4ik^2t} + \Phi_{2\times 2}(t,k) = c_{2\times 2}(t,k), \quad k \in D_1 \cup D_2, \tag{4.6b}$$

The functions $c_{1j}(t,k), c_{2\times 2}(t,k)$ are analytic and bounded in $D_1 \cup D_2$ away from the possible zeros of $m_{11}(k)$ and of order $O(\tfrac{1}{k})$ as $k \to \infty$.

From the asymptotic of $\mu_j(x,t,k)$ in (4.1a) we have

$$\mu_2(0,t,k) = \mathbb{I} + \frac{1}{k}\begin{pmatrix} \int_{(0,0)}^{(0,t)}\Delta_{11} & \frac{1}{2i}Q \\ -\frac{1}{2i}\bar{Q}^T & \int_{(0,0)}^{(0,t)}\Delta^{(1)} \end{pmatrix}$$
$$+ \frac{1}{k^2}\begin{pmatrix} \int_{(0,0)}^{(0,t)}\eta_2 & -\frac{i}{2}Q\int_{(0,0)}^{(0,t)}\Delta^{(1)} + \frac{1}{4}Q_x \\ \frac{i}{2}\bar{Q}^T\int_{(0,0)}^{(0,t)}\Delta_{11} + \frac{1}{4}\bar{Q}_x^T & \int_{(0,0)}^{(0,t)}\Delta^{(2)} \end{pmatrix} + O(\tfrac{1}{k^3}) \tag{4.7}$$

where $Q = (q_1, q_2)$, Δ_{11} is defined by first identities of (4.2a), η_2 is defined by (4.2b), $\Delta^{(1)}$ and $\Delta^{(2)}$ are 2×2 matrices defined as following,

$$\Delta^{(j)} = \begin{pmatrix} \Delta_{22}^{(j)} & \Delta_{23}^{(j)} \\ \Delta_{32}^{(j)} & \Delta_{33}^{(j)} \end{pmatrix}, \quad j = 1, 2, \tag{4.8}$$

here, $\{\Delta_{kl}^{(j)}\}_{k,l=2,3}, j = 1, 2$ are defined as (4.2a) and (4.2c). Also, we have

$$\Phi_{1j}(t, k) = \frac{\Phi_{1j}^{(1)}(t)}{k} + \frac{\Phi_{1j}^{(2)}(t)}{k^2} + O(\frac{1}{k^3}), \quad k \to \infty, k \in D_1 \cup D_2 \tag{4.9a}$$

$$\Phi_{2\times2}(t, k) = \mathbb{I}_{2\times2} + \frac{\Phi_{2\times2}^{(1)}(t)}{k} + \frac{\Phi_{2\times2}^{(2)}(t)}{k^2} + O(\frac{1}{k^3}), \quad k \to \infty, k \in D_1 \cup D_2. \tag{4.9b}$$

where

$$\Phi_{1j}^{(1)}(t) = \frac{1}{2i}g_0(t), \quad \Phi_{1j}^{(2)}(t) = \frac{1}{4}g_1(t) - \frac{i}{2}g_0 \int_{(0,0)}^{(0,t)} \Delta^{(1)}$$
$$\Phi_{2\times2}^{(1)}(t) = \int_{(0,0)}^{(0,t)} \Delta^{(1)}, \Phi_{2\times2}^{(2)}(t) = \int_{(0,0)}^{(0,t)} \Delta^{(2)},$$

here $g_0(t)$ and $g_1(t)$ are vector boundary functions defined by the boundary data of (1.3) as $g_0(t) = (g_{01}(t), g_{02}(t))$ and $g_1(t) = (g_{11}(t), g_{12}(t))$.

In particular, we find the following expressions for the boudary values:

$$g_0 = 2i\Phi_{1j}^{(1)}(t), \tag{4.10a}$$

$$g_1 = 2ig_0\Phi_{2\times2}^{(1)}(t) + 4\Phi_{1j}^{(2)}(t), \tag{4.10b}$$

We will also need the asymptotic of $c_{1j}(t, k)$,

Lemma 4.1. *The global relation (4.6) implies that the large k behavior of $c_{1j}(t,k), c_{2\times2}(t,k)$ satisfies*

$$c_{1j}(t, k) = \frac{\Phi_{1j}^{(1)}(t)}{k} + \frac{\Phi_{1j}^{(2)}(t)}{k^2} + O(\frac{1}{k^3}), \quad k \to \infty, k \in D_1. \tag{4.11}$$

Proof. Let

$$\mu_2(0, t, k) = \begin{pmatrix} \Phi_{11} & \Phi_{1j} \\ \Phi_{j1} & \Phi_{2\times2} \end{pmatrix}, \quad j = 2, 3.$$

The global relation shows that

$$\Phi_{11}(t, k)s_{1j}(k)s_{2\times2}^{-1}(k)e^{-4ik^2t} + \Phi_{1j}(t, k) = c_{1j}(t, k). \tag{4.12}$$

And from equation

$$\mu_t - 2ik^2[\Lambda, \mu] = V_2\mu. \tag{4.13}$$

From the second column of the equation (4.13) we get

$$\begin{cases} \Phi_{1jt} + 4ik^2\Phi_{1j} = 2kQ\Phi_{2\times2} - ikQ\bar{Q}^T\Phi_{1j} + iQ_x\Phi_{2\times2} \\ \Phi_{2\times2t} = 2k\bar{Q}\Phi_{1j} + i\bar{Q}^TQ\Phi_{2\times2} - i\bar{Q}_x^T\Phi_{1j}. \end{cases} \tag{4.14}$$

where $Q = (q_1, q_2)$ is a 1×2 vector function.

Suppose

$$\begin{pmatrix} \Phi_{1j} \\ \Phi_{2\times2} \end{pmatrix} = (\alpha_0(t) + \frac{\alpha_1(t)}{k} + \frac{\alpha_2(t)}{k^2} + \cdots) + (\beta_0(t) + \frac{\beta_1(t)}{k} + \frac{\beta_2(t)}{k^2} + \cdots)e^{-4ik^2t} \tag{4.15}$$

where the coefficients $\alpha_l(t)$ and $\beta_l(t)$, $l \geq 0$, are independent of k. To determine these coefficients, we substitute the above equation into equation (4.14) and use the initial conditions

$$\alpha_0(0) + \beta_0(0) = (0_{1\times2}, \mathbb{I}_{2\times2})^T, \quad \alpha_1(0) + \beta_1(0) = (0_{1\times2}, 0_{2\times2})^T.$$

Then we get

$$\begin{aligned} \begin{pmatrix} \Phi_{1j} \\ \Phi_{2\times2} \end{pmatrix} &= \begin{pmatrix} 0_{1\times2} \\ \mathbb{I}_{2\times2} \end{pmatrix} + \frac{1}{k}\begin{pmatrix} \Phi_{1j}^{(1)} \\ \Phi_{2\times2}^{(1)} \end{pmatrix} + O(\frac{1}{k^2}) \\ &+ \left[\frac{1}{k}\begin{pmatrix} \frac{-Q(0)}{2i} \\ 0 \end{pmatrix} + O(\frac{1}{k^2}) \right] e^{-4ik^2t} \end{aligned} \tag{4.16}$$

From the first column of the equation (4.13) we get

$$\begin{cases} \Phi_{11t} = 2kQ\Phi_{j1} - iQ\bar{Q}^T\Phi_{11} + iQ_x\Phi_{j1} \\ \Phi_{j1t} - 4ik^2\Phi_{j1} = 2k\bar{Q}^T\Phi_{11} + i\bar{Q}^TQ\Phi_{j1} - i\bar{Q}_x^T\Phi_{11}. \end{cases} \tag{4.17}$$

Suppose

$$\begin{pmatrix} \Phi_{2\times2} \\ \Phi_{3j} \end{pmatrix} = (\xi_0(t) + \frac{\xi_1(t)}{k} + \frac{\xi_2(t)}{k^2} + \cdots) + (\nu_0(t) + \frac{\nu_1(t)}{k} + \frac{\nu_2(t)}{k^2} + \cdots)e^{8ik^3t} \tag{4.18}$$

where the coefficients $\xi_l(t)$ and $\nu_l(t)$, $l \geq 0$, are independent of k. To determine these coefficients, we substitute the above equation into equation (4.17) and use the initial conditions

$$\xi_0(0) + \nu_0(0) = (1, 0_{2\times1})^T,$$

Then we get

$$
\begin{pmatrix} \Phi_{11} \\ \Phi_{j1} \end{pmatrix} = \begin{pmatrix} 1 \\ 0_{2\times 1} \end{pmatrix} + \frac{1}{k} \begin{pmatrix} \Phi_{11}^{(1)} \\ \Phi_{j1}^{(1)} \end{pmatrix} + O(\tfrac{1}{k^2})
$$
$$
+ \left[\frac{1}{k} \begin{pmatrix} 0 \\ \frac{\bar{Q}^T(0)}{2i} \end{pmatrix} + O(\tfrac{1}{k^2}) \right] e^{4ik^2 t} \tag{4.19}
$$

So, from the equation (4.12) and the asymptotic of $s_{j3}(k)$ and $s_{33}(k)$, we get the asymptotic behavior of $c_j(t,k)$ as $k \to \infty$,

$$
c_j(t,k) = \frac{\Phi_{j3}^{(1)}}{k} + \frac{\Phi_{j3}^{(2)}}{k^2} + \frac{\Phi_{j3}^{(3)}}{k^3} + \cdots . \tag{4.20}
$$

$\square$

4.2. *The Dirichlet and Neumann problems*

We can now derive effective characterizations of spectral function $S(k)$ for the Dirichlet (g_0 prescribed), the Neumann (g_1 prescribed) problems.

Define functions as

$$
\Omega(t,k) = \Phi_{1j}(t,k) - \Phi_{1j}(t,-k), \quad \hat{\Omega}(t,k) = \Phi_{2\times 2}(t,k) - \Phi_{2\times 2}(t,-k), \tag{4.21a}
$$

and

$$
\omega(t,k) = \Phi_{1j}(t,k) + \Phi_{1j}(t,-k), \quad \tilde{\omega}(t,k) = \Phi_{2\times 2}(t,k) + \Phi_{2\times 2}(t,-k) \tag{4.21b}
$$

Theorem 4.1. *Let $T < \infty$. Let $q_0(x) = (q_{10}(x), q_{20}(x)), x \geq 0$, be a vector function of Schwartz class.*

For the Dirichlet problem it is assumed that the function $g_0(t), 0 \leq t < T$, has sufficient smoothness and is compatible with $q_0(x)$ at $x = t = 0$.

For the Neumann problem it is assumed that the function $g_1(t), 0 \leq t < T$, has sufficient smoothness and is compatible with $q_0(x)$ at $x = t = 0$.

Suppose that $m_{11}(k)$ has a finite number of simple zeros in D_1.

Then the spectral function $S(k)$ is given by

$$
S(k) = \begin{pmatrix} A(k) & e^{-4ik^2 T} B(k) & e^{-4ik^2 T} C(k) \\ e^{4ik^2 T} D(k) & E(k) & F(k) \\ e^{4ik^2 T} G(k) & H(k) & I(k) \end{pmatrix} \tag{4.22}
$$

where

$$A(k) = \Phi_{22}(k)\Phi_{33}(k) - \Phi_{23}(k)\Phi_{32}(k) \quad B(k) = \Phi_{13}(k)\Phi_{22}(k) - \Phi_{12}(k)\Phi_{33}(k)$$

$$C(k) = \Phi_{12}(k)\Phi_{23}(k) - \Phi_{13}(k)\Phi_{22}(k) \quad D(k) = \Phi_{23}(k)\Phi_{31}(k) - \Phi_{21}(k)\Phi_{33}(k)$$

$$E(k) = \Phi_{11}(k)\Phi_{33}(k) - \Phi_{13}(k)\Phi_{31}(k) \quad F(k) = \Phi_{21}(k)\Phi_{13}(k) - \Phi_{11}(k)\Phi_{23}(k)$$

$$G(k) = \Phi_{21}(k)\Phi_{32}(k) - \Phi_{22}(k)\Phi_{31}(k) \quad H(k) = \Phi_{12}(k)\Phi_{31}(k) - \Phi_{11}(k)\Phi_{32}(k)$$

$$I(k) = \Phi_{11}(k)\Phi_{22}(k) - \Phi_{12}(k)\Phi_{21}(k)$$

and the complex-value functions $\{\Phi_{l3}(t,k)\}_{l=1}^3$ satisfy the following system of integral equations:

$$\Phi_{13}(t,k) = \int_0^t e^{4ik^2(t-t')} \left[-i(|g_{01}|^2 + |g_{02}|^2)\Phi_{13} \right.$$
$$\left. + (2kg_{01} + ig_{11})\Phi_{23} + (2kg_{02} + ig_{12})\Phi_{33} \right](t',k)dt' \tag{4.23a}$$

$$\Phi_{23}(t,k) = \int_0^t \left[(2k\bar{g}_{01} - i\bar{g}_{11})\Phi_{13} + i|g_{01}|^2\Phi_{23} + i\bar{g}_{01}g_{02}\Phi_{33} \right](t',k)dt' \tag{4.23b}$$

$$\Phi_{33}(t,k) = 1 + \int_0^t \left[(2k\bar{g}_{02} - i\bar{g}_{12})\Phi_{13} + ig_{01}\bar{g}_{02}\Phi_{23} + i|g_{02}|^2\Phi_{33} \right](t',k)dt' \tag{4.23c}$$

and $\{\Phi_{l1}(t,k)\}_{l=1}^3, \{\Phi_{l2}(t,k)\}_{l=1}^3$ satisfy the following system of integral equations:

$$\Phi_{11}(t,k) = 1 + \int_0^t \left[-i(|g_{01}|^2 + |g_{02}|^2)\Phi_{11} \right.$$
$$\left. + (2kg_{01} + ig_{11})\Phi_{21} + (2kg_{02} + ig_{12})\Phi_{31} \right](t',k)dt' \tag{4.24a}$$

$$\Phi_{21}(t,k) = \int_0^t e^{-4ik^2(t-t')} \left[(2k\bar{g}_{01} - i\bar{g}_{11})\Phi_{11} + i|g_{01}|^2\Phi_{21} + i\bar{g}_{01}g_{02}\Phi_{31} \right](t',k)dt' \tag{4.24b}$$

$$\Phi_{31}(t,k) = \int_0^t e^{-4ik^2(t-t')} \left[(2k\bar{g}_{02} - i\bar{g}_{12})\Phi_{11} + ig_{01}\bar{g}_{02}\Phi_{21} + i|g_{02}|^2\Phi_{31} \right](t',k)dt' \tag{4.24c}$$

$$\Phi_{12}(t,k) = \int_0^t e^{4ik^2(t-t')} \left[-i(|g_{01}|^2 + |g_{02}|^2)\Phi_{12} \right.$$
$$\left. + (2kg_{01} + ig_{11})\Phi_{22} + (2kg_{02} + ig_{12})\Phi_{32} \right](t',k)dt' \tag{4.25a}$$

$$\Phi_{22}(t,k) = 1 + \int_0^t \left[(2k\bar{g}_{01} - i\bar{g}_{11})\Phi_{12} + i|g_{01}|^2\Phi_{22} + i\bar{g}_{01}g_{02}\Phi_{32} \right](t',k)dt' \tag{4.25b}$$

$$\Phi_{32}(t,k) = \int_0^t \left[(2k\bar{g}_{02} - i\bar{g}_{12})\Phi_{12} + ig_{01}\bar{g}_{02}\Phi_{22} + i|g_{02}|^2\Phi_{32} \right](t',k)dt' \tag{4.25c}$$

(i) For the Dirichlet problem, the unknown Neumann boundary value $g_1(t)$ is given by

$$g_1(t) = \frac{2}{\pi i} \int_{\partial D_3} (k\Omega(t,k) + ig_0(t))dk + \frac{2g_0(t)}{\pi} \int_{\partial D_3} \hat{\Omega}(t,k)dk$$
$$- \frac{4}{\pi i} \int_{\partial D_3} k e^{-4ik^2t} \Phi_{11}(-k)s_{1j}(-k)s_{2\times2}^{-1}(-k)dk \qquad (4.26)$$
$$+ 8\sum_{k_j \in D_1} k_j e^{-4ik_j^2 t} Res(\Phi_{11}s_{1j}s_{2\times2}^{-1})(k_j).$$

(ii) For the Neumann problem, the unknown boundary values $g_0(t)$ is given by

$$g_0(t) = \frac{1}{\pi} \int_{\partial D_3} \omega(t,k)dk + \frac{2}{\pi} \int_{\partial D_3} e^{-4ik^2t} \Phi_{11}(-k)s_{1j}(-k)s_{2\times2}^{-1}(-k)dk \qquad (4.27)$$
$$+ 4i\sum_{k_j \in D_1} e^{-4ik_j^2 t} Res(\Phi_{11}s_{1j}s_{2\times2}^{-1})(k_j).$$

Proof. The representations (4.22) follow from the relation $S(k) = e^{4ik^2T}\mu_2^A(0,T,k)^T$. And the system (4.23) is the direct result of the Volteral integral equations of $\mu_2(0,t,k)$.

(i) In order to derive (4.26) we note that equation (4.10b) expresses g_1 in terms of $\Phi_{2\times2}^{(1)}$ and $\Phi_{1j}^{(2)}$. Furthermore, equation (4.9) and Cauchy theorem imply

$$-\frac{\pi i}{2}\Phi_{2\times2}^{(1)}(t) = \int_{\partial D_2} [\Phi_{2\times2}(t,k) - \mathbb{I}_{2\times2}]dk = \int_{\partial D_4} [\Phi_{2\times2}(t,k) - \mathbb{I}_{2\times2}]dk$$

and

$$-\frac{\pi i}{2}\Phi_{1j}^{(2)}(t) = \int_{\partial D_2} \left[k\Phi_{1j}(t,k) - \frac{g_0(t)}{2i}\right]dk = \int_{\partial D_4} \left[k\Phi_{1j}(t,k) - \frac{g_0(t)}{2i}\right]dk.$$

Thus,

$$i\pi\Phi_{2\times2}^{(1)}(t) = -\left(\int_{\partial D_2} + \int_{\partial D_4}\right)[\Phi_{2\times2}(t,k) - \mathbb{I}_{2\times2}]dk$$
$$= \left(\int_{\partial D_1} + \int_{\partial D_3}\right)[\Phi_{2\times2}(t,k) - \mathbb{I}_{2\times2}]dk$$
$$= \int_{\partial D_3}[\Phi_{2\times2}(t,k) - \mathbb{I}_{2\times2}]dk - \int_{\partial D_3}[\Phi_{2\times2}(t,-k) - \mathbb{I}_{2\times2}]dk \qquad (4.28)$$
$$= \int_{\partial D_3}(\Phi_{2\times2}(t,k) - \Phi_{2\times2}(t,-k))dk.$$

Similarly,

$$i\pi\Phi_{1j}^{(2)}(t) = \left(\int_{\partial D_3} + \int_{\partial D_1}\right)\left[k\Phi_{1j}(t,k) - \frac{g_0(t)}{2i}\right]dk$$
$$= \left(\int_{\partial D_3} - \int_{\partial D_1}\right)\left[k\Phi_{1j}(t,-k) - \frac{g_0(t)}{2i}\right]dk + I(t) \qquad (4.29)$$
$$= \int_{\partial D_3}[k(\Phi_{1j}(t,k) - \Phi_{1j}(t,-k)) + ig_0(t)]dk + I(t).$$

where $I(t)$ is defined by

$$I(t) = 2\int_{\partial D_1}\left[k\Phi_{1j}(t,k) - \frac{g_0(t)}{2i}\right]dk$$

The last step involves using the global relation to compute $I(t)$

$$I(t) = \int_{\partial D_1}\left[k(\mu_{3,1j}s_{2\times2}^{-1} - \Phi_{11}s_{1j}e^{-4ik^2t}s_{2\times2}^{-1}) - \frac{g_0(t)}{2i}\right]dk \qquad (4.30)$$

Using the asymptotic (4.11) and Cauchy theorem to compute the first term on the right-hand side of equation (4.30), we find

$$
\begin{aligned}
I(t) = {} & -i\pi\Phi_{1j}^{(2)} - 2\int_{ptD_3} k\Phi_{11}(-k)s_{1j}(-k)s_{2\times2}^{-1}(-k)e^{-4ik^2t}dk \\
& + 4\pi i\sum_{k_j\in D_1} k_j e^{-4ik_j^2 t} Res(\Phi_{11}s_{1j}s_{2\times2}^{-1})(k_j).
\end{aligned}
\tag{4.31}
$$

Equations (4.29) and (4.31) imply

$$
\begin{aligned}
\Phi_{1j}^{(2)}(t) = {} & \tfrac{1}{2\pi i}\int_{\partial D_3}[k(\Phi_{1j}(t,k)-\Phi_{1j}(t,-k))+ig_0(t)]\,dk \\
& - \tfrac{1}{\pi i}\int_{ptD_3} k\Phi_{11}(-k)s_{1j}(-k)s_{2\times2}^{-1}(-k)e^{-4ik^2t}dk \\
& + 2\sum_{k_j\in D_1} k_j e^{-4ik_j^2 t} Res(\Phi_{11}s_{1j}s_{2\times2}^{-1})(k_j).
\end{aligned}
$$

This equation together with (4.10b) and (4.28) yields (4.26).

(ii) In order to derive the representations (4.27) relevant for the Neumann problem, we note that equation (4.10a) expresses g_0 in terms of $\Phi_{1j}^{(1)}$. Furthermore, equation (4.9a) and Cauchy's theorem imply

$$
-\tfrac{\pi i}{2}\Phi_{1j}^{(1)}(t) = \int_{\partial D_2}\Phi_{1j}(t,k)dk = \int_{\partial D_4}\Phi_{1j}(t,k)dk,
\tag{4.32}
$$

Thus,

$$
\begin{aligned}
i\pi\Phi_{1j}^{(1)}(t) &= \left(\int_{\partial D_3}+\int_{\partial D_1}\right)\Phi_{1j}(t,k)dk \\
&= \left(\int_{\partial D_3}-\int_{\partial D_1}\right)\Phi_{1j}(t,k)dk + 2\int_{\partial D_1}\Phi_{1j}(t,k)dk \\
&= \int_{\partial D_3}(\Phi_{1j}(t,k)+\Phi_{1j}(t,-k))\,dk + 2\int_{\partial D_1}\Phi_{1j}(t,k)dk,
\end{aligned}
\tag{4.33}
$$

and using the global relation, we have

$$
\begin{aligned}
2\int_{\partial D_1}\Phi_{1j}(t,k)dk &= 2\int_{\partial D_1}(\mu_{3,1j}s_{2\times2}^{-1}-\Phi_{11}s_{1j}s_{2\times2}^{-1}e^{-4ik^2t})dk \\
&= -i\pi\Phi_{1j}^{(1)}(t)+2\int_{\partial D_3}\Phi_{11}(-k)s_{1j}(-k)s_{2\times2}^{-1}(-k)e^{-4ik^2t}dk \\
&\quad + 4\pi i\sum_{k_j\in D_1} e^{-4ik_j^2 t} Res(\Phi_{11}s_{1j}s_{2\times2}^{-1})(k_j)
\end{aligned}
\tag{4.34}
$$

Equations (4.10a), (4.33) and (4.34) yields (4.27).

$\square$

4.3. *Effective characterizations*

Substituting into the system (4.23) the expressions

$$
\Phi_{ij} = \Phi_{ij,0}+\varepsilon\Phi_{ij,1}+\varepsilon^2\Phi_{ij,2}+\cdots,\quad i,j=1,2,3.
\tag{4.35a}
$$

$$
g_{01} = \varepsilon g_{01}^{(1)}+\varepsilon^2 g_{01}^{(2)}+\cdots,\quad g_{02}=\varepsilon g_{02}^{(1)}+\varepsilon^2 g_{02}^{(2)}+\cdots,
\tag{4.35b}
$$

$$g_{11} = \varepsilon g_{11}^{(1)} + \varepsilon^2 g_{11}^{(2)} + \cdots, \quad g_{12} = \varepsilon g_{12}^{(1)} + \varepsilon^2 g_{12}^{(2)} + \cdots, \tag{4.35c}$$

where $\varepsilon > 0$ is a small parameter, we find that the terms of $O(1)$ give

$$O(1): \begin{cases} \Phi_{13,0} = 0 \ \Phi_{23,0} = 0 \ \Phi_{33,0} = 1, \\ \Phi_{11,0} = 1 \ \Phi_{21,0} = 0 \ \Phi_{31,0} = 0, \\ \Phi_{12,0} = 0 \ \Phi_{22,0} = 1 \ \Phi_{32,0} = 0. \end{cases} \tag{4.36}$$

Moreover, the terms of $O(\varepsilon)$ give

$$O(\varepsilon): \begin{cases} \Phi_{33,1} = 0 \quad \Phi_{23,1} = 0, \\ \Phi_{13,1}(t,k) = \int_0^t e^{4ik^2(t-t')}(2kg_{02}^{(1)} + ig_{12}^{(1)})(t')dt', \\ \Phi_{11,1} = 0, \\ \Phi_{21,1} = \int_0^t e^{-4ik^2(t-t')}(2k\bar{g}_{01}^{(1)} - i\bar{g}_{11}^{(1)})(t')dt', \\ \Phi_{31,1} = \int_0^t e^{-4ik^2(t-t')}(2k\bar{g}_{02}^{(1)} - i\bar{g}_{12}^{(1)})(t')dt', \\ \Phi_{12,1} = \int_0^t e^{4ik^2(t-t')}(2kg_{01}^{(1)} + ig_{11}^{(1)})(t')dt', \\ \Phi_{22,1} = 0, \quad \Phi_{32,1} = 0. \end{cases} \tag{4.37}$$

and the terms of $O(\varepsilon^2)$ give

$$O(\varepsilon^2): \begin{cases} \Phi_{13,2} = \int_0^t e^{4ik^2(t-t')}(2kg_{02}^{(2)} + ig_{12}^{(2)})(t')dt', \\ \Phi_{23,2} = \int_0^t [(2k\bar{g}_{01}^{(1)} - i\bar{g}_{11}^{(1)})(t')\Phi_{13,1}(t',k) + i\bar{g}_{01}^{(1)}(t')g_{02}^{(1)}(t')]dt', \\ \Phi_{33,2} = \int_0^t [(2k\bar{g}_{02}^{(1)} - i\bar{g}_{12}^{(1)})(t')\Phi_{13,1}(t',k) + i|g_{02}^{(1)}(t')|^2]dt', \\ \Phi_{11,2} = \int_0^t \Big[-i(|g_{01}^{(2)}|^2 + |g_{02}^{(2)}|^2)(t') + (2kg_{01}^{(1)} + ig_{11}^{(1)})(t')\Phi_{21,1}(t',k) \\ \quad + (2kg_{02}^{(1)} + ig_{12}^{(1)})(t')\Phi_{31,1}(t',k) \Big] dt', \\ \Phi_{21,2} = \int_0^t e^{-4ik^2(t-t')}(2k\bar{g}_{01}^{(2)} - i\bar{g}_{11}^{(2)})(t')dt', \\ \Phi_{31,2} = \int_0^t e^{-4ik^2(t-t')}(2k\bar{g}_{02}^{(2)} - i\bar{g}_{12}^{(2)})(t')dt', \\ \Phi_{12,2} = \int_0^t e^{4ik^2(t-t')}(2kg_{01}^{(2)} + ig_{11}^{(2)})(t')dt', \\ \Phi_{22,2} = \int_0^t \Big[(2k\bar{g}_{01}^{(1)} - i\bar{g}_{11}^{(1)})(t')\Phi_{12,1}(t',k) + i|g_{01}^{(1)}(t')|^2 \Big] dt', \\ \Phi_{32,2} = \int_0^t \Big[(2k\bar{g}_{02}^{(1)} - i\bar{g}_{12}^{(1)})(t')\Phi_{12,1}(t',k) + ig_{01}^{(1)}(t')\bar{g}_{02}^{(1)}(t') \Big] dt'. \end{cases} \tag{4.38}$$

On the other hand, expanding (4.26) and assuming for simplicity that $m_{11}(k)$ has no zeros, we find

$$g_1^{(1)}(t) = \frac{1}{\pi i}\int_{\partial D_3}(k\Omega^{(1)}(t,k) + ig_0^{(1)})dk - \frac{4}{\pi i}\int_{\partial D_3}ke^{-4ik^2t}s_{1j}^{(1)}(-k)s_{2\times 2}^{(1)-1}(-k)dk \tag{4.39}$$

where $\Omega = \varepsilon\Omega^{(1)} + \varepsilon^2\Omega^{(2)} + O(\varepsilon^3)$ and $s_{1j}s_{2\times 2}^{-1} = \varepsilon s_{1j}^{(1)}s_{2\times 2}^{(1)-1} + O(\varepsilon^2)$.

We also find that

$$\Omega^{(1)} = 4k \int_0^t e^{4ik^2(t-t')} g_{01}^{(1)}(t') dt'. \tag{4.40}$$

The Dirichlet problem can now be solved perturbatively as follows: assuming for simplicity that $m_{11}(k)$ has no zeros and given $g_{01}^{(1)}$ and $g_{02}^{(1)}$, we can use equation (4.40) to determine $\Omega^{(1)}$. We can then compute $g_1^{(1)}$ from (4.39) and then $\Phi_{1j,1}, j = 2,3$ from (4.37). In the same way we can determine $\Omega^{(2)}$ and $\hat{\Omega}^{(2)}$ by $g_0^{(2)}$, then compute $g_1^{(2)}$. And these arguments can be extended to the higher order and also can be extended to the systems (4.24a), (4.25a) and (4.23a), thus yields a constructive scheme for computing $S(k)$ to all orders.

Similarly, these arguments also can be used to the Neumann problem. That is to say, in all cases, the system can be solved perturbatively to all orders.

Acknowledgments

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