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On the dynamics of a class of rational Kolmogorov systems

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In this paper we are interested in studying the existence of a First integral and the non-existence of limit cycles of rational Kolmogorov systems of the form

$$\begin{cases} x' = x \left(P(x,y) + \frac{R(x,y)}{S(x,y)} \right), \\ y' = y \left(Q(x,y) + \frac{R(x,y)}{S(x,y)} \right), \end{cases}$$

where $P(x,y)$, $Q(x,y)$, $R(x,y)$, $S(x,y)$ are homogeneous polynomials of degree n , n , m , a respectively.

Keywords: Kolmogorov System, First Integral, Periodic Orbits, Limit Cycle.

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1. Introduction

The autonomous differential system on the plane given by

$$\begin{cases} x' = \frac{dx}{dt} = xF(x,y), \\ y' = \frac{dy}{dt} = yG(x,y), \end{cases} \quad (1.1)$$

is known as Kolmogorov system, the derivatives are performed with respect to the time variable, and F , G are two functions in the variables x and y . It is frequently used to model the interaction of two species occupying the same ecological niche, see [11, 15, 18]. There are many natural phenomena which can be modeled by the Kolmogorov systems such as mathematical ecology and population dynamics see [13, 19, 20] chemical reactions, plasma physics see [14], hydrodynamics see [6], economics, etc. In the classical Lotka-Volterra-Gause model, F and G are linear and it is well known that there are no limit cycles. There can, of course, only be one critical point in the interior of the realistic quadrant ($x > 0, y > 0$) in this case, but this can be a center; however, there are no isolated periodic solutions. We remind that in the phase plane, a limit cycle of system (1.1) is an isolated periodic orbit in the set of all periodic orbits of system (1.1).

In the qualitative theory of planar dynamical systems see [5, 8, 9, 10, 16, 17], one of the most important topics is related to the second part of the unsolved Hilbert 16th problem. There is a huge literature about limit cycles, most of them deal essentially with their detection, their number and their stability and rare are papers concerned by giving them explicitly see [1, 2, 3, 4, 12].

Let Ω be a non-empty open and dense subset of \mathbb{R}^2 . We say that a non-locally constant C^1 function $H : \Omega \rightarrow \mathbb{R}$ is a first integral of the differential system (1.1) in Ω if H is constant on the trajectories of the system (1.1) contained in Ω , i.e. if

$$\frac{dH(x,y)}{dt} = \frac{\partial H(x,y)}{\partial x}x F(x,y) + \frac{\partial H(x,y)}{\partial y}y G(x,y) \equiv 0 \text{ in the points of } \Omega.$$

Moreover, $H = h$ is the general solution of this equation, where h is an arbitrary constant. It is well known that for differential systems defined on the plane \mathbb{R}^2 the existence of a first integral determines their phase portrait see [7].

In this paper we are interested in studying the existence of a First integral and the non-existence of limit cycles of the 2-dimensional rational Kolmogorov systems of the form

$$\begin{cases} x' = x \left(P(x,y) + \frac{R(x,y)}{S(x,y)} \right), \\ y' = y \left(Q(x,y) + \frac{R(x,y)}{S(x,y)} \right), \end{cases} \quad (1.2)$$

where $P(x,y)$, $Q(x,y)$, $R(x,y)$, $S(x,y)$ are homogeneous polynomials of degree n, n, m, a respectively.

We define the trigonometric functions

$$f_1(\theta) = P(\cos \theta, \sin \theta) \cos^2 \theta + Q(\cos \theta, \sin \theta) \sin^2 \theta, \quad f_2(\theta) = \frac{R(\cos \theta, \sin \theta)}{S(\cos \theta, \sin \theta)},$$

$$f_3(\theta) = Q(\cos \theta, \sin \theta) \cos \theta \sin \theta - P(\cos \theta, \sin \theta) \cos \theta \sin \theta.$$

2. Main result

Our main result on the existence of a First integral and the periodic orbits of the rational Kolmogorov system (1.2) is the following.

Theorem 2.1. *Consider a rational Komogorov system (1.2). Then the following statements hold.*

(a) *if $f_3(\theta) \neq 0$, $S(\cos \theta, \sin \theta) \neq 0$ and $n + a \neq m$, then system (1.2) has the first integral*

$$H(x,y) = (x^2 + y^2)^{\frac{n-m+a}{2}} \exp \left((m-n-a) \int^{\arctan \frac{y}{x}} A(\omega) d\omega \right) -$$

$$(n-m+a) \int^{\arctan \frac{y}{x}} \exp \left((m-n-a) \int^w A(\omega) d\omega \right) B(w) dw,$$

where $A(\theta) = \frac{f_1(\theta)}{f_3(\theta)}$, $B(\theta) = \frac{f_2(\theta)}{f_3(\theta)}$, and the curves which are formed by the trajectories of the differential system (1.2), are written in Cartesian coordinates as

$$x^2 + y^2 = \left(\begin{array}{l} h \exp \left((n-m+a) \int^{\arctan \frac{y}{x}} A(\omega) d\omega \right) + \\ (n-m+a) \exp \left((n-m+a) \int^\theta A(\omega) d\omega \right) \\ \int^{\arctan \frac{y}{x}} \exp \left((m-n-a) \int^w A(\omega) d\omega \right) B(w) dw \end{array} \right)^{\frac{2}{n-m+a}}$$

where $h \in \mathbb{R}$. Moreover, the system (1.2) has no limit cycle.

(b) if $f_3(\theta) \neq 0$, $S(\cos \theta, \sin \theta) \neq 0$ and $n + a = m$, then system (1.2) has the first integral

$$H(x, y) = (x^2 + y^2)^{\frac{1}{2}} \exp \left(- \int^{\arctan \frac{y}{x}} (A(\omega) + B(\omega)) d\omega \right),$$

and the curves which are formed by the trajectories of the differential system (1.2), are written in Cartesian coordinates as

$$(x^2 + y^2)^{\frac{1}{2}} - h \exp \left(\int^{\arctan \frac{y}{x}} (A(\omega) + B(\omega)) d\omega \right) = 0,$$

where $h \in \mathbb{R}$. Moreover, the system (1.2) has no limit cycle.

(c) if $f_3(\theta) = 0$ for all $\theta \in \mathbb{R}$, then system (1.2) has the first integral $H = \frac{y}{x}$, and the curves which are formed by the trajectories of the differential system (1.2), are written in Cartesian coordinates as $y = hx$ where $h \in \mathbb{R}$. Moreover, system (1.2) has no limit cycle.

Proof. In order to prove our results we write the rational differential system (1.2) in polar coordinates (r, θ) , defined by $x = r \cos \theta$, and $y = r \sin \theta$, then system (1.2) becomes

$$\begin{cases} r' = f_1(\theta) r^{n+1} + f_2(\theta) r^{m-a+1}, \\ \theta' = f_3(\theta) r^n, \end{cases} \quad (2.1)$$

where the trigonometric functions $f_1(\theta)$, $f_2(\theta)$ and $f_3(\theta)$ already given in introduction, $r' = \frac{dr}{dt}$ and $\theta' = \frac{d\theta}{dt}$.

If $f_3(\theta) \neq 0$ and $n + a \neq m$: Taking as independent variable the coordinate θ , this differential system (2.1) writes as the Bernoulli equation

$$\frac{dr}{d\theta} = A(\theta) r + B(\theta) r^{1+m-n-a}, \quad (2.2)$$

where $A(\theta) = \frac{f_1(\theta)}{f_3(\theta)}$ and $B(\theta) = \frac{f_2(\theta)}{f_3(\theta)}$.

By introducing the standard change of variables $\rho = r^{n-m+a}$, we obtain the linear equation

$$\frac{d\rho}{d\theta} = (n - m + a) (A(\theta) \rho + B(\theta)). \quad (2.3)$$

The general solution of linear equation (2.3) is

$$\rho(\theta) = \exp \left((n - m + a) \int^{\theta} A(\omega) d\omega \right) \left(\alpha + (n - m + a) \int^{\theta} \exp \left((m - n - a) \int^w A(\omega) d\omega \right) B(w) dw \right),$$

where $\alpha \in \mathbb{R}$, which has the first integral

$$H(x, y) = (x^2 + y^2)^{\frac{n-m+a}{2}} \exp \left((m - n - a) \int^{\arctan \frac{y}{x}} A(\omega) d\omega \right) - (n - m + a) \int^{\arctan \frac{y}{x}} \exp \left((m - n - a) \int^w A(\omega) d\omega \right) B(w) dw.$$

Let γ be a periodic orbit surrounding an equilibrium located in the realistic quadrant ($x > 0, y > 0$), and let $h_\gamma = H(\gamma)$.

The curves $H = h$ with $h \in \mathbb{R}$, which are formed by trajectories of the differential system (1.2), are written as

$$r(\theta) = \left(\begin{array}{l} h \exp \left((n-m+a) \int^\theta A(\omega) d\omega \right) + \\ (n-m+a) \exp \left((n-m+a) \int^\theta A(\omega) d\omega \right) \\ \int^\theta \exp \left((m-n-a) \int^w A(\omega) d\omega \right) B(w) dw \end{array} \right)^{\frac{1}{n-m+a}}.$$

In Cartesian coordinates $r^2 = x^2 + y^2$ and $\theta = \arctan\left(\frac{y}{x}\right)$, we have

$$x^2 + y^2 = \left(\begin{array}{l} h \exp \left((n-m+a) \int^{\arctan \frac{y}{x}} A(\omega) d\omega \right) + \\ (n-m+a) \exp \left((n-m+a) \int^\theta A(\omega) d\omega \right) \\ \int^{\arctan \frac{y}{x}} \exp \left((m-n-a) \int^w A(\omega) d\omega \right) B(w) dw \end{array} \right)^{\frac{2}{n-m+a}}$$

Therefore the periodic orbit γ is contained in the curve

$$r(\theta) = \left(\begin{array}{l} h_\gamma \exp \left((n-m+a) \int^\theta A(\omega) d\omega \right) + \\ (n-m+a) \exp \left((n-m+a) \int^\theta A(\omega) d\omega \right) \\ \int^\theta \exp \left((m-n-a) \int^w A(\omega) d\omega \right) B(w) dw \end{array} \right)^{\frac{1}{n-m+a}}$$

But this curve cannot contain the periodic orbit γ , consequently no limit cycle is contained in the realistic quadrant ($x > 0, y > 0$), because this curve at most have a unique point on every ray $\theta = \theta^*$ for all $\theta^* \in [0, 2\pi)$. Hence statement (a) of Theorem 1 is proved.

Suppose now that $f_3(\theta) \neq 0$, and $n+a = m$. Taking as independent variable the coordinate θ , this differential system (2.1) writes

$$\frac{dr}{d\theta} = (A(\theta) + B(\theta))r. \tag{2.4}$$

The general solution of equation (2.4) is

$$r(\theta) = \alpha \exp \left(\int^\theta (A(\omega) + B(\omega)) d\omega \right),$$

where $\alpha \in \mathbb{R}$, which has the first integral

$$H(x, y) = (x^2 + y^2)^{\frac{1}{2}} \exp \left(- \int^{\arctan \frac{y}{x}} (A(\omega) + B(\omega)) d\omega \right).$$

Let γ be a periodic orbit surrounding an equilibrium located in the realistic quadrant ($x > 0, y > 0$), and let $h_\gamma = H(\gamma)$. The curves $H = h$ with $h \in \mathbb{R}$, which are formed by trajectories of the differential

system (1.2), are written as

$$r(\theta) = h \exp \left(\int^{\theta} (A(\omega) + B(\omega)) d\omega \right)$$

In Cartesian coordinates $r^2 = x^2 + y^2$ and $\theta = \arctan \left(\frac{y}{x} \right)$, we have

$$(x^2 + y^2)^{\frac{1}{2}} - h \exp \left(\int^{\arctan \frac{y}{x}} (A(\omega) + B(\omega)) d\omega \right) = 0.$$

Therefore the periodic orbit γ is contained in the curve

$$r(\theta) = h_{\gamma} \exp \left(\int^{\arctan \frac{y}{x}} (A(\omega) + B(\omega)) d\omega \right).$$

But this curve cannot contain the periodic orbit γ , consequently no limit cycle is contained in the realistic quadrant ($x > 0, y > 0$), because this curve at most have a unique point on every ray $\theta = \theta^*$ for all $\theta^* \in [0, 2\pi)$. Hence statement (b) of Theorem 1 is proved.

Assume now that $f_3(\theta) = 0$ for all $\theta \in \mathbb{R}$, then from (2.1) it follows that $\theta' = 0$. So the straight lines through the origin of coordinates of the differential system (1.2) are invariant by the flow of this system. Hence, $\frac{y}{x}$ is a first integral of the system(1.2), since all the straight lines through the origin are formed by trajectories. Then the curves $H = h$ with $h \in \mathbb{R}$, which are formed by trajectories of the differential system (1.2), in Cartesian coordinates written as $y = hx$ where $h \in \mathbb{R}$. Clearly the system (1.2) has no periodic orbits, consequently no limit cycle.

This completes the proof of statement (c) of Theorem 1. □

3. Examples

The following examples are given to illustrate our result.

Example 1. If we take $P(x, y) = 2xy - y^2$, $Q(x, y) = 2xy + x^2$, $R(x, y) = x - y$ and $S(x, y) = x + y$, then system (1.2) reads

$$\begin{cases} x' = x \left(2xy - y^2 + \frac{x-y}{x+y} \right), \\ y' = y \left(x^2 + 2xy + \frac{x-y}{x+y} \right). \end{cases} \quad (3.1)$$

The rational differential system (3.1) in Polar coordinates (r, θ) becomes

$$\begin{cases} r' = (\sin 2\theta) r^3 + \frac{\cos \theta - \sin \theta}{\cos \theta + \sin \theta} r, \\ \theta' = \frac{1}{2} (\sin 2\theta) r^2, \end{cases}$$

here $f_1(\theta) = \sin 2\theta$, $f_2(\theta) = \frac{\cos \theta - \sin \theta}{\cos \theta + \sin \theta}$ and $f_3(\theta) = \frac{1}{2} \sin 2\theta$. In the realistic quadrant ($x > 0, y > 0$) it is the case (a) of the Theorem 1, then the rational differential system (3.1) has the first integral

$$H(x, y) = (x^2 + y^2) \exp \left(-4 \arctan \frac{y}{x} \right) - \int^{\arctan \frac{y}{x}} \exp(-4w) \left(\frac{4 \cos w - 4 \sin w}{(\cos w + \sin w) \sin 2w} \right) dw.$$

The curves $H = h$ with $h \in \mathbb{R}$, which are formed by trajectories of the differential system (3.1), in Cartesian coordinates are written as

$$x^2 + y^2 = h \exp\left(4 \arctan \frac{y}{x}\right) + \exp\left(4 \arctan \frac{y}{x}\right) \int^{\arctan \frac{y}{x}} \exp(-4w) \left(\frac{4 \cos w - 4 \sin w}{(\cos w + \sin w) \sin 2w}\right) dw,$$

where $h \in \mathbb{R}$. Moreover, the system (3.1) has no limit cycle.

Example 2. If we take $P(x, y) = x + y$, $Q(x, y) = x + 2y$, $R(x, y) = -2y^3 - x^3 + 5x^2y + 8xy^2$ and $S(x, y) = x^2 + y^2$, then system (1.2) reads

$$\begin{cases} x' = x \left(x + y + \frac{-2y^3 - x^3 + 5x^2y + 8xy^2}{x^2 + y^2} \right), \\ y' = y \left(x + 2y + \frac{-2y^3 - x^3 + 5x^2y + 8xy^2}{x^2 + y^2} \right). \end{cases} \quad (3.2)$$

The rational differential system (3.2) in Polar coordinates (r, θ) becomes

$$\begin{cases} r' = (9 \cos \theta \sin^2 \theta + 6 \cos^2 \theta \sin \theta) r^2 \\ \theta' = (\cos \theta) (\sin^2 \theta) r \end{cases}$$

here $f_1(\theta) = \cos \theta + \frac{7}{4} \sin \theta - \frac{1}{4} \sin 3\theta$, $f_2(\theta) = -2 \sin^3 \theta - \cos^3 \theta + 5 \cos^2 \theta \sin \theta + 8 \cos \theta \sin^2 \theta$, $f_3(\theta) = \frac{1}{4} \cos \theta - \frac{1}{4} \cos 3\theta$. In the realistic quadrant ($x > 0, y > 0$) it is the case (b) of the Theorem 1, the rational differential system (3.2) has the first integral

$$H(x, y) = (x^2 + y^2)^{\frac{1}{2}} \exp\left(-\int^{\arctan \frac{y}{x}} \left(\frac{9 \sin w + 6 \cos w}{\sin w}\right) dw\right).$$

The curves $H = h$ with $h \in \mathbb{R}$, which are formed by trajectories of the differential system (3.2), in Cartesian coordinates are written as

$$(x^2 + y^2)^{\frac{1}{2}} = h \exp\left(\int^{\arctan \frac{y}{x}} \left(\frac{9 \sin w + 6 \cos w}{\sin w}\right) dw\right),$$

where $h \in \mathbb{R}$. Moreover, the system (3.2) has no limit cycle.

Example 3. If we take $P(x, y) = x + y$, $Q(x, y) = x + y$, $R(x, y) = 2x + 3y$ and $S(x, y) = x^2 + y^2$, then system (1.2) reads

$$\begin{cases} x' = x \left(x + y + \frac{2x + 3y}{x^2 + y^2} \right), \\ y' = y \left(x + y + \frac{2x + 3y}{x^2 + y^2} \right). \end{cases} \quad (3.3)$$

The rational differential system (3.3) in Polar coordinates (r, θ) becomes

$$\begin{cases} r' = 2 \cos \theta + 3 \sin \theta + (\cos \theta + \sin \theta) r^2, \\ \theta' = 0, \end{cases}$$

here $f_1(\theta) = \cos \theta + \sin \theta$, $f_2(\theta) = 2 \cos \theta + 3 \sin \theta$, $f_3(\theta) = 0$ for all $\theta \in \mathbb{R}$, so it is the case (c) of the Theorem 1. Hence, $\frac{y}{x}$ is a first integral of the system (3.3). Then the curves $H = h$ with $h \in \mathbb{R}$, which are formed by trajectories of the differential system (3.3), are written in Cartesian

coordinates as $y = hx$ where $h \in \mathbb{R}$. Clearly the system (3.3) has no periodic orbits, consequently no limit cycle.

4. Conclusion

The elementary method used in this paper seems to be fruitful to investigate more general planar rational differential systems of ODEs in order to obtain explicit expression for a first integral which characterizes its trajectories. This is one of the classical tools in the classification of all trajectories of dynamical systems.

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