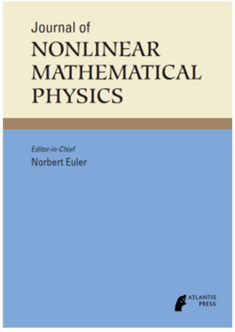

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On the well-posedness of the Holm-Staley b-family of equations

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In this paper we consider the Holm-Staley b-family of equations in the Sobolev spaces $H^s(\mathbb{R})$ for $s > 3/2$. Using a geometric approach we show that, for any value of the parameter b, the corresponding solution map,$u(0) \mapsto u(T)$, is nowhere locally uniformly continuous.

Keywords: diffeomorphism group, solution map

2000 Mathematics Subject Classification: 35Q35

1. Introduction

Holm and Staley introduced in [22] the following family of equations

$$u_t - u_{xxt} + (b+1)uu_x = bu_x u_{xx} + uu_{xxx} \tag{1.1}$$

or rewritten

$$u_t + uu_x = (1-\partial_x^2)^{-1}(-buu_x + (b-3)u_x u_{xx}) \tag{1.2}$$

related to shallow water, where

$$u(t,x) \in \mathbb{R}, \quad (t,x) \in \mathbb{R}^2$$

denotes the velocity field and $b \in \mathbb{R}$ is a parameter. For $b = 2$ we get the Camassa-Holm equation (see e.g. [5]) and for $b = 3$ the Degasperis-Procesi equation (see e.g. [14]). Both equations arise as integrable model equations for the propagation of shallow water waves of moderate amplitude. For the physical aspects see [9] and for the integrability aspects see [10, 11]. In his seminal paper [3], Arnold showed that the Euler equation can be interpreted as an equation for a flow on groups of diffeomorphisms of the underlying space. It turns out that quite a few nonlinear evolution equations such as the KdV (see [30]), the Camassa-Holm equation (see [12, 28]) or various other equations of mathematical physics (see [16, 18, 19]) can be treated by such a geometric approach. Recently, in [17], it has been shown that this is also the case for the Holm-Staley b-family. In [7], Constantin used such a geometric approach to get local well-posedness and blow-up results for the Camassa-Holm equation on $\mathbb{R}$. In this paper we will use these methods to prove our results.

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The Cauchy problem for (1.1) in $H^s(\mathbb{R})$, $s > 3/2$, with initial value $u_0 \in H^s(\mathbb{R})$, is to find $u \in C^0([0,T],H^s(\mathbb{R}))$ for some $T > 0$, such that we have the following identity in $H^{s-1}(\mathbb{R})$

$$u(t) = u_0 + \int_0^t (1-\partial_x^2)^{-1}(-buu_x + (b-3)uu_{xx}) - uu_x \, ds \tag{1.3}$$

for all $t \in [0,T]$. Here we regard $u_x u_{xx}$ as an element of $H^{s-2}(\mathbb{R})$ even if $3/2 < s < 2$ – see Appendix A. With this in mind and the fact that $H^{s-1}(\mathbb{R})$ is a Banachalgebra we see that the integrand in (1.3) is an element of $C^0([0,T],H^{s-1}(\mathbb{R}))$. Hence (1.3) makes sense and we have actually $u \in C^1([0,T],H^{s-1}(\mathbb{R}))$.

Concerning the well-posedness of (1.1) we have the following result – see also [29]

Theorem 1.1. *Let $s > 3/2$. For any given $u_0 \in H^s(\mathbb{R})$ there is a $T > 0$ and a unique solution $u \in C^0([0,T],H^s(\mathbb{R})$ to the Cauchy problem* (1.1) *with initial value $u(0) = u_0$. The T can be chosen to be the same in a neighborhood $U \subseteq H^s(\mathbb{R})$ of u_0. Moreover the map*

$$U \to C^0([0,T],H^s(\mathbb{R})), \quad u_0 \mapsto u$$

is continuous.

Now one asks whether the map mentionned in the above theorem is more than continuous, e.g. C^1 or at least locally lipschitz. We have the following negative answer.

Theorem 1.2. *Denote by $U_T \subseteq H^s(\mathbb{R})$ the set of initial values for which we have existence up to at least T. Then the map $U_T \to H^s(\mathbb{R})$, $u_0 \mapsto u(T)$, mapping the initial value to the time T value of the solution, is nowhere locally uniformly continuous.*

Results saying that the solution map $u_0 \mapsto u(T)$ has not the property to be uniformly continuous on bounded sets is known. On the circle this was proved in the case $b = 2$ (Camassa-Holm equation) in [20] and in the case $b = 3$ (Degasperis-Procesi equation) in [13] for $s \geq 2$. For the b-family on the line this was proved in [32].

2. The geometric framework

In [23] we considered for $s \in \mathbb{R}$, $s > 3/2$, the space $\mathscr{D}^s(\mathbb{R})$ (cf M. Cantor [6]) given by

$$\begin{aligned}\mathscr{D}^s(\mathbb{R}) &:= \left\{\varphi : \mathbb{R} \to \mathbb{R} \ C^1-\text{diffeomorphism} \;\middle|\; \varphi_x > 0 \text{ and } \varphi(x) - x \in H^s(\mathbb{R})\right\} \\ &= \left\{\varphi(x) = x + f(x) \;\middle|\; f \in H^s(\mathbb{R}) \text{ and } \varphi_x > 0\right\}\end{aligned}$$

where $H^s(\mathbb{R})$ denotes the space of Sobolev functions on $\mathbb{R}$ of class s. In terms of the Fourier transform this reads as (see e.g. [2])

$$H^s(\mathbb{R}) = \left\{f \in L^2(\mathbb{R}) \;\middle|\; (1+\xi^2)^{s/2}\hat{f}(\xi) \in L^2(\mathbb{R};\mathbb{C})\right\}$$

where $\hat{f}$ is the Fourier transform of f. Equipped with the scalar product (taking the real part)

$$\langle f,g\rangle_s = \Re \int_{\mathbb{R}} (1+\xi^2)^s \hat{f}(\xi)\overline{\hat{g}(\xi)} d\xi$$

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it becomes a Hilbert space. Then

$$\mathscr{D}^s(\mathbb{R}) - id = \left\{ \varphi(x) - x \,\middle|\, \varphi \in \mathscr{D}^s(\mathbb{R}) \right\} \subseteq H^s(\mathbb{R})$$

is open and thus has naturally the structure of a analytic Hilbert manifold (cf e.g. [23]). Moreover $\mathscr{D}^s(\mathbb{R})$ is a topological group under composition. More precisely, for any $k \in \mathbb{Z}_{\geq 0}$,

$$H^{s+k}(\mathbb{R}) \times \mathscr{D}^s(\mathbb{R}) \to H^s(\mathbb{R}), (f, \varphi) \mapsto f \circ \varphi$$

is a C^k-map. In the literature the partial maps of the compostion map are referred to as the α- resp. the ω-lemma – see [15].
In the following we need the notion of sprays. These are special vectorfields on the tangent bundle – see e.g. [27]. In our case we have the following identification for the tangent bundle of $\mathscr{D}^s(\mathbb{R})$

$$T\mathscr{D}^s(\mathbb{R}) = \mathscr{D}^s(\mathbb{R}) \times H^s(\mathbb{R}).$$

Thus a spray can be defined by a map S with the following structure

$$\begin{aligned} S : \mathscr{D}^s(\mathbb{R}) \times H^s(\mathbb{R}) &\to H^s(\mathbb{R}) \times H^s(\mathbb{R}) \\ (\varphi, v) &\mapsto \big(v, \Gamma_\varphi(v,v)\big) \end{aligned}$$

where Γ, called the Christoffel map of the spray S, is a map

$$\begin{aligned} \Gamma : \mathscr{D}^s(\mathbb{R}) &\to L\big(H^s(\mathbb{R}), H^s(\mathbb{R}); H^s(\mathbb{R})\big) \\ \varphi &\mapsto \Gamma_\varphi(\cdot,\cdot) \end{aligned}$$

with values in the continuous $H^s(\mathbb{R})$-valued bilinear forms on $H^s(\mathbb{R})$. Since we are just interested in the quadratic form $\Gamma_\varphi(v,v)$ we assume Γ_φ to be symmetric. The integral curves of S projected on $\mathscr{D}^s(\mathbb{R})$ are called the geodesics of S. Like in the case of a Riemannian manifold we have also here the notion of an exponential map – see e.g. [27]. More precisely, the equation of the geodesics reads as

$$\varphi_{tt} = \Gamma_\varphi(\varphi_t, \varphi_t) \tag{2.1}$$

where the subscript t denotes differentiation with respect to t. For analytic S the Picard iteration gives local solutions of (2.1) with initial data $\varphi(0) = id \in \mathscr{D}^s(\mathbb{R})$ and $\varphi_t(0) = v \in H^s(\mathbb{R})$. Because of the scaling properties of (2.1) there exists a neighborhood V of $0 \in H^s(\mathbb{R})$ such that the initial value problem

$$\begin{cases} \varphi_{tt} = \Gamma_\varphi(\varphi_t, \varphi_t) \\ \varphi(0) = id, \varphi_t(0) = v \end{cases} \tag{2.2}$$

admits a solution on the time interval $[0,1]$ for all $v \in V$. This allows us to define the exponential map exp as

$$\begin{aligned} \exp : V &\to \mathscr{D}^s(\mathbb{R}) \\ v &\mapsto \varphi_v(1) \end{aligned}$$

where φ_v is the solution of (2.2). Because of the analytic dependence of solutions of (2.2) on the initial values, see [27], we know that exp is a smooth map. Moreover the derivative of exp at $0 \in$

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$H^s(\mathbb{R})$ is the identity, i.e.

$$d_0 \exp : H^s(\mathbb{R}) \to H^s(\mathbb{R}), \quad v \mapsto v$$

where we have identified $T_{id}\mathscr{D}^s(\mathbb{R})$ with $H^s(\mathbb{R})$. By the inverse function theorem for Banach spaces, see [27]), exp is an analytic diffeomorphism between a neighborhood U of $0 \in H^s(\mathbb{R})$ and a neighborhood V of $id \in \mathscr{D}^s(\mathbb{R})$, i.e.

$$\exp : U \to V$$

is an analytic diffeomorphism.
For our purpose we define Γ at $id \in \mathscr{D}^s(\mathbb{R})$ for $v \in H^s(\mathbb{R})$ with $s > 3/2$ by

$$\Gamma_{id}(v,v) = (1-\partial_x^2)^{-1}(-bvv_x + (b-3)v_x v_{xx}) \tag{2.3}$$

which is a continuous $H^s(\mathbb{R})$-valued quadratic form on $H^s(\mathbb{R})$. For $\varphi \in \mathscr{D}^s(\mathbb{R})$ arbitrary, Γ_φ is defined by

$$\Gamma_\varphi(v,v) = \big(\Gamma_{id}(v \circ \varphi^{-1}, v \circ \varphi^{-1})\big) \circ \varphi. \tag{2.4}$$

In view of the poor regularity properties of the composition map, apriori it is not clear if Γ defines a smooth spray. In the next section we verify that this is indeed the case. In the following we will make some formal computations to show how the geodesics of S and solutions to (1.1) are related. Assume that $\varphi : [0,T] \to \mathscr{D}^s(\mathbb{R})$ solves the initial value problem (2.2). Then we have for $u := \varphi_t \circ \varphi^{-1}$

$$\begin{aligned} \varphi_{tt} &= (u \circ \varphi)_t = u_t \circ \varphi + u_x \circ \varphi \cdot \varphi_t \\ &= u_t \circ \varphi + u_x \circ \varphi \cdot u \circ \varphi. \end{aligned}$$

Substituting this expression into equation (2.1) we get

$$\begin{aligned} u_t \circ \varphi + u_x \circ \varphi \cdot u \circ \varphi &= \Gamma_\varphi(\varphi_t, \varphi_t) \\ &= \Gamma_{id}(\varphi_t \circ \varphi^{-1}, \varphi_t \circ \varphi^{-1}) \circ \varphi \end{aligned}$$

or by (2.3) equivalently

$$u_t + uu_x = (1-\partial_x^2)^{-1}(-buu_x + (b-3)u_x u_{xx})$$

which is equation (1.2). In the next section we show that by this approach one gets local well-posedness results for equation (1.1).

3. Local wellposedness of the b-family of equations

In this section we establish local existence and uniqueness for the Cauchy problem

$$u_t + uu_x = (1-\partial_x^2)^{-1}(-buu_x + (b-3)uu_{xx}), \quad u(0) = u_0 \in H^s(\mathbb{R}) \tag{3.1}$$

in $H^s(\mathbb{R})$, $s > 3/2$.

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Theorem 3.1. *The spray S given by*

$$S : \mathscr{D}^s(\mathbb{R}) \times H^s(\mathbb{R}) \to H^s(\mathbb{R}) \times H^s(\mathbb{R}), \quad (\varphi, u) \mapsto \big(v, \Gamma_\varphi(u,u)\big)$$

where

$$\Gamma_\varphi(u,u) = R_\varphi(1-\partial_x^2)^{-1}\big(-b(R_{\varphi^{-1}}u)\cdot(R_{\varphi^{-1}}u)_x + (b-3)(R_{\varphi^{-1}}u)_x \cdot (R_{\varphi^{-1}}u)_{xx}\big)$$

is analytic.

Recall that we use the notation R_φ for right translation $H^s(\mathbb{R}) \to H^s(\mathbb{R}), f \mapsto f \circ \varphi$. Before proving Theorem 3.1 we show the following lemma.

Lemma 3.1. *Let $s > 3/2$. For $k = 1,2$ the map*

$$\delta^{(k)} : \mathscr{D}^s(\mathbb{R}) \times H^s(\mathbb{R}) \to H^{s-k}(\mathbb{R}), \quad (\varphi, f) \mapsto R_\varphi \partial_x^k R_{\varphi^{-1}} f$$

is analytic.

Proof. Consider first the case $k = 1$. Then we have

$$\delta^{(1)}(\varphi, f) = R_\varphi \partial_x R_{\varphi^{-1}} f = \frac{f_x}{\varphi_x}$$

and this is an analytic expression in φ and f. Similarly for $k = 2$ we have we have

$$\delta^{(2)}(\varphi, f) = R_\varphi \partial_x^2 R_{\varphi^{-1}} f = \frac{f_{xx}}{\varphi_x} - \frac{f_x \cdot \varphi_{xx}}{(\varphi_x)^3}$$

which also holds in the case $s < 2$ – see Appendix A for the conventions in this case. We see from the expressions for $\delta^{(2)}(\varphi, f)$ that it is indeed analytic. □

Now we can give the proof Theorem 3.1.

Proof of Theorem 3.1. Consider the continuous symmetric bilinear map

$$\begin{aligned} B : H^s(\mathbb{R}) \times H^s(\mathbb{R}) &\to H^{s-2}(\mathbb{R}) \\ (u,v) &\mapsto \frac{1}{2}\big(-buv_x - bvu_x\big) + \frac{1}{2}\big((b-3)uv_{xx} + (b-3)vu_{xx}\big). \end{aligned}$$

That the range of this expression is in $H^{s-2}(\mathbb{R})$ and its continuity follow from the Banach algebra properties of $H^s(\mathbb{R})$ – see e.g. [2] (for $s < 2$ see Appendix A). From Lemma 3.1 we know that the

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map

$$\begin{aligned}\mathscr{D}^s(\mathbb{R}) &\to L\big(H^s(\mathbb{R}),H^s(\mathbb{R});H^{s-2}(\mathbb{R})\big)\\ \varphi &\mapsto \big[(u,v)\mapsto R_\varphi B(R_{\varphi^{-1}}u,R_{\varphi^{-1}}v)\big]\end{aligned}$$

is analytic. Again using Lemma 3.1 we get that

$$\begin{aligned}\mathscr{D}^s(\mathbb{R}) &\to L\big(H^s(\mathbb{R});H^{s-2}(\mathbb{R})\big)\\ \varphi &\mapsto \big[u\mapsto R_\varphi(1-\partial_x^2)R_{\varphi^{-1}}u\big]\end{aligned}$$

is analytic. Hence

$$\begin{aligned}\mathscr{D}^s(\mathbb{R}) &\to L\big(H^s(\mathbb{R}),H^{s-2}(\mathbb{R})\big)\\ \varphi &\mapsto \big[u\mapsto R_\varphi(1-\partial_x^2)R_{\varphi^{-1}}u\big]\end{aligned}$$

is analytic. Note that $u\mapsto R_\varphi(1-\partial_x^2)R_{\varphi^{-1}}u$ is invertible with inverse given by $v\mapsto R_\varphi(1-\partial_x^2)^{-1}R_{\varphi^{-1}}v$. Now as inversion of linear operators is an analyitc process we see that the map

$$\begin{aligned}\mathscr{D}^s(\mathbb{R}) &\to L\big(H^{s-2}(\mathbb{R}),H^s(\mathbb{R})\big)\\ \varphi &\mapsto \big[v\mapsto R_\varphi(1-\partial_x^2)^{-1}R_{\varphi^{-1}}v\big]\end{aligned}$$

is analytic. Piecing the maps together we get from the identity

$$\begin{aligned}\Gamma_\varphi(u,v) &= R_\varphi(1-\partial_x^2)^{-1}B(R_{\varphi^{-1}}u,R_{\varphi^{-1}}v)\\ &= R_\varphi(1-\partial_x^2)^{-1}R_{\varphi^{-1}}R_\varphi B(R_{\varphi^{-1}}u,R_{\varphi^{-1}}u)\end{aligned}$$

that $\Gamma:\mathscr{D}^s(\mathbb{R})\to L(H^s(\mathbb{R}),H^s(\mathbb{R});H^s(\mathbb{R}))$ is analytic. This completes the proof of the theorem. □

Now consider the initial value problem (IVP)

$$\begin{cases}\varphi_{tt}=\Gamma_\varphi(\varphi_t,\varphi_t)\\ \varphi(0)=id\in\mathscr{D}^s(\mathbb{R}),\quad \varphi_t(0)=u_0\in H^s(\mathbb{R}).\end{cases}\tag{3.2}$$

The Picard theorem gives us local solutions to the IVP (3.2). With this and the discussion at the and of section 2 we get the following local existence result.

Lemma 3.2. *Let $s>3/2$. Given $u_0\in H^s(\mathbb{R})$, there exists $u\in C^0\big([0,T],H^s(\mathbb{R})\big)$, for some $T>0$, such that*

$$u(t)=u_0+\int_0^t(1-\partial_x^2)^{-1}(-buu_x+(b-3)u_xu_{xx})-uu_x\tag{3.3}$$

holds for all $t\in[0,T]$.

Proof. Consider the IVP (3.2). Since Γ is smooth, there exists $T>0$, such that we have a

$$\varphi\in C^\infty\big([0,T],\mathscr{D}^s(\mathbb{R})\big)$$

solving (3.2). We claim that $u:=\varphi_t\circ\varphi^{-1}$ is a solution to (3.3). From [23] we know that $u\in C^0([0,T],H^s(\mathbb{R}))$. We also have $u(0)=\varphi_t(0)=u_0$. From the Sobolev imbedding theorem we see

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that u resp. φ are in $C^1([0,T]\times\mathbb{R})$. Taking the t-derivative of $u\circ\varphi$ we get

$$u_t\circ\varphi+u_x\circ\varphi\cdot\varphi_t=u_t\circ\varphi+u_x\circ\varphi\cdot u\circ\varphi.$$

On the other hand we have

$$(u\circ\varphi)_t=\varphi_{tt}=\Gamma_\varphi(\varphi_t,\varphi_t)$$

as functions on $[0,T]\times\mathbb{R}$. Thus we get

$$u_t\circ\varphi+u_x\circ\varphi\cdot u\circ\varphi=\Gamma_\varphi(\varphi_t,\varphi_t)$$

or

$$\begin{aligned}u_t&=\Gamma_\varphi(\varphi_t,\varphi_t)\circ\varphi^{-1}-uu_x\\&=(1-\partial_x^2)^{-1}(-buu_x+(b-3)u_xu_{xx})-uu_x\end{aligned}$$

as functions on $[0,T]\times\mathbb{R}$. As both sides are continuous functions, we get by the fundamental lemma of calculus for $t\in[0,T]$

$$u(t)=u_0+\int_0^t(1-\partial_x^2)^{-1}(-buu_x+(b-3)u_xu_{xx})-uu_x. \tag{3.4}$$

But as the integrand is in $C^0([0,T],H^{s-1}(\mathbb{R}))$, the identity (3.4) holds also in $H^{s-1}(\mathbb{R})$. □

To get uniqueness we use the fact that for $u\in C([0,T];H^s(\mathbb{R}))$ there is a unique flow $\varphi\in C^1([0,T];\mathscr{D}^s(\mathbb{R}))$, i.e. φ solving

$$\varphi_t=u\circ\varphi,\quad \varphi(0)=id$$

– see [24]. With this we will prove

Lemma 3.3. *Let $s>3/2$. Assume that we have two solutions $u,w\in C^0\big([0,T],H^s(\mathbb{R})\big)$ to the Cauchy problem* (1.3) *with $u(0)=w(0)=u_0\in H^s(\mathbb{R})$. Then we have actually $u=w$ on $[0,T]$.*

Proof. Consider the flows φ resp. ψ in $C^1\big([0,T],\mathscr{D}^s(\mathbb{R})\big)$ corresponding to u resp. w as discussed above. We will show that φ resp. ψ are geodesics. By the uniqueness of geodesics with the same initial condition we will get $\varphi=\psi$ resp. $\varphi_t\circ\varphi^{-1}=\psi_t\circ\psi^{-1}$ or equivalently $u=w$. Now consider $u\circ\varphi$ which is C^1 in $[0,T]\times\mathbb{R}$. Taking the t-derivative we get pointwise

$$\frac{d}{dt}(u\circ\varphi)=u_t\circ\varphi+u_x\circ\varphi\cdot u\circ\varphi.$$

Since u is a solution of the Cauchy problem we have

$$\begin{aligned}\frac{d}{dt}(u\circ\varphi)&=R_\varphi\big((1-\partial_x^2)^{-1}(-buu_x+(b-3)u_xu_xx)\big)\\&=\Gamma_\varphi(\varphi_t\circ\varphi^{-1},\varphi_t\circ\varphi^{-1})\end{aligned}$$

where Γ is as in (2.4). From the fundamental lemma of calculus we get

$$\varphi_t(t)=u_0+\int_0^t\Gamma_\varphi(\varphi_t\circ\varphi^{-1},\varphi_t\circ\varphi^{-1}).$$

But as the integrand is in $C^0\big([0,T],H^s(\mathbb{R})\big)$ we see that $\varphi\in C^2\big([0,T],\mathscr{D}^s(\mathbb{R})\big)$ and that it is a geodesic. Hence the claim. □

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Using Lemma 3.2 and Lemma 3.3 we get

Theorem 3.2. *Let $s > 3/2$. The Cauchy problem* (1.3) *is locally well-posed in $H^s(\mathbb{R})$, i.e. given $u_0 \in H^s(\mathbb{R})$ there exists a solution $u \in C^0\big([0,T], H^s(\mathbb{R})\big)$ for some $T > 0$. Further u is unique on $[0,T]$ and the correspondence $u_0 \to u$ is continuous.*

Remark 3.1. The correspondence $u_0 \to u$ is meant as a map

$$U_T \to C^0\big([0,T], H^s(\mathbb{R})\big), \quad u_0 \mapsto u$$

This result is not new. This was e.g. done in [29]. They looked at a regularized version of (3.1) and took the limit. Also in [31] there is a similar result. They use Kato's abstract semigroup method. The same method is also used in [13] for the periodic case. We used the geometric setting as was e.g. done in [7] to achieve local well-posedness via the Picard theorem.

4. Non-uniform dependence of the solution map

In this section we will prove the non-uniform dependence of the solution map, i.e. we will prove Theorem 1.2. Recall that we denoted for $T > 0$ the set $U_T \subseteq H^s(\mathbb{R})$ to be those u_0 for which we have existence beyond T. Note also that we have the following scaling property for equation (1.1): If u is a solution then for $\lambda > 0$

$$v(x,t) := \lambda u(x, \lambda t)$$

is also a solution and $U_{\lambda T} = \frac{1}{\lambda} U_T$. Therefore it suffices to consider the case $T = 1$ to prove Theorem 1.2. Hence the theorem will follow from

Proposition 4.1. *Let $s > 3/2$ and $U := U_T|_{T=1}$. Denote by Φ the time-one map $\Phi : U \to H^s(\mathbb{R}), u_0 \mapsto u(1)$ for the Cauchy problem* (1.1). *Then Φ is nowhere locally uniformly continuous, i.e. for any non-empty $V \subseteq U$ the restriction $\Phi|_V$ is not uniformly continuous.*

To prove Proposition 4.1 we will use a conserved quantity (cf [17], Proposition 9). Consider equation (1.2) and let u be a solution, φ its corresponding flow. Then we have, omitting the arguments (t,x),

Lemma 4.2. *Let $y := (1 - \partial_x^2)u$. Then we have for all t the following identity in $H^{s-2}(\mathbb{R})$*

$$y \circ \varphi \cdot (\varphi_x)^b = y(t = 0) \tag{4.1}$$

Proof. Taking the t-derivate of $y \circ \varphi$ we get

$$\frac{d}{dt} y \circ \varphi = \varphi_{tt} - \frac{\varphi_{ttxx}}{\varphi_x^2} + 2\frac{\varphi_{txx} \cdot \varphi_{tx}}{\varphi_x^3} + \frac{\varphi_{ttx} \cdot \varphi_{xx}}{\varphi_x^3} + \frac{\varphi_{tx} \cdot \varphi_{txx}}{\varphi_x^3} - 3\frac{\varphi_{tx} \cdot \varphi_{xx} \cdot \varphi_{tx}}{\varphi_x^4} \tag{4.2}$$

On the other hand equation (3.2) gives

$$R_\varphi (1 - \partial_x^2)(\varphi_{tt} \circ \varphi^{-1}) = R_\varphi\big(-b(\varphi_t \circ \varphi^{-1}) \cdot (\varphi_t \circ \varphi^{-1})_x + (b-3)(\varphi_t \circ \varphi^{-1})_x \cdot (\varphi_t \circ \varphi^{-1})_{xx}\big)$$

Expanding this equation we get

$$\varphi_{tt} - \frac{\varphi_{ttxx}}{\varphi_x^2} + \frac{\varphi_{ttx} \cdot \varphi_{xx}}{\varphi_x^3} = -b\frac{\varphi_t \cdot \varphi_{tx}}{\varphi_x} + (b-3)\frac{\varphi_{tx}}{\varphi_x}\left(\frac{\varphi_{txx}}{\varphi_x^2} - \frac{\varphi_{tx} \cdot \varphi_{xx}}{\varphi_x^3}\right) \tag{4.3}$$

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Hence from (4.2)

$$\frac{d}{dt}\left[(R_\varphi y)\cdot\varphi_x^b\right] = \left[\varphi_{tt} - \frac{\varphi_{ttxx}}{\varphi_x^2} + 3\frac{\varphi_{txx}\cdot\varphi_{tx}}{\varphi_x^3} + \frac{\varphi_{ttx}\cdot\varphi_{xx}}{\varphi_x^3} - 3\frac{\varphi_{tx}^2\cdot\varphi_{xx}}{\varphi_x^4}\right]\cdot\varphi_x^b + \left[\varphi_t - \frac{\varphi_{txx}}{\varphi_x^2} + \frac{\varphi_{tx}\cdot\varphi_{xx}}{\varphi_x^3}\right]\cdot b\varphi_x^{b-1}\varphi_{tx} \quad (4.4)$$

Combining (4.4) and (4.3) we get

$$\frac{d}{dt}\left[(R_\varphi y)\cdot\varphi_x^b\right] = 0$$

hence $[0,T]\to H^{s-2}(\mathbb{R}), t\mapsto y\circ\varphi\cdot\varphi_x^b$ is constant, i.e. (4.1) holds. □

As $1-\partial_x^2 : H^s(\mathbb{R})\to H^{s-2}(\mathbb{R})$ is an isomorphism, it will be enough to establish that $y_0\mapsto y(1)$ is nowhere locally uniformly continuous in order to show Proposition 4.1. We will use (4.1) in the form

$$y(1) = \left(\frac{y_0}{\varphi_x(1)^b}\right)\circ\varphi(1)^{-1} \quad (4.5)$$

The approach is as in [26]. The idea is to produce a slight change $\tilde\varphi(1)$ so that $\tilde y_0$ doesn't change much, but $\tilde y(1)$ does. Since composition behaves bad this works. To make such perturbations we will employ the properties of the exponential map. But first we have to establish some preleminary lemmas.

Lemma 4.3. *Given $\varphi_\bullet\in\mathscr{D}^s(\mathbb{R})$ there exists a neighborhood W of $\varphi_\bullet$ in $\mathscr{D}^s(\mathbb{R})$, such that for some constant $C>0$ we have*

$$\frac{1}{C}||y||_{s-2}\le ||R_\varphi^{-1}(y/\varphi_x^b)||_{s-2}\le C||y||_{s-2}$$

for all $y\in H^{s-2}(\mathbb{R})$ and $\varphi\in W$.

Proof. First we establish the second inequality. For $\varphi_n\to\varphi_\bullet$ in $\mathscr{D}^s(\mathbb{R})$ we have (see Remark A.2) $R_{\varphi_n}^{-1}(y/\varphi_x^b)$ converges weakly to y in $H^{s-2}(\mathbb{R})$. By the uniform boundedness principle we get a neighborhood W_1 of $\varphi_\bullet$ in $\mathscr{D}^s(\mathbb{R})$ and $C_1>0$ such that

$$||R_\varphi^{-1}(y/\varphi_x^b)||_{s-2}\le C||y||_{s-2}$$

for all $\varphi\in W_1$ and $y\in H^{s-2}(\mathbb{R})$. Now consider the first inequality. As in the first part we get a W_2 and $C_2>0$ with

$$||\varphi_x^b R_\varphi y||_{s-2}\le C_2||y||_{s-2}$$

for all $\varphi\in W_2$ and $y\in H^{s-2}(\mathbb{R})$. Taking for y the expression $R_\varphi^{-1}(y/\varphi_x^b)$ gives the first inequalty. □

Lemma 4.4. *Let $s>3/2$. The set of $u_0\in U\cap H^{s+1}$ with $d_{u_0}\exp\neq 0$ is dense in U.*

Proof. As $d_0\exp$ is the identity map and $u_0\mapsto d_{u_0}\exp$ is analytic the claim follows immediately. □

Now we can give a proof for Proposition 4.1

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Proof of Proposition 4.1. Take $u_0 \in U$. We will show that Φ is not uniformly continuous on any neighborhood of u_0. As easily seen we can restrict ourselves to a dense subset of U. So we can assume $u_0 \in H^{s+1}$ and $d_{u_0} \exp \neq 0$ by Lemma 4.4. In particular we can fix $v \in H^s(\mathbb{R}) \setminus \{0\}$ and $x_0 \in \mathbb{R}$ with

$$(d_{u_0} \exp(v))\,(x_0) > m||v||_s, \quad m > 0$$

First we choose $R_1 > 0$ such that Lemma 4.3 holds for $\varphi_\bullet = \exp(u_0)$ in the ball $B_{R_1}(u_0)$, i.e.

$$\frac{1}{C_1}||y||_{s-2} \leq ||R_\varphi^{-1}(y/\varphi_x^b)||_{s-2} \leq C_1||y||_{s-2} \tag{4.6}$$

for all $y \in H^{s-2}(\mathbb{R})$ and $\varphi \in B_{R_1}(u_0)$. Taking $R_2 \leq R_1$ we can ensure additionally

$$||R_\varphi^{-1} y||_{s-2} \leq C_2||y||_{s-2}$$

for all $y \in H^{s-2}(\mathbb{R})$ and $\varphi \in B_{R_2}(u_0)$. By choosing $R_3 \leq R_2$ we can establish the conditions of Lemma A.5 and A.6 for all $\varphi \in B_{R_3}$ where in the following we denote the constant appearing in both lemmas by C_3. Further we denote by $C > 0$ the constant from the Sobolev imbedding

$$||f||_\infty \leq C||f||_s$$

Take arbitrary $w, h \in H^s(\mathbb{R})$ with $w, w+h \in B_{R_3}(u_0)$ and consider the Taylor expansion

$$\exp(w+h) = \exp(w) + d_w \exp(w) + \int_0^1 (1-t) d^2_{w+th} \exp(h,h)dt$$

Choosing $0 < R_4 \leq R_3$ we can garantuee

$$||d_w^2 \exp(h,h)||_s \leq K||h||_s^2$$

$$||d_{w_1}^2 \exp(h,h) - d_{w_2}^2 \exp(h,h)||_s \leq K||w_1 - w_2||_s||h||_s^2$$

for all $w, w_1, w_2 \in B_{R_4}(u_0)$ and some constant $K > 0$. By further decreasing $R_5 \leq R_4$ we can ensure $\max\{C \cdot K \cdot R_5, C \cdot K \cdot R_5^2\} < m/2$. Finally by choosing $R_* \leq R_5$ we can ensure

$$|\varphi(x) - \varphi(y)| \leq L|x-y|$$

for all $\varphi \in \exp(B_{R_*}(u_0))$. The goal is now to prove that Φ is not uniformly continuous on $B_R(u_0)$ for $0 < R \leq R_*$. So we fix $R \leq R_*$. In order to apply Lemma B.2 resp. Lemma B.3 we define the sequence of numbers

$$r_n = \frac{m}{8n}||v||_s, \quad n \geq 1$$

and choose $w_n \in C_c^\infty(\mathbb{R})$ with support in $(x_0 - \frac{1}{L}r_n, x_0 + \frac{1}{L}r_n)$ and $||w_n||_s = R/4$. Further we define $v_n := v/n$ and let $N \geq 1$ such that $||v_n||_s \leq R/4$ for $n \geq N$. With this preliminary work we define for

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$n \geq N$ two sequences of initial data:

$$x_n = u_0 + w_n \quad \text{and} \quad \tilde{x}_n = x_n + v_n = u_0 + w_n + v_n$$

We clearly have $x_n, y_n \in B_R(u_0)$ for $n \geq N$ and $||x_n - \tilde{x}_n||_s \to 0$ for $n \to \infty$. Correspondingly we define

$$\varphi_n = \exp(x_n) \quad \text{and} \quad \tilde{\varphi}_n = \exp(\tilde{x}_n)$$

We claim that $\limsup_{n\to\infty} ||\Phi(x_n) - \Phi(\tilde{x}_n)||_s > 0$. With $y_n = (1-\partial_x^2)x_n$ and $\tilde{y}_n = (1-\partial_x^2)\tilde{x}_n$ and using the conservation law (4.5) it is enough to prove

$$\limsup_{n\to\infty} ||R_{\varphi_n}^{-1}\left(y_n/(\varphi_n)_x^b\right) - R_{\tilde{\varphi}_n}^{-1}\left(\tilde{y}_n/(\tilde{\varphi}_n)_x^b\right)||_{s-2} > 0$$

We consider the parts of $y_n, \tilde{y}_n$ seperately

$$y_n = (1-\partial_x^2)(u_0 + w_n) \quad \text{and} \quad \tilde{y}_n = (1-\partial_x^2)(u_0 + w_n + v_n)$$

For the u_0-part we have, denoting $y_0 = (1-\partial_x^2)u_0 \in H^{s-1}$,

$$\begin{aligned} ||R_{\varphi_n}^{-1}(y_0/(\varphi_n)_x^b) - R_{\tilde{\varphi}_n}^{-1}(y_0/(\tilde{\varphi}_n)_x^b||_{s-2} \leq{}& ||R_{\varphi_n}^{-1}(y_0/(\varphi_n)_x^b) - R_{\varphi_n}^{-1}(y_0/(\tilde{\varphi}_n)_x^b)||_{s-2} \\ &+ ||R_{\varphi_n}^{-1}(y_0/(\tilde{\varphi}_n)_x^b) - R_{\tilde{\varphi}_n}^{-1}(y_0/(\tilde{\varphi}_n)_x^b)||_{s-2} \end{aligned}$$

The first term on the right can be estimated by

$$||R_{\varphi_n}^{-1}(y_0/(\varphi_n)_x^b) - R_{\varphi_n}^{-1}(y_0/(\tilde{\varphi}_n)_x^b)||_{s-2} \leq C_2||y_0/(\varphi_n)_x^b - y_0/(\tilde{\varphi}_n)_x^b||_{s-2}$$

The latter goes to 0 as $n \to \infty$ as dividing by φ_x^b is an analytic process. For the second term we have

$$\begin{aligned} ||R_{\varphi_n}^{-1}(y_0/(\tilde{\varphi}_n)_x^b) - R_{\tilde{\varphi}_n}^{-1}(y_0/(\tilde{\varphi}_n)_x^b||_{s-2} &\leq C_3||y_0/(\tilde{\varphi}_n)_x^b||_{s-1}||\varphi_n^{-1} - \tilde{\varphi}_n^{-1}||_{s-1} \\ &\leq C_3^2||y_0/(\tilde{\varphi}_n)_x^b||_{s-1}||\varphi_n - \tilde{\varphi}_n||_s \end{aligned}$$

which goes to 0 as $y_0/(\tilde{\varphi}_n)_x^b$ is bounded in H^{s-1}. For the v_n-term we have

$$||R_{\tilde{\varphi}_n}^{-1}((1-\partial_x^2)v_n/(\tilde{\varphi}_n)_x^b)||_{s-2} \leq C_2||(1-\partial_x^2)v_n/(\tilde{\varphi}_n)_x^b||_{s-2}$$

which by the definition of v_n goes to zero. Hence we conclude

$$\begin{aligned} &\limsup_{n\to\infty} ||R_{\varphi_n}^{-1}\left(y_n/(\varphi_n)_x^b\right) - R_{\tilde{\varphi}_n}^{-1}\left(\tilde{y}_n/(\tilde{\varphi}_n)_x^b\right)||_{s-2} \\ &\qquad = \limsup_{n\to\infty} ||R_{\varphi_n}^{-1}\left((1-\partial_x^2)w_n/(\varphi_n)_x^b\right) - R_{\tilde{\varphi}_n}^{-1}\left((1-\partial_x^2)w_n/(\tilde{\varphi}_n)_x^b\right)||_{s-2} \end{aligned}$$

The claim is now that the latter two terms have disjoint support. To establish this we estimate $|\varphi_n(x_0) - \tilde{\varphi}_n(x_0)|$. By the Taylor expansion we have

$$\varphi_n = \exp(u_0) + d_{u_0}\exp(w_n) + \int_0^1 (1-t) d_{u_0+tw_n}^2(w_n, w_n)dt$$

and similarly

$$\tilde{\varphi}_n = \exp(u_0) + d_{u_0}\exp(w_n + v_n) + \int_0^1 (1-t) d_{u_0+t(w_n+v_n)}^2(w_n+v_n, w_n+v_n)dt$$

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For the latter quadratic term we have

$$d^2_{u_0+t(w_n+v_n)}(w_n+v_n, w_n+v_n) = d^2_{u_0+t(w_n+v_n)}(w_n, w_n) + 2d^2_{u_0+t(w_n+v_n)}(w_n, v_n) + d^2_{u_0+t(w_n+v_n)}(v_n, v_n)$$

Thus we can write

$$\varphi_n - \tilde{\varphi}_n = d_{u_0} \exp(v_n) + \mathscr{R}_1 + \mathscr{R}_2 + \mathscr{R}_3$$

where

$$\mathscr{R}_1 = \int_0^1 (1-t) \left(d^2_{u_0+t(w_n+v_n)}(w_n, w_n) - d^2_{u_0+t(w_n+v_n)}(w_n, w_n) \right) dt$$

and

$$\mathscr{R}_2 = 2 \int_0^1 (1-t) d^2_{u_0+t(w_n+v_n)}(w_n, v_n) dt$$

and

$$\mathscr{R}_3 = \int_0^1 (1-t) d^2_{u_0+t(w_n+v_n)}(v_n, v_n) dt$$

For these we have

$$||\mathscr{R}_1||_\infty \leq C||\mathscr{R}_1||_s \leq CK||v_n||_s||w_n||_s^2 \leq \frac{1}{n}CK||v||_s(R/4)^2 \leq \frac{1}{4n}CKR^2||v||_s$$

and

$$||\mathscr{R}_2||_\infty \leq C||\mathscr{R}_2||_s \leq 2CK||v_n||_s||w_n||_s \leq \frac{1}{n}CK||v||_s(R/4) \leq \frac{2}{4n}CKR||v||_s$$

and

$$||\mathscr{R}_3||_\infty \leq C||\mathscr{R}_3||_s \leq CK||v_n||_s^2 \leq \frac{1}{n}CK||v||_s(R/4) \leq \frac{1}{4n}CKR||v||_s$$

Therefore

$$\begin{aligned} |\varphi(x_0) - \tilde{\varphi}(x_0)| &\geq |d_{u_0} \exp(v_n)| - ||\mathscr{R}_1||_\infty - ||\mathscr{R}_2||_\infty - ||\mathscr{R}_3||_\infty \\ &\geq \frac{1}{n}m||v||_s - \frac{1}{n}\frac{m}{2}||v||_s = \frac{m}{2n}||v||_s \end{aligned}$$

The support of $R^{-1}_{\varphi_n}\left((1-\partial_x^2)w_n/(\varphi_n)_x^b\right)$ is contained in $(\varphi_n(x_0) - r_n, \varphi_n(x_0) + r_n)$ taking into account the lipschitz property of φ_n with lipschitz constant L and the definition of w_n. Analogously the support of $R^{-1}_{\tilde{\varphi}_n}\left((1-\partial_x^2)w_n/(\tilde{\varphi}_n)_x^b\right)$ is contained in $(\tilde{\varphi}_n(x_0) - r_n, \tilde{\varphi}_n(x_0) + r_n)$. Note that the conditions

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of Lemma B.2 resp. B.3 are fullfilled (with $s-2$ playing the role of s in the Lemma) as

$$r_n \leq |\varphi_n(x_0) - \tilde{\varphi}_n(x_0)|/4$$

Thus we have

$$\begin{aligned}
&\limsup_{n\to\infty} ||R_{\varphi_n}^{-1}\left((1-\partial_x^2)w_n/(\varphi_n)_x^b\right) - R_{\tilde{\varphi}_n}^{-1}\left((1-\partial_x^2)w_n/(\tilde{\varphi}_n)_x^b\right)||_{s-2}^2 \\
&\quad \geq \limsup_{n\to\infty} \tilde{C}(||R_{\varphi_n}^{-1}\left((1-\partial_x^2)w_n/(\varphi_n)_x^b\right)||_{s-2}^2 + ||R_{\tilde{\varphi}_n}^{-1}\left((1-\partial_x^2)w_n/(\tilde{\varphi}_n)_x^b\right)||_{s-2}^2) \\
&\qquad \geq \limsup_{n\to\infty} \tilde{C}\frac{2}{C^2}||(1-\partial_x^2)w_n||_{s-2}^2 \geq \limsup_{n\to\infty} \tilde{K}||w_n||_s^2 = \tilde{K}R^2/4
\end{aligned}$$

So for any $R \leq R_*$ we have constructed $(x_n)_{n\geq 1}, (\tilde{x}_n)_{n\geq 1} \subseteq B_R(u_0)$ with $\lim_{n\to\infty} ||x_n - \tilde{x}_n||_s = 0$ and $\limsup_{n\to\infty} ||\Phi(x_n) - \Phi(\tilde{x}_n)||_s \geq C \cdot R$ for some constant $C > 0$ independent of R showing the claim. □

Appendix A. Sobolev spaces with negative indices

In this section we derive the formulas for the expressions which involve Sobolev spaces with negative indices.

Lemma A.1. *Let $1/2 < s_1 < 1$ and $-1/2 < s_2 < 0$. Then multiplication*

$$H^{s_1}(\mathbb{R}) \times H^{s_1}(\mathbb{R}) \to H^{s_1}(\mathbb{R}), \quad (f,g) \mapsto f \cdot g$$

extends to a continuous map

$$H^{s_1}(\mathbb{R}) \times H^{s_2}(\mathbb{R}) \to H^{s_2}(\mathbb{R})$$

where $H^{s_2}(\mathbb{R})$ denotes the dual of $H^{-s_2}(\mathbb{R})$.

Proof. For $f \in H^{s_1}(\mathbb{R})$, $g \in H^{s_2}(\mathbb{R})$ we define $f \cdot g \in H^{s_2}(\mathbb{R})$ by its action on a testfunction ψ as

$$\langle f \cdot g, \psi \rangle := \langle g, f \cdot \psi \rangle$$

where $\langle \cdot, \cdot \rangle$ denotes the duality pairing of $H^{s_2}(\mathbb{R})$ and $H^{-s_2}(\mathbb{R})$. As $f \cdot \psi \in H^{-s_2}(\mathbb{R})$ this definition makes sense. Further we have

$$|\langle g, f \cdot \psi \rangle| \leq ||g||_{s_2}\, ||f \cdot \psi||_{-s_2} \leq C||g||_{s_2}\, ||f||_{s_1}\, ||\psi||_{-s_2}$$

where we have used that multiplication

$$H^{s_1}(\mathbb{R}) \times H^{-s_2}(\mathbb{R}) \to H^{-s_2}(\mathbb{R})$$

is continuous – see e.g. [23]. This shows that $f \cdot g \in H^{s_2}(\mathbb{R})$. But it also shows that

$$||f \cdot g||_{s_2} \leq C||f||_{s_1}\, ||g||_{s_2}.$$

Hence the claim. □

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For the product we have the following Leibniz rule.

Lemma A.2. *Let* $1/2 < \tilde{s} < 1$ *and* $f, g \in H^{\tilde{s}}(\mathbb{R})$. *Then* $f \cdot g \in H^{\tilde{s}}(\mathbb{R})$ *with*

$$\partial_x(f \cdot g) = f_x \cdot g + f \cdot g_x$$

where the subscript refers to differentiation.

Proof. We just have to prove the formula for the derivative. For test functions ψ, ϕ we have

$$\begin{aligned} \langle \partial_x(f \cdot \phi), \psi \rangle &= -\langle f \cdot \phi, \psi_x \rangle = -\langle f, \phi \cdot \psi_x \rangle = -\langle f, (\phi \cdot \psi)_x - \phi_x \cdot \psi \rangle \\ &= \langle \phi \cdot f_x, \psi \rangle + \langle f \cdot \phi_x, \psi \rangle. \end{aligned}$$

Therefore we have the following identity in $H^{\tilde{s}-1}(\mathbb{R})$

$$\partial_x(f \cdot \phi) = f_x \cdot \phi + f \cdot \phi_x.$$

Now letting ϕ tend to g in $H^{1-\tilde{s}}(\mathbb{R})$ we get by Lemma A.1

$$\partial_x(f \cdot g) = f_x \cdot g + f \cdot g_x$$

as elements in $H^{\tilde{s}-1}(\mathbb{R})$. □

Now we extend right translation R_φ to negative Sobolev spaces

Lemma A.3. *Let* $s > 3/2$ *and* $-1/2 < \tilde{s} < 0$. *For* $\varphi \in \mathscr{D}^s(\mathbb{R})$ *the map*

$$R_\varphi : H^s(\mathbb{R}) \to H^s(\mathbb{R}), \quad f \mapsto f \circ \varphi$$

extends to a continuous map $H^{\tilde{s}}(\mathbb{R}) \to H^{\tilde{s}}(\mathbb{R})$.

Proof. Let $f \in H^{\tilde{s}}(\mathbb{R})$ ans ψ a testfunction. We define

$$\langle R_\varphi f, \psi \rangle := \langle f, \frac{\psi \circ \varphi^{-1}}{\varphi_x \circ \varphi^{-1}} \rangle. \tag{A.1}$$

We know – see e.g. [23] – that

$$\left|\left| \frac{\psi \circ \varphi^{-1}}{\varphi_x \circ \varphi^{-1}} \right|\right|_{-\tilde{s}} \leq C ||\psi||_{-\tilde{s}}$$

holds. Therefore $R_\varphi f \in H^{\tilde{s}}(\mathbb{R})$ and further

$$|\langle R_\varphi f, \psi \rangle| = \left| \langle f, \frac{\psi \circ \varphi^{-1}}{\varphi_x \circ \varphi^{-1}} \rangle \right| \leq C ||f||_{\tilde{s}} \, ||\psi||_{-\tilde{s}}$$

which shows that $R_\varphi : H^{\tilde{s}}(\mathbb{R}) \to H^{\tilde{s}}(\mathbb{R})$ is a continuous linear map. □

Remark A.1. We will sometimes write $f \circ \varphi$ instead of $R_\varphi f$ even if f is in a negative Sobolev space.

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Remark A.2. The composition map

$$H^{\tilde{s}}(\mathbb{R}) \times \mathscr{D}^s(\mathbb{R}) \to H^{\tilde{s}}(\mathbb{R}), \quad (f,\varphi) \mapsto f \circ \varphi$$

is not continuous. But as can be seen from (A.1) it is weakly continuous, i.e

$$H^{\tilde{s}}(\mathbb{R}) \times \mathscr{D}^s(\mathbb{R}) \to \mathbb{R}, \quad (f,\varphi) \mapsto \langle f \circ \varphi, \psi \rangle$$

is continuous for any testfunction ψ.

There is also the following chain rule

Lemma A.4. *Let $s > 3/2$ and $1/2 < \tilde{s} < 1$. For $\varphi \in \mathscr{D}^s(\mathbb{R})$ and $f \in H^{\tilde{s}}(\mathbb{R})$ we have*

$$\partial_x (f \circ \varphi) = f_x \circ \varphi \cdot \varphi_x.$$

as an identity in $H^{\tilde{s}-1}(\mathbb{R})$.

Proof. Let ψ be a testfunction. Then we have

$$\begin{aligned}\langle \partial_x (f \circ \varphi), \psi \rangle &= -\langle f \circ \varphi, \psi_x \rangle = -\langle f, \frac{\psi_x \circ \varphi^{-1}}{\varphi_x \circ \varphi^{-1}} \rangle = -\langle f, \partial_x (\psi \circ \varphi^{-1}) \rangle \\ &= \langle f_x, \psi \circ \varphi^{-1} \rangle = \langle f_x \circ \varphi, \psi \cdot \varphi_x \rangle = \langle f_x \circ \varphi \cdot \varphi_x, \psi \rangle\end{aligned}$$

which shows the claim. □

We will also need the following Lipschitz type estimate.

Lemma A.5. *Let $s > 3/2$ and $\varphi_\bullet \in \mathscr{D}^s(\mathbb{R})$. There is a neighborhood $W \subseteq \mathscr{D}^s(\mathbb{R})$ of $\varphi_\bullet$ and a constant $C > 0$ with*

$$||f \circ \varphi_1 - f \circ \varphi_2||_{s-2} \leq C ||f||_{s-1} ||\varphi_1 - \varphi_2||_{s-1}$$

for all $\varphi_1, \varphi_2 \in W$ and for all $f \in H^{s-1}(\mathbb{R})$.

Proof. Let $f \in C_c^\infty(\mathbb{R})$. We have from the fundamental lemma of calculus

$$f(\varphi_2(x)) = f(\varphi_1(x)) + \int_0^1 f'\left(\varphi_1(x) + t(\varphi_2(x) - \varphi_1(x))\right)(\varphi_2(x) - \varphi_1(x))dt$$

Taking W small enough we can ensure that $\varphi_1 + t(\varphi_2 - \varphi_1) \in \mathscr{D}^s(\mathbb{R})$ for $0 \leq t \leq 1$. We thus see that

$$t \to \left(f' \circ (\varphi_1 + t(\varphi_2 - \varphi_1))\right) \cdot (\varphi_2 - \varphi_1)$$

is continuous from $[0,1]$ to $H^s(\mathbb{R})$. As evaluation at $x \in \mathbb{R}$ is a continuous linear map on $H^s(\mathbb{R})$ we have the identity in $H^s(\mathbb{R})$

$$f \circ \varphi_2 - f \circ \varphi_1 = \int_0^1 \left(f' \circ (\varphi_1 + t(\varphi_2 - \varphi_1))\right) \cdot (\varphi_2 - \varphi_1) dt$$

where the integral is understood as a Riemann integral. We thus have

$$||f \circ \varphi_2 - f \circ \varphi_1||_{s-2} \leq \int_0^1 ||f' \circ (\varphi_1 + t(\varphi_2 - \varphi_1))||_{s-2} ||\varphi_2 - \varphi_1||_{s-1} dt$$

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For W small enough we can ensure that

$$||f' \circ (\varphi_1 + t(\varphi_2 - \varphi_1))||_{s-2} \le C||f'||_{s-2}$$

Thus we get

$$||f \circ \varphi_2 - f \circ \varphi_1||_{s-2} \le C||f||_{s-1}||\varphi_2 - \varphi_1||_{s-1}$$

For general $f \in H^{s-1}(\mathbb{R})$ we get the inequality by taking approximations f_n because $f \mapsto f \circ \varphi$ is continuous in $H^{s-1}(\mathbb{R})$. □

For the inversion map $\varphi \mapsto \varphi^{-1}$ we have

Lemma A.6. *Let $s > 3/2$ and $\varphi_\bullet \in \mathscr{D}^s(\mathbb{R})$. There is a neighborhood $W \subseteq \mathscr{D}^s(\mathbb{R})$ of $\varphi_\bullet$ and a constant $C > 0$ with*

$$||\varphi_2^{-1} - \varphi_1^{-1}||_{s-1} \le C||\varphi_2 - \varphi_1||_s$$

for all $\varphi_1, \varphi_2 \in W$.

Proof. If $s > 5/2$ then the lemma follows from [23] where it was shown that $\mathscr{D}^{s+1}(\mathbb{R}) \mapsto \mathscr{D}^s(\mathbb{R})$ is C^1. So it remains to check the case $s \le 5/2$. Consider first the case $3/2 < s < 2$. We have

$$\int_{\mathbb{R}} |\varphi^{-1} - \tilde{\varphi}^{-1}|^2 dx = \int_{\mathbb{R}} |\varphi^{-1} - \varphi^{-1} \circ \varphi \circ \tilde{\varphi}^{-1}|^2 dx$$

As by the Sobolev imbedding the C^1-norm is bounded by the H^s-norm we have

$$|\varphi^{-1}(x) - \varphi^{-1}(y)| \le C|x - y|$$

with a uniform C in a neighborhood of $\varphi_\bullet$. Hence

$$\int_{\mathbb{R}} |\varphi^{-1} - \tilde{\varphi}^{-1}|^2 dx \le C \int_R |x - \varphi(\tilde{\varphi}^{-1}(x)|^2 dx$$

By a change of variables we can bound the latter by

$$C \int_{\mathbb{R}} |\tilde{\varphi}(x) - \varphi(x)|^2 \tilde{\varphi}_x(x) dx$$

As φ_x is uniformly bounded in a neighborhood of $\varphi_\bullet$ we get

$$||\varphi^{-1} - \tilde{\varphi}^{-1}||_{L^2} \le C||\varphi - \tilde{\varphi}||_{s-1}$$

Let us estimate the fractional part. Using the Sobolev-Slobodecki $[\varphi^{-1} - \tilde{\varphi}^{-1}]_\lambda$ norm with $\lambda = s - 1$ we have

$$\int_{\mathbb{R}\times\mathbb{R}} \frac{|[\varphi^{-1}(x) - \tilde{\varphi}^{-1}(x)] - [\varphi^{-1}(y) - \tilde{\varphi}^{-1}(y)]|^2}{|x-y|^{1+2\lambda}} dxdy = \int_{\mathbb{R}\times\mathbb{R}} \frac{|\Phi(\tilde{\varphi}^{-1}(x)) - \Phi(\tilde{\varphi}^{-1}(y))|^2}{|x-y|^{1+2\lambda}} dxdy$$

where

$$\Phi(x) = \varphi^{-1}(\tilde{\varphi}(x)) - \varphi^{-1}(\varphi(x))$$

Change of variables gives

$$\int_{\mathbb{R}\times\mathbb{R}} \frac{|\Phi(x)-\Phi(y)|^2}{|\tilde{\varphi}(x)-\tilde{\varphi}(y)|^{1+2\lambda}} \tilde{\varphi}_x(x)\tilde{\varphi}_x(y)dxdy \leq C \int_{\mathbb{R}\times\mathbb{R}} \frac{|\Phi(x)-\Phi(y)|^2}{|x-y|^{1+2\lambda}} \frac{|x-y|^{1+2\lambda}}{|\tilde{\varphi}(x)-\tilde{\varphi}(y)|^{1+2\lambda}} dxdy$$

which by the fact that $K|\tilde{\varphi}(x)-\tilde{\varphi}(y)| \geq |x-y|$ holds is bounded by

$$C \int_{\mathbb{R}\times\mathbb{R}} \frac{|\Phi(x)-\Phi(y)|^2}{|x-y|^{1+2\lambda}} dxdy$$

By the fundamental lemma of calculus

$$\Phi(x) = \left(\int_0^1 \frac{dt}{\varphi_x \circ \varphi^{-1}(\varphi(x)+t(\tilde{\varphi}(x)-\varphi(x)))} \right) \cdot (\tilde{\varphi}(x)-\varphi(x)) = \Psi(x)\cdot(\tilde{\varphi}(x)-\varphi(x))$$

Writing

$$\Phi(x)-\Phi(y) = \Psi(x)\left([\tilde{\varphi}(x)-\varphi(x)]-[\tilde{\varphi}(y)-\varphi(y)]\right) + (\Psi(x)-\Psi(y))(\tilde{\varphi}(y)-\varphi(y))$$

Thus we can estimate

$$[\Phi]_\lambda \leq \sup_{x\in\mathbb{R}} |\Psi(x)| [\tilde{\varphi}-\varphi]_\lambda + \sup_{x\in\mathbb{R}} |\tilde{\varphi}(x)-\varphi(x)| [\Psi]_\lambda$$

Note that $[\Psi]_\lambda < \infty$ as

$$t \mapsto \frac{1}{\varphi_x \circ \varphi^{-1}(\varphi(x)+t(\tilde{\varphi}(x)-\varphi(x)))} - 1$$

is a continuous path in $H^{s-1}(\mathbb{R})$. Therefore we get

$$[\varphi^{-1}-\tilde{\varphi}^{-1}]_\lambda \leq C||\varphi-\tilde{\varphi}||_{s-1}$$

Now consider the case $2 \leq s \leq 5/2$. Taking the derivative we get

$$(\varphi^{-1}-\tilde{\varphi}^{-1})' = \varphi_x \circ \varphi^{-1} - \tilde{\varphi}_x \circ \tilde{\varphi}^{-1}$$

which rewritten is

$$\varphi_x \circ \varphi^{-1} - \varphi_x \circ \tilde{\varphi}^{-1} + \varphi_x \circ \tilde{\varphi}^{-1} - \tilde{\varphi}_x \circ \tilde{\varphi}^{-1}$$

For the last two terms we have

$$||\varphi_x \circ \tilde{\varphi}^{-1} - \tilde{\varphi}_x \circ \tilde{\varphi}^{-1}||_{s-2} \leq C||\varphi_x - \tilde{\varphi}_x||_{s-2} \leq C||\varphi-\tilde{\varphi}||_{s-1}$$

For the first two terms we can argue as in the proof of Lemma A.5 and write

$$||\varphi_x \circ \varphi^{-1} - \varphi_x \circ \tilde{\varphi}^{-1}||_{s-2} \leq C||\varphi_x - 1||_{s-1} ||\varphi^{-1}-\tilde{\varphi}^{-1}||_{s-2}$$

Hence using the estimate from above for $||\varphi^{-1}-\tilde{\varphi}^{-1}||_{s-2}$ as $0 \leq s-2 < 1$ we get the claim. □

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Appendix B. Inequalities for fractional Sobolev functions

In this section we will establish inequalities of the form

$$||f+g|| \geq C(||f||_s + ||g||_s)$$

for functions f,g with disjoint support. For fractional s this causes some difficulties as the norm $||\cdot||_s$ is defined in a non-local way. For fixed supports we have

Lemma B.1. *Let $s \in \mathbb{R}$. There is a constant $C > 0$ such that for all $f,g \in C_c^\infty(\mathbb{R})$ with* $\operatorname{supp} f \subseteq (-3,-1)$ *and* $\operatorname{supp} g \subseteq (1,3)$ *we have*

$$||f+g||_s^2 \geq C(||f||_s^2 + ||g||_s^2)$$

Proof. We take $\varphi, \psi \in C_c^\infty(\mathbb{R})$ with $\operatorname{supp}\varphi \subseteq (-3.5,-0.5)$ and $\operatorname{supp}\psi \subseteq (0.5,3.5)$ such that $\varphi|_{(-3,-1)} \equiv 1$ and $\psi|_{(1,3)} \equiv 1$. We then have

$$||f||_s = ||\varphi(f+g)||_s \leq C_1||f+g||_s$$

and similarly

$$||g||_s = ||\psi(f+g)||_s \leq C_2||f+g||_s$$

giving the desired result. □

In the following we will use the fact that the H^s-norm is equivalent to the homogeneous $\dot{H}^s$-norm if we restrict ourselves to functions with support in a fixed compact $K \subseteq \mathbb{R}$ (see e.g. [4] p. 39). Recall

$$||f||_{\dot{H}^s}^2 = \int_{\mathbb{R}} |\xi|^{2s}|\hat{f}(\xi)|^2 d\xi$$

We often also use $f^\lambda(x) := f(x/\lambda)$ for which we have the following scaling property

$$||f^\lambda||_{\dot{H}^s}^2 = \lambda^{1-2s}||f||_{\dot{H}^s}^2$$

We have

Lemma B.2. *Let $s \geq 0$. Then there is a constant $C > 0$ with the following property: For x,y in $\mathbb{R}$ with $0 < r := |x-y|/4 < 1$ we have*

$$||f+g||_s^2 \geq C(||f||_s^2 + ||g||_s^2)$$

for all functions $f,g \in C_c^\infty(\mathbb{R})$ with $\operatorname{supp} f \subseteq (x-r,x+r)$, $\operatorname{supp} g \subseteq (y-r,y+r)$

Proof. We use the homogeneous norm. Now scaling with $\lambda = (4r)^{-1}$ gives a situation as in Lemma B.1. We have

$$||f+g||_{\dot{H}^s}^2 = \lambda_n^{2s-1}||f^\lambda + g^\lambda||_{\dot{H}^s}^2$$

Now by Lemma B.1 we then get

$$||f+g||_{\dot{H}^s}^2 \geq C\lambda^{2s-1}\left(||f^\lambda||_{\dot{H}^s}^2 + ||g^\lambda||_{\dot{H}^s}^2\right)$$

Scaling back gives

$$||f+g||_{\dot{H}^s}^2 \geq C(||f||_{\dot{H}^s}^2 + ||g||_{\dot{H}^s}^2)$$

This establishes the lemma. □

We will encounter Lemma B.2 also for some negative values of s. In these cases we will use

Lemma B.3. *Let $s < 0$ and the same situation as in Lemma B.2. Then we have*

$$||f+g||_s^2 \geq C(||f||_s^2 + ||g||_s^2)$$

for all functions $f,g \in C_c^\infty(\mathbb{R})$ with $\operatorname{supp} f \subseteq (x-r,x+r)$, $\operatorname{supp} g \subseteq (y-r,y+r)$

Proof. We claim that for functions with support in some fixed compact set $K \subseteq \mathbb{R}$ the homogeneous norm

$$||f||_{\dot{H}^s}^2 = \int_{\mathbb{R}} |\xi|^{2s}|\hat{f}(\xi)|^2 d\xi$$

is equivalent to the non-homegeneous norm $||\cdot||_s$. One then can argues as in Lemma B.2 by scaling to a fixed situation as in Lemma B.1. So it remains to show the equivalence of the norms. We clearly have $||\cdot||_s \leq ||\cdot||_{\dot{H}^s}$ since

$$\int_{\mathbb{R}} (1+\xi^2)^s|\hat{f}(\xi)|^2 d\xi \leq \int_{\mathbb{R}} \xi^{2s}|\hat{f}(\xi)|^2 d\xi$$

For the other direction we use the dual definition of the Sobolev norm

$$||f||_s = \sup_{||g||_{-s}\leq 1} |\langle f,g\rangle|$$

and analogously for the homogeneous norm. Taking $\psi \in C_c^\infty(\mathbb{R})$ with $\psi = 1$ on K we have for f with support in K

$$||f||_{\dot{H}^s} = \sup_{||g||_{\dot{H}^{-s}}\leq 1} |\langle f, \psi\cdot g\rangle|$$

Now note that we have equivalence of the norms $||\cdot||_{-s}$ and $||\cdot||_{\dot{H}^{-s}}$ for functions with support in some fixed compact. Therefore

$$\begin{aligned} ||f||_{\dot{H}^s} &= \sup_{||g||_{\dot{H}^s}\leq 1} |\langle f, \psi\cdot g\rangle| \leq \sup_{g,||\psi\cdot g||_{-s}\leq C_1} |\langle f, \psi\cdot g\rangle| \\ &\leq C_1 \sup_{||g||_{-s}\leq 1} |\langle f,g\rangle| = ||f||_{-s} \end{aligned}$$

showing the equivalence. □

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