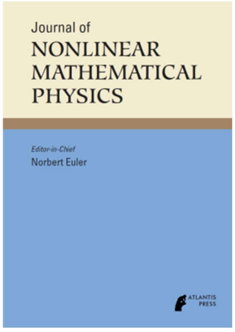

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New extension of dispersionless Harry Dym hierarchy

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The symmetry constraint for dispersionless Harry Dym (dHD) hierarchy is derived for the first time by taking dispersionless limit of that for 2+1 dimensional Harry Dym hierarchy. Then, the dHD is extended by means of the symmetry constraint which we derived. From the zero-curvature equation of the new extended dHD hierarchy, two types of dHD equations with self-consistent sources (dHDESCS) together with their associated conservation equations are obtained. Moreover, the hodograph solutions to the first type of dHDESCS are given. Finally, Bäcklund transformation between the extended dispersionless mKP hierarchy and extended dHD Hierarchy are also constructed.

Keywords: Symmetry constraint; Extended dispersionless Harry Dym hierarchy; Dispersionless Harry Dym equation with self-consistent sources; Hodograph solutions; Bäcklund transformation.

2000 Mathematics Subject Classification: 37K10.

1. Introduction

Dispersionless integrable systems (DIS) can be viewed as quasi-classical limit of the ordinary integrable systems, in which the dispersion effect had been dropped. It is shown that these systems have important applications in conformal maps, hydrodynamics, topological field theory [1,8,9,11,12,13,17,20,21,22]. In the Lax equations, the operators are replaced by phase space functions for dispersionless hierarchies. In addition, the commutator is replaced by the Poisson bracket and the role of Lax pair equations by the conservation equations. Note that these dispersionless systems can be solved by using twistorial method [21,22], hodograph reduction method [12], and the quasi-classical $\overline{\partial}$-method [13].

It is well known that dispersionless KP (dKP) hierarchy, dispersioness modified KP (dmKP) hierarchy and dHD hierarchy are three important classes of DIS within the Sato approach. These three systems have been widely investigated in the past twenty years. In 1999, the dHD hierarchy was defined by a given classical r-matrix on a Poisson algebra[16]. Some progresses have been made for the dHD hierarchy such as hodograph solutions [6], Miura map and bi-Hamiltonian formulation [7], additional symmetries and Bäcklund transformation [5] and Hydrodynamic reduction [4]. We notice that the Miura map between the dmKP hierarchy and the dHD hierarchy is triggered by the "eigenfunctions "of the dmKP hierarchy and depends on a transformation of independent variables.

As a kind of generalization of integrable soliton hierarchies, soliton equations with self-consistent sources (SESCS) have been one of hot topics in mathematical physics [10,15,18,23]. There are two types of SESCS. The first type of SESCS consist of a soliton equation with some additional terms and eigenvalue problem, while the second type consist of a soliton equation with some additional terms and time evolution equations of eigenfunction of the soliton equation. In 2006, the first types of dKP, dmKP hierarchies with self-consistent sources (dKPHSCS, dmKPHSCS) were investigated by treating the constrained integrable hierarchy as the stationary system of the corresponding hierarchy, and their hodograph solutions were given as well [25,26]. In addition, the Bäcklund transformation between dKPHSCS and dmKPHSCS was constructed in [27]. However, we can easily find that the construction of dHD hierarchy with self-consistent sources (dHDHSCS) and the Bäcklund transformation between dmKPHSCS and dHDHSCS still remain unsolved. In the present paper, we focus on the new extension of dHD hierarchy and the construction of Bäcklund transformation between dmKPHSCS and dHDHSCS. We find that the zero curvature representation of this new extended dKP hierarchy gives rise to dHDHSCS. Our research results will fill the gap of the mentioned above and give an important supplement to dispersionless Sato theory.

The outline of this paper is as follows. In section 2, the 2+1 dimensional Harry Dym hierarchy and the dHD hierarchy are briefly reviewed. In section 3, the symmetry constraint for dHD hierarchy is derived by taking dispersionless limit of that for 2+1 dimensional Harry Dym hierarchy. In section 4, based on the symmetry constraint for dHD hierarchy, a new extended dHD hierarchy is constructed. And two types of dHDESCS together with their associated conservation equations are obtained. In section 5, the hodograph solutions to the first type of dHDESCS are given. In section 6, the Bäcklund transformation between the extended dmKP hierarchy (exdmKPH) and extended dHD hierarchy (exdHDH) is constructed. As its byproduct, the Bäcklund transformation between dmKPHSCS and dHDHSCS is also obtained. Section 7 is devoted to a brief summary.

2. The 2+1 dimensional Harry Dym hierarchy and the dispersionless Harry Dym hierarchy

Let's briefly review the 2+1 dimensional Harry Dym hierarchy and the dHD hierarchy. The 2+1 dimensional Harry Dym hierarchy is defined by the Lax equation [2, 14]

$$L_{t_n} = [B_n, L]\ ,\ n \geq 1, \tag{2.1}$$

in which the Lax operator is given by

$$L = u_1\partial + u_0 + u_{-1}\partial^{-1} + u_{-2}\partial^{-2} + \cdots\ , \tag{2.2}$$

where ∂ denotes $\frac{\partial}{\partial x}$, the coefficient functions u_i depend on $t = (t_1, t_2, t_3, \cdots)$ with $t_1 = x$, $i = 0, 1, 2, \cdots$, and $B_n = (L^n)_{\geq 2}$.

By taking the dispersionless limit of (2.1), the dHD hierarchy is obtained as follows [16]

$$\partial_{T_n}\mathscr{L} = \{\mathscr{B}_n, \mathscr{L}\}, \tag{2.3}$$

where the sato function $\mathscr{L}$ is given by

$$\mathscr{L} = \sum_{i=-1}^{\infty} U_{-i}(T)p^{-i}, \tag{2.4}$$

The commutativity of (2.3) leads to

$$\partial_{T_m}\mathscr{B}_n - \partial_{T_n}\mathscr{B}_m + \{\mathscr{B}_n, \mathscr{B}_m\} = 0, \tag{2.5a}$$

The associated conservation equation of (2.5a) reads

$$\partial_{T_n} p = \partial_X \mathscr{B}_n(p) \ \ , \ \ \partial_{T_m} p = \partial_X \mathscr{B}_m(p), \tag{2.5b}$$

where $\mathscr{B}_n = (\mathscr{L}^n)_{\geq 2}$ denotes the polynomial part of $\mathscr{L}^n$ as a function of p and the Poisson bracket is defined as

$$\{A(p,X), B(p,X)\} = \frac{\partial A}{\partial p}\frac{\partial B}{\partial X} - \frac{\partial A}{\partial X}\frac{\partial B}{\partial p},$$

Note that when $m = 3, n = 2$, (2.5a) becomes the dHD equation [3, 6]

$$U_T = \frac{3}{4}U^{-1}[U^2\partial_X^{-1}(\frac{U_Y}{U^2})]_Y, \tag{2.6a}$$

The conservation equation of (2.6a) is

$$\begin{aligned} p_Y &= (U^2p^2)_X, \\ p_T &= (U^3p^3)_X + 3(U^2Vp^2)_X, \end{aligned} \tag{2.6b}$$

where $U_0 = V, U_1 = U, T_2 = Y, T_3 = T$.

3. The symmetry constraint of dispersionless Harry Dym hierarchy

In order to get the exdHDH, we firstly derive the symmetry constraint for dHD hierarchy by taking the dispersionless limit of that for 2+1 dimensional Harry Dym hierarchy.
It is known that the symmetry constraint for Harry Dym hierarchy is given by [20]

$$L^n = B_n + \sum_{i=1}^{N} q_i\partial^{-1}r_i\partial^2 \ \ , \ \ B_n = (L^n)_{\geq 2} \ , \ n \geq 1, \tag{3.1}$$

where q_i and r_i satisfy

$$q_{i,t_n} = B_n(q_i), r_{i,t_n} = -\partial^{-2}B_n^*\partial^2(r_i),$$

where B_n^* is the adjoint operator of B_n. Following the standard procedure of dispersionless limit, we take $T_n = \varepsilon t_n$ and think of $u_n(\frac{T}{\varepsilon}) = U_n(T) + O(\varepsilon)$ as $\varepsilon \longrightarrow 0$, then L in (2.2) changes into

$$L_\varepsilon = \sum_{i=-1}^{\infty} u_{-i}(\frac{T}{\varepsilon})(\varepsilon\partial)^{-i} = \sum_{i=-1}^{\infty}(U_{-i}(T) + O(\varepsilon))(\varepsilon\partial)^{-i}, \partial = \partial_X, X = \varepsilon x, \tag{3.2}$$

and constraint (3.1) becomes

$$L_\varepsilon^n = B_{\varepsilon n} + \sum_{i=1}^{N} q_i(\frac{T}{\varepsilon})(\varepsilon\partial)^{-1} r_i(\frac{T}{\varepsilon})(\varepsilon\partial)^2, B_{\varepsilon n} = (L_\varepsilon^n)_{\geq 2}, \quad (3.3)$$

where $q_i(\frac{T}{\varepsilon})$ and $r_i(\frac{T}{\varepsilon})$ satisfy

$$\varepsilon[q_i(\frac{T}{\varepsilon})]_{T_n} = B_{\varepsilon n} q_i(\frac{T}{\varepsilon}) \ , \ \varepsilon[r_i(\frac{T}{\varepsilon})]_{T_n} = -(\varepsilon\partial)^{-2} B_{\varepsilon n}^* (\varepsilon\partial)^2 r_i(\frac{T}{\varepsilon}) \quad (3.4)$$

It is easy to prove that

$$\mathscr{L} = \sigma^\varepsilon(L_\varepsilon) = \sum_{i=-1}^{\infty} U_{-i}(T) p^{-i}, \quad (3.5)$$

is a solution of the dHD hierarchy, i.e., satisfies

$$\partial_{T_n} \mathscr{L} = \{\mathscr{B}_n, \mathscr{L}\},$$

where σ^ε denotes the principal symbol [21] , and $\mathscr{B}_n = (\mathscr{L}^n)_{\geq 2}$ refers to powers of p. Regarding

$$\begin{aligned} &q_i(\frac{T}{\varepsilon}) \backsim exp[\frac{S(T,\lambda_i)}{\varepsilon} + \alpha_{i1} + O(\varepsilon)] \ , \ \varepsilon \longrightarrow 0, \\ &[\varepsilon^2 r_i(\frac{T}{\varepsilon})]_{XX} \backsim exp[-\frac{S(T,\lambda_i)}{\varepsilon} + \alpha_{i2} + O(\varepsilon)] \ , \ \varepsilon \longrightarrow 0 \ , \ i = 1 \cdots N, \end{aligned} \quad (3.6)$$

and noticing that

$$\begin{aligned} &q_i(\frac{T}{\varepsilon})(\varepsilon\partial)^{-1} r_i(\frac{T}{\varepsilon})(\varepsilon\partial)^2 \\ &= q_i(\frac{T}{\varepsilon})(\varepsilon\partial)^{-1}\varepsilon^2\{\partial^2 r_i(\frac{T}{\varepsilon}) - 2\partial[r_i(\frac{T}{\varepsilon})]_X + [r_i(\frac{T}{\varepsilon})]_{XX}\} \\ &= q_i(\frac{T}{\varepsilon}) r_i(\frac{T}{\varepsilon})(\varepsilon\partial) - q_i(\frac{T}{\varepsilon})\varepsilon[r_i(\frac{T}{\varepsilon})]_X + q_i(\frac{T}{\varepsilon})(\varepsilon\partial)^{-1}[\varepsilon^2 r_i(\frac{T}{\varepsilon})]_{XX}, \end{aligned} \quad (3.7)$$

by a tedious computation, we will find that when $\varepsilon \longrightarrow 0$

$$\begin{aligned} q_i(\frac{T}{\varepsilon})(\varepsilon\partial)^{-1} r_i(\frac{T}{\varepsilon})(\varepsilon\partial)^2 = &\frac{exp(\alpha_{i1}+\alpha_{i2})(\varepsilon\partial)}{[\frac{\partial S(T,\lambda_i)}{\partial X}]^2} + \frac{exp(\alpha_{i1}+\alpha_{i2})}{\frac{\partial S(T,\lambda_i)}{\partial X}} + exp(\alpha_{i1}+\alpha_{i2})\{(\varepsilon\partial)^{-1} \\ &+ \frac{\partial S(T,\lambda_i)}{\partial X}(\varepsilon\partial)^{-2} + \cdots\}. \end{aligned} \quad (3.8)$$

Setting

$$a_i = exp(\alpha_{i1}+\alpha_{i2}), p_i = \frac{\partial S(T,\lambda_i)}{\partial X}, p = \frac{\partial S(T,\lambda)}{\partial X},$$

and substituting (3.8) into (3.3), then we have by taking the principal symbol of the both sides of (3.3)

$$\mathscr{L}^n = \mathscr{B}_n + \sum_{i=1}^{N}(\frac{a_i p}{p_i^2} + \frac{a_i}{p_i} + \frac{1}{p} + \frac{p_i}{p^2} + \ldots) = \mathscr{B}_n + \sum_{i=1}^{N}(\frac{a_i p}{p_i^2} + \frac{a_i}{p_i} + \frac{a_i}{p-p_i}) \ , \ \mathscr{B}_n = (\mathscr{L}^n)_{\geq 2}. \quad (3.9)$$

From (3.4), (3.6) and (3.9b), a direct tedious computation leads to the following the equations of hydrodynamical type

$$p_{i,T_n} = (\mathscr{B}_n(p_i))_X \ , \ a_{i,T_n} = [a_i(\frac{\partial \mathscr{B}_n(p_i)}{\partial p_i})]_X \ , \ i = 1 \cdots N.$$

Remark 3.1. The symmetry constraints of the dKP, dmKP hierarchy was obtained by dispersionless limit method [25, 26]. But up to now, the symmetry constraint for the dHD hierarchy has never been investigated. So here the constraint symmetry of the dHD hierarchy we obtained is given for the first time.

4. New extension of dispersionless Harry Dym hierarchy

In this section, based on the symmetry constraint for dHD hierarchy, the new extension of dHD hierarchy is considered by introducing a new time evolution of $\mathscr{L}$ given by

$$\mathscr{L}_{\tau_k} = \{\mathscr{B}_k + \sum_{i=1}^{N}(\frac{a_i p}{p_i^2} + \frac{a_i}{p_i} + \frac{a_i}{p - p_i}), \mathscr{L}\}, \tag{4.1a}$$

where

$$p_{i,T_n} = (\mathscr{B}_n(p_i))_X \ , \ a_{i,T_n} = [a_i(\frac{\partial \mathscr{B}_n(p_i)}{\partial p_i})]_X \ , \ i = 1 \cdots N, \tag{4.1b}$$

Definition 4.1. The new extended dHD hierarchy (exdHDH) is defined by

$$\mathscr{L}_{\tau_k} = \{\mathscr{B}_k + \sum_{i=1}^{N}(\frac{a_i p}{p_i^2} + \frac{a_i}{p_i} + \frac{a_i}{p - p_i}), \mathscr{L}\}, \tag{4.2a}$$

$$\mathscr{L}_{T_n} = \{\mathscr{B}_n, \mathscr{L}\}, n \neq k, \tag{4.2b}$$

$$p_{i,T_n} = [\mathscr{B}_n(p)|_{p=p_i}]_X, i = 1, 2 \cdots N, \tag{4.2c}$$

$$a_{i,T_n} = [a_i(\frac{\partial \mathscr{B}_n(p)}{\partial p})|_{p=p_i}]_X. \tag{4.2d}$$

In order to verify the commutativity of (4.2a) and (4.2b) under (4.2c) and (4.2d), we need to show the following lemma.

Lemma 4.1. *There holds the identity*

$$(\frac{a_i p}{p_i^2} + \frac{a_i}{p_i} + \frac{a_i}{p - p_i})_{T_n} = \{\mathscr{B}_n, \frac{a_i p}{p_i^2} + \frac{a_i}{p_i} + \frac{a_i}{p - p_i}\}_{\leq 1} \ , \ i = 1 \cdots N.$$

Proof. Noting that the proof of Lemma 4.1 is similar to [24], so we omit the details here. □

Next, we will use Lemma 4.1 to show the following theorem . We can easily find from Theorem 4.1 that the zero-curvature representation of the new exdHDH (4.2) is dHD hierarchy with self-consistent sources.

Theorem 4.1. *Under (4.2c, d), the commutativity of (4.2a) and (4.2b) leads to the zero-curvature equation of the extended dHD hierarchy (4.2).*

$$\mathscr{B}_{n,\tau_k}-[\mathscr{B}_k+\sum_{i=1}^{N}(\frac{a_ip}{p_i^2}+\frac{a_i}{p_i}+\frac{a_i}{p-p_i})]_{T_n}+\{\mathscr{B}_n,\mathscr{B}_k+\sum_{i=1}^{N}(\frac{a_ip}{p_i^2}+\frac{a_i}{p_i}+\frac{a_i}{p-p_i})\}=0, \qquad (4.4a)$$

or equivalently

$$\mathscr{B}_{n,\tau_k}-\mathscr{B}_{k,T_n}+\{\mathscr{B}_n,\mathscr{B}_k\}+\sum_{i=1}^{N}[\{\mathscr{B}_n,(\frac{a_ip}{p_i^2}+\frac{a_i}{p_i}+\frac{a_i}{p-p_i})\}-(\frac{a_ip}{p_i^2}+\frac{a_i}{p_i}+\frac{a_i}{p-p_i})_{T_n}]=0, \quad (4.4a')$$

$$p_{i,T_n}=[\mathscr{B}_n(p_i)]_X \;\;,\;\; a_{i,T_n}=[a_i(\frac{\partial\mathscr{B}_n(p_i)}{\partial p_i})]_X \;,\; i=1\cdots N. \qquad (4.4b)$$

Under(4.4b), the conservation equation associated with (4.4a) or (4.4a')

$$p_{T_n}=[\mathscr{B}_n(p)]_X \;\;,\;\; p_{\tau_k}=[\mathscr{B}_k(p)+\sum_{i=1}^{N}(\frac{a_ip}{p_i^2}+\frac{a_i}{p_i}+\frac{a_i}{p-p_i})]_X. \qquad (4.5)$$

Proof. We will show under (4.4b), (4.2a) and (4.2b) gives rise to (4.4a). By (4.2a),(4.2b) and lemma 4.1,we have

$$\begin{aligned}
&\mathscr{B}_{n,\tau_k}=(\mathscr{L}^n_{\tau_k})_{\geq 2}=\{\mathscr{B}_k+\sum_{i=1}^{N}(\frac{a_ip}{p_i^2}+\frac{a_i}{p_i}+\frac{a_i}{p-p_i}),\mathscr{L}^n\}_{\geq 2}\\
&=\{\mathscr{B}_k+\sum_{i=1}^{N}(\frac{a_ip}{p_i^2}+\frac{a_i}{p_i}+\frac{a_i}{p-p_i}),\mathscr{B}_n\}_{\geq 2}+\{\mathscr{B}_k+\sum_{i=1}^{N}(\frac{a_ip}{p_i^2}+\frac{a_i}{p_i}+\frac{a_i}{p-p_i}),(\mathscr{L}^n)_{\leq 1}\}_{\geq 2}\\
&=\{\mathscr{B}_k+\sum_{i=1}^{N}(\frac{a_ip}{p_i^2}+\frac{a_i}{p_i}+\frac{a_i}{p-p_i}),\mathscr{B}_n\}-\{\mathscr{B}_k+\sum_{i=1}^{N}(\frac{a_ip}{p_i^2}+\frac{a_i}{p_i}+\frac{a_i}{p-p_i}),\mathscr{B}_n\}_{\leq 1}+\{\mathscr{B}_k,(\mathscr{L}^n)_{\leq 1}\}_{\geq 2}\\
&=\{\mathscr{B}_k+\sum_{i=1}^{N}(\frac{a_ip}{p_i^2}+\frac{a_i}{p_i}+\frac{a_i}{p-p_i}),\mathscr{B}_n\}+\{\mathscr{B}_n,\sum_{i=1}^{N}(\frac{a_ip}{p_i^2}+\frac{a_i}{p_i}+\frac{a_i}{p-p_i})\}_{\leq 1}+\{\mathscr{B}_k,(\mathscr{L}^k)_{\leq 1}\}_{\geq 2}\\
&=\{\mathscr{B}_k+\sum_{i=1}^{N}(\frac{a_ip}{p_i^2}+\frac{a_i}{p_i}+\frac{a_i}{p-p_i}),\mathscr{B}_n\}+\{\mathscr{B}_n,\sum_{i=1}^{N}(\frac{a_ip}{p_i^2}+\frac{a_i}{p_i}+\frac{a_i}{p-p_i})\}_{\leq 1}+\mathscr{B}_{k,\tau_n}\\
&=\{\mathscr{B}_k+\sum_{i=1}^{N}(\frac{a_ip}{p_i^2}+\frac{a_i}{p_i}+\frac{a_i}{p-p_i}),\mathscr{B}_n\}+(\mathscr{B}_k+\sum_{i=1}^{N}(\frac{a_ip}{p_i^2}+\frac{a_i}{p_i}+\frac{a_i}{p-p_i}))_{T_n}
\end{aligned}$$

This completes the proof. □

Next we derive some important equations from (4.4). When $n=2,k=3$, (4.4) yields **the first type of dHD equation with self-consistent sources (dHDESCS)**

$$U_T-\frac{3}{4}U^{-1}[U^2\partial_X^{-1}(\frac{U_Y}{U^2})]_Y+2U\sum_{i=1}^{N}(\frac{a_i}{p_i})_X-2U_X\sum_{i=1}^{N}\frac{a_i}{p_i}=0, \qquad (4.6a)$$

$$p_{i,Y}=(U^2p_i^2)_X \;\;,\;\; a_{i,Y}=(2a_iU^2p_i)_X \;,\; i=1\cdots N, \qquad (4.6b)$$

where $T_2 = Y, \tau_3 = T$, $U = U_1$, and U_0 is eliminated by $2U_1^2 U_{0,X} = U_{1,Y}$.
The associated conservation equation of (4.6) reads

$$\begin{aligned} p_Y &= (U^2 p^2)_X, \\ p_T &= [U^3 p^3 - \frac{3}{2} U^2 p^2 \partial_X^{-1} (\frac{1}{U})_Y + \sum_{i=1}^{N} (\frac{a_i p}{p_i^2} + \frac{a_i}{p_i} + \frac{a_i}{p - p_i})]_X. \end{aligned} \tag{4.7}$$

When $n = 3, k = 2$, (4.4) becomes **the second type of dHDESCS**

$$2VU_Y + UV_Y - \frac{2}{3} U_T + U^2 \sum_{i=1}^{N} (\frac{a_i}{p_i})_X + 2UV \sum_{i=1}^{N} (\frac{a_i}{p_i^2})_X - (2VU_X + V_X U) \sum_{i=1}^{N} (\frac{a_i}{p_i^2}) = 0,$$
$$p_{i,T} = (U^3 p_i^3 + 3VU^2 p_i^2)_X \ , \ a_{i,T} = [a_i (3U^3 p_i^2 + 6VU^2 p_i)]_X \ , \ i = 1, \cdots, N. \tag{4.8}$$

The associated conservation equation of (4.8) is given by

$$\begin{aligned} p_Y &= [U^2 p^2 + \sum_{i=1}^{N} (\frac{a_i p}{p_i^2} + \frac{a_i}{p_i} + \frac{a_i}{p - p_i})]_X, \\ p_T &= [U^3 p^3 + 3U^2 V p^2]_X, \end{aligned} \tag{4.9}$$

where $\tau_2 = Y, T_3 = T$, $U_1 = U, U_0 = V$, and U_0 is determined by

$$U_0 = -\frac{1}{2} \partial_X^{-1} (\frac{1}{U})_Y + \frac{1}{2} \partial_X^{-1} (\frac{1}{U} \sum_{i=1}^{N} (\frac{a_i}{p_i^2})_X) - \frac{1}{2} \partial_X^{-1} (\frac{U_X}{U} \sum_{i=1}^{N} \frac{a_i}{p_i^2}). \tag{4.10}$$

5. The Hodograph solution for the first type dHDESCS

In this section, using M-reduction method together with the hodograph transformation, we derive the hodograph solutions to the first type of dHDESCS (4.6). Following [12] , one can consider the M-reduction of the conservation equation (4.7) so that the momentum function p, the auxiliary potentials a_i and p_i , $i = 1 \cdots N$ only depend on a set of functions $W = (W_1, \cdots, W_M)$ with $W_1 = U$, and $(W_1, \cdots, W_M)$ satisfy commuting flows

$$\frac{\partial W}{\partial T_n} = A_n(W) \frac{\partial W}{\partial X} \ , \ n \geq 2, \tag{5.1}$$

where the $N \times N$ matrice A_n are only the functions of $(W_1 \cdots W_M)$. In the following, we will take the first type of dHDESCS (4.6) for example and show its hodograph solutions in the case of $M = 1$ and $M = 2$.

1. $M = 1$

In this case, we will get

$$p = p(U, \mathscr{L}) \ , \ a_i = a_i(U) \ , \ p_i = p_i(U), \tag{5.2}$$

and

$$U_Y = A(U) U_X \ , \ U_T = B(U) U_X. \tag{5.3}$$

(5.3) together with (4.6b) and (4.7) imply that

$$
\begin{aligned}
&(A-2U^2p_i)\frac{dp_i}{dU}=2Up_i^2,\\
&(A-2U^2p)\frac{\partial p}{\partial U}=2Up^2,\\
&(A-2U^2p_i)\frac{da_i}{dU}=4a_ip_iU+2a_iU^2\frac{dp_i}{dU},\\
&(B-3U^2p^2-6U^2Vp)\frac{\partial p}{\partial U}=3U^2p^3+6UVp^2+3U^2\frac{dV}{dU}p^2+(\sum_{i=1}^{N}\frac{a_i}{p_i^2}-U^2\sum_{i=1}^{N}\frac{a_i}{p_i^4}(\frac{dp_i}{dU})^2)\frac{\partial p}{\partial U}.
\end{aligned}
\tag{5.4}
$$

Eqs. (5.4) implies

$$
A=2U^2\frac{dV}{dU}\quad,\quad B=3VA+\frac{3}{4}U^{-1}A^2+\sum_{i=1}^{N}\frac{a_i}{p_i^2}-U^2\sum_{i=1}^{N}\frac{a_i}{p_i^4}(\frac{dp_i}{dU})^2,
\tag{5.5}
$$

where $U_0=V$. It is very easy to verify that with (5.5) and (5.4), (5.3) are compatible. Making the hodograph transformations with the change of variables $(X,Y,T)\to(U,Y,T)$ with $X=X(U,Y,T)$. The hodograph equations for X are given by

$$
\frac{\partial X}{\partial Y}=-A\quad,\quad \frac{\partial X}{\partial T}=-B=-3VA-\frac{3A^2}{4U}-\sum_{i=1}^{N}\frac{a_i}{p_i^2}+U^2\sum_{i=1}^{N}\frac{a_i}{p_i^4}(\frac{dp_i}{dU})^2,
\tag{5.6}
$$

which can be easily integrated as

$$
X+A(U)Y+(3VA+\frac{3A^2}{4U}+\sum_{i=1}^{N}\frac{a_i}{p_i^2}-U^2\sum_{i=1}^{N}\frac{a_i}{p_i^4}(\frac{dp_i}{dU})^2)T=F(U),
\tag{5.7}
$$

where $F(U)$ is an arbitrary function of U.
If we chose $A(U)=-1,F(U)=0$, we can get from (5.4)

$$
p_i=-\frac{1}{U^2},a_i=C_0
\tag{5.8a}
$$

where C_0 is an arbitrary non-zero constant. Combining (5.7) and (5.8a), we obtain

$$
12NC_0TU^5+4(Y-X)U+3T=0
\tag{5.8b}
$$

We find that (5.8b) is a quintic equation of u. It is well known that a general quintic equation with one unknown has no radical solution, but elliptic modular function solution. Thus, we can obtain the hodograph solution of dHDESCS (4.6), in which p_i,a_i,U are determined by (5.8a) and (5.8b), respectively.

2. $M=2$

In the case, we denote $W_1=U,W_2=W$, then $a_i=a_i(U,W),p_i=p_i(U,W),p=p(U,W)$, with the commuting flow

$$
\begin{pmatrix}U\\W\end{pmatrix}_Y=A\begin{pmatrix}U\\W\end{pmatrix}_X\quad,\quad\begin{pmatrix}U\\W\end{pmatrix}_T=B\begin{pmatrix}U\\W\end{pmatrix}_X,
\tag{5.9}
$$

where $A=(A)_{ij}$ and $B=(B)_{ij}$ are 2×2 matrix functions of U and W. By requiring that U_X and W_X are in dependent, (4.6b) and (4.7) give rise to the following equations for $a_i(U,W),p_i(U,W)$ and

$p(U,W)$,

$$
\begin{aligned}
&(\frac{\partial p_i}{\partial U},\frac{\partial p_i}{\partial W})A = 2Up_i^2(1,0)+2U^2p_i(\frac{\partial p_i}{\partial U},\frac{\partial p_i}{\partial W}),\\
&(\frac{\partial p}{\partial U},\frac{\partial p}{\partial W})A = 2Up^2(1,0)+2U^2p_i(\frac{\partial p}{\partial U},\frac{\partial p}{\partial W}),\\
&(\frac{\partial a_i}{\partial U},\frac{\partial a_i}{\partial W})A = 2U^2p_i(\frac{\partial a_i}{\partial U},\frac{\partial a_i}{\partial W})+4a_iUp_i(1,0)+2a_iU^2(\frac{\partial p_i}{\partial U},\frac{\partial p_i}{\partial W}),\\
&(\frac{\partial p}{\partial U},\frac{\partial p}{\partial W})B = (3U^2p^3+6UVp^2)(1,0)+3U^3p^2(\frac{\partial p}{\partial U},\frac{\partial p}{\partial W})+3U^2p^2(\frac{\partial V}{\partial U},\frac{\partial V}{\partial W})+\\
&\qquad 6U^2Vp(\frac{\partial p}{\partial U},\frac{\partial p}{\partial W})+\sum_{i=1}^{N}[(\frac{p}{p_i^2}+\frac{1}{p_i}+\frac{1}{p-p_i})(\frac{\partial a_i}{\partial U},\frac{\partial a_i}{\partial W})-\\
&\qquad \frac{2a_ip}{p_i^3}(\frac{\partial p_i}{\partial U},\frac{\partial p_i}{\partial W})+(\frac{a_i}{p_i^2}-\frac{a_i}{(p-p_i)^2})(\frac{\partial p}{\partial U}-\frac{\partial p_i}{\partial U},\frac{\partial p}{\partial W}-\frac{\partial p_i}{\partial W})].
\end{aligned}
\tag{5.10}
$$

We can easily find from (5.10) that $A(U,W)$ and $B(U,W)$ must satisfy

$$
\begin{aligned}
B =& 3VA+\frac{3A^2}{4U}+\sum_{i=1}^{N}(\frac{a_i}{p_i^2}-\frac{U}{p_i^2}\frac{\partial a_i}{\partial U}+\frac{2Ua_i}{p_i^3}\frac{\partial p_i}{\partial U})I+\\
&(\frac{U}{p_i^2}\frac{\partial a_i}{\partial W}-\frac{2Ua_i}{p_i^3}\frac{\partial p_i}{\partial W})\begin{bmatrix} 0 & 1\\ -\frac{A_{21}}{A_{12}} & \frac{A_{22}-A_{11}}{A_{12}}. \end{bmatrix}
\end{aligned}
\tag{5.11}
$$

where $U_0 = V$, and I is the 2×2 identity matrix, $A_{11} = 2U^2\frac{\partial V}{\partial U}$ and $A_{12} = 2U^2\frac{\partial V}{\partial W}$.
For simplicity, we assume $\frac{U}{p_i^2}\frac{\partial a_i}{\partial W} = \frac{2Ua_i}{p_i^3}\frac{\partial p_i}{\partial W}, i = 1,\cdots,N$. By using the formula

$$
A^2 = (trA)A-(detA)I,
\tag{5.12}
$$

we have

$$
B = (3V+\frac{3trA}{4U})A-[\frac{3detA}{4U}-\sum_{i=1}^{N}(\frac{a_i}{p_i^2}-\frac{U}{p_i^2}\frac{\partial a_i}{\partial U}+\frac{2Ua_i}{p_i^3}\frac{\partial p_i}{\partial U})]I,
\tag{5.13}
$$

where $detA = A_{11}A_{22}-A_{12}A_{21}$ and $trA = A_{11}+A_{22}$.
With (5.13), the compatibility condition for (5.9) requires A to satisfy

$$
\begin{pmatrix}
-\frac{\partial}{\partial W}(\frac{3detA}{4U})+\sum_{i=1}^{N}\frac{\partial}{\partial W}(\frac{a_i}{p_i^2}-\frac{U}{p_i^2}\frac{\partial a_i}{\partial U}+\frac{2Ua_i}{p_i^3}\frac{\partial p_i}{\partial U})\\
\frac{\partial}{\partial U}(\frac{3detA}{4U})-\sum_{i=1}^{N}\frac{\partial}{\partial U}(\frac{a_i}{p_i^2}-\frac{U}{p_i^2}\frac{\partial a_i}{\partial U}+\frac{2Ua_i}{p_i^3}\frac{\partial p_i}{\partial U})
\end{pmatrix}
= A\begin{pmatrix}
-\frac{\partial}{\partial W}(3V+\frac{3trA}{4U})\\
\frac{\partial}{\partial U}(3V+\frac{3trA}{4U})
\end{pmatrix}.
\tag{5.14}
$$

To solve (5.9), we use the hodograph transformation by changing the independent variables (X,Y,T) to (U,W,T) with the dependent variables $X = X(U,W,T)$ and $Y = Y(U,W,T)$. In terms of the new variables, (5.9) becomes

$$\begin{pmatrix} -X_W \\ X_U \end{pmatrix} = A \begin{pmatrix} Y_W \\ -Y_U \end{pmatrix}, \begin{pmatrix} \partial(X,Y)/\partial(W,T) \\ -\partial(X,Y)/\partial(U,T) \end{pmatrix} = B \begin{pmatrix} Y_W \\ -Y_U \end{pmatrix}, \tag{5.15}$$

where $\partial(X,Y)/\partial(W,T) = X_W Y_T - X_T Y_W$. It can be easily find that (5.15) has solutions in the form

$$\begin{aligned} &X - [\frac{3detA}{4U} - \sum_{i=1}^{N}(\frac{a_i}{p_i^2} - \frac{U}{p_i^2}\frac{\partial a_i}{\partial U} + \frac{2Ua_i}{p_i^2}\frac{\partial p_i}{\partial U}]T = F(U,W), \\ &Y + (3V + \frac{3trA}{4U})T = G(U,W). \end{aligned} \tag{5.16}$$

where Y_U and Y_W are required to be independent, and F and G are two arbitrary functions satisfying the linear equations

$$\begin{pmatrix} -F_W \\ F_U \end{pmatrix} = A \begin{pmatrix} G_W \\ -G_U \end{pmatrix}, \tag{5.17}$$

An example of solution is given by

$$A = \begin{bmatrix} 2U^2W & 2U^3 + 2U^2 \\ 2UW^2 & 2U^2W \end{bmatrix}, \tag{5.18a}$$

and

$$p_i = W\ ,\ a_i = c_i(U+1)W^2\ ,\ i = 1, \cdots, N, \tag{5.18b}$$

where $c_i\ ,\ i = 1, \cdots, N$ are constants.
It can be found that (5.18a) implies $V = (U+1)W$. Then (5.16) becomes

$$\begin{aligned} &X + (3U^2W^2 + \sum_{i=1}^{N} c_i)T = F(U,W), \\ &Y + (6U+3)WT = G(U,W). \end{aligned} \tag{5.19}$$

From (5.17) and $F_{UW} = F_{WU}$, G must satisfy

$$(4U+2)G_U + (U^2+U)G_{UU} - 4WG_W - W^2G_{WW} = 0. \tag{5.20}$$

We notice that $G = \frac{1}{W^3}$ is a particular solution of (5.20). From (5.17), we get $F = -\frac{3U^2}{W^2}$.
When we choose $G = \frac{1}{W^3}$ and $F = -\frac{3U^2}{W^2}$, we obtain from (5.17)

$$\begin{aligned} &X + (3U^2W^2 + \sum_{i=1}^{N} c_i)T = -\frac{3U^2}{W^2}, \\ &Y + (6U+3)WT = \frac{1}{W^3}. \end{aligned} \tag{5.21}$$

By solving the equation set (5.21) of U and W, we obtain an implicit solution of (4.6), in which $p_i = W, a_i = c_i(U+1)W^2\ ,\ i = 1, \cdots, N$, and U, W are determined by (5.21).

6. Bäcklund transformation between exdmKPH and exdHDH

It is known that the exdmKPH is defined by [24]

$$\overline{L}_{\overline{\tau}_k} = \{Q_k - \sum_{i=1}^{N}(\frac{\alpha_i}{\beta_i} + \frac{\alpha_i}{p'-\beta_i}), \overline{L}\}, \tag{6.1a}$$

$$\overline{L}_{\overline{T}_n} = \{Q_n, \overline{L}\}, n \neq k, \tag{6.1b}$$

$$\beta_{i,\overline{T}_n} = [Q_n(p') \mid_{p'=\beta_i}]_{\overline{X}}, \tag{6.1c}$$

$$\alpha_{i,\overline{T}_n} = [\alpha_i(\frac{\partial Q_n(p')}{\partial p'}) \mid_{p'=\beta_i}]_{\overline{X}}, \tag{6.1d}$$

with $Q_n = (\overline{L}^n)_{\geq 1}$.

We now explore the Bäcklund transformation between exdmKPH and exdHDH, which is given by the following theorem.

Theorem 6.1. *Suppose that $\overline{L}, \alpha_i, \beta_i$ satisfy the exdmKPH. If ϕ is function of T satisfying*

$$\phi_{\overline{\tau}_k} = \{Q_k, \phi\}_{[0]} + \sum_{i=1}^{N} \frac{\alpha_i}{\beta_i^2}\phi_X,$$

$$\phi_{\overline{T}_n} = \{Q_n, \phi\}_{[0]}, n \neq K,$$

then $\mathscr{L}(X,t_n) = \overline{L}(\overline{X},\overline{t}_n), X = \phi(\overline{X},\overline{t}), t_n = \overline{t}_n, a_i = -\phi_{\overline{X}}^{-1}\alpha_i, p_i = \phi_{\overline{X}}^{-1}\beta_i$ satisfy the exdHDH (4.2). Here we use a notation: $(\Lambda)_{[0]} = a_0$, $\Lambda = \sum_i a_i p^i$.

Proof. Using the same method as in [7], we can show that $\mathscr{L}(X,t_n) = \overline{L}(\overline{X},\overline{t}_n)$ satisfy (4.2b). Therefore,we only need to prove $\mathscr{L}(X,t_n), a_i, p_i$ defined above satisfy (4.2a),(4.2c) and (4.2d). The detailed proofs can be found in the Appendix A. □

Remark 6.1. Theorem 6.1 presents the Bäcklund transformation between exdmKPH (6.1) and exdHDH (4.2). In addition, noting that the Bäcklund transformation between exdKPH and exdmKPH was given in [24], we attain the Bäcklund transformation between exdKPH and exdHDH.

Remark 6.2. As the byproduct of theorem 6.1, we get the the Bäcklund transformation between the first type of dmKPESCS and of dHDESCS. Let

$$\mathscr{L}(X,t) = Up + U_0 + U_{-1}p^{-1} + \cdots + U_{-m}p^{-m} + \cdots,$$
$$\overline{L}(\overline{X},\overline{t}) = p' + V_0 + V_{-1}p'^{-1} + V_{-2}p'^{-2} + \cdots + V_{-m}p'^{-m} + \cdots,$$

from $p' = \phi_{\overline{X}}p$ and $\overline{L}(\overline{X},\overline{t}) = \mathscr{L}(X,t)$, we can get

$$U_{-m}(X,t) = V_{-m}(\overline{X},\overline{t})\phi_{\overline{X}}^{-m}\ ,\ m = -1,0,1,\cdots,U_1 = U.$$

Specially, we get $U = \phi_{\overline{X}}$ when $m = -1$.
When $n = 2, k = 3$, from theorem 6.1, the Bäcklund transformation between the first type of dmKPESCS and dHDESCS is given by

$$U = \phi_{\overline{X}} \ , \ a_i = -\phi_{\overline{X}}^{-1}\alpha_i \ , \ p_i = \phi_{\overline{X}}^{-1}\beta_i \ , \ i = 1,2,\cdots N,$$

in which ϕ satisfies that

$$\begin{aligned}
\phi_{\overline{T}_2} &= \{Q_2,\phi\}_{[0]} = \{p'^2 + 2p'V,\phi\}_{[0]} \\
&= [\frac{\partial(p'^2+2p'V)}{\partial p'}\cdot\frac{\partial\phi}{\partial_{\overline{X}}}]_{[0]} - \frac{\partial(p'^2+2p'V)}{\partial_{\overline{X}}}\cdot\frac{\partial\phi}{\partial p'} = 2V\phi_{\overline{X}}, \\
\phi_{\overline{\tau}_3} &= \{Q_3,\phi\}_{[0]} + \sum_{i=1}^{N}\frac{\alpha_i}{\beta_i^2}\phi_{\overline{X}} \\
&= (\frac{3}{2}\partial_{\overline{X}}^{-1}(V_{\overline{Y}}) + \frac{3}{2}V^2)\phi_{\overline{X}} + \sum_{i=1}^{N}\frac{\alpha_i}{\beta_i^2}\phi_{\overline{X}},
\end{aligned} \tag{6.2}$$

with $V_0 = V$.

Similarly, we can also derive the Bäcklund transformation between the second type of dmKPESCS and dHDHESCS. But we omit the details here.

7. Summary

In this article, the symmetry constraint of dHD hierarchy is derived for the first time by taking dispersionless limit of that for 2+1 dimensional Harry Dym hierarchy. In addition, the new extension of the dHD hierarchy is considered. We can easily find that the new exdHDH is Lax integrable, and that its the zero-curvature equation contains two types of dHDESCS. The hodograph solutions to the first type of dHDESCS are obtained by the reduction method together with hodograph transformation. The Bäcklund transformation between the exdmKPH and exdHDH are finally constructed. Our results give a supplement to the previous studies about the dHD hierarchy.

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Appendix A. The proof of Theorem 6.1

Proof. Step 1: We firstly show $\mathscr{L}(X,t_n), a_i, p_i$ satisfy (4.2a).
Noting that

$$\frac{\partial}{\partial\overline{\tau}_k} = \frac{\partial}{\partial\tau_k} + \phi_{\overline{T}_n}\frac{\partial}{\partial X} \Rightarrow \frac{\partial}{\partial\tau_k} = \frac{\partial}{\partial\overline{\tau}_k} - \phi_{\overline{T}_n}\frac{\partial}{\partial X},$$

we have

$$\begin{aligned}\mathscr{L}_{\tau_k} &= \frac{\partial \mathscr{L}}{\partial \tau_k} = \frac{\partial \overline{L}}{\partial \overline{\tau}_k} - \phi_{\overline{T}_n}\frac{\partial \mathscr{L}}{\partial X} = \overline{L}_{\overline{\tau}_k} - \phi_{\overline{T}_n}\{p,\mathscr{L}\} \\ &= \overline{L}_{\overline{\tau}_k} - \{\phi_{\overline{T}_n}p,\mathscr{L}\} = \{Q_k - \sum_{i=1}^{N}(\frac{\alpha_i}{\beta_i} + \frac{\alpha_i}{p'-\beta_i}),\overline{L}\} - \{\phi_{\overline{T}_n}p,\mathscr{L}\} \\ &= \{Q_k,\overline{L}\} - \{\sum_{i=1}^{N}(\frac{\alpha_i}{\beta_i} + \frac{\alpha_i}{p'-\beta_i}),\overline{L}\} - \{\phi_{\overline{T}_n}p,\mathscr{L}\}.\end{aligned} \tag{A.1}$$

Noting that

$$\mathscr{B}_k = (\mathscr{L}^k)_{\geq 2} = Q_k - \{Q_k,\phi\}_{[0]}p,$$

we obtain

$$\begin{aligned}&\{\mathscr{B}_k + \sum_{i=1}^{N}(\frac{a_i p}{p_i^2} + \frac{a_i}{p_i} + \frac{a_i}{p-p_i},\mathscr{L}\} \\ &= \{Q_k,\mathscr{L}\} - \{\{Q_k,\phi\}_{[0]}p,\mathscr{L}\} + \{\sum_{i=1}^{N}(\frac{a_i p}{p_i^2} + \frac{a_i}{p_i} + \frac{a_i}{p-p_i}),\mathscr{L}\}.\end{aligned} \tag{A.2}$$

Combining (A.1)and (A.2), we have

$$\begin{aligned}&\mathscr{L}_{\tau_k} - \{\mathscr{B}_k + \sum_{i=1}^{N}(\frac{a_i p}{p_i^2} + \frac{a_i}{p_i} + \frac{a_i}{p-p_i}),\mathscr{L}\} \\ &= \{(\phi_{\tau_k} - \{Q_k,\phi\}_{[0]} + \sum_{i=1}^{N}\frac{a_i}{p_i^2})p,\mathscr{L}\} + \sum_{i=1}^{N}\{(\frac{a_i}{p_i} + \frac{\alpha_i}{\beta_i} + (\frac{a_i}{p-p_i} + \frac{\alpha_i}{p'-\beta_i}),\mathscr{L}\}.\end{aligned}$$

Since

$$\phi_{\overline{\tau}_k} = \{Q_k,\phi\}_{[0]} + \sum_{i=1}^{N}\frac{\alpha_i}{\beta_i^2}\phi_{\overline{X}} = \{Q_k,\phi\}_{[0]} - \sum_{i=1}^{N}\frac{a_i\phi_{\overline{X}}^2}{p_i^2\phi_{\overline{X}}^2} = \{Q_k,\phi\}_{[0]} - \sum_{i=1}^{N}\frac{a_i}{p_i^2},$$

and

$$\frac{a_i}{p_i} + \frac{\alpha_i}{\beta_i} = -\frac{\phi_{\overline{X}}^{-1}\alpha_i}{\phi_{\overline{X}}^{-1}\beta_i} + \frac{\alpha_i}{\beta_i} = -\frac{\alpha_i}{\beta_i} + \frac{\alpha_i}{\beta_i} = 0,$$

$$\frac{a_i}{p-p_i} + \frac{\alpha_i}{p'-\beta_i} = \frac{-\phi_{\overline{X}}^{-1}\alpha_i}{\phi_{\overline{X}}^{-1}p' - \phi_{\overline{X}}^{-1}\beta_i} + \frac{\alpha_i}{p'-\beta_i} = 0,$$

with $p = \phi_{\overline{X}}^{-1}p'$, we have

$$\mathscr{L}_{\tau_k} - \{\mathscr{B}_k + \sum_{i=1}^{N}(\frac{a_i p}{p_i^2} + \frac{a_i}{p_i} + \frac{a_i}{p-p_i}),\mathscr{L}\} = 0 \Rightarrow$$

$$\mathscr{L}_{\tau_k} = \{\mathscr{B}_k + \sum_{i=1}^{N}(\frac{a_i p}{p_i^2} + \frac{a_i}{p_i} + \frac{a_i}{p-p_i}),\mathscr{L}\}.$$

Step 2. Then we show $\mathscr{L}(X,t_n), a_i, p_i$ satisfy (4.2c).

Noticing that when $p = p_i = \phi_{\overline{X}}^{-1}\beta_i$, we have

$$\phi_{\overline{X}}^{-1}p' = \phi_{\overline{X}}^{-1}\beta_i \Rightarrow p' = \beta_i.$$

In addition, we also find that $\mathscr{B}_n(p) = Q_n(p') - \{Q_n(p'),\phi\}_{[0]}p$, so we have

$$\begin{aligned}
&[\mathscr{B}_n(p)\mid_{p=p_i}]_X = [Q_n(p') - \{Q_n(p'),\phi\}_{[0]}p\mid_{p=p_i}]_X \\
&= \phi_{\overline{X}}^{-1}[Q_n(p')\mid_{p'=\beta_i}]_X - p_i\phi_{\overline{X}}^{-2}\{Q_n(p'),\phi\}_{[0],\overline{X}} - \phi_{\overline{X}}^{-1}p_{i,X}\{Q_n(p'),\phi\}_{[0]} \\
&+ \phi_{\overline{X}}^{-3}\{Q_n(p'),\phi\}_{[0]}p_i\phi_{\overline{X},\overline{X}}.
\end{aligned} \tag{A.3}$$

Noting that

$$\begin{aligned}
p_{i,T_n} &= (\partial_{\overline{T}_n} - \phi_{\overline{T}_n}\partial_X)(p_i) = (\partial_{\overline{T}_n} - \phi_{\overline{X}}^{-1}\phi_{\overline{T}_n}\partial_{\overline{X}})(\beta_i\phi_{\overline{X}}^{-1}) \\
&= \beta_{i,\overline{T}_n}\phi_{\overline{X}}^{-1} - \beta_i\phi_{\overline{X}}^{-2}\phi_{\overline{X},\overline{T_n}} - \phi_{\overline{X}}^{-1}\phi_{\overline{T}_n}\beta_{i,\overline{X}} + \phi_{\overline{T}_n}\phi_{\overline{X}}^{-3}\beta_i\phi_{\overline{X},\overline{X}},
\end{aligned} \tag{A.4}$$

we have

$$\begin{aligned}
p_{i,T_n} - [\mathscr{B}_n(p)\mid_{p=p_i}]_X = {}& \phi_{\overline{X}}^{-1}[\beta_{i,\overline{T}_n} - (Q_n(p')\mid_{p'=\beta_i})_{\overline{X}}] - \beta_i\phi_{\overline{X}}^{-2}[\phi_{\overline{T}_n} - \{Q_n(p'),\phi\}_{[0]}]_{\overline{X}} + \\
& \beta_{i,\overline{X}}\phi_{\overline{X}}^{-1}(\phi_{\overline{T}_n} - \{Q_n(p'),\phi\}_{[0]}) + \phi_{\overline{X}}^{-3}\beta_i\phi_{\overline{X},\overline{X}}[\phi_{\overline{T}_n} - \{Q_n(p'),\phi\}_{[0]}].
\end{aligned}$$

Since $\beta_{i,\overline{T}_n} = [Q_n(p')\mid_{p'=\beta_i}]_{\overline{X}}$ and $\phi_{\overline{T}_n} = \{Q_n,\phi\}_{[0]}$, we have

$$p_{i,T_n} - [\mathscr{B}_n(p)\mid_{p=p_i}]_X = 0 \Rightarrow p_{i,T_n} = [\mathscr{B}_n(p)\mid_{p=p_i}]_X.$$

Step 3. We finally show $\mathscr{L}(X,t_n), a_i, p_i$ satisfy (4.2d).

We have shown in 2 that when $p = p_i$, we have $p' = \beta_i$.
Since

$$\frac{\partial\mathscr{B}_n(p)}{\partial p}\mid_{p=p_i} = \phi_{\overline{X}}\frac{\partial Q_n(p')}{\partial(p')}\mid_{p'=\beta_i} - \{Q_n,\phi\}_{[0]},$$

and

$$\partial_X = \phi_{\overline{X}}^{-1}\partial_{\overline{X}},$$

accordingly

$$\begin{aligned}
&[a_i(\frac{\partial\mathscr{B}_n(p)}{\partial p})\mid_{p=p_i}]_X = \phi_{\overline{X}}^{-1}[-\phi_{\overline{X}}^{-1}\alpha_i(\phi_{\overline{X}}(\frac{\partial Q_n(p')}{\partial p'})\mid_{p'=\beta_i} - \{Q_n,\phi\}_{[0]})]_{\overline{X}} \\
&= -\phi_{\overline{X}}^{-1}(\alpha_i(\frac{\partial Q_n(p')}{\partial p'})\mid_{p'=\beta_i})_{\overline{X}} + \phi_{\overline{X}}^{-1}(\phi_{\overline{X}}^{-1}\alpha_i\{Q_n,\phi\}_{[0]})_{\overline{X}} \\
&= -\phi_{\overline{X}}^{-1}(\alpha_i(\frac{\partial Q_n(p')}{\partial p'})\mid_{p'=\beta_i})_{\overline{X}} - \phi_{\overline{X}}^{-3}\alpha_i\{Q_n,\phi\}_{[0]}\phi_{\overline{X},\overline{X}} + \\
&\phi_{\overline{X}}^{-2}\alpha_{i,\overline{X}}\{Q_n,\phi\}_{[0]} + \phi_{\overline{X}}^{-2}\alpha_i\{Q_n,\phi\}_{[0],\overline{X}},
\end{aligned} \tag{A.5}$$

and

$$a_{i,T_n} = (\partial_{\overline{T}_n} - \phi_{\overline{T}_n}\partial_X)a_i = (\partial_{\overline{T}_n} - \phi_{\overline{T}_n}\partial_X)(-\alpha_i\phi_{\overline{X}}^{-1}) \\ = -\alpha_{i,\overline{T}_n}\phi_{\overline{X}}^{-1} + \alpha_i\phi_{\overline{X}}^{-2}\phi_{\overline{X},\overline{T}_n} + \phi_{\overline{X}}^{-2}\alpha_{i,\overline{X}}\phi_{\overline{T}_n} - \phi_{\overline{T}_n}\phi_{\overline{X}}^{-3}\alpha_i\phi_{\overline{XX}}. \qquad (A.6)$$

From (A.5) and (A.6), we have

$$a_{i,T_n} - [a_i(\frac{\partial \mathscr{B}_n(p)}{\partial p})\mid_{p=p_i}]_X \\ = -\phi_{\overline{X}}^{-1}[\alpha_{i,\overline{T}_n} - (\alpha_i(\frac{\partial Q_n(p')}{\partial p})\mid_{p'=\beta_i})_{\overline{X}}] - \phi_{\overline{X}}^{-3}\alpha_i\phi_{\overline{XX}}(\phi_{\overline{T}_n} - \{Q_n,\phi\}_{[0]}) \\ + \phi_{\overline{X}}^{-2}\alpha_{i,\overline{X}}(\phi_{\overline{T}_n} - \{Q_n,\phi\}_{[0]}) + \phi_{\overline{X}}^{-2}\alpha_i(\phi_{\overline{T}_n} - \{Q_n,\phi\}_{[0]})_{\overline{X}}.$$

Since $\alpha_{i,\overline{T}_n} = [\alpha_i(\frac{\partial Q_n(p')}{\partial p})\mid_{p'=\beta_i}]_{\overline{X}}$ and $\phi_{\overline{T}_n} = \{Q_n,\phi\}_{[0]}$, respectively,

$$a_{i,T_n} - [a_i(\frac{\partial \mathscr{B}_n(p)}{\partial p})\mid_{p=p_i}]_X = 0 \Rightarrow a_{i,T_n} = [a_i(\frac{\partial \mathscr{B}_n(p)}{\partial p})\mid_{p=p_i}]_X.$$

This completes the proof. □

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