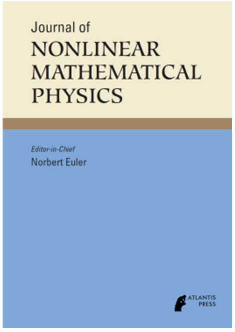

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New *solvable* dynamical systems

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New *solvable* dynamical systems are identified and the properties of their solutions are tersely discussed.

1. Introduction

We call *solvable* a dynamical system if its solution can be *explicitly* exhibited in terms of elementary functions or obtained by *algebraic* operations, typically by finding the N zeros $z_n(t)$ of a time-dependent polynomial $p_N(z;t)$ of degree N in z which is known—again, possibly via *algebraic* operations—in terms of the initial data of the dynamical system. Recently a somewhat novel technique to identify such models has been introduced [1], and quite a few such models have been identified and investigated [2–8].

In the following Section 2 new *solvable* models are presented and their solutions are tersely discussed. Proofs of these findings are provided in Section 3.

Let us end this introductory section by mentioning that all these models generally belong to the class of so-called "goldfish-type" dynamical systems and many-body problems, the *solvability* of which generally emerges from the relations among the N *coefficients* and the N *zeros* of a time-dependent (monic) polynomial of degree N in the (*complex*) variable z: an approach introduced almost four decades ago [9], to which the term "goldfish" was appended about 15 years ago [10], and which has been rather extensively investigated in the following years, see for instance [11–22].

Since a Referee kindly suggested that the above outline of our approach was a bit too terse, let us add here an overview of the general context in which the results reported in the present paper should be situated. The modern interest in *solvable/integrable* models can be traced—of course, somewhat arbitrarily: there always are precursors of precursors—to the discovery/invention in 1967 of the Inverse Spectral Transform technique to solve the KdV equation, which is an evolution Partial Differential Equation in one-plus-one dimensions: time t and space x. [23] This major achievement was soon followed by another major achievement: the discovery/invention in 1968 of the Lax pair method to identify *solvable/integrable* models [24]. Let us hereafter focus on dynamical systems, i. e. on nonlinear systems of Ordinary Differential Equations, which are the topic treated in the present paper. In this context the Lax pair technique allows to reformulate certain dynamical system in terms of two $(N \times N)$-matrices (defined in terms of the variables of the dynamical system), with one of these two matrices evolving in time in a simple manner implying that its N eigenvalues are *constants of motion*. In the special case when the dynamical system so identified is *Hamiltonian*, this generally implies that it is *integrable*. More generally, the Lax breakthrough opened the way to the identification of *solvable* dynamical systems (as defined above): which therefore evolve nonchaotically and indeed, in some cases, remarkably simply, for instance *periodically* or even *isochronously*. Two general techniques emerged over time to identify the equations of motion of such systems. The first one takes as starting point a *solvable* $(N \times N)$-matrix evolution equation and then focusses on the evolution of its N eigenvalues, which are then interpreted as the dynamical variables: its origin can be traced to a seminal 1976 paper which introduced a convenient technique to compute the corresponding equations of motion [25]. It was thereby shown that even a *trivially solvable linear* evolution of the matrix leads to an *interesting nonlinear* evolution of its eigenvalues, such as that described by the so-called Calogero-Moser many-body problem; opening thereby a line of investigation that has been extensively pursued during the last few decades (see for instance [10] and [20] and references therein; but there are also may other books covering this material, and thousands of papers...). An alternative technique (1978) focussed on the connection among the N *coefficients* $c_m(t)$ and the N *zeros* $z_n(t)$ of a time-dependent (monic) polynomial of degree N in its argument z (generally, a complex number), relying on the observation that a *solvable* evolution of the N coefficients $c_m(t)$ implies of course that the corresponding evolution of the N *zeros* $z_n(t)$ is as well *solvable* [9]. As mentioned above, a recent observation [1] has significantly facilitated the identification of *solvable* systems via this approach, leading to the investigation of several such models, each of course with its own distinctive features (see references above). The present paper fits into this development. Let us finally mention—again, in connection with an issue raised by the Referee—that the two technologies to identify and investigate *solvable* dynamical systems outlined above have some common features—for instance, the computation of the eigenvalues of an $(N \times N)$-matrix entails of course the solution of a polynomial equation of degree N—but they are quite different, so that in each specific case it is less than obvious how a model whose *solvability* has been identified within one approach—allowing a detailed study of its behavior—can also be investigated via the other approach. And since both approaches eventually involve the solution of a polynomial equation of degree N it is of course plain that only for $N \leq 4$ the *solution* of almost *all solvable* dynamical systems featuring N dynamical variables can be *explicitly* expressed in terms of *elementary* functions.

2. Results

A first, rather trivial, dynamical system is characterized by the following simple system of N Ordinary Differential Equations (ODEs):

$$\gamma_1'(\tau) \ = a_1 \ [\gamma_1(\tau)]^{1-1/r} \ , \tag{2.1a}$$

$$\gamma_m'(\tau) = a_m \ \gamma_{m-1}(\tau) \ , \quad m = 2, ..., N \ . \tag{2.1b}$$

Notation 2.1. Here and hereafter N is an arbitrary *positive integer* ($N \geq 2$), the N functions $\gamma_m \equiv \gamma_m(\tau)$ with $m = 1, 2, ..., N$ are the dependent variables, τ is the independent variable, the number r in the exponent in the right-hand side of (2.1a) is hereafter assumed to be an arbitrary *real rational* number—$r = p/q$ with p and q two nonvanishing *coprime integers* ($q > 0$ for definiteness, and $p \neq 0$ so that $r \neq 0$), and the N parameters a_m are N arbitrarily assigned (generally *complex*, nonvanishing) numbers. The N dependent variables $\gamma_m \equiv \gamma_m(\tau)$ should as well be generally considered *complex* numbers, and in the following the independent variable τ shall also be *complex*, although when considering this system of ODEs as a dynamical system one might prefer to consider τ as a *real* variable ("time"). Note that in the following the *explicit* indication of the dependence on the independent variable τ will be occasionally omitted. Primes denote of course differentiations with respect to the argument of the function they are appended to. Indices such as m and n generally take *integer* values in the range from 1 to N (unless otherwise indicated, see for instance (2.1b)). And let us finally recall—for their relevance to formulas written below—the standard convention according to which an *empty* sum *vanishes* and an *empty* product equals *unity*: $\sum_{s=S_-}^{S_+} = 0$ if $S_- > S_+$, $\prod_{j=J_-}^{J_+} = 1$ if $J_- > J_+$.

As shown in the following Section 3 the solution of the *initial-values* problem of this dynamical system—with the N *initial* values $\gamma_m(0)$ assigned, and of course $\gamma_1(0) \neq 0$ if $1/r > 1$ (see (2.1a))—is provided by the following formula:

$$\gamma_m(\tau) = \sum_{\ell=0}^{m-2} \left\{ \frac{\tau^\ell}{\ell!} \, [A_{m,m-\ell+1} \, \gamma_{m-\ell}(0) - A_{m,2} \, \beta_{m-\ell} \, \gamma_1(0)] \right\}$$
$$+ A_{m,2} \, \beta_m \, \gamma_1(0) \, (1+\rho\,\tau)^{r+m-1} \, , \quad m = 1,2,...,N \, , \tag{2.2a}$$

with, above and hereafter (see **Notation 2.1**),

$$A_{m,\ell} = \prod_{s=\ell}^{m} (a_s) \, , \quad \beta_m = \rho^{1-m} \prod_{j=1}^{m-1} \left[(r+j)^{-1} \right] \, , \tag{2.2b}$$

$$\rho = \frac{a_1}{r} \, [\gamma_1(0)]^{-1/r} \, . \tag{2.2c}$$

A, perhaps more interesting, dynamical system obtains by introducing new independent and dependent variables via the positions

$$\tau \equiv \tau(t) = \frac{\exp(\mathbf{i}\,\omega\,t) - 1}{\mathbf{i}\,\omega} \, , \tag{2.3a}$$

implying

$$\dot{\tau}(t) = \exp(\mathbf{i}\,\omega\,t) \, , \tag{2.3b}$$

and

$$c_m \equiv c_m(t) = \exp[(-\mathbf{i}\,\lambda_m\,\omega\,t)] \, \gamma_m(\tau) \, , \quad m = 1,2,...,N \, , \tag{2.4a}$$

with, above and hereafter,

$$\lambda_m = r + m - 1 \, . \tag{2.4b}$$

Notation 2.2. Here and hereafter t ("time") is a *real* variable, ω is an arbitrary *real* (*nonvanishing*) parameter to which the basic period

$$T_0 = 2\pi/|\omega| \tag{2.5}$$

is associated, and note that in (2.3b) we used—and we will do so hereafter—the standard notation according to which superimposed dots denote differentiations with respect to the *real* variable t ("time").

It is then plain that the time-dependent variables $c_m(t)$ evolve according to the *autonomous* dynamical system

$$\dot{c}_1(\tau) = -\mathbf{i}\,\lambda_1\,\omega\,c_1(t) + a_1\,[c_1(t)]^{1-1/r}\,, \tag{2.6a}$$

$$\dot{c}_m(\tau) = -\mathbf{i}\,\lambda_m\,\omega\,c_m(t) + a_m\,c_{m-1}(t)\,,\quad m = 2,\ldots,N\,, \tag{2.6b}$$

and that the solution of this dynamical system is provided by the formula

$$\begin{aligned} c_m(t) = \exp(-\mathbf{i}\,\lambda_m\,\omega\,t)\, & \left\{ \sum_{\ell=0}^{m-2} \left(\left\{ \frac{[\exp(\mathbf{i}\,\omega\,t)-1]^\ell}{(\mathbf{i}\,\omega)^\ell\,\ell!} \right\} \cdot \right.\right. \\ & \cdot [A_{m,m-\ell+1}\,c_{m-\ell}(0) - A_{m,2}\,\beta_{m-\ell}\,c_1(0)]\,) \\ & \left. + A_{m,2}\,\beta_m\,c_1(0)\,\left[1+\rho\,\frac{\exp(\mathbf{i}\,\omega\,t)-1}{\mathbf{i}\,\omega}\right]^{\lambda_m} \right\}\,,\quad m = 1,2,\ldots,N\,, \end{aligned} \tag{2.7a}$$

with $A_{m,\ell}$, β_m and λ_m defined again by (2.2b) and (2.4b), but now of course with

$$\rho = \frac{a_1}{r}\,[c_1(0)]^{-1/r}\,. \tag{2.7b}$$

Remark 2.1. It is plain that this solution—for *arbitrary* initial data—is *nonsingular* and *periodic* with period T_0, see (2.5), if r *is a nonnegative integer*, i. e. if $q = 1$ and $p \geq 0$ (see **Notation 2.1**), implying that all the numbers λ_m are as well *nonnegative integers*. Otherwise the solution might become *singular*, but only for the *nongeneric* initial datum $c_1(0)$ such that there hold the equality $|(\mathbf{i}\,\omega/\rho) - 1| = 1$ (see (2.7b) and (2.3a)) namely

$$\left| \left(\frac{\mathbf{i}\,\omega\,r}{a_1} \right)\,[c_1(0)]^{1/r} - 1 \right| - 1 = 0\,. \tag{2.8a}$$

It is moreover plain from (2.7) that if there holds instead the *inequality*

$$\left| \left(\frac{\mathbf{i}\,\omega\,r}{a_1} \right)\,[c_1(0)]^{1/r} - 1 \right| - 1 \neq 0 \tag{2.8b}$$

all the functions $c_m(t)$ are (*nonsingular* and) *periodic* with period $T = qT_0$,

$$c_m(t+T) = c_m(t)\,,\quad T = \frac{2\,q\,\pi}{|\omega|}\,,\quad m = 1,2,\ldots,N\,, \tag{2.9}$$

of course with q the denominator of the rational number r, see **Notation 2.1** and the definition (2.4b) of λ_m. Note that this outcome holds in spite of the fact that the time evolution of $(1+\rho\,\tau)^{\lambda_m}$ with (2.3a) is somewhat different—due to the rational branch point at $\tau = -1/\rho$—depending on the sign of the left-hand side of (2.8b).

Proposition 2.1. *The following set of N* nonlinearly coupled *ODEs characterizing the time-evolution of the N time-dependent variables* $z_n \equiv z_n(t)$,

$$\dot{z}_n = -\left[\prod_{\ell=1,\ \ell\neq n}^{N} (z_n - z_\ell)^{-1}\right] [\left\{-\mathbf{i}\,\lambda_1\ \omega\ c_1(t) + a_1\ [c_1(t)]^{1-1/r}\right\} (z_n)^{N-1}$$
$$+ \sum_{m=2}^{N} \left\{[-\mathbf{i}\,\lambda_m\ \omega\ c_m(t) + a_m\ c_{m-1}(t)]\ \ (z_n)^{N-m}\right\}]\ , \tag{2.10}$$

—where λ_m *is defined by* (2.4b) *and*

$$c_1(t) = -\sum_{n=1}^{N} [z_n(t)]\ , \tag{2.11a}$$

$$c_m(t) = (-1)^m \sum_{\substack{n_1,n_2,\ldots,n_m=1;\\ n_1<n_2<\ldots<n_m}}^{N} [z_{n_1}(t)\ z_{n_2}(t)\cdots z_{n_m}(t)]\ ,$$
$$m = 2,3,\ldots,N\ , \tag{2.11b}$$

—is a new *example of* solvable *dynamical system: indeed its* solution *is provided by the N* zeros $z_n(t)$ *of the polynomial* (*see* (3.2))

$$p_N(z;\ \vec{c}(t);\ \underline{z}(t)) = z^N + \sum_{m=1}^{N} \left[c_m(t)\ z^{N-m}\right]\ , \tag{2.12}$$

the coefficients $c_m(t)$ *of which are provided by the formulas* (2.7), *with the N initial data* $c_m(0)$ *expressed of course in terms of the N initial data* $z_n(0)$ *by the formulas* (2.11) *at* $t = 0$.

Remark 2.2. This dynamical system, (2.10), is *isochronous*, *all* its *generic* (hence *nonsingular*) solutions being *periodic* with period

$$\tilde{T} = N!\ T = N!\ \frac{2\,q\,\pi}{|\omega|}\ , \tag{2.13a}$$

$$z_n\left(t + \tilde{T}\right) = z_n(t)\ ,\quad n = 1,\ ,2,\ldots,N\ . \tag{2.13b}$$

The (*nongeneric*) *singular* solutions are those characterized by the *nongeneric* set of initial data $z_n(0)$ such that the equality (2.8a) hold with $c_1(0) = -\sum_{n=1}^{N}[z_n(0)]$, or such that the polynomial (2.12) feature at least one *multiple zero* (in which case the time-derivative $\dot{z}_n(t)$ eventually blows up, see (2.10)).

We trust the reader to consider this **Remark 2.2** obvious on the basis of the developments reported above: see in particular the right-hand side of (2.10)), and the well-known fact that, if a time-dependent polynomial $p_N(z;t)$ of degree N in z is *periodic* in t, $p_N(z;t+T) = p_N(z;t)$, the *unordered* set $\underline{z}(t)$ of its N *zeros* is of course as well *periodic* with the same period, $\underline{z}(t+T) = \underline{z}(t)$, while each of the N zeros $z_n(t)$, considered as a *continuous* function of t, is also *periodic* but possibly with a larger period νT (with ν a positive integer, $\nu \leq N!$) due to the possibility that the *zeros*, as it were, exchange their roles over their time evolution. Of course the *genericity* of the *initial* data $z_n(0)$ is in the context of evolutions of the dynamical system (2.10) taking place in the *complex* z-plane, corresponding to the fact that a *generic* polynomial of degree N in the *complex* variable z features N *distinct* zeros z_n. And it is also clear that the set of initial data $\underline{z}(0)$ can be partitioned in the *complex* z-plane into $N!$ *different* ensembles of initial data yielding solutions *periodic* with periods $\tilde{T}/\nu$ with ν a positive integer, $\nu \leq N!$, these ensembles being separated from each other by *special* (i. e., *nongeneric*) sets of initial data $\underline{z}(0)$ yielding solutions of the dynamical system which are *singular* due to "particle collisions", i. e. such that, at some time $t = t_c \bmod(T)$, two, or more, *different* coordinates $z_n \equiv z_n(t)$ *coincide*, say $z_n(t_c) = z_\ell(t_c)$ with $\ell \neq n$.

Let us end this section by suggesting that assignments for interested researchers might be: (*i*) to write out more explicitly than we did above the equations of motion (2.10) for $N = 2$, $N = 3$ and perhaps also $N = 4$, and their explicit solutions as detailed in **Proposition 2.1**; (*ii*) to make graphs—for various assignments of the parameters and of the initial data $z_n(0)$—of the *real* and *imaginary* parts of the coordinates $z_n(t)$ as functions of time and of the trajectories of these coordinates in the complex z-plane; (*iii*) to compare these data with analogous data obtained via *numerical* solutions of the equations of motion (2.10). Note that for $N = 2, 3, 4$ the *zeros* of a polynomial of degree N can be *explicitly* expressed in terms of the *coefficients* of the polynomial, although the resulting expressions are somewhat cumbersome for $N = 3$ and especially for $N = 4$. And it is of course always possible to manufacture additional *solvable* dynamical systems by iterating the technique that has allowed the transition from the *solvable* model (2.6) to the *solvable* model (2.10)—as generally described in [2].

3. Proofs

In this Section 3 we prove the results reported without their proofs in the preceding Section 2.

Our first task is to prove that the formula (2.2) provides the solution of the dynamical system (2.1).

The first step to this end is to integrate from 0 to τ the ODE (2.1a), or equivalently the ODE

$$\gamma_1'(\tau)\ \ [\gamma_1(\tau)]^{-1+1/r} = a_1\ . \tag{3.1a}$$

This clearly yields (see (2.2c))

$$\gamma_1(\tau) = \gamma_1(0)\ \ (1+\rho\ \tau)^r\ , \tag{3.1b}$$

which coincides with the $m = 1$ case of (2.2) (see **Notation 2.1**).

A proof by recursion of the formula (2.2) is then achieved by showing—as the diligent reader will easily verify—that the assumed validity of this formula (with m replaced by $m-1$) to replace $\gamma_{m-1}(\tau)$ in the right-hand side of the ODE (2.1b) implies, by a standard integration of this ODE from 0 to τ, the validity of the solution (2.2) for m. □

Next, let us derive the equations of motion of the dynamical system (2.10) in order to prove **Proposition 2.1**.

To this end it is convenient to introduce the following time-dependent (monic) polynomial of degree N in z,

$$p_N(z;\ \vec{c}(t);\ \underline{z}(t)) = z^N + \sum_{m=1}^{N} \left[c_m(t)\ z^{N-m}\right] = \prod_{n=1}^{N} [z - z_n(t)]\ , \tag{3.2}$$

where clearly the N *coefficients* $c_m(t)$ of this polynomial are the N components of the time-dependent N-vector $\vec{c}(t)$ and the N *zeros* $z_n(t)$ of this polynomial are the N components of the *unordered* set $\underline{z}(t)$: see the left-hand side of (3.2), and note that the notation $p_N(z;\ \vec{c}(t);\ \underline{z}(t))$ is somewhat redundant, since this monic polynomial is equally well defined by assigning *either* its N *coefficients* $c_m(t)$ *or* its N *zeros* $z_n(t)$ (indeed the N *coefficients* $c_m(t)$ are themselves defined in terms of the N *zeros* $z_n(t)$ by the well-known formulas (2.11)).

Let us now assume that the N *coefficients* $c_m(t)$ of the polynomial $p_N(z;\ \vec{c}(t);\ \underline{z}(t))$ evolve according to the dynamical system (2.6); it is then plain that—as implied by the formula

$$\dot{z}_n = -\left[\prod_{\ell=1,\ \ell\neq n}^{N}(z_n - z_\ell)\right]^{-1} \sum_{m=1}^{N}\left[\dot{c}_m\ z_n^{N-m}\right] \tag{3.3}$$

which relates the time evolution of the N *zeros* $z_n \equiv z_n(t)$ of *any* time-dependent polynomial such as (3.2) to the time evolution of its N *coefficients* $c_m(t)$ (if need be see [1] for a proof)—the corresponding evolution of the N zeros $z_n(t)$ is characterized just by the set of N *coupled nonlinear* ODEs (2.10). The rest of **Proposition 2.1** clearly follows. □

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