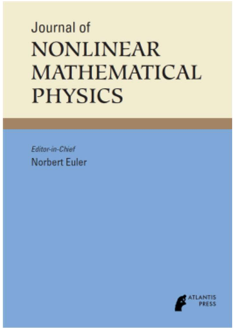

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Linearisable mappings, revisited

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We examine the growth properties of second-order mappings which are integrable by linearisation and which generically exhibit a linear growth of the homogeneous degree of initial conditions. We show that for Gambier-type mappings for which the growth proceeds generically with a step of 1 there exist cases where the degree increase by unity every two steps. We examine also mappings belonging to the family known as "of third kind" in relation to the approach of Diller and Favre concerning the regularisable or not character of mappings and show that the anticonfined singularities of these mappings exhibit a linear growth with step 1. (The term anticonfined is used for singularities where the singular values extend all the way to infinity on both sides with just a few regular values in the middle). Moreover we construct specific examples of Gambier-type mappings which have anticonfined singularities and where the degree of the singularity increases linearly but where the average slope can be adjusted so as to be arbitrarily small.

Keywords: linearisation, Gambier mapping, degree growth, anticonfinement

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1. Introduction

Linearisable systems are a special case of integrable ones. Calogero [1] has coined the term C-integrable in order to distinguish systems linearisable by a Change of dependent variables from those integrable by Spectral methods, designated as S-integrable. While linearisable systems are simpler than their S-integrable brethren when it comes to the precise methods of their integration, they are also richer in a certain sense. In fact, a feature that characterises most linearisable systems, be they continuous or discrete, is the presence of free functions of the independent variable in their non-autonomous forms, to the point that this feature may be considered as an indication of linearisability. The latter is not the only point on which S- and C-integrable systems differ. Another important difference resides in their singularity structure. While the solutions of continuous S-integrable systems possess the Painlevé property [2], the latter is not necessary for C-integrability [3]. The same sentence can be repeated almost verbatim for discrete systems replacing just "Painlevé property" by "singularity confinement" [4].

In this paper we shall revisit discrete lineariable systems, concentrating on mappings of the plane. Before proceeding further it is useful to present the three families of second-order mappings which can be linearised through birational transformations. (As we have showed in [5], this is not the only possibility: when the limitations of the authorised transformations are lifted, one can obtain substantially richer results). The simplest case of linearisablity is that of the projective mappings [6] which have the general, canonical [7], form

$$x_{n+1}x_nx_{n-1}+a_nx_nx_{n-1}+b_nx_{n-1}+c_n=0 \tag{1}$$

where a_n,b_n,c_n are functions of the independent variable. The linearisation of (1) is obtained through a Cole-Hopf transformation $x_n=w_{n+1}/w_n$ resulting to the linear equation

$$w_{n+2}+a_nw_{n+1}+b_nw_n+c_nw_{n-1}=0 \tag{2}$$

The second case is that of the Gambier mappings [8] which are two coupled homographic mappings in cascade. Their generic form is

$$\frac{a_nx_nx_{n-1}+b_nx_n+c_nx_{n-1}+d_n}{e_nx_nx_{n-1}+f_nx_n+g_nx_{n-1}+h_n}=y_n \tag{3}$$

where, without loss of generality, the equation for y can assume the simplest possible form $y_{n+1}=y_n$, i.e. y_n is constant. The interpretation of (3) is now simple: the Gambier mapping can be considered as the derivative of the discrete Riccati equation. The third case comprises the mappings known under the moniker of “third kind”. They were first discovered in [9] where we have given the general framework for their linearisation. In this case we have a nonlinear mapping and an associated linear equation

$$F(x_{n+1},x_n,x_{n-1})=m \tag{4a}$$

and

$$\frac{a_nx_{n+1}+b_nx_n+c_nx_{n-1}+d_n}{e_nx_{n+1}+f_nx_n+g_nx_{n-1}+h_n}=k \tag{4b}$$

where m,k are constant and F is a ratio of two polynomials linear separately in x_{n+1},x_{n-1} with coefficients depending on x_n. Taking the discrete derivatives of (4a) and (4b) we require that the third-order mappings to which they lead be the same up to nonessential factors. In [10] we have examined all known third-kind mappings which belong to the QRT family [11] and showed that they can be explicitly integrated in analogy to what happens at the autonomous limit (where the solution of all linearisable QRT mappings can be given in terms of an exponential function).

In what follows we shall first review the growth properties of linearisable mappings concentrating on the case of Gambier and third-kind mappings. (The case of projective mappings is almost trivial: the homogeneous degree of their iterates does not present any growth). Moreover, given that the linearisable mappings possess anticonfined singularities, we are going to study the growth of these singularities as well and the variety of their behaviour.

2. Growth properties of linearisable mappings

As we showed in [12], linearisability is intimately related to low-growth properties. The standard procedure for computing the degree growth for second degree rational mappings is to introduce homogeneous coordinates for the inital conditions with the additional simplification that x_0 is a constant with homogeneity 0 while x_1 is of the form p/q where p,q have homogeneity 1. Iterating the mapping one obtains the degrees of the (numerator or denominator of the) successive irreducible fractions x_n. For linearisable mappings one finds that this growth is either zero or linear in n.

Let us illustrate this with examples chosen among all three families of linearisable second-order mappings. We start with the projective case, the canonical form of which is

$$x_{n+1} = \frac{\alpha x_n x_{n-1} + \beta x_{n-1} + \gamma}{x_n x_{n-1}} \tag{5}$$

We find the following succession of degrees, 0,1,1,1,$\cdots$, i.e. zero degree growth.

Next we turn to the mapping

$$(x_{n+1} + x_n)(x_n + x_{n-1}) = \alpha(x_n^2 - 1) \tag{6}$$

with $\alpha \neq 1$, which belongs to the Gambier family while being of QRT type. Here the degree growth we obtain is 0,1,2,2,$\cdots$, i.e. again zero degree growth after the second iteration. However this is not true for all Gambier mappings. Suppose we consider the mapping, already analysed in [13].

$$x_{n+1} = \frac{x_{n-1}x_n(1+\gamma^2\beta^2+\alpha\beta x_n) - \beta(\beta\gamma^4 + \alpha\gamma^2 x_n + \beta x_n^2)}{x_{n-1}\beta^2(1+\alpha\beta x_n + x_n^2) - \beta^2(\alpha\beta\gamma^2 + (\gamma^2+\beta^2)x_n)} \tag{7}$$

We find that the degree grows as 0,1,2,3,4,$\cdots$, i.e. the growth is linear but does never saturate.

Finally we examine the case of a mapping of the third kind

$$x_{n+1}x_{n-1} = (x_n - \alpha)(x_n - \beta) \tag{8}$$

We find the following succession of degrees: 0,1,2,4,6,8,$\cdots$, i.e. the degree grows indefinitely (albeit linearly).

An important remark is in order here. Given the different rates of growth of the various families of linearisable mappings one can use the rate obtained by direct calculation in order to identify the family the mapping belongs to. A total absence of growth signals a projective mapping, a growth with step 1 is characteristic of Gambier mappings while for third-kind mappings the growth step is 2. It goes without saying that the precise values of the steps, 1 and 2 respectively, are due to the special choice of initial conditions. Had we expressed not only x_1 but also x_0 as a ratio of two quantities, say r/s, of homogeneity 1, we would have found for the Gambier mapping a growth with step 2, in all four homogeneous coordinates, while for the third-kind mappings we would have obtained a step of 4. What is important is that the step of the linear growth of the third-kind mappings is twice as large as that of the Gambier ones.

At this point one can wonder whether the growths we obtained above are the only possible ones. It turns out that while for the two extreme cases, projective and third-kind, the growth is indeed zero and a step-2 one, for the Gambier mappings one can have a greater variety. We illustrate this with a few selected examples. We start with a very simple linearisable mapping of the form

$$x_{n+1} + x_{n-1} = \frac{\theta}{x_n} \tag{9}$$

Defining $y_n = x_{n+1}x_n$ we obtain a linear mapping $y_n + y_{n-1} = \theta$ and thu s (9) is a mapping of Gambier type. Starting from initial conditions x_0 and $x_1 = p/q$ we find that the homogeneity degree in p, q of the successive iterates is 0,1,1,2,2,3,3,4,4,,5,$\cdots$. The mean step is thus 1/2.

The second example is based on a generalisation of a mapping introduced in [14].

$$x_{n+1} = \frac{\alpha x_{n-1}(x_n+\beta)+(\alpha-\beta)x_n}{x_n+1} \tag{10}$$

We define $y_n = x_{n-1}x_n + x_n + \beta x_{n-1}$ and find for y the linear equation $y_{n+1} = \alpha y_n$. Thus the mapping is indeed of Gambier type. Iterating from initial conditions x_0 and $x_1 = p/q$ we find exacly the same succession of degrees as in the case of mapping (9). So, here again the mean step of growth is 1/2. (The mapping studied in [14] corresponds to $\alpha = \beta$ and leads exactly to the same results as far as the degree growth is concerned).

Finally we turn to the case of (7). Putting $y_n = \beta(\gamma^2 - x_n x_{n+1})/(\beta^2 x_{n+1} - x_n)$ we find that y obeys the equation $y_n - 1/y_{n-1} = \alpha$. It is straightforward to verify that when α, β, γ satisfy the relations $\alpha = \pm(\gamma + 1/\gamma)$ and $\beta^2 = 1$ the growth of the degree stops at 1, while a degree saturating at 2 necessitates the constraints $(\alpha\gamma \pm 1)(\alpha \pm \gamma) + \gamma = 0$ and $\beta^4 = 1$. Taking now $\alpha = \gamma - 1/\gamma$ and $\beta = i$ we obtain exactly the same sequence of degrees as for mapping (9), i.e. again a growth with mean step 1/2.

Thus a growth rate smaller than one is possible for Gambier mappings. Whether rates different from 1/2 are possible is an open question but, although we do not know of any such case, their existence is not a priori impossible.

3. Singularity properties and the anticonfinement notion

Having examined the growth of the homogeneous degree of the various mappings we turn now to the study of the their singularities. The projective mappings have a very simple confined singularity pattern, $\{0, \infty\}$. For the Gambier mappings the situation is more complicated. In some cases the singularity is confined as in the case of mapping (6) where we have the pattern $\{1, -1\}$. In other cases, like that of mapping (7), the singularity is unconfined, unless special relations of parameters do hold. In fact the constraints leading to a saturating homogeneous degree growth are precisely those that lead to a singularity confined after one, two, three, etc. steps. This is a general result that can be understood in the light of a theorem due to Diller and Favre [15], who have shown that if a linearisable mapping has confined singularities its degree growth is nil. This explains the behaviour of the three mappings above.

A remark is in order here. In the case of the Gambier mapping (7) when the confinement conditions are satisfied at the very first step, we obtain the sequence of degrees $0,1,1,1,\cdots$, i.e. the same as the one obtained in the case of projective mappings. This is no mere coincidence and one can show in fact that the mapping under the appropriate confinement conditions is indeed projective. Given that the mapping (6), which has also zero growth after an initial growth spurt, was also shown to be equivalent to a projective one [16], one could be tempted to conjecture that the Gambier mappings with confined singularities can indeed be transformed into projective ones.

Third-kind mappings are special in the sense that all examples of mappings of this type known to date belong to the QRT family when autonomous. This is the case indeed for (8). Being a QRT mapping ensures that its singularities are confined the patterns being $\{\alpha, 0, \beta\}$ and $\{\beta, 0, \alpha\}$. However a word of caution is necessary here. In our singularity confinement approach we focus only on what we call "movable" singularities, which means singularities which appear at some iteration due to specific initial conditions. Our tacit conjecture from the outset has been that the confinement constraints are linked to these movable singularities. However a mapping may also possess singularities which extend all the way to infinity on both sides which we have been simply ignoring in our singularity analysis. For Diller and Favre these singularities and their behaviour do play an important role when one attempts to regularise the mapping by a succession of blow-ups. It is the presence of such singularities that explains why for third-kind mappings the homogeneous degree grows indefinitely while the degree growth does saturate for the QRT-Gambier mapping (6).

We shall not go here into the algebro-geometric analysis of the linearisable mappings but just show how by performing a standard singularity analysis one can find a significant difference between Gambier and third-kind mappings. This difference is to be found in singularities of a type that we have dubbed "anticonfined" [16]. A confined singularity is a sequence of singular values bracketed by regular ones extending all the way to infinity on both sides. An anticonfined singularity is a sequence of regular values bracketed by singular ones which extend all the way to infinity on both sides.

For the Gambier-QRT mapping $(x_{n+1}+x_n)(x_n+x_{n-1}) = \alpha(x_n^2-1)$ we introduce the initial conditions $x_n = \kappa$, $x_{n+1} = -\kappa+\varepsilon$ and we find the following succession of values

$$\cdots, \varepsilon^{-1}, \varepsilon^{-1}, \varepsilon^{-1}, \varepsilon^{-1}, \kappa, -\kappa+\varepsilon, \varepsilon^{-1}, \varepsilon^{-1}, \varepsilon^{-1}, \varepsilon^{-1}, \cdots$$

where the symbol ε^{-1} is a shorthand designating terms proportional to ε^{-1} with coefficients depending on α and κ. We remark that the power of the singular term remains the same throughout. For Diller and Favre this signals a singularity that can be regularised, something that in our terminology should be considered as confined. This means that according to their theorem the Gambier-QRT mapping (6) should have zero degree growth, which indeed it has (after the second iteration step, which is necessary for the "standard" singularity $x = 1$ to be confined).

We turn now to the third-kind-QRT mapping $x_{n+1}x_{n-1} = (x_n-\alpha)(x_n-\beta)$ and introduce the initial conditions $x_n = \kappa$, $x_{n+1} = \varepsilon$. Iterating we find the succession of values

$$\cdots, \varepsilon^{-4}, \varepsilon^{-3}, \varepsilon^{-2}, \varepsilon^{-1}, \kappa, \varepsilon, \kappa^{-1}, \varepsilon^{-1}, \varepsilon^{-2}, \varepsilon^{-3}, \varepsilon^{-4}, \cdots,$$

with the same convention as in the previous paragraph. Here the singularities extending to infinity in both ways become more and more singular. Thus according to Diller and Favre this singularity should be considered as unconfined and no regularisation is possible.

It thus appears that whenever one studies linearisable systems it is important to consider not only the usual, confined, singularities but also the anticonfined, ones. (This statement has an even wider stature. Our recent work [17] on the justification of the singularity confinement approach by algebrogeometric methods shows that one must not neglect the study of the antoconfined singularities contrary to what have been till now the standard practice).

Next we examine the mapping $x_{n+1}+x_{n-1} = \theta/x_n$. In our standard singularity analysis approach the mapping does present a problem since one cannot enter the singular value $x = 0$ coming from finite values. However concluding that the mapping does not possess unconfined singularities would be a hasty conclusion. In fact introducing the initial conditions $x_n = \kappa$, $x_{n+1} = \varepsilon$ and obtain, iterating forwards and backwards the values (with the same conventions as in the previous paragraphs)

$$\cdots, \varepsilon^{-3}, \varepsilon^{3}, \varepsilon^{-2}, \varepsilon^{2}, \varepsilon^{-1}, \varepsilon, \kappa^{-1}, \kappa, \varepsilon, \varepsilon^{-1}, \varepsilon^{2}, \varepsilon^{-2}, \varepsilon^{3}, \varepsilon^{-3}, \cdots$$

i.e. the singularities extend to infinity in both directions while becoming more and more singular, increasing by a unit every two steps. Thus the singularity of the mapping is not confined and given that the degree grows we can conclude that the mapping cannot be regularised.

The mapping just examined is not an isolated occurence. It belongs to a larger family of mappings written as a system

$$x_n x_{n+1} = y_n + 1, \qquad y_n = a y_{n-1} \tag{11}$$

(and the mapping (9) for $\theta = -2$ is revovered when $a = -1$). When a is generic a choice of inital conditions x_0 and $x_1 = p/q$ leads to an homogeneity degree in p,q for the successive iterates of 0,1,1,2,2,3,3,4,4,5,$\cdots$, i.e. again a mean step of 1/2. For generic a the singularity $x = 0$ of (11) is not confined, in agreement with the unsaturated growth of the homogeneity degree. Moreover starting from $x_n = \kappa$, $x_{n+1} = 1/\varepsilon$ we obtain an anticonfined singularity where a finite value alternates with a singular one proportional to ε^{-1}.

Things become more interesting when a is a root of -1. For instance if we take $a = i$ we find that the homogeneity degree growth is still the same but now an anticonfined singularity has made its appearance. We start from $x_n = \kappa$, $x_{n+1} = \varepsilon$ and obtain the following sequence

$$\cdots, \varepsilon^2, \varepsilon^{-1}, \varepsilon, \varepsilon^{-1}, \varepsilon, -\kappa^{-1}, (1-i)\kappa, (i-1)\kappa^{-1}, \kappa, \varepsilon, \varepsilon^{-1}, \varepsilon, \varepsilon^{-1}, \varepsilon^2, \varepsilon^{-2}, \varepsilon^2, \varepsilon^{-2}, \varepsilon^3 \cdots$$

The singularities extend to infinity forwards and backwards with degrees increasing by a unit every 4 steps. Taking $a^3 = -1$ we find similar results with an anticonfined singularity the degree of which increases by a unit every 6 steps. We surmise that similar results will hold for higher roots of -1 i.e. if $a^k = -1$, with integer k, we expect the degree of the anticonfined singularity to increase by a unit every $2k$ steps.

Finally we extend the mapping (11) to

$$x_n x_{n+1} = \frac{y_n - b}{y_n - 1}, \qquad y_n = a y_{n-1} \tag{12}$$

Quite expectedly, when both a and b are generic the homogeneity degree grows with steps of 1 and the singularities $x = 0$ and $x = \infty$ of the mapping do not confine. Thus the mapping behaves like a typical Gambier one. Still when a is a root of -1 anticonfined singularities do exist. For instance when $a = -1$ starting from $x_n = \kappa$, $x_{n+1} = 1/\varepsilon$ we obtain

$$\cdots, \varepsilon^3, \varepsilon^{-3}, \varepsilon^2, \varepsilon^{-2}, \varepsilon, \varepsilon^{-1}, \frac{b+1}{2\kappa}, \kappa, \varepsilon^{-1}, \varepsilon, \varepsilon^{-2}, \varepsilon^2, \varepsilon^{-3}, \varepsilon^3, \cdots$$

i.e. the degree of the singularity increases by one unit every two steps. Similarly if we take $a = i$ we find the sequence

$$\cdots, \varepsilon^{-2}, \varepsilon, \varepsilon^{-1}, \varepsilon, \varepsilon^{-1}, \frac{b^2+1}{\kappa(b+1)}, \kappa\frac{1+i+b+ib}{2(b+i)}, \frac{1+i+b-ib}{2\kappa}, \kappa, \varepsilon^{-1}, \varepsilon, \varepsilon^{-1}, \varepsilon, \varepsilon^{-2}, \varepsilon^2, \varepsilon^{-2}, \varepsilon^2, \varepsilon^{-3} \cdots$$

and thus the degree of the singularity increases by a unit every four steps. Again we surmise that for higher roots of -1 the degree of the anticonfined singularity will increase slower just as in the case of mapping (11).

The upshot of this analysis is that there exist mappings belonging to the Gambier family with anticonfined singularities of increasing degree, although the growth of this degree is slower than in the case of the third-kind mappings. Be that as it may, all those mappings are not regularisable.

4. Conclusion

In this paper we have revisited the question of the growth properties of linearisable mappings concentrating on rational mappings of the plane. As we have shown in [12] second-order linearisable mappings have a growth slower than that of mappings of the same order integrable through spectral methods. In fact while for the latter the homogeneous degree of x_n grows like n^2, in the case of linearisable mappings the growth is at most linear in n. This is an important result in itself, since this property is a handy detector of linearisability, something which, at least to the authors' knowledge, does not exist in the case of differential systems. However the degree growth of linearisable systems encapsulates even more information since it gives an indication as to which family the system belongs.

Projective systems have zero homogeneous degree growth. The mappings belonging to the Gambier family have generically a linear growth with step 1 (with the initial conditions we specified in Section 2) but their growth is arrested if and when their singularities are confined. However, as we have shown here, there may exist Gambier-type mappings where the growth is even slower. In the examples we constructed we have found a linear growth with step 1/2 by which we mean that the homogeneous degree grows by a full unit every two steps. The possibility of existence of linear growth with even smaller step cannot be discarded although no such examples were found. The mappings we have dubbed "of third kind" grow linearly with a step of 2 (again with the appropriate initial conditions as explained in Section 2). Thus the details of the growth give a reliable information as to which family the linearisable mapping belongs.

The third-kind mappings pose a particular problem. In fact all known examples of such mappings belong, when autonomous, to the QRT family. As such, and according to our definitions, they have confined singularities and from a theorem by Diller and Favre we would expect the homogeneous degree growth to saturate at some step. The fact that for third-kind mappings the homogeneous degree grows indefinitely would present a paradox. However this is not the case and the explanation is to be found in the singularity analysis. In our approach we focused on singularities where a sequence of singular values is bracketed by regular ones extending all the way to infinity in both directions. However, the theorem of Diller and Favre covers also singularities which extend all the way to infinity both forwards and backwards. We were thus led to examine singularities consisting of a sequence of regular values bracketed by singular ones which extend all the way to infinity. We have dubbed these singularities "anticonfined". For the homogeneous degree of a mapping to saturate the degree of the singularity in the anticonfined pattern must not grow. We have illustrated this by several examples from the Gambier family and have shown that it is possible to tune the degree of the anticonfined singularity so as to present an arbitrarily small growth leading invariably to an non-saturating homogeneous degree growth.

The present work had focused on mappings of second-order. The case of mappings of higher order remains essentially open despite the existence of specific examples. It would be interesting to extend the approach presented in this paper to higher-order discrete systems and we hope to be able to do so in some future work of ours.

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