Products of distributions and collision of a δ-wave with a δ'-wave in a turbulent model

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We study the possibility of collision of a δ-wave with a stationary δ'-wave in a model ruled by equation $f(t)u_t + [u^2 - \beta(x - \gamma(t))u]_x = 0$, where f, β and γ are given real functions and $u = u(x,t)$ is the state variable. We adopt a solution concept which is a consistent extension of the classical solution concept. This concept is defined in the setting of a distributional product, which is not constructed by approximation processes. By a convenient choice of f, β and γ, we are able to distinguish three distinct dynamics for that collision, to which correspond phenomena of solitonic behaviour, scattering, and merging. Also, as a particular case, taking $f = 2$ and $\beta = 0$ we prove that the referred collision is impossible to arise in the setting of the inviscid Burgers equation. To show how this framework can be applied to other physical models, we included several results already obtained.

1. Introduction and contents

Let *I* be an interval of $\mathbb{R}$ with more than one point and let us consider the nonlinear equation

$$f(t)u_t + [u^2 - \beta(x - \gamma(t))u]_x = 0, \tag{1.1}$$

where $t \in I$ is the time variable, $x \in \mathbb{R}$ is the space variable, $f \in C^0(I)$, $\gamma \in C^1(I)$ and $\beta \in C^1(\mathbb{R})$ are real functions, and $u = u(x,t)$ stands for the unknown state variable.

In a generalized sense, we can consider this equation as a conservation law with flux $\phi = \frac{u^2(x,t) - \beta(x - \gamma(t))u(x,t)}{f(t)}$ (eventually singular, if $f(t) = 0$ for some value of t), and we study the possibility of propagation of the wave

$$u(x,t) = \delta(x - \gamma(t)) + \delta'(x), \tag{1.2}$$

where δ stands for the Dirac measure concentrated at the origin. The goal is to investigate the collision of the δ-wave $u_1(x,t) = \delta(x - \gamma(t))$ with the stationary wave $u_2(x,t) = \delta'(x)$ and to analyze the dynamic of this collision. A necessary and sufficient condition for the propagation of the wave (1.2) is presented and, using this condition, we will prove that:

(a) for $f = 2$ and $\beta = 0$, (1.2) may be a solution of inviscid Burgers equation

$$u_t + \left(\frac{u^2}{2}\right)_x = 0, \tag{1.3}$$

but in this case, if $t \in I = [t_0, +\infty[$, $t_0 \in \mathbb{R}$ and $\gamma(t_0) \neq 0$, then $\gamma(t) \neq 0$ for all $t \in I$; this means that in models ruled by this equation a collision of u_1 with u_2 cannot happen;

(b) it is possible to choose f, β and γ in such a way that (1.2) is a solution of (1.1) and a collision of u_1 with u_2, takes place;

(c) three distinct collision dynamics can be explicitly provided:

- u_1 passes through u_2 and recovers its initial profile: a solitonic phenomenon;
- after the collision, u_1 turns back as if it has been repelled by u_2: a scattering phenomenon;
- u_1 collides with u_2 and both remain stationary after the instant of collision: a merging phenomenon.

Collisions of δ-waves were already studied in several models ruled by nonlinear equations [5, 6, 12, 17]. As far as we know, collisions of δ-waves with δ'-waves have not been addressed in the literature. Regarding δ'-waves, we must remember that they were first introduced by E. Yu. Panov and V. M. Shelkovich for certain systems of conservation laws [13, 25]. The results show that these systems subjected to piecewise continuous initial data may develop not only δ-waves, but also δ'-waves (see also [19, 26, 27]).

For equation (1.1), we will adopt a solution concept (defined within the framework of a distributional product) which is a consistent extension of the classical solution concept. In this framework, the product of two distributions is a distribution which depends on the choice of a certain function α that encodes the indeterminacy inherent to such products. This indeterminacy generally is not avoidable and in many cases it also has a physical meaning; concerning this point let us mention [1, 2, 10, 15]. We stress that our solution concept does not depend on approximation processes; however the solutions of differential equations containing such products may depend (or not) of α. We call such solutions α-solutions. The possibility of their occurrence depends on the physical system: in certain cases we cannot previously know the behavior of the system, possibly due to physical features omitted in the formulation of the model with the goal of simplifying it. Thus, the mathematical indetermination sometimes observed may have this origin.

To introduce the reader into the realm of distributional waves and to show how this framework can be applied to other physical models, let us recall some results we have obtained.

For the conservation law

$$u_t + [\phi(u)]_x = \psi(u), \tag{1.4}$$

where ϕ, ψ are entire functions taking real values on the real axis, we have established [18] necessary and sufficient conditions for the propagation of a travelling wave with a given distributional profile and we also have computed its speed. For example, for LeVeque and Yee equation

$$u_t + u_x = \mu u(1-u)(u - \frac{1}{2}), \tag{1.5}$$

where $\mu \neq 0$, we have proved that there exist six travelling waves with profile $c_1 + (c_2 - c_1)H$ (c_1, c_2 are constants and H stands for the Heaviside function), all of them with speed 1. When

$\psi = 0$ and $\phi'' \neq 0$ in (1.4), we were able to conclude that the only continuous travelling waves are the constant states. Thus, if we ask for nonconstant travelling waves for the conservative equation $u_t + [\phi(u)]_x = 0$, with $\phi'' \neq 0$, we have to seek them among distributions that are not continuous functions; for C^1-wave profiles with one jump discontinuity, our methods easily lead to the well known Rankine-Hugoniot conditions.

Conditions for the propagation of travelling waves with profiles $\beta + m\delta$ and $\beta + m\delta'$ (where β is a continuous function, $m \in \mathbb{R}$ and $m \neq 0$) were also obtained, as well as their speeds [19]. For example, for the diffusionless Burgers-Fischer equation

$$u_t + a\left(\frac{1}{2}u^2\right)_x = ru\left(1 - \frac{u}{k}\right),$$

where $a > 0$, $r > 0$, and $k > 0$, the profiles $m\delta$ and $k + m\delta$ may arise as travelling waves with speed $\frac{ak}{2}$ in both cases. For LeVeque and Yee equation (1.5), the profile $\frac{1}{2} + m\delta'$ may also propagate as a travelling wave with speed 1.

Gas dynamics phenomena, known as "infinitely narrow soliton solutions", discovered by Maslov and his collaborators [3, 7–9], can be obtained directly in distributional form [16].

In a Riemann problem for the 2×2 system of conservation laws

$$u_t + [\phi(u)]_x = 0,$$
$$v_t + [\psi(u)v]_x = 0,$$

the so called generalized pressureless gas dynamics, only assuming $\phi, \psi : \mathbb{R} \to \mathbb{R}$ continuous, we were able to show the formation of a delta-shock wave solution [21]. In this case, we arrived, in a more general setting, to the same result of Danilov and Mitrovic [4], which have employed the weak asymptotic method, and also to the same result of Mitrovic et al. [11], which have used a different approach, based on a linearization process. Thus, there exists loss of regularity relative to initial conditions. As a consequence, the space of all functions is not sufficient to contain all α-solutions of this problem. Also remark that our α-solutions can be considered, in this case, as limits of classical solutions in the sense of [4]. For our present interaction phenomena represented by (1.2), this is still more difficult to prove and we do not know the answer.

In the Brio system

$$u_t + \left(\frac{u^2 + v^2}{2}\right)_x = 0,$$
$$v_t + (uv - v)_x = 0,$$

a simplified model for the study of plasmas, we have subjected $u(x,t)$ and $v(x,t)$ to the initial conditions

$$u(x,0) = c_0\delta(x),$$
$$v(x,0) = h_0\delta(x),$$

with $c_0, h_0 \in \mathbb{R}\backslash\{0\}$. Under certain assumptions, we have obtained, as solutions, travelling delta-waves with speed $\frac{c_0^2 + h_0^2}{c_0^2 - h_0^2}$ and certain singular perturbations (which are not measures) propagating with speed 1 [23]. The space of measures is not sufficient to contain all α-solutions; there exists also loss of regularity relative to initial conditions.

For the Hunter-Saxton equation

$$\left[u_t + \left(\frac{u^2}{2} \right)_x \right]_x = \frac{1}{2} u_x^2,$$

which models the director field's propagation

$$n(x,t) = (\cos u(x,t), \sin u(x,t)),$$

in a one-dimensional nematic liquid crystal, we have considered the Riemann problem corresponding to the initial condition

$$u(x,0) = a + (b-a)H(x),$$

where $a, b \in \mathbb{R}$, and $a \neq b$; in a convenient space of discontinuous functions, all solutions were evaluated. Also a new set of distributional travelling waves was identified [22]; this set contains many δ-waves and also many discontinuous waves.

Let us now summarize the present paper's contents. In Section 2, we present the main ideas of our method for multiplying distributions, displaying all formulas required in the sequel. In Section 3, we define the concept of α-solution for equation (1.1). In Section 4, we give a necessary and sufficient condition for the propagation, according to (1.1), of waves of the form (1.2) and we analyze the case $f = 2$, $\beta = 0$, which corresponds to inviscid Burgers equation. In Sections 5 and 6, we present examples of collisions exhibiting distinct dynamics: solitonic, scattering, and merging.

2. The multiplication of distributions

Let $\mathscr{C}^\infty$ be the space of indefinitely differentiable real or complex-valued functions defined on $\mathbb{R}^N$, $N \in \{1, 2, 3, ...\}$, and $\mathscr{D}$ the subspace of C^∞ consisting of those functions which have compact support. Let $\mathscr{D}'$ be the space of Schwartz distributions and $L(\mathscr{D})$ the space of continuous linear operators $\phi : \mathscr{D} \to \mathscr{D}$, where we suppose $\mathscr{D}$ endowed with the usual topology. We will sketch the main ideas of our distributional product which can be seen as an extension of the classical Schwartz product of a distribution with a C^∞-function. For proofs and other details concerning this product see [14].

First we define a map $\tilde{\zeta} : L(\mathscr{D}) \to \mathscr{D}'$ where the image of ϕ is the distribution $\tilde{\zeta}(\phi)$ given by

$$\langle \tilde{\zeta}(\phi), \xi \rangle = \int \phi(\xi),$$

for all $\xi \in \mathscr{D}$ (in the present paper, all integrals are extended all over $\mathbb{R}^N$). This map is an epimorphism because it is linear and given a distribution $S \in \mathscr{D}'$ there exists always an operator $\phi \in L(\mathscr{D})$ such that $\tilde{\zeta}(\phi) = S$: taking $\alpha \in \mathscr{D}$ such that $\int \alpha = 1$ and $\phi \in L(\mathscr{D})$ defined by $\phi(\xi) = \alpha \langle S, \xi \rangle$ for all $\xi \in \mathscr{D}$, we have

$$\langle \tilde{\zeta}(\phi), \xi \rangle = \int [\alpha \langle S, \xi \rangle] = \langle S, \xi \rangle \int \alpha = \langle S, \xi \rangle,$$

which means that $\tilde{\zeta}(\phi) = S$. Since we have a lot of α satisfying $\int \alpha = 1$, it is also clear that, given $S \in \mathscr{D}'$, the operator ϕ such that $\tilde{\zeta}(\phi) = S$ is not unique. When $\tilde{\zeta}(\phi) = S$ we say that ϕ is a

representative operator for the distribution S. For example, if $\beta \in C^{\infty}$ is seen as a distribution, the operator $\phi_{\beta} \in L(\mathscr{D})$ defined by $\phi_{\beta}(\xi) = \beta \xi$ for all $\xi \in \mathscr{D}$, is a representative operator for β because

$$\langle \tilde{\zeta}(\phi_{\beta}), \xi \rangle = \int \phi_{\beta}(\xi) = \int \beta \xi = \langle \beta, \xi \rangle,$$

for all $\xi \in \mathscr{D}$, which means that $\tilde{\zeta}(\phi_{\beta}) = \beta$.

Now, we can define an operation that makes $\mathscr{D}'$ a right $L(\mathscr{D})$-module. This operation is the product $T\phi \in \mathscr{D}'$, for $T \in \mathscr{D}'$ and $\phi \in L(\mathscr{D})$ defined by

$$\langle T\phi, \xi \rangle = \langle T, \phi(\xi) \rangle,$$

for all $\xi \in \mathscr{D}$. For example,

$$\langle T\phi_{\beta}, \xi \rangle = \langle T, \phi_{\beta}(\xi) \rangle = \langle T, \beta \xi \rangle = \langle T\beta, \xi \rangle,$$

for all $\xi \in \mathscr{D}$, and we conclude that $T\beta = T\phi_{\beta}$, being $T\beta$ the usual classical product of $T \in \mathscr{D}'$ with $\beta \in C^{\infty}$.

Thus, given $T, S \in \mathscr{D}'$ we are tempted to define a natural product by setting $TS := T\phi$, where $\phi \in L(\mathscr{D})$ is a representative operator for S. Unfortunately, this product is not well defined, because TS depends on the representative $\phi \in L(\mathscr{D})$ of $S \in \mathscr{D}'$.

This difficulty can be overcome, if we fix $\alpha \in \mathscr{D}$ with $\int \alpha = 1$ and define $s_{\alpha} : L(\mathscr{D}) \to L(\mathscr{D})$ by

$$[(s_{\alpha}\phi)(\xi)](y) = \int \phi[(\tau_y \check{\alpha})\xi], \tag{2.1}$$

for all $\xi \in \mathscr{D}$ and all $y \in \mathbb{R}^N$, where $\tau_y \check{\alpha}$ is given by $(\tau_y \check{\alpha})(x) = \check{\alpha}(x-y) = \alpha(y-x)$ for all $x \in \mathbb{R}^N$. It can be proved that for each $\alpha \in \mathscr{D}$ with $\int \alpha = 1$, $s_{\alpha}(\phi) \in L(\mathscr{D})$, s_{α} is linear, $s_{\alpha} \circ s_{\alpha} = s_{\alpha}$ (s_{α} is a projector of $L(\mathscr{D})$), $\ker s_{\alpha} = \ker \tilde{\zeta}$, and $\tilde{\zeta} \circ s_{\alpha} = \tilde{\zeta}$.

Now, for each $\alpha \in \mathscr{D}$, we can define a general α-product $\underset{\alpha}{\odot}$, of $T \in \mathscr{D}'$ with $S \in \mathscr{D}'$ by setting

$$T \underset{\alpha}{\odot} S := T(s_{\alpha}\phi) \tag{2.2}$$

where $\phi \in L(\mathscr{D})$ is a representative operator for $S \in \mathscr{D}'$. This α-product is independent of the representative ϕ of S, because if ϕ, ψ are such that $\tilde{\zeta}(\phi) = \tilde{\zeta}(\psi) = S$, then $\phi - \psi \in \ker \tilde{\zeta} = \ker s_{\alpha}$. Hence,

$$T(s_{\alpha}\phi) - T(s_{\alpha}\psi) = T[s_{\alpha}(\phi - \psi)] = 0.$$

Since ϕ in (2.2) satisfies $\tilde{\zeta}(\phi) = S$, we have $\int \phi(\xi) = \langle S, \xi \rangle$ for all $\xi \in \mathscr{D}$, and by (2.1)

$$[(s_{\alpha}\phi)(\xi)](y) = \langle S, (\tau_y \check{\alpha})\xi \rangle = \langle S\xi, \tau_y \check{\alpha} \rangle = (S\xi * \alpha)(y),$$

for all $y \in \mathbb{R}^N$, which means that $(s_{\alpha}\phi)(\xi) = S\xi * \alpha$. Therefore, for all $\xi \in \mathscr{D}$,

$$\langle T \underset{\alpha}{\odot} S, \xi \rangle = \langle T(s_{\alpha}\phi), \xi \rangle = \langle T, (s_{\alpha}\phi)(\xi) \rangle = \langle T, S\xi * \alpha \rangle$$

$$= [T * (S\xi * \alpha)\check{\ }](0) = [(S\xi)\check{\ } * (T * \check{\alpha})](0) = \langle (T * \check{\alpha})S, \xi \rangle,$$

and we obtain an easy formula for the general product (2.2),

$$T \underset{\alpha}{\odot} S = (T * \check{\alpha})S \tag{2.3}$$

In general, this α-product is neither commutative nor associative but it is bilinear and satisfies the usual differentiation rules provided that the Leibniz rule is written in the form

$$D_k(T \underset{\alpha}{\odot} S) = (D_k T) \underset{\alpha}{\odot} S + T \underset{\alpha}{\odot} (D_k S),$$

where D_k is the usual k-partial derivative operator in distributional sense ($k = 1, 2, ..., N$).

Recall that the usual Schwartz products of distributions are not associative and the commutative property is a convention inherent to the definition of such products (see the classical monograph of Schwartz [24] pp. 117, 118, and 121, where these products are defined). Unfortunately the α-product (2.3) is not in general consistent with the classical Schwartz products of distributions with functions.

In order to obtain consistency with the usual product of a distribution with a C^∞-function, we are going to introduce some definitions and single out a certain subspace H_α of $L(\mathscr{D})$.

An operator $\phi \in L(\mathscr{D})$ is said to vanish on an open set $\Omega \subset \mathbb{R}^N$ if and only if $\phi(\xi) = 0$ for all $\xi \in \mathscr{D}$ with support contained in Ω. The support of an operator $\phi \in L(\mathscr{D})$ will be defined as the complement of the largest open set in which ϕ vanishes.

Let $\mathscr{N}$ be the set of operators $\phi \in L(\mathscr{D})$ whose support has Lebesgue measure zero, and $\rho(C^\infty)$ the set of operators $\phi \in L(\mathscr{D})$ defined by $\phi(\xi) = \beta\xi$ for all $\xi \in \mathscr{D}$, with $\beta \in C^\infty$. For each $\alpha \in \mathscr{D}$, with $\int \alpha = 1$, let us consider the space $H_\alpha = \rho(C^\infty) \oplus s_\alpha(\mathscr{N}) \subset L(\mathscr{D})$. It can be proved that $\zeta_\alpha := \tilde{\zeta}|_{H_\alpha} : H_\alpha \to C^\infty \oplus \mathscr{D}'_\mu$ is an isomorphism ($\mathscr{D}'_\mu$ stands for the space of distributions whose support has Lebesgue measure zero). Therefore, if $T \in \mathscr{D}'$ and $S = \beta + f \in C^\infty \oplus \mathscr{D}'_\mu$, a new α-product, $_{\dot\alpha}$, can be defined by $T_{\dot\alpha}S := T\phi_\alpha$, where for each α, $\phi_\alpha = \zeta_\alpha^{-1}(S) \in H_\alpha$. Hence,

$$\begin{aligned} T_{\dot\alpha}S &= T\zeta_\alpha^{-1}(S) = T[\zeta_\alpha^{-1}(\beta + f)] \\ &= T[\zeta_\alpha^{-1}(\beta) + \zeta_\alpha^{-1}(f)] = T\beta + T \underset{\alpha}{\odot} f = T\beta + (T * \check{\alpha})f, \end{aligned}$$

and putting α instead of $\check{\alpha}$ (to simplify), we get

$$T_{\dot\alpha}S = T\beta + (T * \alpha)f. \tag{2.4}$$

Thus, the referred consistency is obtained when the C^∞-function is placed at the right-hand side: if $S \in C^\infty$, then $f = 0$, $S = \beta$ and $T_{\dot\alpha}S = T\beta$.

The α-product (2.4) can be easily extended for $T \in \mathscr{D}'^p$ and $S = \beta + f \in C^p \oplus \mathscr{D}'_\mu$, where $p \in \{0, 1, 2, ..., \infty\}$, $\mathscr{D}'^p$ is the space of distributions of order $\le p$ in the sense of Schwartz ($\mathscr{D}'^\infty$ means $\mathscr{D}'$), $T\beta$ is the Schwartz product of a $\mathscr{D}'^p$-distribution with a C^p-function, and $(T * \alpha)f$ is the usual product of a C^∞-function with a distribution. This extension is clearly consistent with the Schwartz products of $\mathscr{D}'^p$-distributions with C^p-functions, if the C^p-functions are placed at the right-hand side, keeps the bilinearity and satisfies the usual rules of differentiation, provided the Leibniz rule is written in the form

$$D_k(T_{\dot\alpha}S) = (D_k T)_{\dot\alpha}S + T_{\dot\alpha}(D_k S),$$

and the α-products are well defined. Moreover, these products are invariant by translations, that is,

$$\tau_a(T_{\dot\alpha}S) = (\tau_a T)_{\dot\alpha}(\tau_a S),$$

where τ_a stands for the usual translation operator in distributional sense. They are also invariant for the action of any group of linear transformations $h : \mathbb{R}^N \to \mathbb{R}^N$, with $|\det h| = 1$, which leaves α invariant .

Thus, for each $\alpha \in \mathcal{D}$ with $\int \alpha = 1$, formula (2.4) allows us to evaluate the product of $T \in \mathcal{D}'^p$ with $S \in C^p \oplus \mathcal{D}'_\mu$; therefore, we have obtained a family of products, one for each α. For instance, in dimension $N = 1$, if β is a continuous function and $a \in \mathbb{R}$, we have for each α, and by applying (2.4),

$$\beta_{\dot{\alpha}}\beta = \beta_{\dot{\alpha}}(\beta + 0) = \beta\beta + (\beta * \alpha)0 = \beta^2$$
$$\delta_{\dot{\alpha}}\beta = \delta_{\dot{\alpha}}(\beta + 0) = \delta\beta + (\delta * \alpha)0 = \beta(0)\delta,$$
$$\beta_{\dot{\alpha}}\delta = \beta_{\dot{\alpha}}(0 + \delta) = \beta 0 + (\beta * \alpha)\delta = [(\beta * \alpha)(0)]\delta,$$
$$\delta_{\dot{\alpha}}\delta = \delta_{\dot{\alpha}}(0 + \delta) = \delta 0 + (\delta * \alpha)\delta = \alpha\delta = \alpha(0)\delta,$$
$$(\tau_a\delta)_{\dot{\alpha}}(D\delta) = [(\tau_a\delta) * \alpha]D\delta = (\tau_a\alpha)D\delta = [\alpha(-a)]D\delta - [\alpha'(-a)]\delta, \qquad (2.5)$$
$$(D\delta)_{\dot{\alpha}}(\tau_a\delta) = [(D\delta) * \alpha](\tau_a\delta) = \alpha'(\tau_a\delta) = \alpha'(a)\tau_a\delta, \qquad (2.6)$$
$$(D\delta)_{\dot{\alpha}}(D\delta) = [(D\delta) * \alpha]D\delta = \alpha'D\delta = \alpha'(0)D\delta - \alpha''(0)\delta, \qquad (2.7)$$
$$H_{\dot{\alpha}}\delta = (H * \alpha)\delta = \left[\int_{-\infty}^{+\infty}\alpha(-\tau)H(\tau)\,d\tau\right]\delta = \left(\int_{-\infty}^{0}\alpha\right)\delta.$$

For each α the support of the α-product (2.4) satisfies $\text{supp}(T_{\dot{\alpha}}S) \subset \text{supp}\,S$, as for usual functions, but it may happen that $\text{supp}(T_{\dot{\alpha}}S) \not\subset \text{supp}\,T$. For instance, if $a, b \in \mathbb{R}$, from (2.4) we have,

$$(\tau_a\delta)_{\dot{\alpha}}(\tau_b\delta) = [(\tau_a\delta) * \alpha](\tau_b\delta) = (\tau_a\alpha)(\tau_b\delta) = \alpha(b - a)(\tau_b\delta). \qquad (2.8)$$

It is also possible to multiply many other distributions, to define powers of distributions, and to compose distributions with functions (see for instance [15, 18, 20]). However, for the present paper this is all we need.

3. The α-solution concept

Let I be an interval of $\mathbb{R}$ with more that one point, and let $\mathcal{F}(I)$ be the space of continuously differentiable maps $\tilde{u} : I \to \mathcal{D}'$ in the sense of the usual topology of $\mathcal{D}'$. For $t \in I$, the notation $[\tilde{u}(t)](x)$ is sometimes used for emphasizing that the distribution $\tilde{u}(t)$ acts on functions $\xi \in \mathcal{D}$ which depend on x.

Let $\Sigma(I)$ be the space of functions $u : \mathbb{R} \times I \to \mathbb{R}$ such that:
(a) for each $t \in I$, $u(x,t) \in L^1_{loc}(\mathbb{R})$;
(b) $\tilde{u} : I \to \mathcal{D}'$, defined by $[\tilde{u}(t)](x) = u(x,t)$ is in $\mathcal{F}(I)$.

The natural injection $u \mapsto \tilde{u}$ from $\Sigma(I)$ into $\mathcal{F}(I)$ identifies any function in $\Sigma(I)$ with a certain map in $\mathcal{F}(I)$. Since $C^1(\mathbb{R} \times I) \subset \Sigma(I)$, we can write the inclusions

$$C^1(\mathbb{R} \times I) \subset \Sigma(I) \subset \mathcal{F}(I).$$

Thus, identifying u with $\tilde{u}$, the equation (1.1) can be read as follows:

$$f(t)\frac{d\tilde{u}}{dt}(t) + D[\tilde{u}(t)_{\dot{\alpha}}\tilde{u}(t) - \tilde{u}(t)\tau_{\gamma(t)}\beta] = 0. \qquad (3.1)$$

Remark 3.1. We could also write $(\tau_{\gamma(t)}\beta)\tilde{u}(t)$ instead of $\tilde{u}(t)\tau_{\gamma(t)}\beta$, since this product is a classical product of a $\mathcal{D}'^1$ distribution with a C^1-function. Meanwhile, if we want to write it under the form of an α-product, we must write $\tilde{u}(t)_{\dot{\alpha}}(\tau_{\gamma(t)}\beta)$, since, as we have said in Section 2, the α-products

of $\mathscr{D}'^p$-distributions with C^p-functions are consistent with the classical Schwartz products when the C^p-functions are placed at the right-hand side.

Definition 3.1. Given α, the function $\tilde{u} \in \mathscr{F}(I)$ will be called an α-solution of the equation (3.1) on I, if the α-product appearing in this equation is well defined and this equation is satisfied for all $t \in I$.

We have the following results:

Theorem 3.1. *If u is a classical solution of (1.1) on $\mathbb{R}\times I$, then, for any α, the function $\tilde{u} \in \mathscr{F}(I)$ defined by $[\tilde{u}(t)](x) = u(x,t)$ is an α-solution of (3.1) on I.*

Notice that by a classical solution of (1) on $\mathbb{R}\times I$ we mean a C^1-function $u(x,t)$ satisfying (1) on $\mathbb{R}\times I$.

Theorem 3.2. *If $u : \mathbb{R}\times I \to \mathbb{R}$ is a C^1-function and, for a certain α, the function $\tilde{u} \in \mathscr{F}(I)$ defined by $[\tilde{u}(t)](x) = u(x,t)$ is an α-solution of (3.1) on I, then u is a classical solution of (1.1) on $\mathbb{R}\times I$.*

For the proof it is enough to observe that any C^1-function $u(x,t)$ can be read as a continuously differentiable function $\tilde{u} \in \mathscr{F}(I)$ defined by $[\tilde{u}(t)](x) = u(x,t)$, and to use the α-products' consistency with classical Schwartz products of distributions with functions.

Definition 3.2. Given α, any α-solution $\tilde{u}$ of equation (3.1) on I, will be called an α-solution of the equation (1.1) on I.

As a consequence, an α-solution $\tilde{u}$ in this sense, read as a usual distribution u, affords a consistent extension of the classical solution concept for equation (1.1). Thus, and for short, we also call the distribution u an α-solution of (1.1).

Remark 3.2. The study of differential equations in the classical distribution theory is, of course, restricted to linear equations owing to the well known difficulties of multiplying distributions. The classical setting usually considers linear partial differential equations with C^∞-coefficients, and a solution $u \in \mathscr{D}'$, also called a weak solution, is defined as satisfying the equation in the sense of distributions.

Concerning a linear partial differential evolution equation in the unknown u, when re-interpreted as an equation with α-products in the unknown $\tilde{u}(t)$, we can place the (re-interpreted) C^∞-coefficients at the right-hand side of $\tilde{u}(t)$ and its derivatives. The reason is that, in this case, our α-products are consistent with the products of distributions with C^∞-functions. Thus, for any weak solution u, the corresponding $\tilde{u}(t)$ is necessarily an α-solution for any α. Conversely, if for a certain α, $\tilde{u}(t)$ is an α-solution then $\tilde{u}(t)$ read as a usual distribution u is a weak solution. In this sense, the α-solution concept coincides with the weak solution concept. In the framework of linear differential evolution equations, the advantage of the α-solutions is that, now, the coefficients can be considered as distributions, if the α-products are well defined.

4. The possibility of the propagation of the wave $u(x,t) = \delta(x - \gamma(t)) + \delta'(x)$

Here we consider the propagation of the wave (1.2) according to equation (1.1). Having in mind the identification $u \mapsto \tilde{u}$, we must substitute the equation (1.1) by equation (3.1) and the wave (1.2) by

$$\tilde{u}(t) = \tau_{\gamma(t)}\delta + D\delta. \tag{4.1}$$

The following result is a necessary and sufficient condition for the propagation of this wave.

Theorem 4.1. *For each α, the function $\tilde{u}$ given by (4.1) is an α-solution of (3.1) on I, if and only if the following two conditions are satisfied:*

(I) *for each $t \in I$ such that $\gamma(t) \neq 0$, we have*

$$f(t)\gamma'(t) = \alpha(0) + \alpha'(\gamma(t)) - \beta(0), \tag{4.2}$$

$$\alpha(-\gamma(t)) - \beta(-\gamma(t)) + \alpha'(0) = 0, \tag{4.3}$$

$$\alpha'(-\gamma(t)) - \beta'(-\gamma(t)) + \alpha''(0) = 0; \tag{4.4}$$

(II) *for each $t \in I$ such that $\gamma(t) = 0$, we have*

$$f(t)\gamma'(t) = \alpha(0) - \beta(0) - \alpha''(0) + \beta'(0), \tag{4.5}$$

$$\alpha(0) + \alpha'(0) - \beta(0) = 0. \tag{4.6}$$

Proof. Let us suppose that $\tilde{u}$ given by (4.1) is an α-solution of (3.1). From (4.1) we have,

$$\frac{d\tilde{u}}{dt}(t) = -\gamma'(t)\tau_{\gamma(t)}D\delta.$$

Using the α-products' bilinearity and formulas (2.5), (2.6), (2.7) and (2.8), we have

$$\tilde{u}(t)_{\dot\alpha}\tilde{u}(t) = [\alpha(0) + \alpha'(\gamma(t))]\tau_{\gamma(t)}\delta +$$
$$+ [\alpha(-\gamma(t)) + \alpha'(0)]D\delta - [\alpha'(-\gamma(t)) + \alpha''(0)]\delta.$$

Since, for each t, $\tilde{u}(t) \in \mathscr{D}'^1$ and $\tau_{\gamma(t)}\beta \in C^1$, it is easy to evaluate the classical product

$$\tilde{u}(t)(\tau_{\gamma(t)}\beta) = \beta(0)\tau_{\gamma(t)}\delta + \beta(-\gamma(t))D\delta - \beta'(-\gamma(t))\delta,$$

and (3.1) turns out to be

$$[-f(t)\gamma'(t) + \alpha(0) + \alpha'(\gamma(t)) - \beta(0)]\tau_{\gamma(t)}D\delta +$$
$$+ [\alpha(-\gamma(t)) + \alpha'(0) - \beta(-\gamma(t))]D^2\delta +$$
$$+ [-\alpha'(-\gamma(t)) - \alpha''(0) + \beta'(-\gamma(t))]D\delta = 0. \tag{4.7}$$

Now, if we suppose $\gamma(t) \neq 0$, we have (4.2), (4.3) and (4.4). If we suppose $\gamma(t) = 0$, we have (4.5) and (4.6). Conversely, suppose that we have (I). Then (4.7) is satisfied, and this means that $\tilde{u}$ given by (4.1) is an α-solution of (3.1). Suppose that we have (II). Then (4.7) is also satisfied, which means that $\tilde{u}$ given by (4.1) is an α-solution of (3.1). Hence, the theorem is proved. $\square$

Since inviscid Burgers equation

$$\frac{d\tilde{u}}{dt}(t) + \frac{1}{2}D[\tilde{u}(t)_{\dot\alpha}\tilde{u}(t)] = 0 \tag{4.8}$$

can be seen as a particular case of (3.1), it is interesting to apply this theorem (with $\beta = 0$ and $f = 2$) to see that the wave (4.1) may propagate according to (4.8). Thus, taking for instance $\gamma : [1, +\infty[\to \mathbb{R}$ defined by $\gamma(t) = t$, it is easy to see that, for all α such that $\operatorname{supp}\alpha \subset [-1, 1]$, $\alpha(0) = 2$, and $\alpha'(0) = \alpha''(0) = 0$, the wave (4.1) is an α-solution of (4.8); indeed, since $\beta = 0$, $f = 2$ and $\gamma(t) \neq 0$ for all $t \in [1, +\infty[$, conditions (4.2), (4.3) and (4.4) are trivially satisfied, because $\operatorname{supp}\alpha' \subset \operatorname{supp}\alpha \subset [-1, 1]$.

As a consequence, the wave $\tilde{u}_1(t) = \tau_{\gamma(t)}\delta$ travels away from $\tilde{u}_2(t) = D\delta$ with speed $\gamma'(t) = 1$. Since in the interval of time $[1,+\infty[$ does not exists t^* such that $\gamma(t^*) = 0$, we conclude that the wave $\tilde{u}_1(t) = \tau_{\gamma(t)}\delta$ does not collide with the stationary wave $\tilde{u}_2(t) = D\delta$.

Indeed, this phenomenon is much more general: it is not possible any collision of $\tilde{u}_1$ with $\tilde{u}_2$ in the setting of inviscid Burgers equation, as the following theorem shows.

Theorem 4.2. *Let $t_0 \in \mathbb{R}$, and $I = [t_0,+\infty[$. Still suppose that $\tilde{u}$ given by (4.1) is an α-solution of the equation (4.8) on I, satisfying $\gamma(t_0) \neq 0$. Then $\gamma(t) \neq 0$ for all $t \in I$.*

Proof. Suppose that there exists $t^* \in I$ such that $\gamma(t^*) = 0$. Then the set

$$K = \{t \in \,]t_0,t^*] : \gamma(t) = 0\}$$

is a compact of $\mathbb{R}$, and $t^* \in K$. Let $t' = \min K$. Then, $\gamma(t') = 0$, $t' \in \,]t_0,t^*]$ and $\gamma(t) \neq 0$ for all $t \in [t_0,t'[$. Since $\tilde{u}$ is an α-solution of (4.8), from (4.2) in theorem 4.1, we can write (remember that $\beta = 0$ and $f = 2$),

$$2\gamma'(t) = \alpha(0) + \alpha'(\gamma(t)), \tag{4.9}$$

for all $t \in [t_0,t'[$. Thus, equation (4.9) with initial condition $\gamma(t_0) = x_0 \neq 0$ has, for each α, an unique solution $\gamma : [t_0,t'[\,\to \mathbb{R}$, which does not vanish identically on $[t_0,t'[$. However, there exists

$$\lim_{t \to t'}\gamma(t) = \gamma(t') = 0,$$

which shows that the solution γ is continuable at the right. This means that there exists $\varepsilon > 0$ and a function $\bar{\gamma} : [t_0,t'+\varepsilon[\,\to \mathbb{R}$, of class C^1, such that

$$2\bar{\gamma}'(t) = \alpha(0) + \alpha'(\bar{\gamma}(t)), \tag{4.10}$$

$$\bar{\gamma}(t_0) = x_0. \tag{4.11}$$

Since $\bar{\gamma}(t') = \lim_{t \to t'}\gamma(t) = \gamma(t') = 0$, from (4.6) in theorem 4.1, we can write

$$\alpha(0) + \alpha'(0) = 0. \tag{4.12}$$

In addition, the Cauchy problem (4.10), (4.11) has, for each α, the same solution of the equation (4.10), with initial condition $\bar{\gamma}(t') = 0$. On the other hand, the equation (4.10) with initial condition $\bar{\gamma}(t') = 0$ has the unique solution $\bar{\gamma} = 0$, because, by direct substitution, (4.10) turns out to be (4.12). This is a contradiction, because the restriction γ of $\bar{\gamma}$ to $[t_0,t'[$ does not vanish identically on $[t_0,t'[$. Hence, the theorem is proved. $\square$

In the sequel, we will see, applying again theorem 4.1, that this collision may arise in models ruled by equation (1.1) for f, β and γ conveniently chosen.

5. Solitonic collision and scattering

Let us consider the interval of time $I = [-1,+\infty[$. Let $A > 0$ be a real number, $n \geq 2$ an integer, and $\beta \in \mathscr{D}(\mathbb{R})$ such that

- $\int_{-\infty}^{+\infty} \beta = 1$ and
- $\beta(x) = \frac{n-1}{3A}x^3$, for $x \in [-A,A]$.

Let us define $f : [-1, +\infty[\to \mathbb{R}$ by $f(t) = t^n$, and let $\gamma : [-1, +\infty[\to \mathbb{R}$ be defined the following way:

- for $t \in [-1, 1[$,

$$\gamma(t) = At^{n-1}; \tag{5.1}$$

- for $t \in [1, +\infty[$, $\gamma(t)$ is the solution of the Cauchy problem

$$t^n \gamma'(t) = \beta'(\gamma(t)), \tag{5.2}$$
$$\gamma(1) = A. \tag{5.3}$$

Lemma 5.1. *The function $\gamma : [-1, +\infty[\to \mathbb{R}$ defined above has the following properties:*
(a) γ is a C^1-function;
(b) $\gamma(t) = 0$ if and only if $t = 0$.

Proof. (a) The function γ is continuous at $t = 1$ because

$$\lim_{t \to 1^-} \gamma(t) = A \quad \text{and} \quad \lim_{t \to 1^+} \gamma(t) = \gamma(1) = A.$$

Thus, γ is continuous on $[-1, +\infty[$ and differentiable on $[-1, 1[\cup]1, +\infty[$. On the other hand, for $t = 1$ we have from (5.2) and (5.1),

$$\gamma'(1^+) = \beta'(\gamma(1)) = \beta'(A) = (n-1)A,$$

and from (5.1),

$$\gamma'(1^-) = (n-1)A.$$

Therefore, there exists $\gamma'(1) = (n-1)A$. Hence, γ' is continuous at $t = 1$ and (a) follows
(b) Clearly, the unique solution of the equation $\gamma(t) = 0$ on $[-1, 1[$ is $t = 0$. For $t \in [1, +\infty[$, $\gamma(t) \neq 0$: indeed, if $\gamma(t_0) = 0$ for a certain $t_0 \in [1, +\infty[$, then the Cauchy problem

$$t^n \gamma'(t) = \beta'(\gamma(t)), \tag{5.4}$$
$$\gamma(t_0) = 0, \tag{5.5}$$

would have the same solution of Cauchy problem (5.2), (5.3), because both problems have a unique solution on $[1, +\infty[$. However, since $\beta'(0) = 0$, the problem (5.4), (5.5) has the unique solution $\gamma(t) = 0$ on $[1, +\infty[$, as we can see by direct substitution; this is a contradiction and (b) follows. $\square$

Theorem 5.1. *Let f, β and γ be defined as above and let $\alpha = \beta$. Then, $\tilde{u}$ defined by (4.1) is an α-solution of (3.1) on $[-1, +\infty[$.*

Proof. This result is a consequence of theorem 4.1 and lemma 5.1. Indeed, if $t \neq 0$, (4.2), (4.3) and (4.4) are clearly satisfied, having in mind the definition of γ, β and f. If $t = 0$, (4.5) and (4.6) are also satisfied. Hence, the theorem is proved. $\square$

Thus, in the interval of time $[-1, +\infty[$, and with the assumptions of this theorem, a collision of $\tilde{u}_1(t) = \tau_{\gamma(t)} \delta$ with $\tilde{u}_2(t) = D\delta$ is possible at the instant $t = 0$.
In the sequel, we will see the consequences of this theorem with n even and with n odd:

(1) Suppose $n \geq 2$ an even integer. Then, the wave $\tilde{u}_1(t) = \tau_{\gamma(t)}\delta$, initially with support at $\gamma(-1) = -A < 0$, begin moving along the x-axis to the right and collides with the stationary wave $\tilde{u}_2(t) = D\delta$ at the instant $t = 0$. At this instant the speed of $\tilde{u}_1$ is

$$\gamma'(0) = \begin{cases} A & \text{if } n = 2 \\ 0 & \text{if } n \geq 4 \end{cases},$$

and the shape of $\tilde{u}$ is

$$\tilde{u}(0) = \delta + D\delta. \tag{5.6}$$

After, that is, for $t > 0$, the wave $\tilde{u}_1$ moves on the positive part of the x-axis, without more collisions in the future. Thus, $\tilde{u}_1$ passes through $\tilde{u}_2$ recovering its initial profile. Clearly this is a solitonic phenomenon.

(2) Suppose $n \geq 3$ an odd integer. Then, the wave $\tilde{u}_1(t) = \tau_{\gamma(t)}\delta$, initially with support at $\gamma(-1) = A > 0$, begin moving along the x-axis to the left while $t \in [-1,0[$. The collision takes place at the instant $t = 0$. The speed of $\tilde{u}_1$ at this instant is $\gamma'(0) = 0$, and the shape of $\tilde{u}$ is also given by (5.6). After, that is, for $t > 0$, the wave $\tilde{u}_1$ turns back as if it has been repelled by $\tilde{u}_2$, and moves on the positive part of the x-axis, without more collisions in the future. Clearly this is a scattering phenomenon.

6. The merging phenomenon

Suppose $n \geq 3$. Let I, A, β, f be as in Section 5 and let us define γ in the following way:

- for $t \in [-1,0[$,

$$\gamma(t) = At^{n-1}; \tag{6.1}$$

- for $t \in [0,+\infty[$,

$$\gamma(t) = 0. \tag{6.2}$$

Clearly, γ is a C^1-function on $[-1,+\infty[$ and we have $\gamma(t) = 0$ if and only if $t \in [0,+\infty[$. It is now also easy to verify, applying theorem 4.1, that theorem 5.1 is also true in this new context. Thus, the wave $\tilde{u}_1(t) = \tau_{\gamma(t)}\delta$, initially with support at $\gamma(-1) = A(-1)^{n-1}$ begin moving along the x-axis and collides, at the instant $t = 0$, with the stationary wave $\tilde{u}_2(t) = D\delta$. At this instant, the speed of $\tilde{u}_1$ is $\gamma'(0) = 0$ and the shape of $\tilde{u}$ is given by (5.6). Meanwhile, for $t > 0$, the wave $\tilde{u}_1$ remains stationary! Clearly this is a merging phenomenon.

Remark also that in Sections 5 and 6, regarding the α-solution (4.1), the function γ depends continuously on $A > 0$ (see (5.1), (5.2), (5.3), (6.1) and (6.2)). Thus, there exists some robustness in this sense. For the general problem of considering all α-solutions of (1.1) dependent of three functional parameters f, β and γ, we do not know what happens.

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