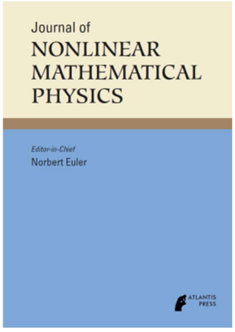

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Integrability properties of some equations obtained by symmetry reductions

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In our recent paper [1], we gave a complete description of symmetry reduction of four Lax-integrable (i.e., possessing a zero-curvature representation with a non-removable parameter) 3-dimensional equations. Here we study the behavior of the integrability features of the initial equations under the reduction procedure. We show that the ZCRs are transformed to nonlinear differential coverings of the resulting 2D-systems similar to the one found for the Gibbons-Tsarev equation in [17]. Using these coverings we construct infinite series of (nonlocal) conservation laws and prove their nontriviality. We also show that the recursion operators are not preserved under reductions.

Keywords: Partial differential equations, symmetry reductions, solutions, the Gibbons-Tsarev equation, Lax-integrable equations

2010 Mathematics Subject Classification: 35B06

Introduction

In [1] we gave a complete description of symmetry reductions for four three dimensional systems: the universal hierarchy equation, the 3D rdDym equation, the modified Veronese web equation, and Pavlov's equation. The result comprised more than 30 equations, but the majority of them were either exactly solvable or linearized by the generalized Legendre transformations. Nevertheless, there were 10 'interesting' reductions, among which two well-known equations, i.e., the Liouville

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and Gibbons-Tsarev equations, [3, 5]. The rest eight can be divided in two groups by their symmetry properties: five equations admit infinite-dimensional Lie algebras of contact symmetries (with functional parameters) and three others possess finite-dimensional symmetry algebras. These are

$$u_y u_{xy} - u_x u_{yy} = e^y u_{xx} \tag{0.1}$$

(reduction of the universal hierarchy equation),

$$u_{yy} = (u_x + x)u_{xy} - u_y(u_{xx} + 2) \tag{0.2}$$

(reduction of the 3D rdDym equation), and

$$u_{xx} = (x - u_y)u_{xy} + (2y + u_x)u_{yy} - u_y \tag{0.3}$$

(reduction of the Pavlov equation)[a]. These equations are pair-wise inequaivalent (see Section 5).

We deal with these three equations below and study how the integrability properties of the initial 3D systems behave under reduction. More precisely, we construct (Section 1) the reductions of the zero-curvature representations for Equations (0.1)–(0.2) and show that they result in differential coverings of the form

$$w_x = \frac{a_2 w^2 + a_1 w + a_0}{w^2 + c_1 w + c_0}, \qquad w_y = \frac{b_2 w^2 + b_1 w + b_0}{w^2 + c_1 w + c_0},$$

where a_i, b_i, c_i are functions in x, y, u, u_x, and u_y. These coverings are similar to the one found in [17] for the Gibbons-Tsarev equation and this resemblance, by all means, reflects the relations between generalized Gibbons-Tsarev equations and integrable 3D-systems [18]. In Section 3, for every nonlinear covering we construct an infinite series of conservation laws and prove their non-triviality.

We also study the behavior of the recursion operators for symmetries of three-dimensional systems and show that these operators do not survive under reduction (Section 4).

In Section 2 local symmetries and cosymmetries of the reduction equations are described. The corresponding conservation laws are presented in the Appendix.

Throughout the text the notion of (differential) covering is understood in the sense of [9].

1. Reduction of the Lax pairs

Using Lax representations of the 3D equations, whose reductions are the equations at hand, we construct here nonlinear coverings of Equations (0.1)–(0.3).

1.1. *Equation* (0.1)

This equation is obtained as the reduction of the universal hierarchy equation[b]

$$u_{yy} = u_z u_{xy} - u_y u_{xz} \tag{1.1}$$

with respect to the symmetry

$$\varphi = u_z + u_x + y u_y + u. \tag{1.2}$$

[a]All the reductions of the modified Veronese web equation were either exactly solvable or linearizable.

[b]To save the notation here and below, we denote by u the dependent and by x, y the independent variables. These are *not* the same as in the initial equation; see the details in [1].

Equivalently, this reduction may be written in the form

$$u_{yy} = u_y u_{xx} - (u_x + u) u_{xy} + u_x u_y \tag{1.3}$$

and Equation (0.1) transforms to (1.3) by the change of variables $x \mapsto y$, $y \mapsto x$, $u \mapsto -e^y u$.

Equation (1.1) admits the following Lax representation

$$\begin{aligned} w_z &= (w u_z - u_y) w^{-2} w_x, \\ w_y &= u_y w^{-1} w_x. \end{aligned} \tag{1.4}$$

The symmetry φ can be extended to a symmetry $\Phi = (\varphi, \chi)$ of (1.4), where

$$\chi = w_z + w_x + y w_y + w$$

and the corresponding reduction leads to the covering

$$\begin{aligned} w_x &= -\frac{w^3}{w^2 - (u_x + u) w - u_y}, \\ w_y &= -\frac{u_y w^2}{w^2 - (u_x + u) w - u_y} \end{aligned} \tag{1.5}$$

of Equation (1.3). Note that the first equation above is cubic in w, but by an appropriate gauge transformation it can be converted to a quadratic one, see Subsection 3.2 below.

Remark 1.1. Equation (0.1) can be written in the potential form

$$\left(\frac{u_y}{u_x}\right)_y = \left(\frac{e^y}{u_x}\right)_x,$$

the corresponding Abelian covering being

$$v_x = \frac{u_y}{u_x}, \qquad v_y = \frac{e^y}{u_x}. \tag{1.6}$$

Then v enjoys the equation

$$v_y - v_{yy} = v_y v_{xx} - v_x v_{xy}, \tag{1.7}$$

which also admits the rational covering

$$\begin{aligned} w_x &= \frac{w v_x - x v_x + v_y}{w^2 + (-2x + v_x) w + x^2 - x v_x + v_y}, \\ w_y &= \frac{w v_y - x v_y}{w^2 + (-2x + v_x) w + x^2 - x v_x + v_y}. \end{aligned}$$

of the same type. □

1.2. *Equation* (0.2)

This equation was obtained as the reduction of the 3D rdDym equation

$$u_{ty} = u_x u_{xy} - u_y u_{xx} \tag{1.8}$$

with respect to the symmetry

$$\varphi = u_t - xu_x - u_y + 2u. \tag{1.9}$$

The Lax representation for Equation (1.8) is

$$\begin{aligned} w_t &= (u_x + w)w_x, \\ w_y &= -u_y w^{-1} w_x. \end{aligned} \tag{1.10}$$

The symmetry φ extends to the one of (1.10): $\Phi = (\varphi, \chi)$, where

$$\chi = w_t - xw_x - w_y + u.$$

Reduction of the covering (1.10) with respect to Φ leads to the covering

$$\begin{aligned} w_x &= -\frac{w^2}{w^2 + (u_x - x)w + u_y}, \\ w_y &= \frac{u_y w}{w^2 + (u_x - x)w + u_y}. \end{aligned} \tag{1.11}$$

over Equation (0.2).

1.3. *Equation* (0.3)

Finally, Equation (0.3) is the reduction of the Pavlov equation

$$u_{yy} = u_{tx} + u_y u_{xx} - u_x u_{xy} \tag{1.12}$$

with respect to the symmetry

$$\varphi = u_t - 2xu_x - yu_y + 3u. \tag{1.13}$$

The Pavlov equation possesses the Lax pair

$$\begin{aligned} w_t &= (w^2 - wu_x - u_y)w_x, \\ w_y &= (w - u_x)w_x. \end{aligned} \tag{1.14}$$

The symmetry φ lifts to the symmetry $\Phi = (\varphi, \chi)$ of (1.14), where

$$\chi = w_t - 2xw_x - yw_y + w.$$

Reduction of the covering (1.14) with respect to this symmetry results in the nonlinear covering

$$\begin{aligned} w_x &= -\frac{w(w - u_y)}{w^2 - (u_y + x)w + xu_y - u_x - 2y}, \\ w_y &= -\frac{w}{w^2 - (u_y + x)w + xu_y - u_x - 2y} \end{aligned} \tag{1.15}$$

of Equation (0.3).

Remark 1.2. Equation (0.3) has a close relative. Namely, if we accomplish reduction of the Pavlov equation using another symmetry

$$\varphi' = u_t - yu_x + 2x$$

the resulting equation will be

$$u_{yy} = (u_y + y)u_{xx} - u_x u_{xy} - 2. \tag{1.16}$$

The symmetry φ' can also be lifted to (1.14) by $\Phi' = (\varphi', \chi')$, where

$$\chi' = w_t - yw_x + 1,$$

and the reduction of (1.14) will be

$$\begin{aligned} w_x &= -\frac{1}{w^2 - u_x w - u_y - y}, \\ w_y &= -\frac{w - u_x}{w^2 - u_x w - u_y - y}. \end{aligned} \tag{1.17}$$

By the change of variables $u \mapsto u - y^2/2$, Equation (1.16) transforms to the Gibbons-Tsarev equation, see [5],

$$u_{yy} = u_y u_{xx} - u_x u_{xy} - 1,$$

while (1.15) becomes

$$\begin{aligned} w_x &= -\frac{1}{w^2 - u_x w - u_y}, \\ w_y &= -\frac{w - u_x}{w^2 - u_x w - u_y}, \end{aligned}$$

cf. [17]. □

Remark 1.3. Equations (0.1), (0.2) and (0.3) are known to admit linear Lax representations with non-removable parameter (see [10,11,19] for the universal hierarchy equation, [13,20] for the 3Drd-Dym equation, and [4, 19] for the Pavlov equation). Nonlinear Lax pairs (1.4), (1.10), and (1.14) can be obtained from their linear counterparts by the standard procedure proposed in [21] or by the methods used in [13, 14].

2. Local symmetries and cosymmetries of the reduced equations

We present here computational results on classical symmetries and cosymmetries of Equations (0.1)–(0.3), i.e., solutions of the equations

$$\ell_{\mathscr{E}}(\varphi) = 0$$

and

$$\ell^*_{\mathscr{E}}(\psi) = 0,$$

where $\ell_{\mathscr{E}}$ is the linearization of the equation at hand and $\ell^*_{\mathscr{E}}$ is its formally adjoint and φ and ψ depend on x, y, u, u_x, u_y (see, e.g., [7]). The conservation laws corresponding to classical cosymmetries are presented in the Appendix below. The spaces of solutions are denoted by $\mathrm{sym_c}(\mathscr{E})$ and $\mathrm{cosym_c}(\mathscr{E})$, respectively.

All the equations under consideration happen to possess a scaling symmetry and thus admit weights (which we denote by $|\cdot|$) with respect to which they become homogeneous.

2.1. *Equation* (0.1)

We consider this equation in the form (1.3), i.e.,

$$u_{yy} = u_y u_{xx} - (u_x + u)u_{xy} + u_x u_y.$$

The weights are

$$|x| = 0, \quad |y| = 1, \quad |u| = -1, \quad |u_x| = -1, \quad |u_y| = -2.$$

Symmetries

The defining equation for symmetries is[c]

$$D_y^2(\varphi) = u_y D_x^2(\varphi) - (u_x + u)D_x D_y(\varphi) + (u_y - u_{xy})D_x(\varphi) + (u_{xx} + u_x)D_y(\varphi) - u_{xy}\varphi.$$

The space $\mathrm{sym_c}(\mathscr{E})$ is generated by the symmetries

$$\varphi_{-1} = u_y, \quad \varphi_0 = yu_y + u, \quad \varphi_0' = u_x, \quad \varphi_1 = e^{-x},$$

where the subscripts coincide with the weights[d].

Cosymmetries

The defining equation for cosymmetries of Equation (0.1) is

$$D_y^2(\psi) = u_y D_x^2(\psi) - (u_x + u)D_x D_y(\psi) + 2(u_{xy} + u_y)D_x(\psi) - 2(u_{xx} + u_x)D_y(\psi) - 3u_{xy}\psi.$$

The space $\mathrm{cosym_c}(\mathscr{E})$ is 6-dimensional and is spanned by the following cosymmetries:

$$\psi_{-3} = e^{4x}(3u_x^2 + 8u^2 + 10uu_x + 2u_y), \quad \psi_{-2} = e^{3x}(3u + 2u_x), \quad \psi_{-1} = e^{2x}$$

and

$$\psi_3 = \frac{1}{u_y^2}, \qquad \psi_4 = \frac{2u_x - yu_y + 2u}{u_y^3},$$
$$\psi_5 = \frac{-4u_x y u_y + 6uu_x + 3u_x^2 - 4yuu_y + 3u^2 + 2u_y + y^2 u_y^2}{u_y^4},$$

where superscript coincides with the weight[e].

2.2. *Equation* (0.2)

The weights are

$$|x| = 1, \quad |y| = 0, \quad |u| = 2, \quad |u_x| = 1, \quad |u_y| = 2.$$

[c] Here and below D_x and D_y denote the total derivatives with respect to x and y.
[d] To a symmetry φ we assign the weight of the corresponding evolutionary vector field $\mathbf{E}_\varphi$.
[e] To every cosymmetry we assign the weight of the corresponding variational form, see [8]

Symmetries

The linearized equation is

$$D_y^2(\varphi) = (u_x + x)D_x D_y(\varphi) - u_y D_x^2(\varphi) + u_{xy} D_x(\varphi) - (u_{xx} + 2)D_y(\varphi).$$

The space $\mathrm{sym}_c(\mathscr{E})$ is generated by the symmetries

$$\varphi_{-2} = 1, \quad \varphi_{-1} = u_x + x, \quad \varphi_0 = u - \frac{1}{2} x u_x, \quad \varphi_0' = u_y.$$

Cosymmetries

The defining equation for cosymmetries reads

$$D_y^2(\psi) = (u_x + x)D_x D_y(\psi) - u_y D_x^2(\psi) - 2u_{xy} D_x(\psi) + (2u_{xx} + 3)D_y(\psi).$$

The space $\mathrm{cosym}_c(\mathscr{E})$ is generated by the cosymmetries

$$\psi_{-3} = \frac{e^{-2y}(u_x + x)}{u_y^3}, \qquad \psi_2 = 1,$$
$$\psi_{-2} = \frac{e^{-y}}{u_y^2}, \qquad \psi_3 = u_x + 2x.$$

2.3. *Equation* (0.3)

The weights of variables are

$$|x| = 1, \quad |y| = 2, \quad |u| = 3, \quad |u_x| = 2, \quad |u_y| = 1.$$

in this case.

Symmetries

The symmetries are defined by the equation

$$D_x^2(\varphi) = (x - u_y)D_x D_y(\varphi) + (2y + u_x)D_y^2(\varphi) - D_y(\varphi)$$

and the space $\mathrm{sym}_c(\mathscr{E})$ is generated the symmetries

$$\varphi_0 = -\frac{1}{3} x u_x - \frac{2}{3} y u_y + u, \qquad \varphi_{-1} = u_x - x u_y + y - \frac{1}{2} x^2,$$
$$\varphi_{-2} = u_y + 2x, \qquad \varphi_{-3} = 1.$$

Cosymmetries

The defining equation for cosymmetries is of the form

$$D_x^2(\psi) = (x - u_y)D_x D_y(\psi) + (2y + u_x)D_y^2 - u_{yy} D_x + 3(2 - u_{xy})D_y.$$

The space $\mathrm{cosym}_c(\mathscr{E})$ is 6-dimensional and and is spanned by the elements

$$\psi_7 = \frac{54}{5} x u_x u_y + \frac{164}{5} x u_y y + \frac{256}{5} x^2 y + 2xu + \frac{4}{5} u u_y + \frac{12}{5} u_y^2 u_x + 4y u_x + \frac{36}{5} u_y^2 y$$

$$
\begin{aligned}
&+\frac{82}{5}x^2u_x+\frac{512}{15}x^3u_y+\frac{32}{5}xu_y^3+\frac{96}{5}x^2u_y^2+\frac{32}{5}y^2+\frac{512}{15}x^4+\frac{3}{5}u_x^2+u_y^4,\\
\psi_6 &= \frac{49}{4}xy+4xu_x+\frac{3}{2}u_yu_x+\frac{9}{2}u_yy+\frac{49}{4}x^2u_y+\frac{21}{4}xu_y^2+\frac{343}{24}x^3+\frac{1}{4}u+u_y^3,\\
\psi_5 &= 4xu_y+6x^2+2y+\frac{2}{3}u_x+u_y^2,\\
\psi_4 &= \frac{5}{2}x+u_y,\\
\psi_3 &= 1,\\
\psi_{-1} &= \frac{1}{(-xu_y+u_x+2y)^2}.
\end{aligned}
$$

3. Hierarchies of nonlocal conservation laws

Using the nonlinear coverings presented in Section 1 we construct here infinite hierarchies of nonlocal conservation laws for Equations (0.1)–(0.1).

3.1. *A general construction*

The initial step of the construction is the so-called *Pavlov reversing*, [21] (see [6] for the invariant geometrical interpretation). Let $\mathscr{E}$ be an equation in two independent variables x and y and unknown function u and

$$w_x = X(x,y,[u],w), \qquad w_y = Y(x,y,[u],w)$$

be a differential covering over $\mathscr{E}$, where $[u]$ denotes u itself and a collection of its derivatives up to some finite order. Then the system

$$\psi_x = -X(x,y,[u],\lambda)\psi_\lambda, \qquad \psi_y = -Y(x,y,[u],\lambda)\psi_\lambda \tag{3.1}$$

is also compatible modulo $\mathscr{E}$ (thus, the nonlocal variable w turns into a formal parameter in the new setting).

Assume now that

$$
\begin{aligned}
X &= X_{-1}\lambda+X_0+\frac{X_1}{\lambda}+\dots+\frac{X_i}{\lambda^i}+\dots,\\
Y &= Y_{-1}\lambda+Y_0+\frac{Y_1}{\lambda}+\dots+\frac{Y_i}{\lambda^i}+\dots,
\end{aligned}
$$

where X_i, Y_i, $i\geq -1$, are functions in x, y and $[u]$, and also expand ψ in formal Laurent series

$$\psi = \psi_{-1}\lambda+\psi_0+\frac{\psi_1}{\lambda}+\dots+\frac{\psi_i}{\lambda^i}+\dots$$

Then (3.1) implies

$$\psi_{i,x} = -\sum_{j+k=i+1} kX_j\psi_k, \qquad \psi_{i,y} = -\sum_{j+k=i+1} kY_j\psi_k,$$

or

$$\psi_{-1,x} = -X_{-1}\psi_{-1}, \qquad \psi_{-1,y} = -Y_{-1}\psi_{-1};$$

$$\begin{aligned}
&\psi_{0,x} = -X_0\psi_{-1}, && \psi_{0,y} = -Y_0\psi_{-1};\\
&\psi_{1,x} = X_{-1} - X_1\psi_{-1}, && \psi_{1,y} = Y_{-1} - Y_1\psi_{-1};\\
&\psi_{2,x} = 2X_{-1}\psi_2 + X_0\psi_1 - X_2\psi_{-1}, && \psi_{2,y} = 2Y_{-1}\psi_2 + Y_0\psi_1 - Y_2\psi_{-1};\\
&\dots && \dots
\end{aligned}$$

and

$$\begin{aligned}
\psi_{k,x} &= kX_{-1}\psi_k + (k-1)X_0\psi_{i-1} + \dots + X_{k-2}\psi_1 - X_k\psi_{-1},\\
\psi_{k,y} &= kY_{-1}\psi_k + (k-1)Y_0\psi_{i-1} + \dots + Y_{k-2}\psi_1 - Y_k\psi_{-1}
\end{aligned}$$

for all $k > 2$.

In general, this system defines an infinite-dimensional non-Abelian covering (which may be trivial generally) over the base equation $\mathscr{E}$, but in the particular case $X_{-1} = Y_{-1} = 0$ the covering becomes Abelian, i.e., transforms to an infinite series of (nonlocal) conservation laws. Indeed, the first pair of equations reads

$$\psi_{-1,x} = 0, \qquad \psi_{-1,y} = 0$$

in this case and without loss of generality we may set $\psi_{-1} = 1$. The rest equations read

$$\begin{aligned}
&\psi_{0,x} = -X_0, && \psi_{0,y} = -Y_0;\\
&\psi_{1,x} = -X_1, && \psi_{1,y} = -Y_1;\\
&\psi_{2,x} = X_0\psi_1 - X_2, && \psi_{2,y} = Y_0\psi_1 - Y_2;\\
&\psi_{3,x} = 2X_0\psi_2 + X_1\psi_1 - X_3, && \psi_{3,x} = 2Y_0\psi_2 + Y_1\psi_1 - Y_3;\\
&\dots && \dots
\end{aligned}$$

and

$$\begin{aligned}
\psi_{k,x} &= (k-1)X_0\psi_{k-1} + (k-2)X_1\psi_{k-2} + \dots + X_{k-2}\psi_1 - X_k,\\
\psi_{k,y} &= (k-1)Y_0\psi_{k-1} + (k-2)Y_1\psi_{k-2} + \dots + Y_{k-2}\psi_1 - Y_k
\end{aligned} \tag{3.2}$$

for all $k > 3$.

Remark 3.1. The first two pairs of equations define local conservation laws (probably, trivial) and the potential ψ_0 does not enter the other equations. This means that the obtained covering is the Whitney product of the one-dimensional Abelian covering τ_0 associated to ψ_0 and the infinite-dimensional τ_* related to ψ_1, $\psi_2, \dots$ We shall deal with τ_* below. □

We now confine ourselves to the case

$$X = \frac{a_2w^2 + a_1w + a_0}{w^2 + c_1w + c_0}, \qquad Y = \frac{b_2w^2 + b_1w + b_0}{w^2 + c_1w + c_0}, \tag{3.3}$$

where a_i, b_i, and c_i are functions in x, y, and $[u]$, and deduce the needed Laurent expansions. One has

$$\frac{a_2\lambda^2 + a_1\lambda + a_0}{\lambda^2 + c_1\lambda + c_0} = \left(a_2 + \frac{a_1}{\lambda} + \frac{a_0}{\lambda^2}\right) \cdot \left(\frac{1}{1 + \frac{c_1\lambda + c_0}{\lambda^2}}\right)$$

$$= \left(a_2 + \frac{a_1}{\lambda} + \frac{a_0}{\lambda^2}\right) \cdot \sum_{i \geq 0} \left(-\frac{c_1\lambda + c_0}{\lambda^2}\right)^i.$$

Let us present temporally the second factor in the form

$$\sum_{i \geq 0} \left(-\frac{c_1\lambda + c_0}{\lambda^2}\right)^i = \sum_{i \geq 0} \frac{d_i}{\lambda^i}.$$

Then

$$\frac{a_2\lambda^2 + a_1\lambda + a_0}{\lambda^2 + c_1\lambda + c_0} = \left(a_2 + \frac{a_1}{\lambda} + \frac{a_0}{\lambda^2}\right) \cdot \sum_{i \geq 0} \frac{d_i}{\lambda^i}$$

$$= a_2 d_0 + \frac{a_2 d_1 + a_1 d_0}{\lambda} + \frac{a_2 d_2 + a_1 d_1 + a_0 d_0}{\lambda^2} + \dots + \frac{a_2 d_i + a_1 d_{i-1} + a_0 d_{i-2}}{\lambda^i} + \dots$$

Compute the coefficients d_i now. One has

$$\left(-\frac{c_1\lambda + c_0}{\lambda^2}\right)^i = (-1)^i \sum_{j=0}^{i} \binom{i}{j} \frac{c_1^j c_0^{i-j}}{\lambda^{2i-j}},$$

from where it follows that

$$d_0 = 1, \qquad d_1 = -c_1$$

and

$$d_i = \begin{cases} \sum_{j=0}^{k} (-1)^{k-j} \binom{k+j}{2j} c_0^{k-j} c_1^{2j} & \text{if } i = 2k, \\ \sum_{j=0}^{k} (-1)^{k-j+1} \binom{k+j+1}{2j+1} c_0^{k-j} c_1^{2j+1} & \text{if } i = 2k+1 \end{cases} \tag{3.4}$$

for $i > 1$, Or, in shorter notation

$$d_i = \sum_{j=0}^{[i/2]} (-1)^{[i/2]-j+p(i)} \binom{[i/2]+j+p(i)}{2j+p(i)} c_0^{[i/2]-j} c_1^{2j+p(i)}, \tag{3.5}$$

where $p(i) = i \bmod 2$ is the parity of i and $[k/2]$ is the integer part.

Gathering together the results of the above computations, one obtains that in the case of coverings (3.3) we have $X_{-1} = Y_{-1} = 0$, while other coefficients are

$$\begin{aligned} &X_0 = a_2, && Y_0 = b_2; \\ &X_1 = a_1 - a_2 c_1, && Y_1 = b_1 - b_2 c_1; \\ &X_2 = a_0 - a_1 c_1 + a_2(c_1^2 - c_0), && Y_2 = b_0 - b_1 c_1 + b_2(c_1^2 - c_0); \\ &\dots && \dots \\ &X_i = a_0 d_{i-2} + a_1 d_{i-1} + a_2 d_i, && Y_i = b_0 d_{i-2} + b_1 d_{i-1} + b_2 d_i; \\ &\dots && \dots, \end{aligned}$$

where the functions d_i are given by (3.4).

Let us now show how these general constructions look like in the particular cases of the equations under consideration.

3.2. *Equation* (0.1)

Note first that the covering (1.5) is not of the form (3.3). Nevertheless, it can be transformed to the needed form by the gauge transformation $w \mapsto we^{-x}$. Then (1.5) acquires the form

$$w_x = \frac{(u_x+u)e^x w^2 - u_y e^{2x} w}{w^2 - (u_x+u)e^x w - u_y e^{2x}}, \qquad w_y = -\frac{u_y e^x w^2}{w^2 - (u_x+u)e^x w - u_y e^{2x}}.$$

We have $|w| = -1$.

Thus,

$$\begin{aligned} &a_0 = 0, && a_1 = -u_y e^{2x} && a_2 = (u_x+u)e^x, \\ &b_0 = 0, && b_1 = 0, && b_2 = -u_y e^x, \\ &c_0 = -u_y e^{2x}, && c_1 = -(u_x+u)e^x. && \end{aligned}$$

Let us compute the coefficients d_i. By (3.4), we have

$$d_{2k} = \sum_{j=0}^{k} (-1)^{k-j} \binom{k+j}{2j} \left(-u_y e^{2x}\right)^{k-j} \left(-(u_x+u)e^x\right)^{2j}$$
$$= e^{2kx} \sum_{j=0}^{k} \binom{k+j}{2j} u_y^{k-j} (u_x+u)^{2j}$$

and

$$d_{2k+1} = \sum_{j=0}^{k} (-1)^{k-j+1} \binom{k+j+1}{2j+1} \left(-u_y e^{2x}\right)^{k-j} \left(-(u_x+u)e^x\right)^{2j+1}$$
$$= e^{(2k+1)x} \sum_{j=0}^{k} \binom{k+j+1}{2j+1} u_y^{k-j} (u_x+u)^{2j+1},$$

or

$$d_i = e^{ix} \sum_{j=0}^{[i/2]} \binom{[i/2]+j+p(i)}{2j+p(i)} u_y^{[i/2]-j} (u_x+u)^{2j+p(i)}. \tag{3.6}$$

Hence,

$$\begin{aligned} &X_0 = (u_x+u)e^x, && Y_0 = -u_y e^x; \\ &X_1 = \left((u_x+u)^2 - u_y\right) e^{2x}, && Y_1 = (u_x+u)u_y e^{2x} \end{aligned}$$

and

$$X_i = e^{(i+1)x} \left((u_x+u)^{i+1} + \sum_{j=1}^{[(i+1)/2]} \left(\binom{i-j}{i-2j} - \binom{i-j}{i-2j+1} \right) u_y^j (u_x+u)^{i-2j+1} \right),$$
$$Y_i = -e^{(i+1)x} \sum_{j=0}^{[i/2]} \binom{[i/2]+j+p(i)}{2j+p(i)} u_y^{[i/2]-j+1} (u_x+u)^{2j+p(i)}$$

for $i > 1$ (we assume $\binom{\alpha}{\beta} = 0$ for $\beta < 0$). Obviously,

$$|X_i| = -i-1, \qquad |Y_i| = -i-2.$$

The functions X_i, Y_i define, by Equations (3.2), the infinite number of nonlocal variables ψ_i for Equation (0.1) with

$$|\psi_i| = -i-1.$$

The corresponding conservation laws have the same weights and the first three of them coincide (up to equivalence) with the local conservation laws ω_{-2}, ω_{-3}, ω_{-4} described in Section 2.1. The first essentially nonlocal one is associated to ψ_3.

3.3. *Equation* (0.2)

Due to Equations (1.11), one has

$$\begin{aligned} &a_0 = 0, && a_1 = 0 && a_2 = -1, \\ &b_0 = 0, && b_1 = u_y, && b_2 = 0, \\ &c_0 = u_y, && c_1 = u_x - x. && \end{aligned}$$

Hence,

$$\begin{aligned} &X_0 = -1, && Y_0 = 0; \\ &X_1 = u_x - x, && Y_1 = u_y; \\ &X_2 = -(u_x-x)^2 + u_y, && Y_2 = -u_y(u_x - x) \end{aligned}$$

and

$$\begin{aligned} X_i = -d_i &= \sum_{j=0}^{[i/2]} (-1)^{[i/2]-j+p(i)+1} \binom{[i/2]+j+p(i)}{2j+p(i)} u_y^{[i/2]-j} (u_x - x)^{2j+p(i)}, \\ Y_i = u_y d_{i-1} &= \sum_{j=0}^{[(i-1)/2]} (-1)^{[(i-1)/2]-j+p(i-1)} \times \\ &\times \binom{[(i-1)/2]+j+p(i-1)}{2j+p(i-1)} u_y^{[(i-1)/2]-j+1} (u_x - x)^{2j+p(i-1)} \end{aligned}$$

for $i > 2$. Consequently,

$$\begin{aligned} &\psi_{0,x} = -X_0 = 1, && \psi_{0,y} = -Y_0 = 0; \\ &\psi_{1,x} = -X_1 = -u_x + x, && \psi_{1,y} = -Y_1 = -u_y \end{aligned}$$

and one may set

$$\psi_0 = x, \qquad \psi_1 = -u + \frac{x^2}{2},$$

while

$$\psi_{2,x} = (u_x - x)^2 + u_y + u - \frac{x^2}{2}, \quad \psi_{2,y} = (u_x - x)u_y$$

and for $i > 2$

$$\psi_{i,x} = -(i-1)\psi_{i-1} + (i-2)X_1\psi_{i-2} + \cdots + X_{i-3}\psi_2 + \left(\frac{x^2}{2} - u\right)X_{i-2} - X_i,$$
$$\psi_{i,y} = (i-2)Y_1\psi_{i-2} + \cdots + Y_{i-3}\psi_2 + \left(\frac{x^2}{2} - u\right)Y_{i-2} - Y_i,$$

where X_k, Y_k are given by the above formulas.

One has

$$|X_i| = i, \quad |Y_i| = i+1, \quad |\psi_i| = i+1.$$

The conservation law corresponding to ψ_i is of the weight $i+1$ and the first two ones, up to equivalence coincide with those described in Section 2.2, while all the others are essentially nonlocal.

3.4. *Equation* (0.3)

By Equation (1.15), we have

$$\begin{aligned} a_0 &= 0, & a_1 &= u_y & a_2 &= -1, \\ b_0 &= 0, & b_1 &= -1, & b_2 &= 0, \\ c_0 &= xu_y - u_x - 2y, & c_1 &= -(u_y + x). \end{aligned}$$

Consequently,

$$\begin{aligned} X_0 &= -1, & Y_0 &= 0; \\ X_1 &= -x, & Y_1 &= -1; \\ X_2 &= -u_x - x^2 - 2y, & Y_2 &= -u_y - x \end{aligned}$$

and

$$X_i = u_y d_{i-1} - d_i, \qquad Y_i = -d_{i-1}$$

for $i > 2$, where

$$d_i = \sum_{j=0}^{[i/2]} (-1)^{[i/2]-j} \binom{[i/2] + j + p(i)}{2j + p(i)} (xu_y - u_x - 2y)^{[i/2]-j} (u_y + x)^{2j+p(i)}.$$

One has

$$|X_i| = i, \qquad |Y_i| = i-1.$$

Thus we have

$$\psi_{1,x} = x, \qquad \psi_{1,y} = 1;$$

$$\psi_{2,x} = u_x + \frac{x^2}{2} + y, \qquad\qquad \psi_{2,y} = u_y + x$$

and we may set

$$\psi_1 = \frac{x^2}{2} + y, \qquad \psi_2 = u + xy + \frac{x^3}{6}.$$

Then the other potentials are defined by

$$\begin{aligned}\psi_{i,x} &= -(i-1)\psi_{i-1} - (i-2)\psi_{i-2}(i-3)X_2\psi_{i-3} + \dots\\ &\quad \dots + 3X_{i-4}\psi_3 + \left(2u + 2xy + \frac{x^3}{3}\right)X_{i-3} + \left(\frac{x^2}{2} + y\right)X_{i-2} - X_i,\\ \psi_{i,y} &= -(i-2)\psi_{i-2}(i-3)Y_2\psi_{i-3} + \dots\\ &\quad \dots + 3Y_{i-4}\psi_3 + \left(2u + 2xy + \frac{x^3}{3}\right)Y_{i-3} + \left(\frac{x^2}{2} + y\right)Y_{i-2} - Y_i,\end{aligned}$$

$i > 2$. We have

$$|\psi_i| = i + 1.$$

The conservation laws associated with $\psi_3, \dots, \psi_7$ are equivalent to $\omega_4, \dots, \omega_8$ introduced in Section 2.3. The first essentially nonlocal conservation law corresponds to ψ_8.

3.5. *Proof of nontriviality*

We shall now prove that the above constructed conservation laws are nontrivial. To this end, introduce the notation $\mathscr{E}_\alpha$, $\alpha = 1, 2, 3$, for Equations (0.1), (0.2) and (0.3), respectively, and

$$\tau_{i,\alpha} \colon \mathscr{E}_{i,\alpha} \to \mathscr{E}_\alpha$$

for the coverings defined by the nonlocal variables $\psi_\alpha, \dots, \psi_i$. Let

$$D_x^{i,\alpha}, \qquad D_y^{i,\alpha}$$

be the total derivatives on $\mathscr{E}_{i,\alpha}$.

Proposition 3.1. *For all $i \geq \alpha$, the only solutions of the system*

$$D_x^{i,\alpha}(f) = 0, \qquad D_y^{i,\alpha}(f) = 0 \tag{3.7}$$

are constants.

Proof. Let us present the total derivatives in the form

$$D_x^{i,\alpha} = D_x^\alpha + X^{i,\alpha}, \quad D_y^{i,\alpha} = D_y^\alpha + Y^{i,\alpha},$$

where D_x^α, D_y^α are the total derivatives on $\mathscr{E}_\alpha$ and $X^{i,\alpha}$, $Y^{i,\alpha}$ are the 'nonlocal tails':

$$X^{i,\alpha} = \sum_{j=\alpha}^{i} X_j^{i,\alpha}\frac{\partial}{\partial\psi_j}, \qquad Y^{i,\alpha} = \sum_{j=\alpha}^{i} Y_j^{i,\alpha}\frac{\partial}{\partial\psi_j},$$

$X_j^{i,\alpha}$, $Y_j^{i,\alpha}$ being the right-hand sides of the defining equations (3.2) for the potentials ψ.

From the constructions of Sections 3.2–3.4 one readily sees that the quantities $X_j^{i,\alpha}$ and $Y_j^{i,\alpha}$ are polynomials in u_x and u_y and, moreover,

$$
\begin{aligned}
X^{i,1} &= \pm e^{(i+1)x} u_x^{i+1} \frac{\partial}{\partial \psi_i} + o, & Y^{i,1} &= \pm e^{(i+1)x} u_x^{i} u_y \frac{\partial}{\partial \psi_i} + o;\\
X^{i,2} &= \pm u_x^{i} \frac{\partial}{\partial \psi_i} + o; & Y^{i,2} &= \pm u_x^{i-1} \frac{\partial}{\partial \psi_i} + o;\\
X^{i,3} &= \pm u_y^{i-2} u_x \frac{\partial}{\partial \psi_i} + o, & Y^{i,3} &= \pm u_y^{i-1} \frac{\partial}{\partial \psi_i} + o,
\end{aligned}
$$

where o denotes terms of lower degree.

Now, the proof goes by induction. For small i's the result follows from the fact that the cosymmetries corresponding to the local conservation laws do not vanish and these conservation laws are of different weights. Assume now that the statement is valid for all $k < i$ and consider Equation (3.7). Then from the above estimates it follows that $\partial f / \partial \psi_i = 0$. □

Evidently, nontriviality of the constructed conservation laws is a direct consequence of the Proposition 3.1.

4. On reductions of the recursion operators

We show here that symmetry reductions of Equations (1.1), (1.8), and (1.12) are incompatible with their recursion operators and thus the latter are not inherited by Equations (0.1), (0.2), and (0.3), respectively.

4.1. *A general construction*

We treat here recursion operators for symmetries as Bäcklund transformations of the tangent coverings, cf. [12]. More precisely, let $\mathscr{E}$ be a differential equation given by the system

$$\mathscr{E} = \{F = 0\}, \quad F = (F^1(x,y,[u]), \ldots, F^s(x,y,[u])),$$

F^j being functions on some jet space, [7]. Here, as above, $[u]$ denotes the collection of u and its derivatives. The tangent covering $\mathrm{t} = \mathrm{t}_{\mathscr{E}} \colon \mathscr{T}\mathscr{E} \to \mathscr{E}$ is the projection $(x,y,[u],[q]) \mapsto (x,y,[u])$ of the system

$$\mathscr{T}\mathscr{E} = \{F(x,y,[u]) = 0,\ \ell_F(x,y,[u],[q]) = 0\}$$

to $\mathscr{E}$. The characteristic property of t is that its sections that preserve the Cartan (higher contact) distribution are identified with symmetries of $\mathscr{E}$.

A Bäcklund transformation between equations $\mathscr{E}_1$ and $\mathscr{E}_2$ is a diagram

$$
\begin{array}{ccc}
 & \mathscr{B} & \\
 {\scriptstyle\tau_1}\swarrow & & \searrow{\scriptstyle\tau_2} \\
\mathscr{E}_1 & & \mathscr{E}_2,
\end{array}
$$

where τ_1 and τ_2 are coverings. It relates solutions of $\mathcal{E}_1$ and $\mathcal{E}_2$ to each other. A recursion operator between symmetries of $\mathcal{E}_1$ and $\mathcal{E}_2$ is a Bäcklund transformation of the form

$$\begin{array}{ccccc} & & \mathcal{T}\mathcal{E}_1 & \xrightarrow{\mathrm{t}_{\mathcal{E}_1}} & \mathcal{E}_1 \\ & \overset{\tau_1}{\nearrow} & & & \\ \mathcal{R} & & & & \\ & \underset{\tau_2}{\searrow} & & & \\ & & \mathcal{T}\mathcal{E}_2 & \xrightarrow{\mathrm{t}_{\mathcal{E}_2}} & \mathcal{E}_2. \end{array}$$

In particular, if $\mathcal{E}_1 = \mathcal{E}_2 = \mathcal{E}$ it relates symmetries of $\mathcal{E}$ to each other. Then $\mathcal{R}$ may be considered as an equation

$$\mathcal{R} \subset \mathcal{T}\mathcal{E} \otimes_{\mathcal{E}} \mathcal{T}\mathcal{E}$$

in the Whitney product of $\mathrm{t}_{\mathcal{E}}$ with itself.

Any symmetry $\varphi = \varphi(x, y, [u])$ of $\mathcal{E}$ admits a natural lift $\Phi = (\varphi, \varphi')$ to $\mathcal{T}\mathcal{E}$. To this end, it suffices to set

$$\varphi' = \frac{\partial \varphi}{\partial u} q + \dots + \frac{\partial \varphi}{\partial u_\sigma} q_\sigma + \dots$$

Choose a symmetry φ of $\mathcal{E}$ and denote by $\mathrm{r}_\varphi \colon \mathcal{E} \to \mathcal{E}_\varphi$ the corresponding reduction map. Then the diagram

$$\begin{array}{ccc} \mathcal{T}\mathcal{E} & \xrightarrow{\mathrm{t}_{\mathcal{E}}} & \mathcal{E} \\ {\scriptstyle \mathrm{r}_\Phi}\Big\downarrow & & \Big\downarrow{\scriptstyle \mathrm{r}_\varphi} \\ (\mathcal{T}\mathcal{E})_\Phi = \mathcal{T}(\mathcal{E}_\varphi) & \xrightarrow{\mathrm{t}_{\mathcal{E}_\varphi}} & \mathcal{E}_\varphi \end{array}$$

is commutative. An immediate consequence of this fact is

Proposition 4.1. *Let $\mathcal{R} \subset \mathcal{T}\mathcal{E} \otimes_{\mathcal{E}} \mathcal{T}\mathcal{E}$ be a recursion operator for symmetries of equation $\mathcal{E}$ and φ be a symmetry of $\mathcal{E}$. If $\mathcal{R}$ is invariant with respect to φ then $\mathcal{R}_\Phi$ is a recursion operator for symmetries of $\mathcal{E}_\varphi$.*

4.2. *Recursion operators for symmetries of 3D systems*

We briefly recall here the results on recursion operators for symmetries of Equation (1.1), (1.8), and (1.12) obtained in [15, 16]

The universal hierarchy equation

Equation (1.1) admits the following recursion operator

$$\begin{aligned} D_y(\tilde{\varphi}) &= u_y D_x(\varphi) - u_{xy}\,\varphi, \\ D_z(\tilde{\varphi}) &= u_z D_x(\varphi) - D_y(\varphi) - u_{xz}\,\varphi \end{aligned} \tag{4.1}$$

that acts on its symmetries.

The 3DrdDym equation

The Bäcklund transformation

$$D_x(\tilde{\varphi}) = u_x D_x(\varphi) - D_t(\varphi) - u_{xx}\,\varphi,$$
$$D_y(\tilde{\varphi}) = u_y D_x(\varphi) - u_{xy}\,\varphi \tag{4.2}$$

is a recursion operator for symmetries of Equation (1.8).

The Pavlov equation

The relations

$$D_x(\tilde{\varphi}) = u_x D_x(\varphi) + D_y(\varphi) - u_{xx}\,\varphi,$$
$$D_y(\tilde{\varphi}) = D_t(\varphi) + u_y D_x(\varphi) - u_{xy}\,\varphi. \tag{4.3}$$

are a recursion operator for symmetries of Equation (1.12).

4.3. *The negative result*

Here we show that the general construction of Section 4.1 produces no recursion operator for the reduced equations under consideration.

Proposition 4.2. *Recursion operators* (4.1), (4.2) *and* (4.3) *are not invariant with respect to the natural lifts of the symmetries* (1.2), (1.9), *and* (1.13), *respectively.*

Proof. By direct check. □

Remark 4.1. The same fact holds for the reduction of the Pavlov equation that leads to the Gibbons-Tsarev equation.

Remark 4.2. We also tried to construct recursion operators for all the equations at hand directly, but this did not lead us to positive results either.

5. Discussion

Let us first establish the following fact:

Proposition 5.1. *Equations* (0.1), (0.2), *and* (0.3) *are pair-wise inequivalent with respect to an arbitrary contact transformation.*

Proof. Let us first compare dimensions (see Table 1). Consequently, only Equations (0.1) and (0.3)

	$\dim \mathrm{sym}_{\mathrm{c}}(\mathscr{E})$	$\dim \mathrm{cosym}_{\mathrm{c}}(\mathscr{E})$
Equation (0.1)	4	6
Equation (0.2)	4	4
Equation (0.3)	4	6

Table 1. Dimensions of symmetry and cosymmetry spaces

may be equivalent. Now, the Lie algebra structure of $\mathrm{sym}_{\mathrm{c}}(\mathscr{E})$ for Equations (0.1) and (0.3) is

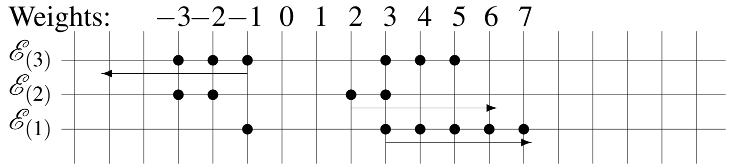

Fig. 1. Distribution of cosymmetries

presented in Table 2. One can see that dimension of the commutant in the first case is 2, while in the second case it equals 3. Thus, the algebras are not isomorphic. □

Remark 5.1. The equations under consideration are not equivalent to the Gibbons-Tsarev equation, because the symmetry algebra of the latter is five-dimensional.

Nevertheless, as we saw, all these equations have several common features. In particular, we would like to indicate how local cosymmetries of our equations are distributed with respect to weights (see Figure 1). In all three cases, they fit into two disjoint groups with certain gaps between them: the first one consist of cosymmetries whose corresponding conservation laws are members of infinite series (these are underlined by arrows, and the arrow itself indicates the direction to which the sequence of conservation laws goes). The second group includes 'standing-alone' cosymmetries.

Remark 5.2. A similar picture is observed in the case of the Gibbons-Tsarev equation. It also possesses a 'standing-alone' cosymmetry of order three.

A natural question arises: does there exist a construction, similar to the one of Section 3, that allows to embed the conservation laws corresponding to the 'standing-alone' cosymmetries into other infinite hierarchies?

Another question relates to the algebras of nonlocal symmetries in the infinite-dimensional coverings constructed above. It seems that such an algebra for Equation (0.3) should be similar (or isomorphic to that of the Gibbons-Tsarev equation), while the algebras for Equations (0.1) and (0.2) are different: all these Lie algebras are graded, but in the first two cases all homogeneous components are one-dimensional and for other equations this is not the case.

Finally, it is interesting to study the structure of symmetries and cosymmetries of the reductions that admit symmetry algebras with functional parameters (see the Introduction) and compare them with the results described here.

All these problems are subject to future research.

6. Appendix: Conservation laws

We present here the conservation laws that correspond to the cosymmetries described above. Everywhere below $|\omega_i| = i$. We also use the notation $\psi_\omega \in \mathrm{cosym_c}(\mathscr{E})$ for the generating function of a conservation law ω.

Eq. (0.1)	φ_0	φ_0'	φ_1
φ_{-1}	φ_{-1}	0	0
φ_0	$*$	0	φ_1
φ_0'	$*$	$*$	$-\varphi_1$

Eq. (0.3)	φ_{-2}	φ_{-1}	φ_0
φ_{-3}	0	0	$-\varphi_{-3}$
φ_{-2}	$*$	$-\varphi_{-3}$	$\frac{2}{3}\varphi_{-2}$
φ_{-1}	$*$	$*$	$-\frac{1}{3}\varphi_{-1}$

Table 2. Commutators in $\mathrm{sym_c}\,\mathscr{E}_{(0.1)}$ and $\mathrm{sym_c}\,\mathscr{E}_{(0.3)}$

Equation (0.1)

The space of corresponding conservation laws is 6-dimensional and is spanned by the following elements $\omega_i = P_i\,dx + Q_i\,dy$:

$$
\begin{aligned}
P_{-4} &= e^{4x}(u_x^2u_y + 8u^3u_x + 13u^2u_x^2 + 2uu_x^3 + 8u^2u_y + u_y^2 - 3uu_x^2u_{xx} + 2uu_xu_y \\
&\quad - 2uu_xu_{xy} - 2uu_{xx}u_y),\\
Q_{-4} &= ue^{4x}(-2u_xu_yu_{xx} + 3u_x^2u_y - u_x^2u_{xy} + 8uu_xu_y + 2uu_xu_{xy} + 4u_y^2 \\
&\quad - 2u_yu_{xy});\\
P_{-3} &= e^{3x}(-uu_{xy} + u_xu_y + 3u^2u_x + uu_x^2 - 2uu_xu_{xx}),\\
Q_{-3} &= ue^{3x}(-u_yu_{xx} - u_xu_{xy} + uu_{xy} + 2u_xu_y);\\
P_{-2} &= -e^{2x}(-u_y + uu_x + uu_{xx}),\\
Q_{-2} &= -ue^{2x}u_{xy};\\
P_2 &= -\frac{1}{u_y},\\
Q_2 &= \frac{1}{u_y}(u_x + u);\\
P_3 &= \frac{1}{u_y^3}(u_y^2y + 2uu_{xy} - uu_y - 2u_xu_y),\\
Q_3 &= -\frac{1}{u_y^3}(uu_y^2y + u_xu_y^2y + 2u^2u_{xy} - u^2u_y + 2uu_xu_{xy} - 4uu_xu_y - 2uu_{xx}u_y \\
&\quad - u_x^2u_y);\\
P_4 &= \frac{1}{u_y^4}(-u_y^3y^2 - 4uu_{xy}u_yy + 2uu_y^2y + 4u_xu_y^2y - u^2u_y + 6uu_xu_{xy} \\
&\quad - 2uu_xu_y - 2uu_{xx}u_y - 3u_x^2u_y - u_y^2),\\
Q_4 &= \frac{1}{u_y^4}(uu_y^3y^2 + u_xu_y^3y^2 + 4u^2u_{xy}u_yy - 2u^2u_y^2y + 4uu_xu_{xy}u_yy \\
&\quad - 8uu_xu_y^2y - 4uu_{xx}u_y^2y - 2u_x^2u_y^2y + u^3u_y - 6u^2u_xu_{xy} + 3u^2u_xu_y \\
&\quad - 6uu_x^2u_{xy} + 9uu_x^2u_y + 6uu_xu_{xx}u_y + u_x^3u_y - 2uu_{xy}u_y + 4uu_y^2).
\end{aligned}
$$

Here $|\psi_\omega| = |\omega| + 1$.

Equation (0.2)

The space of conservation laws is 4-dimensional and is generated by $\omega_i = P_i\,dx + Q_i\,dy$ of the form

$$
\begin{aligned}
P_{-2} &= \frac{1}{2}(2uu_{xy} - 2u_xu_y - u_yx)\frac{e^{-2y}}{u_y^3},\\
Q_{-2} &= \frac{1}{2}(2uu_xu_{xy} - 2uu_{xx}u_y + 2uu_{xy}x - u_x^2u_y - 2u_xu_yx - u_yx^2 - 2uu_y)\frac{e^{-2y}}{u_y^3};
\end{aligned}
$$

$$P_{-1} = -\frac{e^{-y}}{u_y},$$
$$Q_{-1} = -(u_x + x)\frac{e^{-y}}{u_y};$$
$$P_3 = uu_{xx} + 3u + u_y,$$
$$Q_3 = uu_{xy} + u_y x;$$
$$P_4 = -\frac{1}{2}uu_{xy} + 2u_y x + \frac{1}{2}u_x u_y + \frac{5}{2}uxu_{xx} + uu_x u_{xx} + 8ux + \frac{1}{2}uu_x,$$
$$Q_4 = 2u_y x^2 + \frac{1}{2}u_x u_y x + 2uu_{xy}x + \frac{1}{2}uu_x u_{xy} + \frac{1}{2}uu_{xx}u_y + uu_y.$$

Again, $|\psi_\omega| = |\omega| - 1$.

Equation (0.3)

The space of conservation laws is 6-dimensional; elements $\omega_i = P_i\,dx + Q_i\,dy$ of a basis are

$$\begin{aligned}
P_8 = {} & u_y^3 u_x u_{yy} u + \frac{1}{5}uxu_y^3 u_{xy} + \frac{116}{5}ux^2 u_x u_{xy} + \frac{162}{5}uxu_x u_y + \frac{229}{15}ux^3 u_y u_{xy} \\
& + \frac{8}{5}ux^2 u_y^2 u_{xy} + \frac{3}{5}u_y^2 u_x u_{xy} u + \frac{379}{15}u_x u_{yy} ux^3 + \frac{758}{15}u_{yy} ux^3 y + 2u_y^3 u_{yy} uy \\
& + \frac{184}{5}u_{yy} uxy^2 + \frac{348}{5}ux^2 yu_{xy} - \frac{48}{5}xyu_x u_y^2 + \frac{6}{5}uyu_y^2 u_{xy} + \frac{72}{5}u_y u_{yy} uy^2 \\
& + \frac{12}{5}u_y u_x^2 u_{yy} u + 80uxyu_y + \frac{36}{5}uyu_x u_{xy} - \frac{164}{5}x^2 yu_x u_y - \frac{6}{5}yu_x u_y^3 - \frac{8}{5}x^2 u_x u_y^3 \\
& - \frac{1024}{15}x^4 yu_y + 43ux^3 u_y + \frac{48}{5}uyu_y^2 + \frac{18}{5}uu_x u_y^2 - \frac{164}{5}xy^2 u_y^2 \\
& - \frac{1}{5}xu_x u_y^4 + \frac{52}{5}uy^2 u_{xy} + \frac{14}{5}x^2 u_x^2 u_y - \frac{64}{5}x^2 yu_y^3 + \frac{2048}{5}ux^2 y + 2yu_x^2 u_y + \frac{16}{5}uxu_y^3 \\
& + \frac{82}{5}uyu_x + \frac{32}{5}u_x u_y^2 u_{yy} ux + \frac{64}{5}u_y^2 u_{yy} uxy + 24uxyu_y u_{xy} + \frac{132}{5}u_x u_{yy} uxy \\
& + 12u_x u_y u_{yy} uy + \frac{96}{5}u_x u_y u_{yy} ux^2 + \frac{192}{5}u_y u_{yy} ux^2 y + \frac{56}{5}uxu_x u_y u_{xy} + \frac{1}{5}u_x^2 u_y^3 \\
& + \frac{3}{5}u_x^3 u_y + \frac{256}{5}uy^2 + \frac{4096}{15}ux^4 - \frac{241}{5}u^2 x + \frac{2}{5}uu_y^4 - \frac{24}{5}y^2 u_y^3 - \frac{64}{5}y^3 u_y - \frac{2}{5}yu_y^5 \\
& + \frac{8}{5}uu_x^2 - \frac{512}{5}x^2 y^2 u_y + \frac{64}{5}ux^2 u_y^2 + \frac{113}{5}ux^2 u_x - \frac{512}{15}x^3 yu_y^2 - \frac{16}{5}xyu_y^4 - 4y^2 u_x u_y \\
& - \frac{32}{5}x^3 u_x u_y^2 + \frac{6}{5}u_x^2 u_{xy} u - \frac{256}{15}x^4 u_x u_y + \frac{127}{3}ux^4 u_{xy} + xu_x^2 u_y^2,
\end{aligned}$$

$$\begin{aligned}
Q_8 = {} & \frac{36}{5}uyu_x u_{yy} - \frac{72}{5}xyu_x u_y + \frac{42}{5}uyu_y^2 u_{yy} + \frac{92}{5}uxyu_{xy} + \frac{32}{5}uxu_y^2 u_{xy} + 4uxu_x u_{xy} \\
& + \frac{256}{5}ux^2 yu_{yy} + \frac{12}{5}u_x u_{xy} u_y u + \frac{64}{3}ux^3 u_y u_{yy} + \frac{72}{5}ux^2 u_y^2 u_{yy} + \frac{36}{5}uyu_y u_{xy} \\
& + \frac{28}{5}uxu_y^3 u_{yy} + \frac{96}{5}ux^2 u_x u_{yy} + \frac{96}{5}ux^2 u_y u_{xy} + 3u_y^2 u_x u_{yy} u + \frac{52}{5}uy^2 u_{yy} \\
& + \frac{6}{5}u_x^2 u_{yy} u + u_y^4 u_{yy} u + \frac{256}{15}ux^4 u_{yy} + \frac{32}{5}xy^2 u_y + \frac{94}{5}uyu_y + \frac{379}{15}ux^3 u_{xy}
\end{aligned}$$

$$
\begin{aligned}
&-\frac{256}{5}x^2yu_x+\frac{82}{5}uxu_x+16uxu_y^2-\frac{17}{5}xu_x^2u_y+u_y^3u_{xy}u+\frac{32}{5}uu_xu_y\\
&-\frac{133}{15}x^3u_xu_y+\frac{256}{5}x^3yu_y+\frac{512}{5}uxy+\frac{176}{5}uxyu_yu_{yy}+\frac{64}{5}uxu_xu_yu_{yy}+\frac{12}{5}uu_y^3\\
&+\frac{2048}{15}ux^3+\frac{512}{15}x^5u_y-2yu_x^2-\frac{32}{5}y^2u_x-\frac{41}{5}x^2u_x^2-\frac{512}{15}x^4u_x-\frac{1}{5}u_x^3;
\end{aligned}
$$

$$
\begin{aligned}
P_7={}&\frac{13}{4}uyu_xu_{yy}-\frac{25}{4}xyu_xu_y+2uyu_y^2u_{yy}+\frac{65}{4}uxyu_{xy}+\frac{1}{4}uxu_y^2u_{xy}+\frac{23}{4}uxu_xu_{xy}\\
&+\frac{65}{4}ux^2yu_{yy}+\frac{3}{2}u_xu_{xy}u_yu+\frac{13}{4}uyu_yu_{xy}+\frac{65}{8}ux^2u_xu_{yy}+\frac{47}{8}ux^2u_yu_{xy}\\
&+u_y^2u_xu_{yy}u+\frac{9}{2}uy^2u_{yy}+\frac{1}{2}u_x^2u_{yy}u-\frac{49}{2}xy^2u_y+\frac{45}{4}uyu_y+\frac{391}{24}ux^3u_{xy}+2uxu_x\\
&+\frac{7}{2}uxu_y^2+\frac{5}{4}xu_x^2u_y+\frac{9}{2}uu_xu_y-\frac{49}{8}x^3u_xu_y-\frac{343}{12}x^3yu_y+\frac{343}{4}uxy-\frac{1}{4}xu_xu_y^3\\
&-yu_xu_y^2+\frac{21}{2}uxyu_yu_{yy}+\frac{21}{4}uxu_xu_yu_{yy}+\frac{1}{2}uu_y^3+\frac{2401}{24}ux^3\\
&-\frac{9}{2}y^2u_y^2+\frac{1}{4}u_x^2u_y^2-\frac{1}{2}yu_y^4-\frac{53}{8}u^2-\frac{7}{2}xyu_y^3-\frac{49}{4}x^2yu_y^2-\frac{7}{4}x^2u_xu_y^2+\frac{131}{8}ux^2u_y,
\end{aligned}
$$

$$
\begin{aligned}
Q_7={}&\frac{21}{4}uxu_xu_{yy}+\frac{21}{4}uxu_yu_{xy}+\frac{11}{2}uyu_yu_{yy}+\frac{35}{4}ux^2u_yu_{yy}+\frac{9}{2}uxu_y^2u_{yy}\\
&+2u_yu_xu_{yy}u+14uxyu_{yy}-\frac{49}{4}xyu_x+\frac{343}{8}ux^2+\frac{49}{4}uy-2xu_x^2-\frac{1}{2}u_x^2u_y-\frac{343}{24}x^3u_x\\
&+2uu_x+\frac{343}{24}x^4u_y+\frac{5}{2}uu_y^2+\frac{1}{2}u_xu_{xy}u+\frac{49}{4}x^2yu_y+\frac{49}{6}ux^3u_{yy}+\frac{65}{8}ux^2u_{xy}\\
&-\frac{9}{4}yu_xu_y+\frac{9}{4}uyu_{xy}-\frac{33}{8}x^2u_xu_y+u_y^3u_{yy}u+u_y^2u_{xy}u;
\end{aligned}
$$

$$
\begin{aligned}
P_6={}&12uy+\frac{2}{3}uu_y^2+36ux^2+\frac{1}{3}u_x^2u_y-\frac{2}{3}yu_y^3+\frac{7}{3}uxu_xu_{yy}+\frac{14}{3}uxyu_{yy}+u_yu_xu_{yy}u\\
&+2uyu_yu_{yy}+\frac{17}{3}uxu_y+2uxu_yu_{xy}-\frac{2}{3}yu_xu_y+\frac{8}{3}uyu_{xy}+u_xu_{xy}u+\frac{19}{3}ux^2u_{xy}\\
&-12x^2yu_y-2x^2u_xu_y-4xyu_y^2-\frac{1}{3}xu_xu_y^2-4y^2u_y-\frac{1}{3}uu_x,
\end{aligned}
$$

$$
\begin{aligned}
Q_6={}&12xu-6x^2u_x-2yu_x+6x^3u_y-\frac{1}{3}u_x^2+\frac{10}{3}uxu_yu_{yy}-\frac{5}{3}xu_xu_y+4ux^2u_{yy}\\
&+u_y^2u_{yy}u+\frac{7}{3}uxu_{xy}+\frac{8}{3}uyu_{yy}+u_xu_{yy}u+u_{xy}u_yu+2xu_yy;
\end{aligned}
$$

$$
\begin{aligned}
P_5={}&-5xu_yy-\frac{5}{2}xu_xu_y-u_y^2y-\frac{1}{2}u_y^2u_x+\frac{25}{2}xu+\frac{1}{2}u_xu_{yy}u+uyu_{yy}-\frac{1}{2}u_{xy}u_yu\\
&+\frac{1}{2}uxu_{xy}-\frac{1}{2}uu_y,
\end{aligned}
$$

$$
Q_5=\frac{1}{2}uu_{xy}+\frac{5}{2}u-\frac{5}{2}xu_x-\frac{1}{2}u_yu_x+\frac{5}{2}x^2u_y-2xu_y^2-\frac{1}{2}u_y^3;
$$

$$
P_4=-u_yu_x-2u_yy+4u,
$$

$$
Q_4=-u_y^2+xu_y-u_x;
$$

$$
P_0=\frac{u_y}{xu_y-u_x-2y},
$$

$$Q_0 = -\frac{1}{xu_y - u_x - 2y}.$$

Here $|\psi_\omega| = |\omega| - 1$.

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