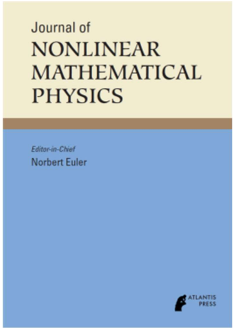

Journal of Nonlinear Mathematical Physics

ISSN (Online): 1776-0852 **ISSN (Print): 1402-9251**
Journal Home Page: https://www.atlantis-press.com/journals/jnmp

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To cite this article: Jianqin Mei, Lijuan Wu (2015) The Bilinear Integrability, N-soliton and Riemann-theta function solutions of B-type KdV Equation, Journal of Nonlinear Mathematical Physics 22:2, 298–307, DOI: https://doi.org/10.1080/14029251.2015.1033241

To link to this article: https://doi.org/10.1080/14029251.2015.1033241

Published online: 04 January 2021

Journal of Nonlinear Mathematical Physics, Vol. 22, No. 2 (2015) 298–307

The Bilinear Integrability, N-soliton and Riemann-theta function solutions of B-type KdV Equation

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Received 25 December 2014

Accepted 4 March 2015

In this paper, the bilinear integrability for B-type KdV equation have been explored. According to the relation to tau function, we find the bilinear transformation and construct the bilinear form with an auxiliary variable of the B-type KdV equation. Based on the truncation form, the Bäcklund transformation has been constructed. Furthermore, the N-soliton solutions and Riemann-theta function 1-periodic solutions of the B-type KdV equation are obtained.

Keywords: Bilinear Integrability; Bäcklund Transformation; Soliton Solution; Riemann-theta function solution.

1. Introduction

Various physical phenomena in physics, engineering, mechanics, biology and chemistry are modeled by nonlinear partial differential equations(NLPDEs). The research on the integrability and exact solution of these equations plays a major role in the study of the nonlinear interaction in the physical phenomena and provide better knowledge of possible applications[1,6,13]. It is well known that there are several kinds of definitions for integrability, Liouville integrability, Lax integrability, inverse scattering integrability, bilinear integrability, Painlevé integrability, symmetry integrability, C-integrability and so on [7,8,9,11]. The Hirota bilinear method is a direct approach to construct the soliton solutions and Riemann-theta function solutions of certain NLPDEs[4,10,12,14].

The B-type KdV equation

$$\begin{cases} u_t = (k+1)v_x \\ v_t = -\dfrac{1}{k+1}(kv_{xx} - v_{xx} - 2w_x) \\ w_t = -\dfrac{k^2+1}{k+1}v_{xxx} + (k+1)(vu_x + 2uv_x) + \dfrac{k-1}{k+1}w_{xx} \end{cases} \tag{1.1}$$

belongs to the B-type KdV hierarchy constructed in the Ref.[5], where we constructed a class of B-type KdV hierarchies by using Lie algebra splitting, and researched the Lax pair, Bäcklund transformation and Hamilton structure of the B-type KdV equation(1.1). However, it is not clear about the bilinear integrability, the existence of N-soliton solutions and Riemann-theta function solutions. In this paper, we mainly discuss the bilinear integrability, and construct the N-soliton solutions and Riemann-theta function solutions of the B-type KdV equation.

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Co-published by Atlantis Press and Taylor & Francis

2. Lax Integrability and Tau function

First we recall the definition of τ_f given by Wilson in [3].

Definition 2.1. Assume that $\mathscr{L}_\pm$ is a splitting of $\mathscr{L}$ compatible with the 2-cocycle that defines a central extension, and that $J = \{J_j | j \geq 1\}$ is a vacuum sequence in $\mathscr{L}_+$. For $f \in \mathscr{L}_-$, the tau function τ_f associated to f is a function of $t = (t_1, \dots, t_N)$ defined by

$$\tau_f(t) = \mu(V(t)f^{-1}) \tag{2.1}$$

where $V(t) = \exp(\Sigma_{j=1}^N t_j J_j)$ is the vacuum frame and μ is the Wilson's μ-function[3].

Theorem 2.1[2]. Let $\mathscr{L}_\pm \in \mathscr{L}$, $(\mathscr{L}_+, \mathscr{L}_-)$ be a splitting, $J = \{J_j | j \geq 1\}$ a vacuum sequence, ω a 2-cocycle on $\mathscr{L}$ compatible with the splitting, and $V(t) = \exp(\Sigma_{j=1}^N t_j J_j)$ the vacuum frame. Let $f \in \mathscr{L}_-$, and

$$V(t)f^{-1} = M^{-1}(t)E(t) \tag{2.2}$$

with $M(t) \in \mathscr{L}_-$ and $E(t) \in \mathscr{L}_+$. Then

(1)$(\ln \tau_f)_{t_j} = \langle J_j, M^{-1}\partial_\lambda M\rangle_{-1} = \langle M J_j M^{-1}, (\partial_\lambda M)M^{-1}\rangle_{-1}$,

(2)$(\ln \tau_f)_{t_1 t_j} = \langle M J_j M^{-1}, \partial_\lambda J_1\rangle_{-1}$.

From Ref.[5], we know that for the Lie algebras

$$\mathscr{L}_+^B = \{\sum_{j\geq 0} A_j \lambda^j | A_j \in sl(4,C)\}, \mathscr{L}_-^B = \{B((A_1)_+) + \sum_{j<0} A_j \lambda^j | A_j \in sl(4,C)\}, \tag{2.3}$$

if

$$B = \begin{pmatrix} 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ \frac{1-k}{k+1} & 0 & 0 & 1 & 0 & \frac{k-1}{k+1} \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \end{pmatrix}, \tag{2.4}$$

then $(\mathscr{L}_+^B, \mathscr{L}_-^B)$ becomes a Lie algebra splitting.

Let e_{ij} be the ij-th elementary matrix, $a = e_{41}, b = e_{12} + e_{23} + e_{34}$, and $J = az + b$. From the Lie algebra splitting $(\mathscr{L}_+^B, \mathscr{L}_-^B)$ and vacuum sequence $J = \{J^i | i \geq 1\}$, the B-type III KdV hierarchy can be constructed.

Let $f \in \mathscr{L}_-^B$, and $u_f = \begin{pmatrix} 0 & 0 & 0 & 0 \\ u & 0 & 0 & 0 \\ v & 0 & 0 & 0 \\ w & kv & u & 0 \end{pmatrix}$, we have $Q(u_f) = MJM^{-1}$. Write $Q(u_f)$ in power series in λ

$$Q(u_f) = MJM^{-1} = a\lambda + \sum_{i\leq 0} Q_i \lambda^i.$$

And the flow generated by J^3 is exact B-type KdV Equation (1.1). By Theorem 2.1 (2), we have $(\ln \tau_f)_{t_1 t_j} = tr(aQ_j)$. According to the expression of Q_1 , we can give explicit formulas of $(\ln \tau_f)_{t_1 t_1}$ in terms of u_f for the B-type KdV hierarchy and

$$(\ln \tau_f)_{t_1 t_1} = tr(aQ_1) = -\frac{1}{2}u. \tag{2.5}$$

3. Bilinear Integrability

Theorem 3.1. *Under the transformation*

$$\begin{cases} u = -2(\ln\phi)_{xx}, \\ v = -\dfrac{2}{k+1}(\ln\phi)_{xt}, \\ w = -(\ln\phi)_{tt} - \dfrac{k-1}{k+1}(\ln\phi)_{xxt}, \end{cases} \tag{3.1}$$

B-type KdV equations (1.1) can be bilinearized into

$$\begin{cases} (\frac{1}{3}D_x^4 + D_t^2 + D_xD_s)\phi\cdot\phi = 0, \\ (\frac{1}{3}D_x^3D_t - \frac{1}{2}D_sD_t)\phi\cdot\phi = 0, \end{cases} \tag{3.2}$$

where s is an auxiliary variable.

Proof. From the formula (2.5), we have

$$u = -2(\ln\phi)_{xx}, \phi = \tau_f. \tag{3.3}$$

Substituting (2.5) into (1.1), we can easily obtain the bilinear transformation (3.1).

Let $p = 2\ln\phi$, (3.1) can be simplified into

$$\begin{cases} u = -p_{xx}, \\ v = -\dfrac{1}{k+1}p_{xt}, \\ w = -\dfrac{1}{2}p_{tt} - \dfrac{k-1}{2(k+1)}p_{xxt}. \end{cases} \tag{3.4}$$

Substituting (3.4) into (1.1), we can represent the resulting equation as follows,

$$\frac{1}{2}(p_{xxxx} + p_{xx}^2 + p_{tt})_t + (p_{xt}p_{xx})_x = 0. \tag{3.5}$$

In order to write (3.5) in a local bilinear form, we introduce an auxiliary variable s and impose subsidiary constraint items, then (3.5) becomes

$$\frac{1}{2}(\frac{1}{3}p_{xxxx} + p_{xx}^2 + p_{tt} + p_{xs})_t + (\frac{1}{3}p_{xxxt} + p_{xt}p_{xx} - \frac{1}{2}p_{st})_x = 0. \tag{3.6}$$

Thus, we can assume that

$$\begin{cases} \frac{1}{3}p_{xxxx} + p_{xx}^2 + p_{tt} + p_{xs} = 0 \\ \frac{1}{3}p_{xxxt} + p_{xt}p_{xx} - \frac{1}{2}p_{st} = 0. \end{cases} \tag{3.7}$$

Finally we have the following bilinear forms

$$\begin{cases} (\frac{1}{3}D_x^4 + D_t^2 + D_xD_s)\phi\cdot\phi = 0 \\ (\frac{1}{3}D_x^3D_t - \frac{1}{2}D_sD_t)\phi\cdot\phi = 0. \end{cases}$$

□

4. Soliton Solution

Next we construct soliton solutions of B-type KdV equation (1.1).

First, expand ϕ in the power series of a small parameter ε as follows

$$\phi = 1 + \varepsilon\phi^{(1)} + \varepsilon^2\phi^{(2)} + \varepsilon^3\phi^{(3)} + \cdots. \tag{4.1}$$

Substituting it into (3.2), we have

$$\varepsilon: \begin{cases} (\frac{1}{3}D_x^4 + D_t^2 + D_xD_s)(\phi^{(1)}\cdot 1 + 1\cdot\phi^{(1)}) = 0, \\ (\frac{1}{3}D_x^3D_t - \frac{1}{2}D_sD_t)(\phi^{(1)}\cdot 1 + 1\cdot\phi^{(1)}) = 0, \end{cases} \tag{4.2}$$

$$\varepsilon^2: \begin{cases} (\frac{1}{3}D_x^4 + D_t^2 + D_xD_s)(\phi^{(2)}\cdot 1 + \phi^{(1)}\cdot\phi^{(1)} + 1\cdot\phi^{(2)}) = 0, \\ (\frac{1}{3}D_x^3D_t - \frac{1}{2}D_sD_t)(\phi^{(2)}\cdot 1 + \phi^{(1)}\cdot\phi^{(1)} + 1\cdot\phi^{(2)}) = 0, \end{cases} \tag{4.3}$$

$$\varepsilon^3: \begin{cases} (\frac{1}{3}D_x^4 + D_t^2 + D_xD_s)(\phi^{(3)}\cdot 1 + \phi^{(2)}\cdot\phi^{(1)} + \phi^{(1)}\cdot\phi^{(2)} + 1\cdot\phi^{(3)}) = 0, \\ (\frac{1}{3}D_x^3D_t - \frac{1}{2}D_sD_t)(\phi^{(3)}\cdot 1 + \phi^{(2)}\cdot\phi^{(1)} + \phi^{(1)}\cdot\phi^{(2)} + 1\cdot\phi^{(3)}) = 0, \end{cases} \tag{4.4}$$

$$\vdots$$

Consider 1-soliton solution, we suppose

$$\phi^{(1)} = e^{\eta_1},\ \eta_1 = \alpha_1 x + \beta_1 t + \gamma_1 s + \eta_0^{(1)}, \tag{4.5}$$

where $\alpha_1,\ \beta_1,\ \gamma_1$ are constants to be determined. Substituting it into (4.2), we get

$$\begin{aligned} &\frac{1}{3}\alpha_1^4 + \beta_1^2 + \alpha_1\gamma_1 = 0, \\ &\frac{1}{3}\alpha_1^3\beta_1 - \frac{1}{2}\beta_1\gamma_1 = 0. \end{aligned} \tag{4.6}$$

From (4.6) we have

$$\gamma_1 = \frac{2}{3}\alpha_1^3,\ \alpha_1^4 + \beta_1^2 = 0. \tag{4.7}$$

Choosing $\phi^{(2)} = \phi^{(3)} = \cdots = 0$, the expansion of ϕ is truncated with a finite sum, and exact solution of (4.2) reads

$$\phi = 1 + \varepsilon e^{\eta_1},\ \ \eta_1 = \alpha_1 x + \beta_1 t + \frac{2}{3}\alpha_1^3 s + \eta_0^{(1)}, \tag{4.8}$$

where $\alpha_1,\ \beta_1$ satisfy condition $\alpha_1^4+\beta_1^2=0$. By using transformation (3.1), 1-soliton solution of B-type KdV equations (1.1) reads(set $\varepsilon=1$)

$$\begin{cases} u=-2\ln(1+e^{\eta_1})_{xx}=-\dfrac{\alpha_1^2}{2}(\sinh\dfrac{\alpha_1x+\beta_1t+\gamma_1s+\eta_0^{(1)}}{2})^2,\\ v=-\dfrac{2}{k+1}\ln(1+e^{\eta_1})_{xt}=-\dfrac{\alpha_1\beta_1}{2(k+1)}(\sinh\dfrac{\alpha_1x+\beta_1t+\gamma_1s+\eta_0^{(1)}}{2})^2,\\ w=-\ln(1+e^{\eta_1})_{tt}-\dfrac{k-1}{k+1}\ln(1+e^{\eta_1})_{xxt}=[-\dfrac{\beta_1^2}{4}+\dfrac{(k-1)\alpha_1^2\beta_1}{4(k+1)}\\ \qquad \tanh\dfrac{\alpha_1x+\beta_1t+\gamma_1s+\eta_0^{(1)}}{2}](\sinh\dfrac{\alpha_1x+\beta_1t+\gamma_1s+\eta_0^{(1)}}{2})^2, \end{cases} \tag{4.9}$$

where $\alpha_1,\ \beta_1,\ \gamma_1$ satisfy the conditions $\gamma_1=\frac{2}{3}\alpha_1^3,\ \alpha_1^4+\beta_1^2=0$.

Next we find 2-soliton solution by assuming

$$\phi^{(1)}=e^{\eta_1}+e^{\eta_2},\ \eta_i=\alpha_ix+\beta_it+\gamma_i+\eta_0^{(i)},\ \ i=1,2. \tag{4.10}$$

From (4.2), we can get

$$\gamma_i=\frac{2}{3}\alpha_i^3,\ \alpha_i^4+\beta_i^2=0,\ i=1,2. \tag{4.11}$$

According to value of $\phi^{(1)}$, solving (4.3), we have

$$\phi^{(2)}=e^{\eta_1+\eta_2+A_{12}},\ e^{A_{12}}=\frac{(\alpha_1-\alpha_2)^2}{\alpha_1^2+\alpha_2^2}. \tag{4.12}$$

By choosing $\phi^{(i)}=0, i\geq 3$, the expansion (4.1) is truncated with a finite sum, and exact solution of (4.3) reads

$$\phi=1+\varepsilon e^{\eta_1}+\varepsilon e^{\eta_2}+\varepsilon^2\frac{(\alpha_1-\alpha_2)^2}{\alpha_1^2+\alpha_2^2}e^{\eta_1+\eta_2}, \tag{4.13}$$

where

$$\eta_i=\alpha_ix+\beta_it+\frac{2}{3}\alpha_i^3s+\eta_0^{(i)},\ \ i=1,2. \tag{4.14}$$

Utilizing transformations (3.1), we have the 2-soliton solution of B-type KdV equations(1.1)

$$\begin{cases} u=-2\ln(1+e^{\eta_1}+e^{\eta_2}+e^{\eta_1+\eta_2+A_{12}})_{xx},\\ v=-\dfrac{2}{k+1}\ln(1+e^{\eta_1}+e^{\eta_2}+e^{\eta_1+\eta_2+A_{12}})_{xt},\\ w=-\ln(1+e^{\eta_1}+e^{\eta_2}+e^{\eta_1+\eta_2+A_{12}})_{tt}-\dfrac{k-1}{k+1}\ln(1+e^{\eta_1}+e^{\eta_2}+e^{\eta_1+\eta_2+A_{12}})_{xxt}. \end{cases} \tag{4.15}$$

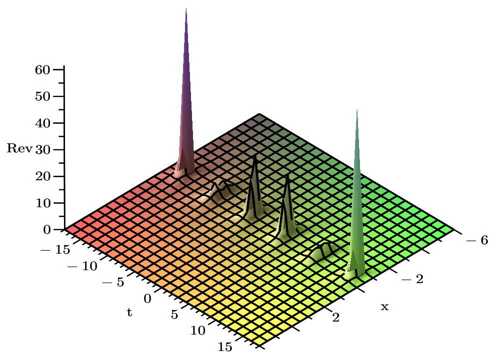
And we can easily have the N-soliton solution of B-type KdV equation (1.1)

$$\begin{cases} u = -2(\ln\phi)_{xx}, \\ v = -\dfrac{2}{k+1}(\ln\phi)_{xt}, \\ w = -(\ln f)_{tt} - \dfrac{k-1}{k+1}(\ln\phi)_{xxt}, \\ \phi = \sum\limits_{\mu=0,1}\exp(\sum\limits_{i=1}^{N}\mu_i\eta_i + \sum\limits_{1\le i<j}^{N} A_{ij}\mu_i\mu_j), \end{cases} \tag{4.16}$$

where

$$\begin{aligned} \eta_i &= \alpha_i x + \beta_i t + \frac{2}{3}\alpha_i^3 s + \eta_0^{(i)}, \\ \gamma_i &= \frac{2}{3}\alpha_i^3, \quad \alpha_i^4 + \beta_i^2 = 0, \\ e^{A_{ij}} &= \frac{(\alpha_i-\alpha_j)^2}{\alpha_i^2+\alpha_j^2}, i,j = 1,2,\cdots,N. \end{aligned} \tag{4.17}$$

the notation $\sum\limits_{\mu=0,1}$ means the sum of all possible combinations of $\mu_1 = 0,1,\ \mu_2 = 0,1,...,\ \mu_n = 0,1$.



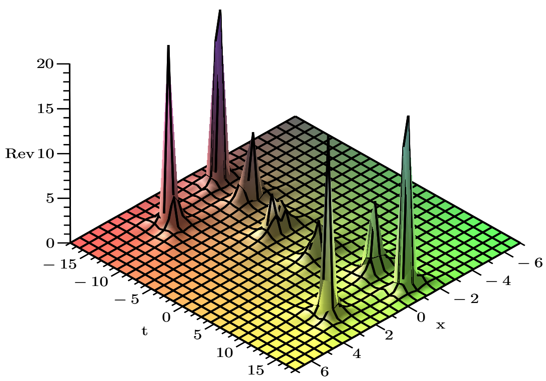


Fig.1 The evolution of soliton solutions (4.16) *Rev* with $\alpha_1 = 1, \alpha_2 = 0.5, s = 1, k = 2$. (a)1-soliton (b)2-soliton.

5. Bäcklund Transformation

Based on the bilinear transformation (3.1), we can obtain the Bäcklund transformation of (1.1).

If $\{u_0, v_0, w_0\}$ is a solution of (1.1), then

$$\begin{aligned} u =& \frac{2\phi_x^2 - 2\phi_{xx}\phi}{\phi^2} + u_0, \\ v =& \frac{2}{k+1}\frac{\phi_x\phi_t - \phi_{xt}\phi}{\phi^2} + v_0, \\ w =& \frac{(k-1)(-2\phi_x^2\phi_t + 2\phi\phi_x\phi_{xt} + \phi\phi_t\phi_{xx} - \phi_{xxt}\phi^2)}{(k+1)\phi^3} + \frac{\phi_t^2 - \phi\phi_{tt}}{\phi^2} + w_0, \end{aligned} \tag{5.1}$$

is another new solution of (1.1), where ϕ satisfies the following conditions

$$\begin{cases} -\phi_{xxxxt} - \phi_{ttt} + 4u_0\phi_{xxt} + 2(k+1)v_0\phi_{xxx} + 2u_{0x}\phi_{xt} + 4(k+1)v_{0x}\phi_{xx} = 0, \\ \phi_{xxxx}\phi_t - 2\phi_{xx}\phi_{xxt} + 4\phi_x\phi_{xxxt} + 3\phi_t\phi_{tt} - 8u_0\phi_x\phi_{xt} - 4u_0\phi_{xx}\phi_t \\ -6(k+1)v_0\phi_x\phi_{xx} - 2u_{0x}\phi_x\phi_t - 4(k+1)v_{0x}\phi_x^2 = 0, \\ 2(k+1)v_0\phi_x^3 + 2\phi_x^2(2u_0\phi_t - \phi_{xxt}) + 2\phi_x(\phi_{xx}\phi_{xt} - \phi_{xxx}\phi_t) + \phi_t(\phi_{xx}^2 - \phi_t^2) = 0. \end{cases} \tag{5.2}$$

Start with zero solution $\{u_0, v_0, w_0\} = \{0,0,0\}$, (5.2) becomes into

$$\begin{cases} \phi_{xxxxt} + \phi_{ttt} = 0, \\ 3\phi_t\phi_{tt} + 4\phi_x\phi_{xxxt} - 2\phi_{xx}\phi_{xxt} + \phi_{xxxx}\phi_t = 0, \\ \phi_{xx}^2\phi_t - 2\phi_x^2\phi_{xxt} - 2\phi_x\phi_t\phi_{xxx} - \phi_t^3 + 2\phi_x\phi_{xt}\phi_{xx} = 0. \end{cases} \tag{5.3}$$

It is not difficult to verify that if ϕ satisfies

$$\phi_t = i\phi_{xx}, \quad \phi_{xx} = c_0\phi_x, \tag{5.4}$$

where c_0 is a constant, the system (5.3) holds automatically.

Remark 5.1: It is obvious that the system (5.3) is different from the bilinear equation (3.2). But these are both special cases of (3.5) from two different angles. Actually, if we look for soliton solutions using this Bäcklund transformation, we can obtain the same result as section 4.

Remark 5.2: These linear differential conditions may be useful to get the Wronskian determinant solutions.

6. Riemann theta Function Solution

In this section, we consider Riemann theta function solution of Eq.(1.1).

In fact, we have the following general bilinear form of Eq.(1.1) based on (3.6),

$$\begin{aligned} \mathfrak{L}_1(D_x, D_t, D_s)\phi\cdot\phi = (\frac{1}{3}D_x^4 + D_t^2 + D_xD_s + c_1)\phi\cdot\phi = 0, \\ \mathfrak{L}_2(D_x, D_t, D_s)\phi\cdot\phi = (\frac{1}{3}D_x^3D_t - \frac{1}{2}D_sD_t + c_2)\phi\cdot\phi = 0. \end{aligned} \tag{6.1}$$

where $c_1 = c_1(x,s), c_2 = c_2(t,s)$ are constants of integration.

In order to find one-periodic wave solutions of (1.1), we investigate the following Riemann theta function with $N = 1$

$$\phi = \vartheta(\xi, \tau) = \sum_{n=-\infty}^{\infty} e^{2\pi i n\xi + \pi n^2 \tau} \tag{6.2}$$

where the phase variable $\xi = \alpha x + \omega t + \gamma s + \delta_0$ and the parameter $\tau < 0$.

Substitute (6.2) into (6.1), by virtue of the similar result in [10,11], we have

$$\begin{aligned}
\sum_{n=-\infty}^{\infty} \mathfrak{L}_1(4n\pi i\alpha, 4n\pi i\omega, 4n\pi i\gamma)e^{2n^2\pi\tau} = 0,\\
\sum_{n=-\infty}^{\infty} \mathfrak{L}_1(2\pi i(2n-1)\alpha, 2\pi i(2n-1)\omega, 2\pi i(2n-1)\gamma)e^{(2n^2-2n+1)\pi\tau} = 0,\\
\sum_{n=-\infty}^{\infty} \mathfrak{L}_2(4n\pi i\alpha, 4n\pi i\omega, 4n\pi i\gamma)e^{2n^2\pi\tau} = 0,\\
\sum_{n=-\infty}^{\infty} \mathfrak{L}_2(2\pi i(2n-1)\alpha, 2\pi i(2n-1)\omega, 2\pi i(2n-1)\gamma)e^{(2n^2-2n+1)\pi\tau} = 0.
\end{aligned} \tag{6.3}$$

That is, ω, γ, c_1, c_2 are determined by the algebraic system

$$2a\omega^2 + 2a\alpha\gamma + c_1 a_1 = a_2\alpha^4$$

$$2b\omega^2 + 2b\alpha\gamma + c_1 b_1 = b_2\alpha^4$$

$$-a\gamma\omega + c_2 a_1 = -a_2\alpha^3\omega$$

$$-b\gamma\omega + c_2 b_1 = -b_2\alpha^3\omega$$

where

$$\begin{aligned}
a = 8\pi^2 \sum_{n=-\infty}^{\infty} n^2\lambda^{2n^2}, b = 2\pi^2 \sum_{n=-\infty}^{\infty} (2n-1)^2\lambda^{2n^2-2n+1},\\
a_1 = \sum_{n=-\infty}^{\infty} \lambda^{2n^2}, b_1 = \sum_{n=-\infty}^{\infty} \lambda^{2n^2-2n+1}, \lambda = e^{\pi\tau}\\
a_2 = \frac{256}{3}\pi^4 \sum_{n=-\infty}^{\infty} n^4\lambda^{2n^2}, b_2 = \frac{16}{3}\pi^4 \sum_{n=-\infty}^{\infty} (2n-1)^4\lambda^{2n^2-2n+1}
\end{aligned} \tag{6.4}$$

Solve this system, we easily get

$$\omega^2 = \frac{a_1 b_2 - a_2 b_1}{2ab_1 - 2a_1 b}\alpha^4, c_1 = \frac{ab_2 - ba_2}{ab_1 - a_1 b}\alpha^4, c_2 = -\frac{ab_2 - ba_2}{ab_1 - a_1 b}\alpha^3\omega.$$

It can be concluded that the B-type KdV equation (1.1) has the following Riemann-theta function 1-periodic solutions

$$\begin{cases}
u = -2(\ln\vartheta(\xi,\tau))_{xx},\\
v = -\dfrac{2}{k+1}(\ln\vartheta(\xi,\tau))_{xt},\\
w = -(\ln\vartheta(\xi,\tau))_{tt} - \dfrac{k-1}{k+1}(\ln\vartheta(\xi,\tau))_{xxt}
\end{cases} \tag{6.5}$$

where $\xi = \alpha x \pm \sqrt{\frac{a_1b_2-a_2b_1}{2ab_1-2a_1b}}\alpha^2 t + \varepsilon_0$, $\varepsilon_0 = \gamma s + \delta_0$ is an arbitrary constant and a, a_1, a_2, b, b_1, b_2 satisfy (6.4).

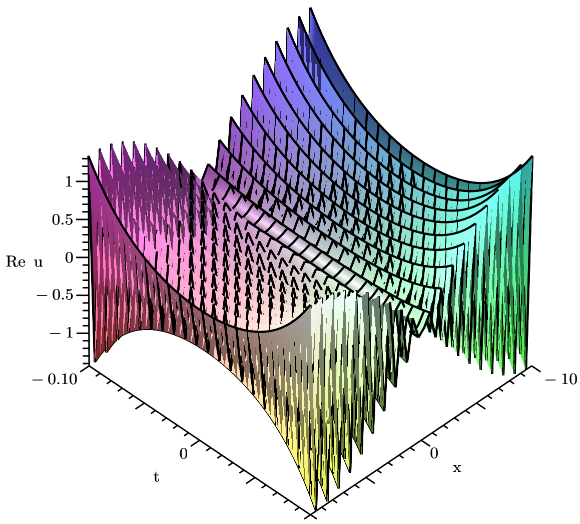


Fig.2 The Riemann-theta function 1-periodic solution (6.5) Reu with $k = -2, \tau = -1, \alpha = 1, \varepsilon_0 = 0$.

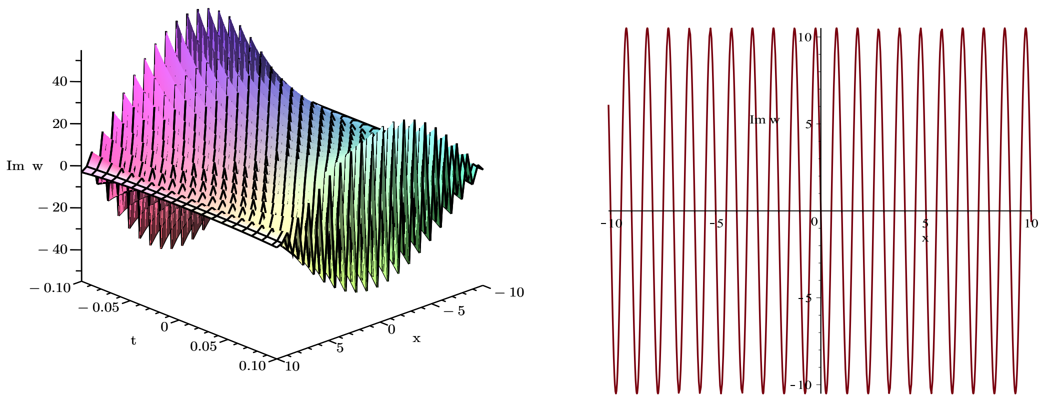


Fig.3 The Riemann-theta function 1-periodic solution (6.5) Imw with $k = -2, \tau = -1, \alpha = 1, \varepsilon_0 = 0$.

Remark 6.1: If we take the limit of Riemann-theta function 1-periodic solution as $\lambda \to 0$, we can get the 1-soliton solution (4.9). It is not surprised, but it is worthwhile to say that both c_1 and c_2 approach to zero as $\lambda \to 0$. That is, c_1 and c_2 are not necessary to get soliton solutions. This is why we only need use the simplest bilinear form (3.2) in section 3 and 4.

7. Conclusion

In sum, we have discussed bilinear integrability of the B-type KdV equation. By using the relation between flow and tau function, we constructed the Hirota bilinear formulation and Bäcklund transformation. In additon, the N-soliton solution and Riemann-theta function 1-periodic solutions have been constructed. This idea can be extended to other B-type KdV hierarchy even more NLPDEs.

Acknowledgement:

The work is supported by the National Natural Science Foundation of China (No.11201048) and the Fundamental Research Funds for the Central Universities.

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