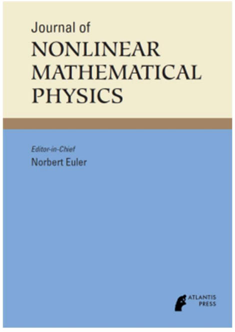

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Continuous Correspondence of Conservation Laws of the Semi-discrete AKNS System

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In this paper we investigate the semi-discrete Ablowitz–Kaup–Newell–Segur (sdAKNS) hierarchy, and specifically their Lax pairs and infinitely many conservation laws, as well as the corresponding continuum limits. The infinitely many conserved densities derived from the Ablowitz-Ladik spectral problem are trivial, in the sense that all of them are shown to reduce to the first conserved density of the AKNS hierarchy in the continuum limit. We derive new and nontrivial infinitely many conservation laws for the sdAKNS hierarchy, and also the explicit combinatorial relations between the known conservation laws and our new ones. By performing a uniform continuum limit, the new conservation laws of the sdAKNS system are then matched with their counterparts of the continuous AKNS system.

Keywords: semi-discrete AKNS hierarchy; Lax pairs; conservation laws; continuum limits.

2000 Mathematics Subject Classification: 35Q51, 35Q55, 37K05, 39A99

1. Introduction

By discretising a continuous spectral problem one may have a discrete spectral problem and then a discrete integrable system. For the well known Ablowitz–Kaup–Newell–Segur (AKNS) spectral problem [1, 2]

$$\begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix}_x = \begin{pmatrix} \eta & q \\ r & -\eta \end{pmatrix} \begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix}, \tag{1.1}$$

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its discretisation reads [3,4]

$$\begin{pmatrix} \phi_{1,n+1} \\ \phi_{2,n+1} \end{pmatrix} = \begin{pmatrix} \lambda & Q_n \\ R_n & \frac{1}{\lambda} \end{pmatrix} \begin{pmatrix} \phi_{1,n} \\ \phi_{2,n} \end{pmatrix}, \tag{1.2}$$

which is referred to as the Ablowitz–Ladik (AL) spectral problem, where the derivative $\phi_{j,x}$ is replaced with a difference $\phi_{j,n+1} - \phi_{j,n}$. If the temporal variable t remains continuous, equations derived from (1.2) are semi-discrete systems which compose the AL hierarchy. However, the AL hierarchy is not the semi-discrete AKNS (sdAKNS) hierarchy although (1.2) is a discretization of the AKNS spectral problem (1.1). In fact, equations in the sdAKNS hierarchy are some combinations of the members of the AL hierarchy [27]. For the AL hierarchy, their integrability characteristics, such as conservation laws, symmetries and Hamiltonian structures, have been well studied [11,13,19,23–27,29]. In principle, these characteristics should be transferred to the sdAKNS hierarchy.

In this paper we will investigate the sdAKNS hierarchy, and specifically their Lax pairs and infinitely many conservation laws, as well as the corresponding continuum limits. The known infinitely many conserved densities derived from the AL spectral problem (cf. [24]) are trivial. We will show that all of them go to the first conserved density of the AKNS hierarchy in the continuum limit. To derive new and nontrivial infinitely many conservation laws, we will rederive the sdAKNS hierarchy and their Lax pairs so that they are ready for constructing conservation laws as well as for considering continuum limits. This will be done in Sec.2.

The paper is organized as follows. In addition to Sec.2 mentioned above, in Sec.3 we derive new infinitely many conservation laws and prove explicit combinatorial relations between the new conservation laws and the known ones. Finally, in Sec.4 we perform a uniform continuum limit, under which the Lax pairs and new conservation laws of the sdAKNS system are matched with their counterparts of the continuous AKNS system.

2. The sdAKNS hierarchy and Lax pairs

2.1. *The AL hierarchy and the sdAKNS hierarchy*

The sdAKNS hierarchy can be derived from the AL hierarchy [27]. Suppose that the AL spectral problem and its time evolution part are

$$E\bar{\Phi} = \bar{M}\Phi, \quad \bar{M} = \begin{pmatrix} \lambda & Q_n \\ R_n & \frac{1}{\lambda} \end{pmatrix}, \quad \bar{U} = \begin{pmatrix} Q_n \\ R_n \end{pmatrix}, \quad \bar{\Phi} = \begin{pmatrix} \bar{\phi}_1(n) \\ \bar{\phi}_2(n) \end{pmatrix}, \tag{2.1a}$$

$$\bar{\Phi}_{\bar{t}_s} = \bar{N}_s^{\text{AL}}\bar{\Phi}, \quad \bar{N}_s^{\text{AL}} = \begin{pmatrix} \bar{A}_s^{\text{AL}} & \bar{B}_s^{\text{AL}} \\ \bar{C}_s^{\text{AL}} & \bar{D}_s^{\text{AL}} \end{pmatrix}, \quad s \in \mathbb{Z}, \tag{2.1b}$$

where E is a shift operator defined as $Ef(n) = f(n+1)$, λ is a spectral parameter and independent of time, $Q_n = Q(n,t)$ and $R_n = R(n,t)$ are potential functions, and $\bar{A}_s^{\text{AL}}, \bar{B}_s^{\text{AL}}, \bar{C}_s^{\text{AL}}$ and $\bar{D}_s^{\text{AL}}$ are Laurent polynomials of λ living on Q_n, R_n and their shifts. From the discrete zero curvature equation of (2.1), i.e.

$$\bar{M}_{\bar{t}_s} = (E\bar{N}_s^{\text{AL}})\bar{M} - \bar{M}\bar{N}_s^{\text{AL}}, \tag{2.2}$$

the whole AL hierarchy can be derived and expressed as (a detailed procedure can be found in [25])

$$\bar{U}_{\bar{t}_s} = \bar{K}_s^{\mathrm{AL}} = \bar{L}^s \begin{pmatrix} Q_n \\ -R_n \end{pmatrix}, \quad s \in \mathbb{Z}, \tag{2.3}$$

where

$$\begin{aligned}\bar{L} = &\begin{pmatrix} E & 0 \\ 0 & E^{-1} \end{pmatrix} + \begin{pmatrix} -Q_nE \\ R_n \end{pmatrix} (E-1)^{-1}(R_nE, Q_nE^{-1}) \\ &+ \mu_n \begin{pmatrix} -EQ_n \\ R_{n-1} \end{pmatrix} (E-1)^{-1}(R_n, Q_n)\frac{1}{\mu_n},\end{aligned} \tag{2.4a}$$

and

$$\begin{aligned}\bar{L}^{-1} = &\begin{pmatrix} E^{-1} & 0 \\ 0 & E \end{pmatrix} + \begin{pmatrix} Q_n \\ -R_nE \end{pmatrix} (E-1)^{-1}(R_nE^{-1}, Q_nE) \\ &+ \mu_n \begin{pmatrix} Q_{n-1} \\ -ER_n \end{pmatrix} (E-1)^{-1}(R_n, Q_n)\frac{1}{\mu_n},\end{aligned} \tag{2.4b}$$

with

$$\mu_n = 1 - Q_nR_n. \tag{2.4c}$$

The first few flows in the hierarchy are

$$\bar{K}_0^{\mathrm{AL}} = \begin{pmatrix} Q_n \\ -R_n \end{pmatrix}, \tag{2.5a}$$

$$\bar{K}_1^{\mathrm{AL}} = \mu_n \begin{pmatrix} Q_{n+1} \\ -R_{n-1} \end{pmatrix}, \tag{2.5b}$$

$$\bar{K}_{-1}^{\mathrm{AL}} = \mu_n \begin{pmatrix} Q_{n-1} \\ -R_{n+1} \end{pmatrix}. \tag{2.5c}$$

The sdAKNS hierarchy can be given through combining the AL flows in a suitable way [27]. Define initial flows

$$\bar{K}_0 = \bar{K}_0^{\mathrm{AL}}, \tag{2.6a}$$

$$\bar{K}_1 = \frac{1}{2}(\bar{K}_1^{\mathrm{AL}} - \bar{K}_{-1}^{\mathrm{AL}}). \tag{2.6b}$$

Then the sdAKNS hierarchy is given by

$$\bar{U}_{\bar{t}_s} = \bar{K}_s = \begin{cases} \bar{\mathscr{L}}^j \bar{K}_0, \ s = 2j, \\ \bar{\mathscr{L}}^j \bar{K}_1, \ s = 2j+1, \end{cases} \quad (j = 0, 1, \cdots), \tag{2.7}$$

where

$$\begin{aligned}\bar{\mathscr{L}} &= \bar{L} - 2I + \bar{L}^{-1} \\ &= \begin{pmatrix} E-2+E^{-1} & 0 \\ 0 & E-2+E^{-1} \end{pmatrix} + \mu_n \begin{pmatrix} Q_{n-1} - Q_{n+1}E \\ R_{n-1} - R_{n+1}E \end{pmatrix} (E-1)^{-1}(R_n, Q_n)\frac{1}{\mu_n} \\ &\quad + \begin{pmatrix} -Q_nE \\ R_n \end{pmatrix} (E-1)^{-1}(R_nE, Q_nE^{-1}) - \begin{pmatrix} -Q_n \\ R_nE \end{pmatrix} (E-1)^{-1}(R_nE^{-1}, Q_nE),\end{aligned} \tag{2.8}$$

L and L^{-1} are given in (2.4) and I is the unit operator. The first few flows of the sdAKNS hierarchy are

$$\bar{K}_0 = \begin{pmatrix} Q_n \\ -R_n \end{pmatrix}, \tag{2.9a}$$

$$\bar{K}_1 = \frac{1}{2}\mu_n \begin{pmatrix} Q_{n+1} - Q_{n-1} \\ R_{n+1} - R_{n-1} \end{pmatrix}, \tag{2.9b}$$

$$\bar{K}_2 = \begin{pmatrix} Q_{n+1} - 2Q_n + Q_{n-1} - Q_nR_n(Q_{n+1} + Q_{n-1}) \\ -R_{n+1} + 2R_n - R_{n-1} + Q_nR_n(R_{n+1} + R_{n-1}) \end{pmatrix}, \tag{2.9c}$$

$$\bar{K}_3 = \frac{1}{2}\mu_n \begin{pmatrix} (E - E^{-1})(Q_{n+1} - 2Q_n + Q_{n-1}) \\ \quad +Q_{n+1}Q_{n+2}R_{n+1} - Q_n(Q_{n+1}R_{n-1} - Q_{n-1}R_{n+1}) \\ \quad -Q_{n-2}Q_{n-1}R_{n-1} - R_n(Q_{n+1}^2 - Q_{n-1}^2) \\ (E - E^{-1})(R_{n+1} - 2R_n + R_{n-1}) \\ \quad +Q_{n-1}R_{n-2}R_{n-1} + R_n(Q_{n+1}R_{n-1} - Q_{n-1}R_{n+1}) \\ \quad -Q_{n+1}R_{n+1}R_{n+2} - Q_n(R_{n+1}^2 - R_{n-1}^2) \end{pmatrix}. \tag{2.9d}$$

We note that the equations $\bar{U}_{\bar{t}_2} = \bar{K}_2$ and $\bar{U}_{\bar{t}_3} = \bar{K}_3$ respectively correspond to the coupled semi-discrete nonlinear Schrödinger equation [21] and the coupled semi-discrete modified Korteweg–de Vries equation [20] in the papers of Tsuchida et al.

2.2. *The sdAKNS hierarchy and Lax pairs: revisit*

In the sdAKNS hierarchy, for the equation

$$\bar{U}_{\bar{t}_s} = \bar{K}_s \tag{2.10}$$

its Lax pair is expressed as

$$E\bar{\Phi} = \bar{M}\bar{\Phi}, \quad \bar{M} = \begin{pmatrix} \lambda & Q_n \\ R_n & \frac{1}{\lambda} \end{pmatrix}, \tag{2.11a}$$

$$\bar{\Phi}_{\bar{t}_s} = \bar{N}_s\bar{\Phi}, \quad \bar{N}_s = \begin{pmatrix} \bar{A}_s & \bar{B}_s \\ \bar{C}_s & \bar{D}_s \end{pmatrix}, \tag{2.11b}$$

where (2.11a) is the AL spectral problem (2.1a). Since (2.7) already provided the combinatorial relation between $\bar{K}_s$ and the AL flows $\{\bar{K}_j^{\text{AL}}\}$, the matrix $\bar{N}_s$ can be accordingly written out as combinations of $\{\bar{N}_j^{\text{AL}}\}$ and $\bar{A}_s$, $\bar{B}_s$, $\bar{C}_s$ and $\bar{D}_s$ will be expressed through the Laurent polynomials in λ. However, it is hard to write out explicit form of these polynomials, which will make difficulty in the consideration continuum limit. In the following, we need to re-derive the sdAKNS hierarchy and their Lax pairs so that they are ready for considering continuum limits as well as for deriving new conservation laws.

Let us start from the compatible condition (zero curvature equation)

$$\bar{M}_{\bar{t}_s} = (E\bar{N}_s)\bar{M} - \bar{M}\bar{N}_s \tag{2.12}$$

and make use of the Gâteaux derivatives. For the given functions $\bar{F} = \bar{F}(\bar{U})$ and $\bar{G} = \bar{G}(\bar{U})$,

$$\bar{F}'[G] = \frac{\partial}{\partial \varepsilon}\bar{F}(\bar{U} + \varepsilon\bar{G})\big|_{\varepsilon=0}$$

is called the Gâteaux derivative of $\bar{F}(\bar{U})$ w.r.t. $\bar{U}$ in the direction $\bar{G}(\bar{U})$. It is easy to see that $\bar{F}_{\bar{t}}(\bar{U}) = \bar{F}'[\bar{U}_{\bar{t}}]$, from which we can rewrite the zero curvature equation (2.12) as

$$\bar{M}'[\bar{K}_s] = (E\bar{N}_s)\bar{M} - \bar{M}\bar{N}_s. \tag{2.13}$$

Now we consider the following equation

$$\bar{M}'\left[\bar{X} - \left(\lambda - \frac{1}{\lambda}\right)^2 \bar{Y}\right] = (E\bar{\mathfrak{N}})\bar{M} - \bar{M}\bar{\mathfrak{N}}, \tag{2.14}$$

where $\bar{X} = (\bar{X}_1, \bar{X}_2)^T$ and $\bar{Y} = (\bar{Y}_1, \bar{Y}_2)^T$ are vector functions of $\bar{U}$ but independent of λ, and

$$\bar{\mathfrak{N}} = \begin{pmatrix} \bar{\mathfrak{A}} & \bar{\mathfrak{B}} \\ \bar{\mathfrak{C}} & \bar{\mathfrak{D}} \end{pmatrix}.$$

When $\bar{Y} = 0$, we assign the following two initial flows

$$\bar{X} = \bar{K}_0 \quad \text{and} \quad \bar{X} = \bar{K}_1,$$

where $\bar{K}_0$ and $\bar{K}_1$ are defined in (2.6). Correspondingly, we can take

$$\bar{\mathfrak{N}} = \bar{N}_0 \quad \text{and} \quad \bar{\mathfrak{N}} = \bar{N}_1,$$

respectively. When $\bar{Y} \neq 0$, we restrict $\bar{\mathfrak{N}}|_{\bar{U}=0} = 0$ and rewrite (2.14) into

$$\bar{X} - \left(\lambda - \frac{1}{\lambda}\right)^2 \bar{Y} = (\lambda \bar{L}_1 - \frac{1}{\lambda}\bar{L}_2)\begin{pmatrix} \bar{\mathfrak{B}} \\ \bar{\mathfrak{C}} \end{pmatrix} \tag{2.15}$$

and

$$\bar{\mathfrak{A}} = -\frac{1}{\lambda}(E-1)^{-1}(R_nE, -Q_n)\begin{pmatrix} \bar{\mathfrak{B}} \\ \bar{\mathfrak{C}} \end{pmatrix}, \tag{2.16a}$$

$$\bar{\mathfrak{D}} = \frac{1}{\lambda}(E-1)^{-1}(R_n, -Q_nE)\begin{pmatrix} \bar{\mathfrak{B}} \\ \bar{\mathfrak{C}} \end{pmatrix}, \tag{2.16b}$$

where

$$\bar{L}_1 = \begin{pmatrix} -1 & 0 \\ 0 & E \end{pmatrix} + \begin{pmatrix} -Q_n \\ R_nE \end{pmatrix}(E-1)^{-1}(R_n, -Q_nE), \tag{2.17a}$$

$$\bar{L}_2 = \begin{pmatrix} -E & 0 \\ 0 & 1 \end{pmatrix} - \begin{pmatrix} -Q_nE \\ R_n \end{pmatrix}(E-1)^{-1}(R_nE, -Q_n), \tag{2.17b}$$

with their inverses

$$\bar{L}_1^{-1} = \begin{pmatrix} -1 & 0 \\ 0 & E^{-1} \end{pmatrix} + \begin{pmatrix} Q_n \\ R_{n-1} \end{pmatrix}(E-1)^{-1}(R_n, Q_n)\frac{1}{\mu_n}, \tag{2.18a}$$

$$\bar{L}_2^{-1} = \begin{pmatrix} -E^{-1} & 0 \\ 0 & 1 \end{pmatrix} - \begin{pmatrix} Q_{n-1} \\ R_n \end{pmatrix}(E-1)^{-1}(R_n, Q_n)\frac{1}{\mu_n}. \tag{2.18b}$$

To solve (2.15), we expand

$$\begin{pmatrix} \bar{\mathfrak{B}} \\ \bar{\mathfrak{C}} \end{pmatrix} = \begin{pmatrix} \bar{\mathfrak{b}}^+ \\ \bar{\mathfrak{c}}^+ \end{pmatrix}\lambda + \begin{pmatrix} \bar{\mathfrak{b}}^- \\ \bar{\mathfrak{c}}^- \end{pmatrix}\frac{1}{\lambda}. \tag{2.19}$$

Then we from (2.15) have

$$\bar{Y} = -\bar{L}_1 \begin{pmatrix} \bar{\mathfrak{b}}^+ \\ \bar{\mathfrak{c}}^+ \end{pmatrix}, \tag{2.20a}$$

$$\bar{X} + 2\bar{Y} = \bar{L}_1 \begin{pmatrix} \bar{\mathfrak{b}}^- \\ \bar{\mathfrak{c}}^- \end{pmatrix} - \bar{L}_2 \begin{pmatrix} \bar{\mathfrak{b}}^+ \\ \bar{\mathfrak{c}}^+ \end{pmatrix}, \tag{2.20b}$$

$$\bar{Y} = \bar{L}_2 \begin{pmatrix} \bar{\mathfrak{b}}^- \\ \bar{\mathfrak{c}}^- \end{pmatrix}, \tag{2.20c}$$

which gives rise to the relation

$$\bar{X} = \bar{\mathscr{L}}\bar{Y} \tag{2.21}$$

and

$$\begin{pmatrix} \bar{\mathfrak{B}} \\ \bar{\mathfrak{C}} \end{pmatrix} = (-\lambda \bar{L}_1^{-1} + \frac{1}{\lambda}\bar{L}_2^{-1})\bar{Y}. \tag{2.22}$$

Here $\bar{\mathscr{L}} = \bar{L} - 2I + \bar{L}^{-1}$ is given in (2.8), which serves as the recursion operator of the sdAKNS hierarchy. In fact, (2.21) and (2.14) indicate the following recursive relation

$$\bar{M}'\Big[\bar{\mathscr{L}}\bar{Y} - \Big(\lambda - \frac{1}{\lambda}\Big)^2 \bar{Y}\Big] = (E\bar{\mathfrak{N}})\bar{M} - \bar{M}\bar{\mathfrak{N}}. \tag{2.23}$$

Repeating such a relation we can reach the form

$$\bar{M}'\Big[\bar{\mathscr{L}}^j\bar{Y} - \Big(\lambda - \frac{1}{\lambda}\Big)^{2j} \bar{Y}\Big] = (E\bar{\mathfrak{N}})\bar{M} - \bar{M}\bar{\mathfrak{N}}, \tag{2.24}$$

where following (2.22), $\bar{\mathfrak{B}}$ and $\bar{\mathfrak{C}}$ in matrix $\bar{\mathfrak{N}}$ are expressed as

$$\begin{pmatrix} \bar{\mathfrak{B}} \\ \bar{\mathfrak{C}} \end{pmatrix} = \sum_{k=1}^{j} \Big(\lambda - \frac{1}{\lambda}\Big)^{2(j-k)} \Big(-\lambda \bar{L}_1^{-1} + \frac{1}{\lambda}\bar{L}_2^{-1}\Big)\bar{\mathscr{L}}^{k-1}\bar{Y}. \tag{2.25}$$

Thus, we can respectively take $\bar{Y} = \bar{K}_0$ and $\bar{Y} = \bar{K}_1$ in (2.24) as initial flows, and obtain the following zero curvature representation of the flow $\bar{K}_s$,

$$\bar{M}'[\bar{K}_s] = (E\bar{N}_s)\bar{M} - \bar{M}\bar{N}_s, \;\; s = 0, 1, \cdots, \tag{2.26}$$

where the elements of the matrix $\bar{N}_s$ are given by

$$\begin{pmatrix} \bar{B}_s \\ \bar{C}_s \end{pmatrix} = \sum_{k=1}^{j} \left(\lambda - \frac{1}{\lambda}\right)^{2(j-k)} \left(-\lambda \bar{L}_1^{-1} + \frac{1}{\lambda}\bar{L}_2^{-1}\right)\bar{\mathscr{L}}^{k-1}\bar{K}_0, \quad s = 2j, \tag{2.27a}$$

$$\begin{aligned} \begin{pmatrix} \bar{B}_s \\ \bar{C}_s \end{pmatrix} = &\sum_{k=1}^{j} \left(\lambda - \frac{1}{\lambda}\right)^{2(j-k)} \left(-\lambda \bar{L}_1^{-1} + \frac{1}{\lambda}\bar{L}_2^{-1}\right)\bar{\mathscr{L}}^{k-1}\bar{K}_1 \\ &+ \frac{1}{2}\left(\lambda - \frac{1}{\lambda}\right)^{2j} \begin{pmatrix} \lambda Q_n + \frac{1}{\lambda}Q_{n-1} \\ \lambda R_{n-1} + \frac{1}{\lambda}R_n \end{pmatrix}, \quad s = 2j+1 \end{aligned} \tag{2.27b}$$

and

$$\bar{A}_s = -\frac{1}{\lambda}(E-1)^{-1}(R_nE, -Q_n)\begin{pmatrix}\bar{B}_s\\ \bar{C}_s\end{pmatrix} + \bar{A}_s^{(0)}, \tag{2.28a}$$

$$\bar{D}_s = \frac{1}{\lambda}(E-1)^{-1}(R_n, -Q_nE)\begin{pmatrix}\bar{B}_s\\ \bar{C}_s\end{pmatrix} + \bar{D}_s^{(0)}, \tag{2.28b}$$

$$\bar{A}_s^{(0)} = -\bar{D}_s^{(0)} = \begin{cases}\frac{1}{2}(\lambda-\frac{1}{\lambda})^s, & s=2j,\\ \frac{1}{4}(\lambda-\frac{1}{\lambda})^{s-1}(\lambda^2-\frac{1}{\lambda^2}), & s=2j+1,\end{cases} \tag{2.28c}$$

for $j = 0, 1, \cdots$. The first three $\bar{N}_s$ are listed in Appendix A.

We note that $\bar{N}_s$ obeys the following boundary condition

$$\bar{N}_s|_{\bar{U}=0} = \begin{pmatrix}\bar{A}_s^{(0)} & 0\\ 0 & \bar{D}_s^{(0)}\end{pmatrix} \tag{2.29}$$

where $\bar{A}_s^{(0)}$ and $\bar{D}_s^{(0)}$ are given in (2.28c). In our procedure $\bar{N}_s$ is uniquely determined by the above boundary condition. This is guaranteed by the following fact (cf. [25, 29]).

Proposition 2.1. *Suppose that $\bar{X} = (\bar{X}_1, \bar{X}_2)^T$ is a vector function of $\bar{U}$ but independent of λ and $\bar{N}$ is a 2×2 matrix Laurent polynomial in λ living on $\bar{U}$. Then the matrix equation*

$$\bar{M}'[\bar{X}] = (E\bar{N})\bar{M} - \bar{M}\bar{N}, \quad \bar{N}|_{\bar{U}=0} = 0 \tag{2.30}$$

has only zero solution $\bar{X} = 0, \bar{N} = 0$.

We note that $\bar{N}_s$ derived from Sec.2.1 as combinations of $\{\bar{N}_j^{\mathrm{AL}}\}$ also satisfies (2.29). Thus two $\bar{N}_s$ are same in light of above proposition.

3. Conservation laws

3.1. *Conservation laws: Trivial in continuum limit*

Infinitely many conservation laws of an integrable system can be constructed from its Lax pair (cf. [22]), and this approach was also generalized to semi-discrete case (e.g. [20, 21, 24]). In this subsection, we will use the same approach to construct trivial (in terms of continuum limit) conservation laws of the sdAKNS hierarchy.

We begin with the spectral problem of the sdAKNS hierarchy, i.e. the AL spectral problem

$$E\bar{\phi}_1 = \lambda\bar{\phi}_1 + Q_n\bar{\phi}_2, \tag{3.1a}$$

$$E\bar{\phi}_2 = R_n\bar{\phi}_1 + \frac{1}{\lambda}\bar{\phi}_2. \tag{3.1b}$$

Setting $\bar{\omega} = \frac{\bar{\phi}_2}{\bar{\phi}_1}$, we arrive at the discrete Riccati equation [24]

$$\lambda E\bar{\omega} = \frac{1}{\lambda}\bar{\omega} - Q_n\bar{\omega}E\bar{\omega} + R_n, \tag{3.2}$$

which is solved by

$$\bar{\omega} = \sum_{j=1}^{\infty}\bar{\omega}^{(j)}\lambda^{-2j+1}, \tag{3.3}$$

with

$$\bar{\omega}^{(1)} = R_{n-1}, \quad \bar{\omega}^{(2)} = R_{n-2}, \tag{3.4a}$$

$$\bar{\omega}^{(j+1)} = E^{-1}\bar{\omega}^{(j)} - Q_{n-1}\sum_{k=1}^{j-1}\bar{\omega}^{(k)}E^{-1}\bar{\omega}^{(j-k)}, \quad j = 2,3,\cdots. \tag{3.4b}$$

From the Lax pair (2.11) we have

$$(E-1)(\ln\bar{\phi}_1) = \ln(1+\lambda^{-1}Q_n\bar{\omega}), \quad (\ln\bar{\phi}_1)_{t_s} = \bar{A}_s + \bar{B}_s\,\bar{\omega},$$

which provides a formal conservation law

$$\left[\ln(1+\lambda^{-1}Q_n\bar{\omega})\right]_{\bar{t}_s} = (E-1)(\bar{A}_s + \bar{B}_s\,\bar{\omega}). \tag{3.5}$$

Then, for the equation $\bar{U}_{\bar{t}_s} = \bar{K}_s$ in the sdAKNS hierarchy, with corresponding $\bar{A}_s$ and $\bar{B}_s$ in the above formula, we can expand (3.5) in terms of λ^2 and get

$$\partial_{\bar{t}_s}\sum_{j=1}^{\infty}\bar{\sigma}^{(j)}\lambda^{-2j} = (E-1)\sum_{j=1}^{\infty}\bar{\mathscr{J}}^{(j)}\lambda^{-2j}. \tag{3.6}$$

The coefficients of λ^{-2j} provide the infinitely many conservation laws for the equation $\bar{U}_{\bar{t}_s} = \bar{K}_s$:

$$\partial_{\bar{t}_s}\bar{\sigma}^{(j)} = (E-1)\bar{\mathscr{J}}^{(j)} \quad j = 1,2,\cdots. \tag{3.7}$$

However, under the continuum limit given in [27], all of them go to the first conservation law (i.e. $j=1$ in (B.13)) of the continuous case (we will show this in Proposition 4.4). In other words, such infinitely many conservation laws are trivial in terms of the continuum limit.

3.2. *Conservation laws: Meaningful in continuum limit*

We need to derive new forms of the conservation laws so that they are meaningful in the continuum limit. To do that, let us introduce

$$\bar{\Omega}(z) = \frac{1}{\lambda}\bar{\omega}(\lambda), \tag{3.8}$$

where λ and z are related through

$$\lambda = \sqrt{\frac{1+z}{z}}. \tag{3.9}$$

Rewriting the discrete Riccati equation (3.2) in terms of $\bar{\Omega}$ and z, we obtain

$$\frac{1}{z}\bar{\Omega} = (E^{-1}-1)\bar{\Omega} - \Big(1+\frac{1}{z}\Big)Q_{n-1}\bar{\Omega}E^{-1}\bar{\Omega} + R_{n-1}. \tag{3.10}$$

Inserting the following expansion

$$\bar{\Omega}(z) = \sum_{j=1}^{\infty}\bar{\Omega}^{(j)}z^j \tag{3.11}$$

yields

$$\bar{\Omega}^{(1)} = R_{n-1}, \quad \bar{\Omega}^{(2)} = R_{n-2}(1 - Q_{n-1}R_{n-1}) - R_{n-1}, \tag{3.12a}$$

$$\bar{\Omega}^{(j+1)} = (E^{-1} - 1)\bar{\Omega}^{(j)} - Q_{n-1}\sum_{k=1}^{j-1}\bar{\Omega}^{(k)}E^{-1}\bar{\Omega}^{(j-k)} - Q_{n-1}\sum_{k=1}^{j}\bar{\Omega}^{(k)}E^{-1}\bar{\Omega}^{(j+1-k)} \tag{3.12b}$$

for $j = 2, 3, \cdots$. The first few $\Omega^{(j)}$ are

$$\bar{\Omega}^{(1)} = R_{n-1}, \tag{3.13a}$$

$$\bar{\Omega}^{(2)} = R_{n-2}(1 - Q_{n-1}R_{n-1}) - R_{n-1}, \tag{3.13b}$$

$$\begin{aligned}\bar{\Omega}^{(3)} = {} & R_{n-1} + 2R_{n-2}(Q_{n-1}R_{n-1} - 1) + Q_{n-1}R_{n-2}^2(Q_{n-1}R_{n-1} - 1) \\ & + R_{n-3}(Q_{n-2}R_{n-2} - 1)(Q_{n-1}R_{n-1} - 1).\end{aligned} \tag{3.13c}$$

Meanwhile, the formal conservation law (3.5) can be written as

$$\left[\ln(1 + Q_n\bar{\Omega})\right]_{\bar{t}_s} = (E - 1)(\bar{\mathscr{A}}_s + \bar{\mathscr{B}}_s\bar{\Omega}), \tag{3.14}$$

where

$$\bar{\mathscr{A}}_s(z) = \bar{A}_s(\lambda)\Big|_{\lambda=\sqrt{\frac{1+z}{z}}}, \quad \bar{\mathscr{B}}_s(z) = \lambda\bar{B}_s(\lambda)\Big|_{\lambda=\sqrt{\frac{1+z}{z}}}. \tag{3.15}$$

Note that we have expansions

$$\ln(1 + Q_n\bar{\Omega}) = \sum_{j=1}^{\infty}\bar{\rho}^{(j)}z^j, \tag{3.16a}$$

$$\bar{\mathscr{A}}_s + \bar{\mathscr{B}}_s\bar{\Omega} = \bar{\mathscr{A}}_s^{(0)} + \sum_{j=1}^{\infty}\bar{J}^{(j)}z^j, \tag{3.16b}$$

where $\bar{\mathscr{A}}_s^{(0)} = \bar{A}_s^{(0)}|_{\lambda=\sqrt{\frac{1+z}{z}}}$ and $\bar{A}_s^{(0)}$ is defined by (2.28c). Then, comparing the coefficients of z^j on the both sides of (3.14), we obtain infinitely many conservation laws

$$\partial_{\bar{t}_s}\bar{\rho}^{(j)} = (E - 1)\bar{J}_s^{(j)}, \quad j = 1, 2, \cdots \tag{3.17}$$

for the equation $\bar{U}_{\bar{t}_s} = \bar{K}_s$. Explicit formulae of $\bar{\rho}^{(j)}$ can be given with the help of the following proposition (see Proposition 2 in [28]).

Proposition 3.1. *The following expansion holds,*

$$\ln\left(1 + \sum_{i=1}^{\infty}y_iz^i\right) = \sum_{j=1}^{\infty}h_j(\mathbf{y})z^j, \tag{3.18a}$$

where

$$h_j(\mathbf{y}) = \sum_{||\boldsymbol{\alpha}||=j}(-1)^{|\boldsymbol{\alpha}|-1}(|\boldsymbol{\alpha}|-1)!\frac{\mathbf{y}^{\boldsymbol{\alpha}}}{\boldsymbol{\alpha}!}, \tag{3.18b}$$

and

$$\mathbf{y}=(y_1,y_2,\cdots),\quad \boldsymbol{\alpha}=(\alpha_1,\alpha_2,\cdots),\quad \alpha_i\in\{0,1,\cdots\}, \tag{3.18c}$$

$$\mathbf{y}^{\boldsymbol{\alpha}}=\prod_{i=1}^{\infty}y_i^{\alpha_i},\quad \boldsymbol{\alpha}!=\prod_{i=1}^{\infty}(\alpha_i!),\quad |\boldsymbol{\alpha}|=\sum_{i=1}^{\infty}\alpha_i,\quad ||\boldsymbol{\alpha}||=\sum_{i=1}^{\infty}i\alpha_i. \tag{3.18d}$$

The first few of $\{h_j(\mathbf{y})\}$ *are*

$$h_1(\mathbf{y})=y_1, \tag{3.19a}$$

$$h_2(\mathbf{y})=-\frac{1}{2}y_1^2+y_2, \tag{3.19b}$$

$$h_3(\mathbf{y})=\frac{1}{3}y_1^3-y_1y_2+y_3, \tag{3.19c}$$

$$h_4(\mathbf{y})=-\frac{1}{4}y_1^4+y_1^2y_2-y_1y_3-\frac{1}{2}y_2^2+y_4. \tag{3.19d}$$

For convenience, we call $\{h_j(\mathbf{y})\}$ h-polynomials and we will see that they play quite helpful roles in our investigation.

With the help of h-polynomials, $\{\bar{\rho}^{(j)}\}$ are expressed as

$$\bar{\rho}^{(j)}=h_j(\mathbf{y}),\quad j=1,2,\cdots, \tag{3.20}$$

with $y_i=Q_n\bar{\Omega}^{(i)}$. The first two $\bar{\rho}^{(j)}$ are

$$\bar{\rho}^{(1)}=Q_nR_{n-1}, \tag{3.21a}$$

$$\bar{\rho}^{(2)}=-\frac{1}{2}Q_n\big[2R_{n-2}(Q_{n-1}R_{n-1}-1)+R_{n-1}(Q_nR_{n-1}+2)\big]. \tag{3.21b}$$

It is known that all the equations in the sdAKNS hierarchy share the same conserved densities, while the associated fluxes depend on the time part of the Lax pairs.

The infinitely many conservation laws are not trivial in the continuum limit (see Sec.4.4).

3.3. *Conservation laws: Combinatorial relation*

In this subsection, we prove that each conserved density $\bar{\rho}^{(j)}$ is a combination of certain conserved densities $\{\bar{\sigma}^{(i)}\}$ and the same combinatorial relation holds for the corresponding conservation laws. To find this relation we first prove the following lemma.

Lemma 3.1. *If*

$$\sum_{j=1}^{\infty}h_j(\mathbf{y})z^j=\sum_{s=1}^{\infty}h_s(\mathbf{x})\left(\sum_{k=1}^{\infty}(-1)^{k-1}z^k\right)^s, \tag{3.22}$$

where $\mathbf{x}=(x_1,x_2,\cdots)$*, then we have*

$$h_j(\mathbf{y})=\sum_{s=1}^{j}(-1)^{j-s}\mathrm{C}_{j-1}^{s-1}h_s(\mathbf{x}),\quad j=1,2,\cdots, \tag{3.23}$$

where

$$\mathrm{C}_m^n=\frac{m!}{n!(m-n)!},\quad m\geq n.$$

Proof. First, expanding the r.h.s. of (3.22) in terms of z and comparing the coefficients of z^j, we find

$$h_1(\mathbf{y}) = h_1(\mathbf{x}), \tag{3.24a}$$
$$h_2(\mathbf{y}) = h_2(\mathbf{x}) - h_1(\mathbf{x}), \tag{3.24b}$$
$$h_3(\mathbf{y}) = h_3(\mathbf{x}) - 2h_2(\mathbf{x}) + h_1(\mathbf{x}), \tag{3.24c}$$
$$h_4(\mathbf{y}) = h_4(\mathbf{x}) - 3h_3(\mathbf{x}) + 3h_2(\mathbf{x}) - h_1(\mathbf{x}), \tag{3.24d}$$

which cope with the formula (3.23) for $j = 1,2,3,4$. Next, let us go to prove that (3.23) holds for generic j. This is equivalent to prove

$$\left(\sum_{k=1}^{\infty}(-1)^{k-1}z^k\right)^s = \sum_{j=s}^{\infty}(-1)^{j-s}\mathrm{C}_{j-1}^{s-1}z^j. \tag{3.25}$$

Based on (3.24), let us suppose that (3.25) is true for $s \le i$. Then, when $s = i+1$ we find

$$\begin{aligned}\left(\sum_{k=1}^{\infty}(-1)^{k-1}z^k\right)^{i+1} &= \left(\sum_{k=1}^{\infty}(-1)^{k-1}z^k\right)\left(\sum_{k=1}^{\infty}(-1)^{k-1}z^k\right)^i \\ &= \left(\sum_{k=1}^{\infty}(-1)^{k-1}z^k\right)\sum_{l=i}^{\infty}(-1)^{l-i}\mathrm{C}_{l-1}^{i-1}z^l \\ &= \sum_{j=i+1}^{\infty}(-1)^{j-i-1}\left(\sum_{l=i}^{j-1}\mathrm{C}_{l-1}^{i-1}\right)z^j.\end{aligned}$$

Then, by the combinatorial formula

$$\sum_{k=0}^{m}\mathrm{C}_{n+k}^{n} = \mathrm{C}_{n+m+1}^{n+1},$$

we immediately obtain

$$\left(\sum_{k=1}^{\infty}(-1)^{k-1}z^k\right)^{i+1} = \sum_{j=i+1}^{\infty}(-1)^{j-(i+1)}\mathrm{C}_{j-1}^{i}\,z^j,$$

which means (3.25) is true for $s = i+1$. Thus, with the help of mathematical inductive method we complete the proof. □

Now, noting that employing h-polynomials $\{h_j(\mathbf{y})\}$ we have $\bar{\rho}^{(j)} = h_j(\mathbf{y})$ and $\bar{\sigma}^{(j)} = h_j(\mathbf{x})$ where $y_i = Q_n\bar{\Omega}^{(i)}$ and $x_i = Q_n\bar{\omega}^{(i)}$, we immediately reach the following relation for $\bar{\rho}^{(j)}$ and $\bar{\sigma}^{(j)}$.

Proposition 3.2. *The conserved densities $\bar{\rho}^{(j)}$ and $\bar{\sigma}^{(j)}$ enjoy the following combinatorial relation,*

$$\bar{\rho}^{(j)} = \sum_{s=1}^{j}(-1)^{j-s}\mathrm{C}_{j-1}^{s-1}\bar{\sigma}^{(s)}, \quad j = 1,2,\cdots. \tag{3.26}$$

4. Continuum limits

Since the integrable discretizations usually break the original dispersion relations, it is not easy, in general, to give a uniform continuum limit which maps the discrete integrable systems together with their integrability characteristics to the continuous counterparts (cf. [12,15–18,30]). In [27] we have presented a unform continuum limit which sends the whole sdAKNS hierarchy to the AKNS hierarchy. In the following we use the scheme to investigate first the Lax pairs and then conservation laws of the sdAKNS hierarchy. The corresponding results of the continuous AKNS hierarchy are listed in Appendix B as reference.

4.1. *Plan*

Our plan for the continuum limit runs below [27]:

- Replacing Q_n and R_n with hq_n and hr_n, where h is the real spacing parameter.
- Let $n \to \infty$ and $h \to 0$ such that nh finite.
- Define continuous variable $x = x_0 + nh$. Then for a scalar function, for example, q_n, one has $q_{n+j} = q(x + jh)$. For convenience we take $x_0 = 0$.
- Define temporal coordinate relation $t_s = h^s \bar{t}_s$ for $s = 0, 1, \cdots$.
- Continuous spectral parameter η is defined by $\lambda = e^{h\eta}$.

Here we remark the following. The above continuum limit scheme means that the discretization from the AKNS hierarchy to the sdAKNS hierarchy is direct. This is different from those discrete systems obtained from the so-called Miwa transformation [14]. Miwa's transformation led to a discretization for the famous Sato theory [5–9]. In that approach discrete independent variables are directly added into the continuous dispersion relation, which essentially breaks the original spatial and temporal independence. As a consequence, when taking continuum limit to recover a integrable continuous nonlinear system from a discrete system obtained through the discrete Sato theory, one needs to allocate independent variables so that the new independent variables coincide with the desired continuous dispersion relation.

4.2. *Hierarchy*

In Ref. [27] we have shown that in the above continuum limit the whole sdAKNS hierarchy (2.7) goes to the continuous AKNS hierarchy (B.5). Let us briefly recall these results.

In the continuum limit described in Sec.4.1, it can be shown that

$$\bar{K}_0 = K_0 h + O(h^2), \tag{4.1a}$$

$$\bar{K}_1 = K_1 h^2 + O(h^3), \tag{4.1b}$$

and

$$\bar{\mathscr{L}} = L^2 h^2 + O(h^3). \tag{4.2}$$

Then, from the recursive structure of the sdAKNS hierarchy (2.7), the continuum limits for the flows are

$$\bar{K}_s = K_s h^{s+1} + O(h^{s+2}), \quad s = 0, 1, \cdots, \tag{4.3}$$

and at the level of equations we have the following.

Proposition 4.1. *Under the continuum limit described in Sec.4.1, we have*

$$\bar{U}_{\bar{t}_s} - \bar{K}_s = (u_{t_s} - K_s) h^{s+1} + O(h^{s+2}), \quad s = 0, 1, \cdots. \tag{4.4}$$

4.3. *Lax pairs*

Based on the continuum limit designed in Sec.4.1, it is easy to find that the continuum limit of the spectral problem (2.11a) is

$$E\bar{\Phi} - \bar{M}\bar{\Phi} = (\Phi_x - M\Phi)h + O(h^2). \tag{4.5}$$

To investigate the relations in the continuum limit between the time parts of the Lax pairs (B.1) and (2.11), we rewrite (B.3b) as the following form:

$$\begin{pmatrix} B_s \\ C_s \end{pmatrix} = \sigma_3 \sum_{k=1}^{j} (2\eta)^{2(j-k)}(L+2\eta)L^{2(k-1)} \begin{pmatrix} q \\ -r \end{pmatrix}, \quad s = 2j, \tag{4.6a}$$

$$\begin{pmatrix} B_s \\ C_s \end{pmatrix} = \sigma_3 \sum_{k=1}^{j} (2\eta)^{2(j-k)}(L+2\eta)L^{2k-1} \begin{pmatrix} q \\ -r \end{pmatrix} + (2\eta)^{2j} \begin{pmatrix} q \\ r \end{pmatrix}, \quad s = 2j+1 \tag{4.6b}$$

for $j = 0, 1, \cdots$. A direct calculation yields

$$\lambda - \frac{1}{\lambda} = 2\eta h + O(h^2), \quad -\lambda \bar{L}_1^{-1} + \frac{1}{\lambda}\bar{L}_2^{-1} = (L+2\eta)h + O(h^2). \tag{4.7}$$

Then, from the expression (2.27) we have

$$\begin{pmatrix} \bar{B}_s \\ \bar{C}_s \end{pmatrix} = \left[\sigma_3 \sum_{k=1}^{j} (2\eta)^{2(j-k)}(L+2\eta)L^{2(k-1)} \begin{pmatrix} q \\ -r \end{pmatrix} \right] h^{2j} + O(h^{2j+1}), \quad s = 2j,$$

$$\begin{pmatrix} \bar{B}_s \\ \bar{C}_s \end{pmatrix} = \left[\sigma_3 \sum_{k=1}^{j} (2\eta)^{2(j-k)}(L+2\eta)L^{2k-1} \begin{pmatrix} q \\ -r \end{pmatrix} + (2\eta)^{2j} \begin{pmatrix} q \\ r \end{pmatrix} \right] h^{2j+1} + O(h^{2j+2}), \quad s = 2j+1$$

for $j = 0, 1, \cdots$, namely

$$\begin{pmatrix} \bar{B}_s \\ \bar{C}_s \end{pmatrix} = \begin{pmatrix} B_s \\ C_s \end{pmatrix} h^s + O(h^{s+1}), \quad s = 0, 1, \cdots. \tag{4.8}$$

Next, substituting (4.8) into (2.28) we find

$$\bar{A}_s = A_s h^s + O(h^{s+1}), \ \ \bar{D}_s = D_s h^s + O(h^{s+1}), \quad s = 0, 1, \cdots. \tag{4.9}$$

Therefore, from the relations (4.8) and (4.9) we conclude that

$$\bar{N}_s = N_s h^s + O(h^{s+1}), \quad s = 0, 1, \cdots. \tag{4.10}$$

As a result, we have the following.

Proposition 4.2. *Under the continuum limit described in Sec.4.1, for Lax pairs we have*

$$E\bar{\Phi} - \bar{M}\bar{\Phi} = (\Phi_x - M\Phi)h + O(h^2), \tag{4.11a}$$

$$\bar{\Phi}_{\bar{t}_s} - \bar{N}_s\bar{\Phi} = (\Phi_{t_s} - N_s\Phi)h^s + O(h^{s+1}), \quad s = 0, 1, \cdots. \tag{4.11b}$$

4.4. *Conservation laws*

In this part we will investigate continuum limit of the infinitely many conservation laws obtained in Sec.3 for the sdAKNS hierarchy (2.7). We need to examine the relations between $\bar{\Omega}^{(j)}$ and $\omega^{(j)}$, $\bar{\Omega}$ and ω, the Riccati equations (3.10) and (B.7), the formal conservation laws (3.14) and (B.11), and the infinitely many conservation laws (3.17) and (B.13), respectively.

First, for the relation between $\bar{\Omega}^{(j)}$ and $\omega^{(j)}$, we have the following result.

Lemma 4.1. *Under the continuum limit described in Sec.4.1, we have*

$$\bar{\Omega}^{(j)} = \omega^{(j)} h^j + O(h^{j+1}), \quad j = 1, 2, \cdots, \tag{4.12}$$

where $\bar{\Omega}$ *and* $\omega^{(j)}$ *are defined in* (3.12) *and* (B.9)*, respectively.*

Proof. We use mathematical inductive method. First, for $j = 1, 2$, we find that

$$\begin{aligned}
&\bar{\Omega}^{(1)} = R_{n-1} = rh + O(h^2), \\
&\bar{\Omega}^{(2)} = R_{n-2}(1 - Q_{n-1}R_{n-1}) - R_{n-1} = r_x h^2 + O(h^3),
\end{aligned}$$

which means the relation (4.12) holds for $j = 1, 2$. Then we suppose

$$\bar{\Omega}^{(i)} = \omega^{(i)} h^i + O(h^{i+1})$$

is true for any $i \leq j$. Then, from the recursive relation (3.12b) we find

$$\begin{aligned}
\bar{\Omega}^{(j+1)} &= (E^{-1} - 1)\bar{\Omega}^{(j)} - Q_{n-1} \sum_{k=1}^{j-1} \bar{\Omega}^{(k)} E^{-1} \bar{\Omega}^{(j-k)} - Q_{n-1} \sum_{k=1}^{j} \bar{\Omega}^{(k)} E^{-1} \bar{\Omega}^{(j+1-k)} \\
&= \Big(-\omega_x^{(j)} - q \sum_{k=1}^{j-1} \omega^{(k)} \omega^{(j-k)} \Big) h^{(j+1)} + O(h^{j+2}) \\
&= \omega^{(j+1)} h^{(j+1)} + O(h^{j+2}),
\end{aligned}$$

where the last equality coincides with the recursive relation (B.9b). Therefore (4.12) holds for any $j \geq 1$. □

Now we come to the relation between $\bar{\Omega}$ and ω. Let us go back to the expansion (3.11), i.e.

$$\bar{\Omega}(z) = \sum_{j=1}^{\infty} \bar{\Omega}^{(j)} z^j. \tag{4.13}$$

Noting that

$$z = \frac{1}{\lambda^2 - 1} = (2\eta)^{-1} h^{-1} + O(1), \tag{4.14}$$

and inserting (4.12) and (4.14) into (4.13), we have

$$\bar{\Omega}(z) = \sum_{j=1}^{\infty} \bar{\Omega}^{(j)} z^j = \sum_{j=1}^{\infty} \omega^{(j)} (2\eta)^{-j} + O(h), \tag{4.15}$$

namely,

$$\bar{\Omega} = \omega + O(h). \tag{4.16}$$

This gives the relation between $\bar{\Omega}$ and ω.

Next, let us look at the relation between the Riccati equations (3.10) and (B.7). Substituting (4.16) into the Riccati equation (3.10) yields

$$\begin{aligned}&\frac{1}{z}\bar{\Omega}-\Big[(E^{-1}-1)\bar{\Omega}-\Big(1+\frac{1}{z}\Big)Q_{n-1}\bar{\Omega}E^{-1}\bar{\Omega}+R_{n-1}\Big]\\&=\big[2\eta\omega-(\omega_x-q\omega^2+r)\big]h+O(h^2),\end{aligned}\tag{4.17}$$

which means the Riccati equation (3.10) goes to the continuous (B.7) in the continuum limit.

We also need to check the formal conservation laws (3.14) and (B.11). For the l.h.s. of (3.14), by using (4.16) it is easy to see that

$$\ln(1+Q_n\bar{\Omega})=(q\omega)h+O(h^2).\tag{4.18}$$

Meanwhile, from the relations (3.15) and (4.10) we immediately reach

$$\bar{\mathscr{A}}_s=Ah^s+O(h^{s+1}),\ \ \bar{\mathscr{B}}_s=Bh^s+O(h^{s+1}),\tag{4.19}$$

which provides

$$\bar{\mathscr{A}}_s+\bar{\mathscr{B}}_s\bar{\Omega}=(A_s+B_s\omega)h^s+O(h^{s+1}).\tag{4.20}$$

Thus, for the relation of the formal conservation laws of (3.14) and (B.11), we have the following.

Lemma 4.2. *Under the continuum limit described in Sec.4.1, we have*

$$\begin{aligned}&\big[\ln(1+Q_n\bar{\Omega}\big]_{\bar{t}_s}-(E-1)(\bar{\mathscr{A}}_s+\bar{\mathscr{B}}_s\bar{\Omega})\\&=\big[(q\omega)_{t_s}-(A_s+B_s\omega)_x\big]h^{s+1}+O(h^{s+2}),\end{aligned}\tag{4.21}$$

which describes the relation of the two formal conservation laws.

Finally, we focus on the continuum limits of the infinitely many conservation laws (3.17), i.e.

$$\partial_{\bar{t}_s}\bar{\rho}^{(j)}=(E-1)\bar{J}_s^{(j)},\quad j=1,2,\cdots.\tag{4.22}$$

The common conserved densities $\bar{\rho}^{(j)}$ are determined by (3.16a) with (3.20) and $y_i\doteq Q_n\bar{\Omega}^{(i)}$. To investigate the continuum limit of $\bar{\rho}^{(j)}$, we introduce *degrees* of functions (cf. [27]).

Definition 4.1. Under the plan described in Sec.4.1, a function $\bar{F}(\bar{U})$ can be expanded as a series of h. The order of leading term of the series is called the *degree* of $\bar{F}(\bar{U})$, denoted by $\deg\bar{F}$.

For example,

$$\deg Q_n=1,\quad \deg\bar{\Omega}^{(i)}=i,\quad \deg y_i=i+1,\quad \deg\bar{t}_s=s,\quad \deg z=-1.$$

Now, looking at the definition (3.18b) of h-polynomials $h_j(\mathbf{y})$ (as examples see (3.19)), since $\deg y_i=i+1$, we can find that $\deg h_j(\mathbf{y})=\deg y_j=j+1$ and

$$\lim_{h\to0}\frac{h_j(\mathbf{y})}{h^{j+1}}=\lim_{h\to0}\frac{y_j}{h^{j+1}}.$$

Since $y_j\doteq Q_n\bar{\Omega}^{(j)}=q\omega^{(j)}h^{j+1}+O(h^{j+2})$, we immediately find

$$\bar{\rho}^{(j)}=h_j(\mathbf{y})=q\omega^{(j)}h^{j+1}+O(h^{j+2}),\tag{4.23}$$

which means in the continuum limit we have $\bar{\rho}^{(j)}\to\rho^{(j)}=q\omega^{(j)}$ and $\deg\bar{\rho}^{(j)}=j+1$. Next, to balance the degrees of both sides of (4.22), $\deg\bar{J}_s^{(j)}$ must be $j+s$. Therefore, we can suppose $\bar{J}_s^{(j)}$ to be the following form

$$\bar{J}_s^{(j)} = W_s^{(j)} h^{j+s} + O(h^{j+s+1}). \tag{4.24}$$

This means, in light of (4.14), we have

$$\bar{J}_s^{(j)} z^j = W_s^{(j)} (2\eta)^{-j} h^s + O(h^{s+1}), \tag{4.25}$$

and further, from (3.16b) we get

$$\bar{\mathscr{A}}_s + \bar{\mathscr{B}}_s \bar{\Omega} - \bar{\mathscr{A}}_s^{(0)} = \sum_{j=1}^{\infty} \bar{J}_s^{(j)} z^j = \left(\sum_{j=1}^{\infty} W_s^{(j)} (2\eta)^{-j} \right) h^s + O(h^{s+1}).$$

On the other hand, from (4.20) and (B.12) we find

$$\begin{aligned} \bar{\mathscr{A}}_s + \bar{\mathscr{B}}_s \bar{\Omega} - \bar{\mathscr{A}}_s^{(0)} &= \left[A_s + B_s \omega - \frac{1}{2}(2\eta)^s \right] h^s + O(h^{s+1}) \\ &= \left(\sum_{j=1}^{\infty} J_s^{(j)} (2\eta)^{-j} \right) h^s + O(h^{s+1}). \end{aligned}$$

Thus, comparing the term of $(2\eta)^{-j}$ immediately yields $W_s^{(j)} = J_s^{(j)}$, which gives rise to

$$\bar{J}_s^{(j)} = J_s^{(j)} h^{j+s} + O(h^{j+s+1}). \tag{4.26}$$

Now we can sum up the discussion of this subsection.

Proposition 4.3. *Under the continuum limit described in Sec.4.1, for the infinitely many conservation laws of the sdAKNS hierarchy we have*

$$\bar{\rho}^{(j)} = \rho^{(j)} h^{j+1} + O(h^{j+2}), \tag{4.27a}$$

$$\bar{J}_s^{(j)} = J_s^{(j)} h^{j+s} + O(h^{j+s+1}), \tag{4.27b}$$

and

$$\partial_{\bar{t}_s} \bar{\rho}^{(j)} - (E-1)\bar{J}_s^{(j)} = (\partial_{t_s} \rho^{(j)} - \partial_x J_s^{(j)}) h^{j+s+1} + O(h^{j+s+2}), \tag{4.28}$$

for $s = 0, 1, \cdots$ and $j = 1, 2, \cdots$, which describe the relation of the two sets of infinitely many conservation laws.

At the end of this part, let us take a look at the continuum limit of the conserved densities $\bar{\sigma}^{(j)} = h_j(\mathbf{x})$ with $x_i \doteq Q_n \bar{\omega}^{(i)}$, which were derived in Sec.3.1 (also see [24]). Obviously,

$$\bar{\omega}^{(1)} = R_{n-1} = hr + O(h^2), \quad \bar{\omega}^{(2)} = R_{n-2} = hr + O(h^2),$$

which means $\deg \bar{\omega}^{(1)} = \deg \bar{\omega}^{(2)} = 1$. Then, from the recursive structure (3.4) we find

$$\deg \bar{\omega}^{(j)} \equiv 1, \quad j = 1, 2, \cdots,$$

and

$$\bar{\omega}^{(j)} = hr + O(h^2), \quad j = 1, 2, \cdots.$$

Thus we get

$$x_j = qrh^2 + O(h^3), \quad j = 1, 2, \cdots,$$

and then from the definition of $h_j(\mathbf{x})$ we obtain

$$h_j(\mathbf{x}) \equiv qrh^2 + O(h^3), \quad j = 1, 2, \cdots.$$

Proposition 4.4. *All the conserved densities $\bar{\sigma}^{(j)} = h_j(\mathbf{x})$ with $x_i \doteq Q_n\bar{\omega}^{(i)}$, which are derived in Sec.3.1 (also see [24]) for the AL hierarchy as well as for the sdAKNS hierarchy, are trivial in the sense that under the continuum limit described in Sec.4.1*

$$\bar{\sigma}^{(j)} \equiv qrh^2 + O(h^3), \quad j = 1, 2, \cdots.$$

In other words, in our continuum limit, all of the conserved densities $\{\bar{\sigma}^{(j)}\}$ go to $\rho^{(1)} = qr$, which is the first conserved density of the AKNS hierarchy.

5. Conclusions

In this paper we have investigated the sdAKNS hierarchy, and specifically their Lax pairs and infinitely many conservation laws, as well as the corresponding continuum limits. The Lax pairs and conservation laws of the sdAKNS hierarchy were rederived so that they cope with their continuous counterparts in continuum limit. We performed a uniform continuum limit in which the sdAKNS hierarchy, their Lax pairs and infinitely many conservation laws go to their continuous counterparts of the AKNS system. In this continuum limit scheme the spatial and temporal independence is kept. The same scheme has been also used to explain the structure deformation of symmetry algebra [27]. Such structure deformation of Lie algebras usually happens in semi-discrete cases and suitable continuum limit schemes are useful in the explanation of the structure deformation (see also the example of the semi-discrete Kadomtsev–Petviashvili hierarchy [10]). For the sdAKNS hierarchy, Hamiltonian structures and their continuum limit will be investigated later.

Finally, a further comment is given for conservation laws. The known infinitely many conservation laws derived in [24] for the AL hierarchy as well as for the sdAKNS hierarchy are trivial in terms the continuum limit. We have shown that all the conserved densities $\bar{\sigma}^{(j)}$ go to the same $\rho^{(1)} = qr$ which is the first conserved density of the AKNS hierarchy. For the sdAKNS hierarchy, we have given *new* forms of the conservation laws which agree with their continuous counterparts. They are related to those trivial ones through explicit combinatorial relation. We made use of degrees of functions to investigate continuum limits and h-polynomials also play quite helpful roles in our investigation.

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Appendix A. The first three $\bar{N}_s$

Here we give the first three of $\bar{N}_s$:

$$\bar{N}_0 = \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & -\frac{1}{2} \end{pmatrix}, \tag{A.1a}$$

$$\bar{N}_1 = \frac{1}{2}\begin{pmatrix} \frac{1}{2}\lambda^2 - Q_nR_{n-1} - \frac{1}{2}\lambda^{-2} & Q_n\lambda + Q_{n-1}\lambda^{-1} \\ R_{n-1}\lambda + R_n\lambda^{-1} & -\frac{1}{2}\lambda^2 - Q_{n-1}R_n + \frac{1}{2}\lambda^{-2} \end{pmatrix}, \tag{A.1b}$$

$$\bar{N}_2 = \begin{pmatrix} \frac{1}{2}\lambda^2 - (1+Q_nR_{n-1}) + \frac{1}{2}\lambda^{-2} & Q_n\lambda - Q_{n-1}\lambda^{-1} \\ R_{n-1}\lambda - R_n\lambda^{-1} & -\frac{1}{2}\lambda^2 + (1+Q_{n-1}R_n) - \frac{1}{2}\lambda^{-2} \end{pmatrix}. \tag{A.1c}$$

Appendix B. Results of the AKNS hierarchy

B.1. *The AKNS hierarchy and Lax pairs*

The AKNS spectral problem coupled with a time evolution part reads [1]

$$\Phi_x = M\Phi, \quad M = \begin{pmatrix} \eta & q \\ r & -\eta \end{pmatrix}, \quad u = \begin{pmatrix} q \\ r \end{pmatrix}, \quad \Phi = \begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix}, \tag{B.1a}$$

$$\Phi_{t_s} = N_s\Phi, \quad N_s = \begin{pmatrix} A_s & B_s \\ C_s & -A_s \end{pmatrix}, \quad s = 0,1,2\cdots, \tag{B.1b}$$

where η is a spectral parameter independent of time, $q = q(x,t)$ and $r = r(x,t)$ are potential functions and A_s, B_s and C_s are polynomials of η depending on q, r and their derivatives. The sub-s is used to indicate the s-th equation in the AKNS hierarchy. To obtain the AKNS hierarchy, starting from the zero curvature equation

$$\partial_{t_s}M - \partial_x N_s + [M, N_s] = 0, \quad s = 1,2,\cdots, \tag{B.2}$$

where $[M,N_s] = MN_s - N_sM$, one can expand

$$\begin{pmatrix} B_s \\ C_s \end{pmatrix} = \sum_{k=1}^{s} \begin{pmatrix} b_k \\ c_k \end{pmatrix} (2\eta)^{s-k}$$

and then rewrite (B.2) as the following,

$$u_{t_s} = K_s = L^s \begin{pmatrix} q \\ -r \end{pmatrix}, \tag{B.3a}$$

$$\begin{pmatrix} B_s \\ C_s \end{pmatrix} = \sigma_3 \sum_{k=1}^{s} L^{k-1} \begin{pmatrix} q \\ -r \end{pmatrix} (2\eta)^{s-k}, \tag{B.3b}$$

$$A_s = -\partial_x^{-1}(r,-q)\begin{pmatrix} B_s \\ C_s \end{pmatrix} + \frac{1}{2}(2\eta)^s, \tag{B.3c}$$

where L is the recursion operator defined as

$$L = -\sigma\partial_x - 2\sigma_3 \begin{pmatrix} q \\ r \end{pmatrix} \partial_x^{-1}(r,q), \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{B.4}$$

(B.3a) is referred to as the AKNS hierarchy. The hierarchy can start from $s=0$ by defining $K_0 = (q,-r)^T$. Thus, the AKNS hierarchy is expressed as

$$u_{t_s} = K_s = L^s \begin{pmatrix} q \\ -r \end{pmatrix}, \quad s = 0,1,\cdots. \tag{B.5}$$

(B.1) is called the Lax pair of the hierarchy (B.5).

B.2. *Conservation laws*

The infinitely many conservation laws of the AKNS hierarchy can be constructed from their Lax pairs (cf. [22]). Starting from the AKNS spectral problem (B.1a), i.e.

$$\phi_{1,x} = \eta\phi_1 + q\phi_2, \tag{B.6a}$$
$$\phi_{2,x} = r\phi_1 - \eta\phi_2, \tag{B.6b}$$

and setting $\omega = \frac{\phi_2}{\phi_1}$, we are able to obtain a Riccati equation

$$2\eta\omega = -\omega_x - q\omega^2 + r, \tag{B.7}$$

which is solved by a series-form

$$\omega = \sum_{j=1}^{\infty} \omega^{(j)} (2\eta)^{-j} \tag{B.8}$$

with

$$\omega^{(1)} = r, \quad \omega^{(2)} = -r_x, \tag{B.9a}$$
$$\omega^{(j+1)} = -\omega_x^{(j)} - q\sum_{k=1}^{j-1} \omega^{(k)}\omega^{(j-k)}, \quad j = 2,3,\cdots. \tag{B.9b}$$

Next, from the Lax pair (B.1) one can find

$$(\ln\phi_2)_x = \eta + q\omega, \ \ (\ln\phi_2)_{t_s} = A_s + B_s\omega, \tag{B.10}$$

where A_s and B_s are expressed by (B.3c) and (B.3b), respectively. The compatibility condition $(\ln\phi_2)_{x,t_s} = (\ln\phi_2)_{t_s,x}$ leads to a formal conservation law

$$(q\omega)_{t_s} = (A_s + B_s\omega)_x. \tag{B.11}$$

To derive infinitely many conservation laws, one needs to insert (B.8) into (B.11) and expand (B.11) into a series in terms of 2η. After expansion one finds

$$A_s + B_s\omega = \frac{1}{2}(2\eta)^s + \sum_{j=1}^{\infty} J_s^{(j)} (2\eta)^{-j}, \tag{B.12}$$

where the term $\frac{1}{2}(2\eta)^s$ comes from $A_s|_{u=0}$ and contributes nothing to the conservation laws. The coefficients of every different power of 2η in (B.11) compose the infinitely many conservation laws

$$\partial_{t_s}\rho^{(j)} = \partial_x J_s^{(j)}, \quad j = 1,2,\cdots. \tag{B.13}$$

We note that the infinitely many conserved densities $\{\rho^{(j)}\}$ are shared by all the equations in the AKNS hierarchy. The first three of the conserved densities are

$$\rho^{(1)} = qr, \;\; \rho^{(2)} = -qr_x, \;\; \rho^{(3)} = qr_{xx} - q^2r^2. \tag{B.14}$$

The associated fluxes depend on which equation is considered in the hierarchy.

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