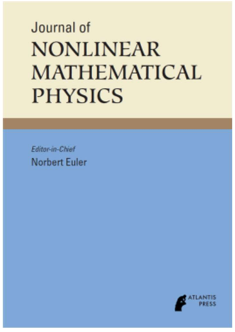

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A generalized Schrödinger equation via a complex Lagrangian of electrodynamics

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In this paper we give a generalized form of the Schrödinger equation in the relativistic case, which contains a generalization of the Klein-Gordon equation. By complex Legendre transformation, the complex Lagrangian of electrodynamics produces a complex relativistic Hamiltonian H of electrodynamics, on the holomorphic cotangent bundle $T'^{*}M$. By a special quantization process, a relativistic time dependent Schrödinger equation, in the adapted frames of $(T'^{*}M,H)$ is obtained. This generalized Schrödinger equation can be expressed with respect to the Laplace operator of the complex Hamilton space $(T'^{*}M,H)$. Finally, under some additional conditions on the proper time s of the complex space-time M and the time parameter t along the quantum state, by the method of separation of variables, we obtain two classes of solutions for the Schrödinger equation, one for the weakly gravitational complex curved space M, and the second in the complex space-time with Schwarzschild metric.

Keywords: Schrödinger equation; complex Lagrangian; $\mathscr{L}-$ duality; complex relativistic Hamiltonian.

2010 Mathematics Subject Classification: 53B40, 53C60, 53Z05

1. Introduction

We begin with a brief introduction to the fundamental equations of quantum mechanics, ([9, 11, 14, 18], etc).

In classical mechanics the dynamic evolution of a particle is given by its trajectory $x(t)$ governed by Newton's law $m\ddot{x} = -\nabla V$. In quantum mechanics the analogue of the Newton's law is the Schrödinger equation

$$-\frac{\hbar^2}{2m}\nabla^2\psi = i\hbar\frac{\partial\psi}{\partial t}, \tag{1.1}$$

where $\psi(\vec{r},t)$ is a complex field, called the wave function (or quantum state) of the quantum system, $\vec{r}(x,y,z)$ is the position vector of the state, ∇^2 is the Laplacian and $\hbar$ is Plank's constant. This (free) Schrödinger equation is linear with constant coefficients and a solution can be derived by Fourier transformation, and it has the form $\psi(\vec{r},t) = A\cdot e^{\frac{i}{\hbar}(\vec{p}\cdot\vec{r}-Et)}$, where $\vec{p} = (p_x, p_y, p_z)$ is the momentum and E is the kinetic energy.

Considering the operator $\hat{H} = -\frac{\hbar^2}{2m}\nabla^2$, which is the free Hamiltonian of the system, and taking into account that in classical mechanics the energy is $E = \frac{p^2}{2m}$, it is easy to see that $\hat{H}\psi = E\psi$. This means that the energy is an eigenvalue of the state ψ for the free Hamiltonian.

Moreover, from $E - \frac{p^2}{2m} = 0$, the Schrödinger equation results exactly by the quantization procedure

$$E \to i\hbar\frac{\partial}{\partial t}; \;\; p_x \to i\hbar\frac{\partial}{\partial x}; \;\; p_y \to i\hbar\frac{\partial}{\partial y}; \;\; p_z \to i\hbar\frac{\partial}{\partial z}. \tag{1.2}$$

The same Schrödinger equation can be derived in terms of Lagrangian or Hamiltonian mechanics (via the Legendre transformation $H = L - \vec{p}\cdot\vec{\dot{r}}$) by a similar quantization $H \to i\hbar\frac{\partial}{\partial t}$; $p_x \to i\hbar\frac{\partial}{\partial x}$; $p_y \to i\hbar\frac{\partial}{\partial y}$; $p_z \to i\hbar\frac{\partial}{\partial z}$, where H is the Hamiltonian function.

An immediate generalization is the Schrödinger equation in the presence of a potential energy $V(\vec{r},t)$,

$$\hat{H}\psi = i\hbar\frac{\partial\psi}{\partial t}, \quad \text{with } \hat{H} = -\frac{\hbar^2}{2m}\nabla^2 + V(\vec{r},t). \tag{1.3}$$

When the potential energy V depends only on $\vec{r}$, (i.e. $V = V(x,y,z)$), (1.3) is known as the time independent Schrödinger equation.

The next generalization is the Schrödinger equation for a particle in an electromagnetic field $(\Phi, -\vec{A}) := (\Phi, -A^1, -A^2, -A^3)$. In this case the Hamiltonian function is $H = \frac{1}{2m}(\vec{p} - q\vec{A})^2 + \frac{q^2}{2m}\Phi^2$, with q the electric charge.

Finally, these things can be designed in a curved space (M, g_{ij}), namely starting from Hamilton-Jacobi equation, in [11], B. Carter found a more general Hamiltonian

$$H_q = \frac{1}{2m}g^{kj}(p_k - qA_k)(p_j - qA_j) \tag{1.4}$$

and then, by the same formal substitutions $H_q \to i\hbar\frac{\partial}{\partial t}$ and $p_k \to i\hbar\frac{\partial}{\partial x^k}$ in (1.4), the corresponding Schrödinger equation was obtained.

We remark that in all these situations it is worked in terms of the classical mechanics, thus the obtained Schrödinger equation is called *nonrelativistic*.

Considering the relativistic mechanics, the problem is more complicated. First of all the relativistic energy and momentum are related by $E^2 = p^2c^2 + m^2c^4$ and, therefore, the same substitutions (1.2) lead to an equation in which the right side of (1.1) contains now the second order derivatives of the wave function ψ. The outcome is the *relativistic* Schrödinger equation, or otherwise known as the Klein-Gordon equation $-\hbar^2\frac{\partial^2}{\partial t^2}\psi = -c^2\hbar^2\nabla^2\psi + c^4m^2\psi$. Moreover, using the D'Alembert operator $\Box = \frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \nabla^2$, it can be rewritten as $(\Box + \frac{m^2c^2}{\hbar^2})\psi = 0$. An extension of the Klein-Gordon equation on holomorphic tangent bundle was analyzed by the authors in [2, 25].

On the other hand, writing the square root of energy in the form

$$\sqrt{p^2c^2 + m^2c^4} = c(\gamma_0 mc + \gamma_1 p_x + \gamma_2 p_y + \gamma_3 p_z),$$

and by the substitutions (1.2), P. Dirac reached the equation $(i\gamma^k\partial_k - m)\psi = 0$, known as Dirac equation.

The main purpose of the present paper is to give a generalized form of the Schrödinger equation in the relativistic case, which obviously contains a generalization of the Klein-Gordon equation.

Let us explain in a few words in what this generalization consists of and what are its advantages. By complexification, the real space-time generates a two dimensional complex manifold M, with the local coordinates (z^1, z^2). In the previous papers [2, 25] we studied the relativistic metrics $g_{i\bar{j}}$ of complex Finsler type, on the holomorphic tangent bundle $T'M$. Two of these proved particularly useful for relativistic applications, namely the complex weakly gravitational metric and the hyperbolic version of the Schwarzschild metric. Although they are purely Hermitian metrics on $T'M$ (i.e. $g_{i\bar{j}}(z)$ depend only on the position on M), they can be viewed as prolongations of Hermitian metrics on M. In a curved space with a relativistic metric $g_{i\bar{j}}$, we consider the complex Lagrangian L of electrodynamics (3.1) on $T'M$, for which a thorough study was made in [23]. The geometry of $(T'M, L)$ is "linearized" by the adapted frames $\{\frac{\delta}{\delta z^k}, \frac{\partial}{\partial \eta^k}\}_{k=1,2}$ of a complex nonlinear connection, according to the general theory of complex Lagrange spaces, [23]. By complex Legendre transformation, the complex Lagrangian of electrodynamics produces complex momentum $(\zeta_k)_{k=1,2}$, which are sections in the dual holomorphic cotangent bundle T'^*M, and also a complex relativistic Hamiltonian H of electrodynamics, given by (3.9). The geometry of (T'^*M, H) is "linearized" in adapted frames $\{\frac{\delta^*}{\delta z^k}, \frac{\partial}{\partial \zeta_k}\}_{k=1,2}$ of a complex nonlinear connection, according to the $\mathscr{L}-$dual process, [23]. Further on, by a special quantization process, a two times dependent Schrödinger equation, in the adapted frames of (T'^*M, H), is obtained. This generalized Schrödinger equation can be expressed with respect to the Laplace operator of the complex Hamilton space (T'^*M, H), (formula (4.2)), and, for a special form of the wave function ψ, it is reduced to a kind of Klein-Gordon equation, (formula (4.5)). Moreover, under the Kähler assumption we get a very closely writing of the Schrödinger equation with the well known equation (1.3). Note that the obtained equation contains the relativistic space geometry introduced by the metric tensor $g_{i\bar{j}}$ and also the electrodynamics induced by the relativistic Lagrangian L.

Of course, the main problem remains that of finding the classes of solutions for this generalized Schrödinger equation. We point this out in the last section of the paper. For some additional conditions on the proper time s of the complex space-time M and the time parameter t along the quantum state, by separation of variables, we have obtained two classes of fundamental solutions for the equation (4.1), one for the weakly gravitational complex curved space M, and the second in the complex space-time with Schwarzschild metric.

Given that the base space is a curved one by a gravitational metric and that the Schrödinger equation is derived from a complex Hamiltonian of electrodynamics, we can say that this theory, exposed in terms of complex Hamilton geometry, describes the motion of a complex quantum particle in relativistic space-time.

2. Preliminaries

The geometries of real Lagrange and Hamilton spaces, (particularly Finsler and Cartan spaces) (see [5,6,21,22]) know many applications in Physics (for more details see [4,7,10,12,13,16,17,20], etc). Complex Lagrange and Hamilton geometries, (particularly Finsler and Cartan geometries) are more recent (see [1–3,23–25,28,29]) and we hope that the subject described in this paper will attract more interest for them.

We begin by setting the main notations and notions from the theory of two-dimensional complex Lagrange and Hamilton geometries, although these are available for any complex dimension n. For more details see [1, 23], etc.

Let M be a two dimensional complex manifold. We consider $z \in M$, and so $z = (z^1, z^2)$ are the complex coordinates in a local chart. Since $z^k = x^k + \sqrt{-1}x^{k+2}$, $k = \overline{1,2}$, the complex coordinates induce the real coordinates $\{x^1, x^2, x^3, x^4\}$ on M. Let $T_R M$ be the real tangent bundle. Its complexified tangent bundle $T_C M$ splits into the sum of holomorphic tangent bundle $T'M$ and its conjugate $T''M$, under the action of the natural complex structure J on M. The holomorphic tangent bundle $T'M$ is itself a complex manifold and the coordinates in a local chart will be denoted by $(z^k, \eta^k)_{k=1,2}$, with $\eta^k = y^k + \sqrt{-1}y^{k+2}$, $k = 1,2$. The dual of $T'M$ is the holomorphic cotangent bundle and it is denoted by T'^*M. On the manifold T'^*M, a point u^* is characterized by the coordinates $(z^k, \zeta_k)_{k=1,2}$, with $\zeta_k = p_k + \sqrt{-1}p_{k+2}$, $k = 1,2$, and a change of these has the form $z'^k = z'^k(z)$ and $\zeta'_k = \frac{\partial^* z^j}{\partial z'^k}\zeta_j$, $rank(\frac{\partial^* z'^k}{\partial z^l}) = n$. Here and further, we use the star notation for the partial derivatives with respect to z, on T'^*M, only to distinguish them from those on $T'M$.

Everywhere in this paper the indices $i, j, k, ...$ run over $\{1,2\}$.

2.1. *Geometry of* $(T'M, L)$

Consider the sections of the complexified tangent bundle of $T'M$. Let $VT'M \subset T'(T'M)$ be the vertical bundle, locally spanned by $\{\frac{\partial}{\partial \eta^k}\}$, and $VT''M$ its conjugate. The idea of complex nonlinear connection, briefly $(c.n.c.)$, is an instrument in 'linearization' of the geometry of the manifold $T'M$. A $(c.n.c.)$ is a supplementary complex subbundle to $VT'M$ in $T'(T'M)$, i.e. $T'(T'M) = HT'M \oplus VT'M$. The horizontal distribution $H_u T'M$ is locally spanned by $\{\frac{\delta}{\delta z^k} = \frac{\partial}{\partial z^k} - N_k^j \frac{\partial}{\partial \eta^j}\}$, where $N_k^j(z,\eta)$ are the coefficients of the $(c.n.c.)$. The pair $\{\delta_k := \frac{\delta}{\delta z^k}, \dot{\partial}_k := \frac{\partial}{\partial \eta^k}\}$ will be called the adapted frame of the $(c.n.c.)$, which obey the change rules $\delta_k = \frac{\partial z'^j}{\partial z^k}\delta'_j$ and $\dot{\partial}_k = \frac{\partial z'^j}{\partial z^k}\dot{\partial}'_j$. By conjugation everywhere we obtain an adapted frame $\{\delta_{\bar{k}}, \dot{\partial}_{\bar{k}}\}$ on $T''_u(T'M)$. The dual adapted bases are $\{dz^k, \delta\eta^k\}$ and $\{d\bar{z}^k, \delta\bar{\eta}^k\}$.

A $2-$ dimensional *complex Lagrange space* is a pair (M, L) where $L : T'M \to \mathbb{R}$ is a smooth function, which satisfies the regularity condition:

$$g_{k\bar{j}} = \frac{\partial^2 L}{\partial \eta^k \partial \bar{\eta}^j} \tag{2.1}$$

is nondegenerate ($\det(g_{k\bar{j}}) \neq 0$, $g_{k\bar{j}}g^{\bar{j}l} = \delta_k^l$) and it is a Hermitian metric of constant signature.

In comparison to complex Finsler spaces, the function L is smooth and defined on the whole $T'M$ and the homogeneity condition is not assumed. Thus all the consequences of the homogeneity condition are not satisfied.

A fundamental problem in the geometry of $T'M$ is that of the existence of a complex nonlinear connection, depending here only on the Lagrangian function L. An interesting $(c.n.c.)$ which is a natural generalization of Chern-Finsler $(c.n.c.)$ is

$$N_j^k = g^{\bar{i}k}\frac{\partial^2 L}{\partial z^j \partial \bar{\eta}^i} \tag{2.2}$$

called by us the Chern-Lagrange $(c.n.c.)$, (see [23]).

Under homogeneity condition $L(z, \lambda\eta) = |\lambda|^2 L(z,\eta)$, $\forall \lambda \in \mathbb{C}$, (i.e., (M, L) is a *complex Finsler space*) there are some different nuances of the Kähler property, in [1]'s terminology. Namely, the space (M, L) is *Kähler* iff $T_{jk}^i \eta^j = 0$ and *weakly Kähler* iff $g_{i\bar{l}}T_{jk}^i \eta^j \bar{\eta}^l = 0$, where $T_{jk}^i := g^{\bar{l}i}(\delta_k g_{j\bar{l}} - \delta_j g_{k\bar{l}})$. We notice that in the particular case of complex Finsler metrics which come from Hermitian metrics on M, so-called *purely Hermitian metrics* in [23], (i.e. $g_{i\bar{j}} = g_{i\bar{j}}(z)$), these two kinds of Kähler structures are the same.

2.2. *Geometry of* (T'^*M, H)

Now we return to the complex manifold T'^*M. The complexified of the real tangent bundle of T'^*M is decomposed in the sum $T_C(T'^*M) = T'(T'^*M) \oplus T''(T'^*M)$.

Let $VT'^*M \subset T'(T'^*M)$ be the vertical bundle, which has the vertical distribution $V_{u^*}(T'^*M)$, locally spanned by $\{\frac{\partial}{\partial \zeta_k}\}$. A $(c.n.c.)$ on T'^*M is a supplementary subbundle in $T'(T'^*M)$ of $V(T'^*M)$, i.e., $T'(T'^*M) = H(T'^*M) \oplus V(T'^*M)$. The horizontal distribution $H_{u^*}(T'^*M)$ is locally spanned by $\{\frac{\delta^*}{\delta z^j}\}$, where $\frac{\delta^*}{\delta z^k} = \frac{\partial^*}{\partial z^k} + N_{jk}\frac{\partial}{\partial \zeta_j}$ and the functions N_{jk} are the coefficients of the $(c.n.c.)$ on T'^*M. The pair $\{\delta^*_k := \frac{\delta^*}{\delta z^k}, \dot{\partial}^k := \frac{\partial}{\partial \zeta_k}\}$ will be called the adapted frame of the $(c.n.c.)$, which obey the change rules $\delta^*_k = \frac{\partial^* z'^j}{\partial z^k}\delta^{*\prime}_j$ and $\dot{\partial}^k = \frac{\partial^* z'^k}{\partial z^j}\dot{\partial}'^j$. By conjugation everywhere we have obtained an adapted frame $\{\delta^*_{\bar{k}}, \dot{\partial}^{\bar{k}}\}$ on $T''_{u^*}(T'^*M)$. The dual adapted frames are $\{d^*z^k,\ \delta\zeta_k = d\zeta_k - N_{kj}dz^j\}$ and $\{d^*\bar{z}^k, \delta\bar{\zeta}_k\}$.

A $2-$ dimensional *complex Hamilton space* is a pair (M,H) where $H : T'^*M \to \mathbb{R}$ is a smooth function, which satisfies the regularity condition:

$$h^{\bar{j}k}(z,\zeta) = \frac{\partial^2 H}{\partial \zeta_k \partial \bar{\zeta}_j} \tag{2.3}$$

is nondegenerate ($\det(h^{\bar{j}k}) \neq 0$, $h^{\bar{j}l}h_{k\bar{j}} = \delta^l_k$) and it is a Hermitian metric of constant signature.

A main problem for a complex Hamilton space is that of determining a complex nonlinear connection depending only on the fundamental function H of the space. Due to [23], the following functions

$$N_{jl} = -h_{l\bar{k}}\frac{\partial^2 H}{\partial z^j \partial \bar{\zeta}_k} \tag{2.4}$$

are the coefficients of a $(c.n.c.)$ on T'^*M, depending only on the complex Hamilton function H.

Moreover, if the function H satisfies the homogeneity condition $H(z,\lambda\zeta) = |\lambda|^2 H(z,\zeta)$, $\forall \lambda \in \mathbb{C}$, then the space (M,H) becomes a complex Cartan space. A usual example of complex Cartan space is the so called *purely Hermitian* complex Cartan space, this means that $h^{\bar{j}i} = h^{\bar{j}i}(z)$. Also, a complex Cartan space (M,H) is called *Kähler-Cartan* iff $T^{*i}_{jk} = 0$ and *weakly Kähler-Cartan* if $T^{*i}_{jk}\zeta_i\zeta^j = 0$, where $T^{*i}_{jk} := H^i_{jk} - H^i_{kj}$, $H^i_{jk} := h^{\bar{m}i}(\delta^*_k h_{j\bar{m}})$ and $\zeta^j := h^{\bar{m}j}\zeta_{\bar{m}}$. Note that these nuances of Kähler-Cartan are the same with $\frac{\partial h_{j\bar{m}}}{\partial z^i} = \frac{\partial h_{i\bar{m}}}{\partial z^j}$, in the particular case of a purely Hermitian complex Cartan metric.

2.3. $\mathscr{L}-$ *dual process*

Another approach of the complex Hamilton spaces is given by the correspondence between the various geometrical objects on a complex Lagrange space (M,L) and those of a complex Hamilton space (M,H), via the complex *Legendre transformation,* (the $\mathscr{L}-$ dual process).

In [23] the *complex Legendre transformation* was introduced as a local diffeomorphism $\Phi \times \bar{\Phi}$ with $\Phi : U \subset T'M \to \bar{U}^* \subset T''^*M$, $\Phi(z^k,\eta^k) = (z^k, \dot{\partial}_{\bar{k}}L)$, and $\bar{\Phi} : \bar{U} \subset T''M \to U^* \subset T'^*M$, $\bar{\Phi}(z^k,\bar{\eta}^k) = (z^k, \dot{\partial}_k L)$. Further on, for simplicity the complex Legendre transformation is denoted only by Φ and the distinction between the open sets U and $\bar{U}$ is not specified, but we have assumed that it is defined on whole $T_C M$. The properties obtained by Φ or by $\Phi^{-1}$are called $\mathscr{L}-$ dual one to another. Further on, we denote by '*' the image by Φ of various geometric objects on $U \subset T'M$ and by ' $^\circ$ ' the image by Φ^{-1} of geometric objects on $U^* \subset T'^*M$, (see [23], p. 163).

Setting the local tangent maps $d\Phi : T_C(T'M) \to T_C(T''^*M)$ and $d\bar{\Phi} : T_C(T''M) \to T_C(T'^*M)$, in [23] we established the conditions under which $d\Phi$ sends the complex tangent vectors in $T'M$ into the complex tangent vectors in T'^*M, such that the image by complex Legendre transformation, of a complex Lagrange space (M,L) is locally a complex Hamilton space (M,H), and conversely ([23], p. 164), i.e.,

$$\Big(L(z^k,\eta^k)\Big)^* = H(z^k,\zeta_k)\ ;\ \Big(H(z^k,\zeta_k)\Big)^\circ = L(z^k,\eta^k),$$
$$(g_{i\bar{j}}(z,\eta))^* = h_{i\bar{j}}(z,\zeta)\ ;\ (h^{\bar{j}k}(z,\zeta))^\circ = g^{\bar{j}k}(z,\eta),$$

with

$$L(z,\eta) = \zeta_i\eta^i + \bar{\zeta}_i\bar{\eta}^i - H(z,\zeta)\ ;\ \eta^k = \dot{\partial}^k H\ ;\ \zeta_k = \dot{\partial}_k L. \tag{2.1}$$

Moreover, the image by complex Legendre transformation of the Chern-Lagrange $(c.n.c.)$ is Chern-Hamilton $(c.n.c.)$, (for more details, see [23], p. 166).

3. The dual of the complex Lagrangian of electrodynamics

Since complex Lagrangians appear frequently in quantum mechanics or in gauge theory, we consider a complex version of a Lagrangian model of electrodynamics ([23, 24]), on the complex manifold $T'M$

$$L(z,\eta) = g_{k\bar{j}}\eta^k\bar{\eta}^j + q[A_k(z)\eta^k + A_{\bar{j}}(z)\bar{\eta}^j] + V(z)\quad ;\quad k,j=1,2, \tag{3.1}$$

where $g_{k\bar{j}} = g_{k\bar{j}}(z)$ is a purely Hermitian metric on M and $A := A_k(z)dz^k$ is a $(1,0)-$ form on M, which comes from the electromagnetic potential (Φ, $-A^1$, $-A^2$, $-A^3$), with $A_1 = \Phi - iA^1$, $A_2 = -A^2 - iA^3$, $A_{\bar{j}} := \bar{A}_j$. $V(z)$ is a real valued function and q is a real number which represents the electric charge. Also, by $\tilde{L} := g_{k\bar{j}}\eta^k\bar{\eta}^j$ is induced a complex Finsler metric or pseudo-Finsler metric on M and L and $\tilde{L}$ have the same fundamental metric tensor, namely this is $g_{k\bar{j}}$. Thus, the local coefficients of the Chern-Lagrange $(c.n.c.)$ associated to the complex Lagrangian (3.1) are the following

$$N^i_j = \tilde{N}^i_j + qg^{\bar{k}i}\frac{\partial A_{\bar{k}}}{\partial z^j}, \tag{3.2}$$

where $\tilde{N}^i_j := g^{\bar{k}i}\frac{\partial g_{l\bar{k}}}{\partial z^j}\eta^l$ are the local coefficients of the Chern-Finsler $(c.n.c.)$ associated to $\tilde{L}$.

Let $c(t)$, with t a real parameter, be a curve on M and $(z^k(t),\eta^k(t) := \frac{dz^k}{dt})$, $k=1,2$, its extension on $T'M$. The Euler-Lagrange equation $\frac{\partial L}{\partial z^j} - \frac{d}{dt}(\frac{\partial L}{\partial \eta^j}) = 0$, corresponding to the complex Lagrangian

(3.1), can be rewritten as

$$\begin{aligned}&\frac{d^2z^k}{dt^2}+N^k_h(z(t),\frac{dz}{dt})\frac{dz^h}{dt} \\ &= \theta^{*k}(z(t),\frac{dz}{dt})+g^{\bar{h}k}(qE_{\bar{h}\bar{r}}\frac{d\bar{z}^r}{dt}+qE_{\bar{h}l}\frac{dz^l}{dt}+\frac{\partial V}{\partial \bar{z}^h});\ k=1,2,\end{aligned} \tag{3.3}$$

where $\theta^{*k} := g^{\bar{h}k}(\frac{\partial g_{l\bar{j}}}{\partial \bar{z}^h}-\frac{\partial g_{l\bar{h}}}{\partial \bar{z}^j})\eta^l\bar{\eta}^j$ and $E_{\bar{h}l} := \frac{\partial A_l}{\partial \bar{z}^h}$, $E_{\bar{h}\bar{r}} := \frac{\partial A_{\bar{r}}}{\partial \bar{z}^h}-\frac{\partial A_{\bar{h}}}{\partial \bar{z}^r}$ are the local coefficients of a complex electromagnetic field

$$\mathbf{E}=\frac{1}{2}E_{jk}dz^j\wedge dz^k+E_{\bar{h}k}d\bar{z}^h\wedge dz^k,$$

satisfying $\mathbf{E}=dA$.

Note that if the charge q vanishes and V is constant valued, then the equations (3.3) lead to the geodesic curves of the complex Finsler metric $\tilde{L}$ on M, (see [1,23]).

In order to point out a physical significance of the complex Lagrangian (3.1), we choose $s(t)$ the arc length of the curve c on $T'M$ with respect to the purely Hermitian metric $g_{k\bar{j}}$, which means that $ds^2=\tilde{L}(z(t),\frac{dz}{dt})dt^2$. If the parameter t is normalized with respect to the proper distance (so called the *proper time*) s, by $s=mt$, then the complex Lagrangian (3.1), together with the equation (3.3), give rise to the equation of motion of a particle of the mass m and the charge q, under the action of the Lorentz force, due to the electromagnetic field $\mathbf{E}$, with an additional force (gravitational or inertial, for example) represented by some derivatives of the function $V(z)$:

$$\begin{aligned}&\frac{d^2z^k}{ds^2}+N^k_h(z(s),\frac{dz}{ds})\frac{dz^h}{ds} \\ &= \theta^{*k}(z(s),\frac{dz}{ds})+\frac{1}{m}g^{\bar{h}k}(qE_{\bar{h}\bar{r}}\frac{d\bar{z}^r}{ds}+qE_{\bar{h}l}\frac{dz^l}{ds}+\frac{\partial V}{\partial \bar{z}^h});\ k=1,2.\end{aligned} \tag{3.4}$$

Thus, an immediate consequences of the normalizing condition is $\tilde{L}(z(t),\frac{dz}{dt})=m^2$.

An important example of purely Hermitian metric is the one used in the study of the weakly gravitational fields in complex Finsler geometry [25], given by the fundamental metric tensor

$$\left(g_{j\bar{k}}\right)_{j,k=1,2}=\begin{pmatrix}1+\frac{2\Phi}{c^2} & -i(1-\frac{2\Phi}{c^2}) \\ i(1-\frac{2\Phi}{c^2}) & -(1-\frac{2\Phi}{c^2})\end{pmatrix},\ \text{with}\ i:=\sqrt{-1}, \tag{3.5}$$

where here $\Phi=\Phi(z)$ is a real valued smooth function, $\Phi>\frac{c^2}{2}$, $c\neq 0$, (i.e., $\det(g_{i\bar{j}})>0$) and the inverse matrix of (3.5) is

$$\left(g^{\bar{k}j}\right)_{j,k=1,2}=\frac{1}{2}\begin{pmatrix}1 & -i \\ i & -\frac{1+\frac{2\Phi}{c^2}}{1-\frac{2\Phi}{c^2}}\end{pmatrix}. \tag{3.6}$$

Note that the purely Hermitian metric (3.5) is Kähler if and only if $i\Phi_2=\Phi_1$, where $\Phi_k := \frac{\partial\Phi}{\partial z^k}$, $k=1,2$.

Also, the hyperbolic version of the Schwarzschild metric induces a Hermitian pseudo-Finsler metric on $M \subset C^2$, called the complex Hermitian Schwarzschild metric (see [25]), given by

$$\left(g_{j\bar{k}}\right)_{j,k=1,2} = \begin{pmatrix} -\varphi^4 & 0 \\ 0 & \varphi^4 \end{pmatrix}, \tag{3.7}$$

with the inverse matrix

$$\left(g^{\bar{k}j}\right)_{j,k=1,2} = \begin{pmatrix} -1/\varphi^4 & 0 \\ 0 & 1/\varphi^4 \end{pmatrix}, \tag{3.8}$$

and $g := \det(g_{i\bar{j}}) = -\varphi^8 < 0$, where $\varphi(r_1) = \frac{1}{\sqrt{kr_1 \pm 1}}$, $k \in R$, $r_1 := \frac{1}{2}\sqrt{(z^1 - \bar{z}^1)^2 + 4|z^2|^2}$.

A suggestion for unifying quantum theory and relativity was given by M. Born in [8], using *Reciprocity Principle* which is based on a primary symmetry between the space-time coordinates (x^1, x^2, x^3, x^4) and momentum-energy coordinates (p_1, p_2, p_3, p_4). Based on the complex Legendre transformation (the $\mathscr{L}-$ dual process), we pass from the complex space-time coordinates $z^k = x^k + \sqrt{-1}x^{k+2}$, $k = \overline{1,2}$, to the complex momentum-energy coordinates $\zeta_k = p_k + \sqrt{-1}p_{k+2}$, $k = 1,2$.

Taking into account that the image of a complex Lagrangian by complex Legendre transformation is locally a complex Hamiltonian $H(z,\zeta) = \zeta_k \eta^k + \bar{\zeta}_k \bar{\eta}^k - L(z,\eta)$, and conversely, where $\zeta_k := \dot{\partial}_k L$, it is trivial to produce the momenta

$$\zeta_k = g_{k\bar{j}} \bar{\eta}^j + qA_k \; ; \; k = 1,2,$$

which yields $\eta^j = g^{\bar{m}j}(\zeta_{\bar{m}} - qA_{\bar{m}})$. Further on, we obtain the complex Hamiltonian

$$H(z,\zeta) = g^{\bar{j}k}(\zeta_k - qA_k)(\zeta_{\bar{j}} - qA_{\bar{j}}) - V(z) \tag{3.9}$$

on the complex manifold T'^*M, with the fundamental tensor metric $h^{\bar{j}k} := g^{\bar{j}k} = \frac{\partial^2 H}{\partial \zeta_k \partial \bar{\zeta}_j}$ which depends only on z, (i.e. it is purely Hermitian). Moreover, the fundamental tensor metric induces a purely Hermitian complex Cartan structure $\tilde{H}(z,\zeta) = h^{\bar{j}k}\zeta_k \zeta_{\bar{j}}$, with $\tilde{N}_{jk} = -h_{j\bar{m}} \frac{\partial h^{\bar{m}l}}{\partial z^k} \zeta_l$ the local coefficients of the Chern-Cartan $(c.n.c.)$. A straightforward computation leads to the local expressions of Chern-Hamilton $(c.n.c.)$ corresponding to (3.9):

$$N_{jk} = \tilde{N}_{jk} + q\left(\frac{\partial A_j}{\partial z^k} - h_{j\bar{m}} \frac{\partial h^{\bar{m}l}}{\partial z^k} A_l\right). \tag{3.10}$$

Subsequently, the adapted frame δ_k^* is with respect to this Chern-Hamilton $(c.n.c.)$ given in (3.10).

By the $\mathscr{L}-$dual process, corresponding to the curves $c : t \to (z^k(t), \eta^k(t) = \frac{dz^k}{dt})_{k=1,2}$ on $T'M$ are the curves $c^* : t \to (z^k(t), \zeta_k(t) = g_{k\bar{j}}(t)\frac{d\bar{z}^j}{dt} + qA_k(t))_{k=1,2}$ on T'^*M, which we can see that satisfy the Hamilton-Jacobi equations iff c is a geodesic, i.e. a solution for (3.3).

4. Schrödinger equation

In quantum mechanics the analogue of Newton's law is Schrödinger equation for quantum system (usually atoms, molecules and subatomic particles whether free, bound or localized). It is a partial differential equation which describes the wave function of the system. The most general form of the Schrödinger equation is time dependent, which gives a description of the system evolving with time.

Among the space-time coordinates (x^1, x^2, x^3, x^4), we consider x^1 as the usual coordinate time τ, and then in the complex coordinate z^1 we have $z^1 = \tau + iu$, where $u := x^3$. Moreover, along the curve $c(t)$ described by (3.4), the proper time s depends on an auxiliary time t, which means a quantum time.

Now, if we substitute the complex Hamiltonian and the complex momenta from (3.9) by the following operators

$$H \to -i\hbar\frac{\partial}{\partial t}\ ;\ \zeta_k \to -\frac{\hbar}{\sqrt{2m}}\delta_k^*\ ;\ k = 1,2,$$

the outcome is a two times (the parameter time t and, by means of z^1, the coordinate time τ) Schrödinger equation

$$i\hbar\frac{\partial\psi}{\partial t} = [-h^{\bar{j}k}(\frac{\hbar}{\sqrt{2m}}\delta_k^* + qA_k)(\frac{\hbar}{\sqrt{2m}}\delta_{\bar{j}}^* + qA_{\bar{j}}) + V(z)]\psi, \tag{4.1}$$

where $\psi = \psi(t, z, \zeta)$ is the wave function.

According to [29] we can write the local expression of the horizontal Laplace operator of the complex Lagrange space (3.1),

$$\Delta = \frac{1}{g}\delta_k(gg^{\bar{j}k}\delta_{\bar{j}}) = g^{\bar{j}k}[\delta_k(\delta_{\bar{j}}) + (L_{kl}^l - L_{lk}^l)\delta_{\bar{j}}],$$

where $g = \det(g_{k\bar{j}})$ and δ_k is with respect to Chern-Lagrange $(c.n.c.)$ (3.2). By $\mathscr{L}-$ dual process, it results that the local expression of the horizontal Laplace operator of the complex Hamilton space (3.9) is

$$\Delta^* = \frac{1}{h}\delta_k^*(hh^{\bar{j}k}\delta_{\bar{j}}^*) = h^{\bar{j}k}[\delta_k^*(\delta_{\bar{j}}^*) + (H_{kl}^l - H_{lk}^l)\delta_{\bar{j}}^*],$$

where $h = \det(h_{k\bar{j}})$.

Taking into account

$$h^{\bar{j}k}\delta_k^*(\delta_{\bar{j}}^*) = \Delta^* - h^{\bar{j}k}(H_{kl}^l - H_{lk}^l)\delta_{\bar{j}}^* = \Delta^* - h^{\bar{j}k}T_{kl}^{*l}\delta_{\bar{j}}^*,$$

the equation (4.1) becomes

$$\begin{aligned} i\hbar\frac{\partial\psi}{\partial t} &= -\frac{\hbar^2}{2m}\Delta^*\psi - \frac{\hbar}{\sqrt{2m}}h^{\bar{j}k}\{[qA_k - \frac{\hbar}{\sqrt{2m}}T_{kl}^{*l}](\delta_{\bar{j}}^*\psi) + qA_{\bar{j}}(\delta_k^*\psi)\} \\ &\quad -\{h^{\bar{j}k}q[\frac{\hbar}{\sqrt{2m}}(\delta_k^*A_{\bar{j}}) + qA_{\bar{j}}A_k] - V(z)\}\psi. \end{aligned} \tag{4.2}$$

In particular, if the charge q vanishes, then the equation (4.2) is reduced to

$$i\hbar\frac{\partial\psi}{\partial t} = -\frac{\hbar^2}{2m}[\Delta^*\psi - h^{\bar{j}k}T_{kl}^{*l}(\delta_{\bar{j}}^*\psi)] + V(z)\psi. \tag{4.3}$$

Moreover, if $h^{\bar{j}k}$ is Kähler, then the Schrödinger equation (4.3) is

$$i\hbar\frac{\partial\psi}{\partial t} = -\frac{\hbar^2}{2m}\Delta^*\psi + V(z)\psi, \tag{4.4}$$

with $\Delta^*\psi = h^{\bar{j}k}\delta_k^*(\delta_{\bar{j}}^*\psi)$.

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Here, some remarks are entailed.

1. The auxiliary time t is a real parameter, while the coordinate time τ is the real part of the complex coordinate z^1, $z^1 = \tau + iu$. These two times may be independent or not. Of course, the most convenient case is that in which we take $\tau = t$, this being to highlight the naturalness generalization of the classical Schrödinger equation and also, from viewpoint of the computation of solutions.

2. Clearly, the generalization (4.1) of the Schrödinger equation is a relativistic one, because the fundamental metric tensor $h^{\bar{j}k}$ coincides with $g^{\bar{j}k}$ which is relativistic ($g^{\bar{j}k}$ is the weakly gravitational metric or Hermitian Schwarzschild metric) and also, the complex momenta and implicitly the adapted frames δ_k^* depend on the geometry of this relativistic space.

3. Moreover, if $g_{i\bar{j}}(z)$ is the Euclidian metric $\delta_{i\bar{j}}$, then $\delta_k^* = \frac{\partial}{\partial z^k}$ and $\Delta^* = \Delta$. So, as we pointed out in the introduction, we obtain $\hat{H}\psi = E\psi$. Therefore, going back to the reasoning, we can say that $\Delta^*\psi$ has the meaning of an energy that depends on the geometry of the relativistic space-time $\left(M, g_{i\bar{j}}(z)\right)$.

4. For this generalized Schrödinger equation we suppose that the wave function ψ is in $\mathscr{C}^1(\mathbb{R}_t, L^2(M, dz^1dz^2))$.

5. Now we use an idea from [11]. Taking the wave function ψ in the form

$$\psi = e^{i\frac{m}{\hbar}s}\Psi,$$

where Ψ is independent of the proper time $s(t) = mt$, the equation (4.1) is reduced to

$$[-h^{\bar{j}k}(\frac{\hbar}{\sqrt{2m}}\delta_k^* + qA_k)(\frac{\hbar}{\sqrt{2m}}\delta_{\bar{j}}^* + qA_{\bar{j}}) + V(z)]\Psi + m^2\Psi = 0, \tag{4.5}$$

which is a generalization of Klein-Gordon equation similar to that studied by us in [2, 25], for $V(z) = 0$. This relation between Klein-Gordon equation and two times Schrödinger equation is similar with the $(t,t')-$method used in the description of the light-matter interaction in atomic and molecular physics, [15, 19, 26, 27].

5. Some solutions for generalized Schrödinger equation

In order to point out some solutions for the Schrödinger equation by the method of separation of variables, we consider the convenient case $t = \tau$ and so, $z^1 = t + iu$. Also, we impose some other additional conditions: the wave function depends only on the space-time coordinates, i.e. $\psi = \psi(z^1, z^2)$, the charge q vanishes and the function V is independent of t, that is $V = V(u, z^2)$. Thus, the equation (4.1) is reduced to

$$i\hbar\frac{\partial\psi}{\partial t} = [-\frac{\hbar^2}{2m}h^{\bar{j}k}\delta_k^*(\delta_{\bar{j}}^*) + V(u, z^2)]\psi(z^1, z^2).$$

Moreover, because $\psi(z^1, z^2)$, we have $\delta_k^*\psi = \frac{\partial\psi}{\partial z^k}$ and so,

$$i\hbar\frac{\partial\psi}{\partial t} = \{-\frac{\hbar^2}{2m}h^{\bar{j}k}\frac{\partial}{\partial z^k}(\frac{\partial}{\partial\bar{z}^j}) + V(u, z^2)\}\psi(z^1, z^2). \tag{5.1}$$

Next, for the Schrödinger equation (5.1) we seek a solution of the form

$$\psi(z^1, z^2) = f(t)g(u, z^2),$$

where $t = \frac{1}{2}(z^1 + \bar{z}^1)$ and $u = \frac{1}{2i}(z^1 - \bar{z}^1)$. Hence

$\frac{\partial\psi}{\partial t}=g(u,z^2)\frac{\partial f}{\partial t}$, $\frac{\partial\psi}{\partial z^1}=\frac{1}{2}(g(u,z^2)\frac{\partial f}{\partial t}-if(t)\frac{\partial g}{\partial u})$,
$\frac{\partial\psi}{\partial z^2}=\frac{\partial g}{\partial z^2}$, $\frac{\partial\psi}{\partial z^1\partial\bar{z}^1}=\frac{1}{4}(g(u,z^2)\frac{\partial^2 f}{\partial t^2}+f(t)\frac{\partial^2 g}{\partial u^2})$,
$\frac{\partial\psi}{\partial z^1\partial\bar{z}^2}=\frac{1}{2}(\frac{\partial g}{\partial\bar{z}^2}\frac{\partial f}{\partial t}-if(t)\frac{\partial^2 g}{\partial u\partial\bar{z}^2})$,
$\frac{\partial\psi}{\partial z^2\partial\bar{z}^2}=f(t)\frac{\partial g}{\partial\bar{z}^j}$, $\frac{\partial^2\psi}{\partial z^2\partial\bar{z}^2}=f(t)\frac{\partial^2 g}{\partial z^2\partial\bar{z}^2}$, etc., and then the equation (5.1) becomes

$$i\hbar g\frac{\partial f}{\partial t}=-\frac{\hbar^2}{2m}[\frac{1}{4}h^{\bar{1}1}(g\frac{\partial^2 f}{\partial t^2}+f\frac{\partial^2 g}{\partial u^2})+\frac{1}{2}h^{\bar{1}2}(\frac{\partial g}{\partial z^2}\frac{\partial f}{\partial t}+if\frac{\partial^2 g}{\partial u\partial z^2}) \\ +\frac{1}{2}h^{\bar{2}1}(\frac{\partial g}{\partial\bar{z}^2}\frac{\partial f}{\partial t}-if\frac{\partial^2 g}{\partial u\partial\bar{z}^2})+fh^{\bar{2}2}\frac{\partial^2 g}{\partial z^2\partial\bar{z}^2}]+V(u,z^2)fg. \tag{5.2}$$

If $f(t)=e^{-i\frac{12m}{\hbar}t}$, then the equation (5.2) can be rewritten as

$$12mg=-\frac{\hbar^2}{2m}[\frac{1}{4}h^{\bar{1}1}(-\frac{144m^2}{\hbar^2}g+\frac{\partial^2 g}{\partial u^2})+\frac{i}{2}h^{\bar{1}2}(-\frac{12m}{\hbar}\frac{\partial g}{\partial z^2}+\frac{\partial^2 g}{\partial u\partial z^2}) \\ +\frac{i}{2}h^{\bar{2}1}(-\frac{12m}{\hbar}\frac{\partial g}{\partial\bar{z}^2}-\frac{\partial^2 g}{\partial u\partial\bar{z}^2})+h^{\bar{2}2}\frac{\partial^2 g}{\partial z^2\partial\bar{z}^2}]+V(u,z^2)g, \tag{5.3}$$

Now, to obtain the solutions for the equation (5.3), we choose two different forms for the fundamental metric tensor $h^{\bar{j}k}$, the first case is the inverse of the weakly gravitational metric (3.5) and then in the second case $h^{\bar{j}k}$ is the inverse matrix of the complex Hermitian Schwarzschild metric (3.7).

Case 1. If we set $h^{\bar{j}k}$ as in (3.6) and $V(u,z^2)=0$, the equation (5.3) becomes

$$-\frac{48m^2}{\hbar^2}g=\frac{1}{4}(-\frac{144m^2}{\hbar^2}g+\frac{\partial^2 g}{\partial u^2})+\frac{1}{2}(-\frac{12m}{\hbar}\frac{\partial g}{\partial z^2}+\frac{\partial^2 g}{\partial u\partial z^2}) \\ -\frac{1}{2}(-\frac{12m}{\hbar}\frac{\partial g}{\partial\bar{z}^2}-\frac{\partial^2 g}{\partial u\partial\bar{z}^2})-\frac{1+\frac{2\Phi}{c^2}}{1-\frac{2\Phi}{c^2}}\frac{\partial^2 g}{\partial z^2\partial\bar{z}^2},$$

which admits the particular solution

$$g(u)=e^{i\frac{2\sqrt{3}m}{\hbar}u}.$$

Thus, a solution of the Schrödinger equation (5.3), with $h^{\bar{j}k}$ as in (3.6) and $V(u,z^2)=0$ is

$$\psi(z^1,z^2)=e^{-i\frac{12m}{\hbar}t+\frac{\sqrt{3}m}{\hbar}(z^1-\bar{z}^1)}.$$

Case 2. With $\left(h^{\bar{j}k}\right)=\begin{pmatrix}-1/\varphi^4 & 0\\ 0 & 1/\varphi^4\end{pmatrix}$, where $\varphi=\varphi(r_1)=\frac{1}{\sqrt{kr_1\pm1}}$, $k\in\mathbb{R}$, and $r_1=\sqrt{|z^2|^2-u^2}$, the equation (5.3) is the following

$$\frac{1}{\varphi^4}(-\frac{144m^2}{\hbar^2}g+\frac{\partial^2 g}{\partial u^2}-4\frac{\partial^2 g}{\partial z^2\partial\bar{z}^2})=\frac{8m}{\hbar^2}[12m-V(r_1)]g. \tag{5.4}$$

To solve (5.4), we look for solutions

$$g(u,z^2)=e^{\Omega(r_1)},$$

which reduces the equation (5.4) to

$$\Omega'^2+\Omega''+\frac{2}{r_1}\Omega' = -\frac{8m}{\hbar^2}[18m+\frac{12m-V(r_1)}{(kr_1\pm 1)^2}], \tag{5.5}$$

where $\Omega' := \frac{d\Omega}{dr_1}$, $\Omega'' := \frac{d^2\Omega}{dr_1^2}$.

The substitution $w := \Omega'$, transforms (5.5) into a Ricatti equation:

$$w' = -w^2-\frac{2}{r_1}w-\frac{8m}{\hbar^2}[18m+\frac{12m-V(r_1)}{(kr_1\pm 1)^2}]. \tag{5.6}$$

If we assume that $V(r_1) = \frac{m[(kr_1\pm1)^2(18k^2r_1^2-\alpha)+12k^2r_1^2]}{k^2r_1^2}$, where $\alpha\in\mathbb{R}$, then the equation (5.6) can be rewritten as

$$w' = -w^2-\frac{2}{r_1}w-\frac{8m\alpha}{\hbar^2k^2}\frac{1}{r_1^2}. \tag{5.7}$$

The Ricatti equation (5.7) admits a particular solution $w_0 = \frac{B}{r_1}$, where B is a solution of the algebraic equation $B^2+B+\frac{8m\alpha}{\hbar^2k^2} = 0$. This leads us to the general solution of the equation (5.7)

$$w = \frac{a(2B+1)^2(r_1)^{2B}}{a(2B+1)(r_1)^{2B+1}-1}-\frac{B+1}{r_1},$$

where a is a constant. Then, $\Omega = \ln b\frac{a(2B+1)(r_1)^{2B+1}-1}{(r_1)^{B+1}}$ and so, the general solution of the equation (5.4) is

$$g(u,z^2) = e^{\Omega(r_1)} = ab(2B+1)r_1-\frac{b}{(r_1)^{B+1}},$$

where b is a constant.

Therefore, a solution of the Schrödinger equation (5.1), with $h^{\bar{j}k}$ as in (3.8) and $V(u,z^2) = \frac{m[(kr_1\pm1)^2(18k^2r_1^2-\alpha)+12k^2r_1^2]}{k^2r_1^2}$ is

$$\psi(z^1,z^2) = b[a(2B+1)r_1-\frac{1}{(r_1)^{B+1}}]e^{-i\frac{12m}{\hbar}t}.$$

In particular, if $\alpha = \frac{\hbar^2k^2}{32m}$, then $B = -\frac{1}{2}$ and

$$\psi(z^1,z^2) = -\frac{be^{-i\frac{12m}{\hbar}t}}{\sqrt{r_1}} = -\frac{b\sqrt{2}e^{-i\frac{6m}{\hbar}(z^1+\bar{z}^1)}}{[(z^1-\bar{z}^1)^2+4|z^2|^2]^{\frac{1}{4}}}.$$

is the solution of the Schrödinger equation (5.1).

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