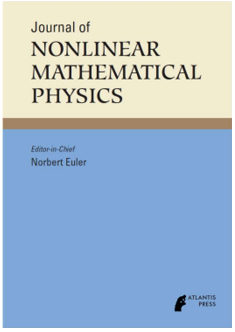

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On generalized Lax equation of the Lax triple of KP Hierarchy

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In terms of the operator Nambu 3-bracket and the Lax pair (L,B_n) of the KP hierarchy, we propose the generalized Lax equation with respect to the Lax triple (L,B_n,B_m). The intriguing results are that we derive the KP equation and another integrable equation in the KP hierarchy from the generalized Lax equation with the different Lax triples (L,B_n,B_m). Furthermore we derive some no integrable evolution equations and present their single soliton solutions.

Keywords: Integrable systems; KP hierarchy; Lax triple.

2000 Mathematics Subject Classification: 37K10, 35Q53

1. Introduction

Nambu mechanics is a generalization of classical Hamiltonian mechanics in which the usual binary Poisson bracket is replaced by the Nambu bracket. Since it was firstly proposed by Nambu [14], this generalized Hamiltonian mechanics has received a lot of attention [7, 11, 15, 18]. In the context of integrable systems, the integrable hydrodynamical systems have been investigated via Nambu mechanics [9, 10]. Moreover various super-integrable systems, such as Calogero-Moser system, Kepler problem, three Hamiltonian structures of Landau problem, have been analyzed in the framework of Nambu mechanics [1, 19], where a super-integrable system means that it is not only an integrable system in the Liouville-Arnold sense, but also possesses more constant of motion than degrees of freedom. With the development of infinite-dimensional 3-algebras [2, 5, 6, 8], more attempts have been made to understand the connection between the infinite-dimensional 3-algebras and the integrable systems. Recently the relation between the infinite-dimensional 3-algebras and the dispersionless KdV hierarchy has been established in the framework of Nambu mechanics [4]. It was found that the dispersionless KdV system is not only a bi-Hamiltonian system, but also a bi-Nambu-Hamiltonian system.

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The KP hierarchy is a paradigm of the hierarchies of integrable systems. It consists of an infinite number of non-linear differential equations. There are different approaches to the description of the algebraical and geometrical properties of the KP hierarchy. One already knew that the $W_{1+\infty}$ algebra is intrinsically related to the first Hamiltonian structure of the KP hierarchy [20, 21]. Recently the relation between the $W_{1+\infty}$ 3-algebra and the KP hierarchy has also been investigated [3]. It is well-known that the KP hierarchy can be represented in terms of a Lax pair (L, B_n). In this paper, we reinvestigate the property of the Lax operators L and B_n of the KP hierarchy in the framework of Nambu mechanics. In terms of the operator Nambu 3-bracket and Lax operators of the KP hierarchy, we present the generalized Lax equation with respect to the Lax triple (L, B_n, B_m) and derive the corresponding (no)integrable nonlinear evolution equations for the cases of the different Lax triples (L, B_n, B_m).

2. KP Hierarchy

In this section, we shall briefly recall the KP hierarchy that will be useful in what follows. The KP hierarchy is a paradigm of the hierarchies of integrable systems. There are several different ways to formulate the mathematical problem of the KP hierarchy equations. In this paper we only focus on the pseudo-differential operator formalism [17].

Let us introduce a pseudo-differential operator

$$L = \partial + \sum_{i=0}^{+\infty} v_i(t)\partial^{-i-1}, \tag{2.1}$$

where $t = (t_1, t_2, \cdots)$ are the time variables and $\partial = \partial/\partial_x$, $x = t_1$, the negative powers of ∂ are to be understood as the formal integration symbols obeying the generalized Leibniz rule

$$\partial^{-n} f = \sum_{l=0}^{\infty} (-1)^l \frac{(n+l-1)!}{l!(n-1)!} f^{(l)} \partial^{-n-l}, \quad (n > 0). \tag{2.2}$$

We define B_n as the differential part of L^n, i.e.,

$$B_n = (L^n)_+, \quad n \geq 1. \tag{2.3}$$

The first few members of B_n are

$$\begin{aligned}
B_1 &= \partial, \\
B_2 &= \partial^2 + 2v_0, \\
B_3 &= \partial^3 + 3v_0\partial + 3v_1 + 3v_{0,x}, \\
B_4 &= \partial^4 + 4v_0\partial^2 + (4v_1 + 6v_{0,x})\partial + 4v_2 + 6v_0^2 + 6v_{1,x} + 4v_{0,xx}, \\
B_5 &= \partial^5 + 5v_0\partial^3 + 5(v_1 + 2v_{0,x})\partial^2 + 5(v_2 + 2v_0^2 + 2v_{1,x} + 2v_{0,xx})\partial \\
&\quad + 5(v_3 + 4v_0v_1 + 4v_0v_{0,x} + 2v_{2,x} + 2v_{1,xx} + v_{0,xxx}), \\
B_6 &= \partial^6 + 6v_0\partial^4 + 3(2v_1 + 5v_{0,x})\partial^3 + (6v_2 + 15v_0^2 + 15v_{1,x} + 20v_{0,xx})\partial^2 \\
&\quad + (6v_3 + 30v_0v_1 + 15v_{2,x} + 45v_0v_{0,x} + 20v_{1,xx} + 15v_{0,xxx})\partial \\
&\quad + 6v_4 + 6v_{0,xxxx} + 20v_{2,xx} + 15v_{1,xxx} + 15v_{3,x} + 30v_0v_2 \\
&\quad + 45v_0v_{1,x} + 30v_1v_{0,x} + 35v_0v_{0,xx} + 20v_0^3 + 25v_{0,x}^2 + 15v_1^2, \\
&\ \vdots,
\end{aligned} \tag{2.4}$$

where the subscript x denotes partial differentiation in x.

Let us consider the following system of linear equations:

$$L\psi = \lambda\psi, \qquad \frac{\partial}{\partial t_n}\psi = B_n\psi, \quad n = 1, 2, \cdots. \tag{2.5}$$

From the compatibility condition of (2.5), it immediately gives the well-known Lax equation,

$$\frac{\partial L}{\partial t_n} = [B_n, L] = B_nL - LB_n, \quad n = 1, 2, \cdots. \tag{2.6}$$

Substituting (2.1) and (2.3) into (2.6), one may derive the KP hierarchy.

3. Lax triple

3.1. *Generalized Lax equation*

The operator Nambu 3-bracket is defined by [14],

$$[\hat{A}, \hat{B}, \hat{C}] = [\hat{A}, \hat{B}]\hat{C} + [\hat{B}, \hat{C}]\hat{A} + [\hat{C}, \hat{A}]\hat{B}, \tag{3.1}$$

where $[\hat{A}, \hat{B}] = \hat{A}\hat{B} - \hat{B}\hat{A}$.

Based on the operator Nambu 3-bracket (3.1), let us consider the the following generalized Lax equation with respect to the Lax triple (L, B_n, B_m):

$$\frac{\partial L}{\partial t_{mn}} = [B_m, B_n, L]_-, \ \ (m, n = 0, 1, 2\cdots), \tag{3.2}$$

where $B_0 = 1$, L and $B_n, n \in Z_+$ are given by (2.1) and (2.3), respectively, the operator Nambu 3-bracket $[, ,]_-$ denotes the formal integration operator part of the derived pseudo-differential operator.

Taking $B_m = B_0$ in (3.2), it is easy to verify that (3.2) leads to the Lax equation (2.6),

$$\frac{\partial L}{\partial t_{0n}} = [B_0, B_n, L]_- = [B_n, L]. \tag{3.3}$$

Thus it is natural to derive the KP hierarchy from (3.3).

Taking $B_n = B_2$ in (3.3) and equating the coefficient of ∂^{-i-1} with the left and right-hands side of (3.3), we have

$$\begin{aligned}
\frac{\partial v_0}{\partial t_{02}} &= 2v_{1,x} + v_{0,xx}, \\
\frac{\partial v_1}{\partial t_{02}} &= 2v_{2,x} + v_{1,xx} + 2v_0v_{0,x}, \\
\frac{\partial v_2}{\partial t_{02}} &= 2v_{3,x} + v_{2,xx} + 4v_1v_{0,x} - 2v_0v_{0,xx}, \\
\frac{\partial v_3}{\partial t_{02}} &= 2v_{4,x} + v_{3,xx} + 6v_2v_{0,x} - 6v_1v_{0,xx} + 2v_0v_{0,xxx}, \\
\frac{\partial v_4}{\partial t_{02}} &= 2v_{5,x} + v_{4,xx} + 8v_3v_{0,x} - 12v_2v_{0,xx} + 8v_1v_{0,xxx} - 2v_0v_{0,xxxx}, \\
&\vdots
\end{aligned} \tag{3.4}$$

From (3.4), we obtain

$$
\begin{aligned}
v_1 &= \frac{1}{2}(\partial_x^{-1}v_{0,y} - v_{0,x}),\\
v_2 &= \frac{1}{4}(\partial_x^{-2}v_{0,yy} - 2v_{0,y} + v_{0,xx} - 2v_0^2),\\
v_3 &= \frac{1}{8}(\partial_x^{-3}v_{0,yyy} - 3\partial_x^{-1}v_{0,yy} + 3v_{0,xy} - v_{0,xxx} + 12v_0v_{0,x} - 8v_0\partial_x^{-1}v_{0,y} + 4\partial_x^{-1}(v_0v_{0,y})),\\
v_4 &= \frac{1}{16}(\partial_x^{-4}v_{0,yyyy} - 4\partial_x^{-2}v_{0,yyy} + 6v_{0,yy} - 4v_{0,xxy} + 32\partial_x^{-1}(v_{0,x}v_{0,y}) - 8\partial_x^{-1}(v_{0,y}\partial_x^{-1}v_{0,y})\\
&\quad -36v_{0,x}^2 - 28\partial_x^{-1}(v_0v_{0,xxx}) + 32\partial_x^{-1}(v_{0,xx}\partial_x^{-1}v_{0,y}) - 8\partial_x^{-1}(v_0\partial_x^{-1}v_{0,yy}) + 4\partial_x^{-2}(v_{0,y}^2)\\
&\quad +4\partial_x^{-2}(v_0v_{0,yy}) + v_{0,xxxx} - 12\partial_x^{-1}(v_{0,x}\partial_x^{-2}v_{0,yy}) + 8v_0^3 + 16v_0v_{0,y}),\\
v_{5x} &= \frac{1}{32}\partial_x^{-4}v_{0,yyyyy} - \frac{5}{32}\partial_x^{-2}v_{0,yyyy} + \frac{5}{16}v_{0,yyy} - \frac{5}{16}v_{0,xxyy} + \frac{9}{8}(v_{0,y})^2 + \frac{5}{8}v_0v_{0,yy}\\
&\quad +\partial_x^{-1}(v_{0,x}v_{0,yy}) - \frac{1}{4}(\partial_x^{-1}v_{0,y})(\partial_x^{-1}v_{0,yy}) - \frac{23}{4}v_{0,x}v_{0,xy} - \frac{7}{8}\partial_x^{-1}(v_{0,y}v_{0,xxx})\\
&\quad -\frac{7}{8}\partial_x^{-1}(v_0v_{0,xxxy}) + \partial_x^{-1}(v_{0,xxy}\partial_x^{-1}v_{0,y}) + \partial_x^{-1}(v_{0,xx}\partial_x^{-1}v_{0,yy}) - \frac{1}{4}\partial_x^{-1}(v_{0,y}\partial_x^{-1}v_{0,yy})\\
&\quad -\frac{1}{4}\partial_x^{-1}(v_0\partial_x^{-1}v_{0,yyy}) + \frac{3}{8}\partial_x^{-2}(v_{0,y}v_{0,yy}) + \frac{5}{32}v_{0,xxxxy} + \frac{17}{8}v_{0,x}\partial_x^{-1}v_{0,yy}\\
&\quad -\frac{3}{8}\partial_x^{-1}(v_{0,xy}\partial_x^{-2}v_{0,yy}) - \frac{3}{8}\partial_x^{-1}(v_{0,x}\partial_x^{-2}v_{0,yyy}) + \frac{3}{4}v_0^2v_{0,y} - \frac{11}{2}v_{0,xx}v_{0,y}\\
&\quad +\frac{1}{4}v_{0,xy}\partial_x^{-1}v_{0,y} + \frac{15}{4}v_{0,xx}^2 + \frac{45}{8}v_{0,x}v_{0,xxx} + \frac{15}{8}v_0v_{0,xxxx} - 3v_{0,xxx}\partial_x^{-1}v_{0,y}\\
&\quad +\frac{1}{8}\partial_x^{-2}(v_0v_{0,yyy}) + \frac{15}{8}v_{0,xx}\partial_x^{-2}v_{0,yy} - \frac{15}{4}v_0^2v_{0,xx} - \frac{15}{2}v_0v_{0,x}^2 - \frac{1}{2}v_{0,x}\partial_x^{-3}v_{0,yyy}\\
&\quad +4v_0v_{0,x}\partial_x^{-1}v_{0,y} - \frac{1}{2}v_0v_{0,xxy} - 2v_{0,x}\partial_x^{-1}(v_0v_{0,y}) - \frac{1}{32}v_{0,xxxxxx},\\
&\quad \vdots,
\end{aligned}
\tag{3.5}
$$

where $y = t_{02}$.

For later convenience, let us list some evolution equations of the KP hierarchy as follows:

- For the case of $B_n = B_3$ in (3.3), we have

$$
\frac{\partial v_0}{\partial t_{03}} = 3v_{2,x} + 3v_{1,xx} + v_{0,xxx} + 6v_0v_{0,x}. \tag{3.6}
$$

Substituting the expressions of v_2 and v_1 in (3.5) into (3.6), we obtain the usual KP equation

$$
3v_{0,yy} = (4v_{0,t} - v_{0,xxx} - 12v_0v_{0,x})_x, \tag{3.7}
$$

where $t = t_{03}$. Under the reduction $y = 0$, (3.7) reduces to the well-known KdV equation.

- For the case of $B_n = B_4$ in (3.3), we have

$$
\frac{\partial v_0}{\partial t_{04}} = 4v_{3,x} + 6v_{2,xx} + 4v_{1,xxx} + v_{0,xxxx} + 6(v_{0,x})^2 + 6v_0v_{0,xx} + 12v_0v_{1,x} + 12v_1v_{0,x}. \tag{3.8}
$$

Substituting the expressions of $v_i, i = 1,2,3$ in (3.5) into (3.8), we obtain the following integrable equation:

$$\frac{\partial v_0}{\partial t} = \frac{1}{2} v_{0,xxy} + \frac{1}{2}\partial_x^{-2} v_{0,yyy} + 2v_{0,x}\partial_x^{-1} v_{0,y} + 4v_0 v_{0,y}, \tag{3.9}$$

where $t = t_{04}$.

• For the case of $B_n = B_5$ in (3.3), we have

$$\begin{aligned}\frac{\partial v_0}{\partial t_{05}} =& 5v_{4,x} + 10v_{3,xx} + 10v_{2,xxx} + 5v_{1,xxxx} + v_{0,xxxxx} + 20v_2 v_{0,x} + 20v_0 v_{2,x} + 20v_1 v_{1,x} \\ &+30v_{0,x}v_{1,x} + 20v_0 v_{1,xx} + 10v_1 v_{0,xx} + 20v_{0,x}v_{0,xx} + 10v_0 v_{0,xxx} + 30v_0^2 v_{0,x}.\end{aligned} \tag{3.10}$$

Substituting the expressions of $v_i, i = 1,2,3,4$ in (3.5) into (3.10), we obtain the following integrable equation:

$$\begin{aligned}\frac{\partial v_0}{\partial t} =& \frac{1}{16} v_{0,xxxxx} + \frac{5}{4}\partial_x^{-1}(v_0 v_{0,yy}) + \frac{5}{4}\partial_x^{-1}(v_{0,y})^2 + \frac{5}{16}\partial_x^{-3} v_{0,yyyy} + \frac{5}{4} v_{0,x}\partial_x^{-2} v_{0,yy} \\ &+\frac{5}{2} v_0 \partial_x^{-1} v_{0,yy} + \frac{5}{2} v_{0,y}\partial_x^{-1} v_{0,y} + \frac{15}{2} v_0^2 v_{0,x} + \frac{5}{2} v_{0,x}v_{0,xx} + \frac{5}{4} v_0 v_{0,xxx} + \frac{5}{8} v_{0,xyy},\end{aligned} \tag{3.11}$$

where $t = t_{05}$.

• For the case of $B_n = B_6$ in (3.3), we have

$$\begin{aligned}\frac{\partial v_0}{\partial t_{06}} =& 6v_{5,x} + 15v_{4,xx} + 20v_{3,xxx} + 15v_{2,xxxx} + 6v_{1,xxxxx} + v_{0,xxxxxx} + 30v_3 v_{0,x} \\ &+30v_0 v_{3,x} + 30v_1 v_{2,x} + 30v_2 v_{1,x} + 60v_{2,x}v_{0,x} + 15v_2 v_{0,xx} + 45v_0 v_{2,xx} \\ &+30v_1 v_{1,xx} + 30(v_{1,x})^2 + 35v_0 v_{1,xxx} + 55v_{1,x}v_{0,xx} + 65v_{0,x}v_{1,xx} + 25v_1 v_{0,xxx} \\ &+60v_0^2 v_{1,x} + 120v_1 v_0 v_{0,x} + 20(v_{0,xx})^2 + 60v_0 v_{0,x}^2 + 30v_0^2 v_{0,xx} + 10v_0 v_{0,xxxx} \\ &+30v_{0,x}v_{0,xxx}.\end{aligned} \tag{3.12}$$

Substituting the expressions of $v_i, i = 1,2,\cdots 5$ in (3.5) into (3.12), we obtain the following integrable equation:

$$\begin{aligned}\frac{\partial v_0}{\partial t} =& 3(v_{0,y})^2 + 9v_0 v_{0,x}\partial_x^{-1} v_{0,y} + 6\partial_x^{-1}(v_{0,x}v_{0,yy}) + 12v_0^2 v_{0,y} + 3v_{0,x}\partial_x^{-1}(v_0 v_{0,y}) \\ &-\frac{21}{4}\partial_x^{-1}(v_0 v_{0,xxxy}) + \frac{3}{4}\partial_x^{-2}(v_0 v_{0,yyy}) + \frac{15}{4}\partial_x^{-2}(v_{0,yy})v_{0,y} + \frac{15}{4} v_0 \partial_x^{-2}(v_{0,yyy}) \\ &+6\partial_x^{-1}(v_{0,xxy}\partial_x^{-1} v_{0,y}) + 6\partial_x^{-1}(v_{0,xx}\partial_x^{-1} v_{0,yy}) - \frac{3}{2}\partial_x^{-1}(v_0 \partial_x^{-1} v_{0,yyy}) - \frac{3}{4} v_{0,x}v_{0,xy} \\ &+\frac{33}{4} v_{0,xx}v_{0,y} + \frac{33}{4} v_0 v_{0,xxy} - 6v_{0,xy}\partial_x^{-1} v_{0,y} + \frac{3}{4} v_{0,xxx}\partial_x^{-1} v_{0,y} - 6v_{0,x}\partial_x^{-1} v_{0,yy} \\ &+\frac{3}{4} v_{0,x}\partial_x^{-3} v_{0,yyy} + \frac{9}{4}\partial_x^{-2}(v_{0,y}v_{0,yy}) - \frac{3}{2}\partial_x^{-1}(v_{0,y}\partial_x^{-1} v_{0,yy}) + \frac{3}{16}\partial_x^{-4}(v_{0,yyyyy}) \\ &+\frac{5}{8} v_{0,yyy} - \frac{21}{4}\partial_x^{-1}(v_{0,y}v_{0,xxx}) - \frac{9}{4}\partial_x^{-1}(v_{0,x}\partial_x^{-2} v_{0,yyy}) + \frac{3}{16} v_{0,xxxxy} \\ &-\frac{9}{4}\partial_x^{-1}(v_{0,xy}\partial_x^{-2} v_{0,yy}) + \frac{9}{4}(\partial_x^{-1} v_{0,yy})(\partial_x^{-1} v_{0,y}),\end{aligned} \tag{3.13}$$

where $t = t_{06}$.

3.2. *Generalized Lax equation and integrable evolution equations*

For the generalized Lax equation (3.2), we note that when $B_m = B_0$, it reduces to the Lax equation. Thus we may derive the KP hierarchy from (3.3). Now encouraged by this result, it would be interesting to study further and see whether one could derive the integrable evolution equations from (3.2) except for $B_m = B_0$. In this subsection we give affirmative answer to this question. Let us turn to deal with (3.2) for the cases of (B_1, B_2) and (B_1, B_3), respectively.

• Taking the operator pair (B_1, B_2) in (3.2), we have

$$\frac{\partial v_0}{\partial t_{12}} = v_{2,x} + v_{1,xx} + 2v_0 v_{0,x}. \tag{3.14}$$

Substituting the expressions of v_2 and v_1 in (3.5) into (3.14), we obtain

$$v_{0,yy} = (4\frac{\partial v_0}{\partial t} + v_{0,xxx} - 4v_0 v_{0,x})_x, \tag{3.15}$$

where $t = t_{12}$. Under the scaling transformation $y \to \frac{\mathbf{i}}{\sqrt{3}}y$, $v_0 \to -3v_0$, and $t \to -t$, where $\mathbf{i} = \sqrt{-1}$, (3.15) becomes the usual KP equation (3.7).

• Taking the operator pair (B_1, B_3) in (3.2), we have

$$\frac{\partial v_0}{\partial t_{13}} = 2v_{3,x} + 3v_{2,xx} + v_{1,xxx} + 6v_1 v_{0,x} + 6v_0 v_{1,x} + 3(v_{0,x})^2 + 3v_0 v_{0,xx}, \tag{3.16}$$

Substituting the expressions of $v_i, i = 1,2,3$ in (3.5) into (3.16), we obtain

$$\frac{\partial v_0}{\partial t} = -\frac{1}{4}v_{0,xxy} + \frac{1}{4}\partial_x^{-2}v_{0,yyy} + v_{0,x}\partial^{-1}(v_{0,y}) + 2v_0 v_{0,y}, \tag{3.17}$$

where $t = t_{13}$. Under the scaling transformation $x \to \mathbf{i}x$, $y \to \mathbf{i}y$, and $t \to 2\mathbf{i}t$, (3.17) becomes the integrable equation (3.9).

It is interesting to note that we derive the KP equation (3.7) and the integrable equation (3.9) in the KP hierarchy from the generalized Lax equation with the different Lax triple (L, B_n, B_m). Due to the generalized Lax equation involving two different operators B_n and B_m, our analysis indicates that there is the deep relationship between the operator pair (B_n, B_m). It should be stressed that there is not such kind of remarkable property in the usual Lax equation.

3.3. *Generalized Lax equation and evolution equations*

In the previous section, we derived two integrable equations in the KP hierarchy. Let us proceed to deal with (3.2) with more general Lax triples.

• Taking the operator pair (B_1, B_4) in (3.2), we have

$$\begin{aligned}\frac{\partial v_0}{\partial t_{14}} &= 3v_{4,x} + 6v_{3,xx} + 4v_{2,xxx} + v_{1,xxxx} + 12v_0 v_{1,xx} + 18v_0^2 v_{0,x} + 6v_1 v_{0,xx} \\ &\quad + 8v_{0,x}v_{0,xx} + 12v_2 v_{0,x} + 6v_0 v_{0,xxx} + 18v_{0,x}v_{1,x} + 12v_1 v_{1,x} + 12v_0 v_{2,x},\end{aligned} \tag{3.18}$$

Substituting the expressions of $v_i, i = 1,2,3,4$ in (3.5) into (3.18), we obtain

$$\begin{aligned}\frac{\partial v_0}{\partial t} &= -\frac{1}{16}v_{0,xxxxx} + \frac{3}{4}\partial_x^{-1}(v_0 v_{0,yy}) + \frac{3}{4}\partial_x^{-1}(v_{0,y})^2 + \frac{3}{16}\partial_x^{-3}v_{0,yyyy} + \frac{3}{4}v_{0,x}\partial_x^{-2}v_{0,yy} \\ &\quad + \frac{3}{2}v_0\partial_x^{-1}v_{0,yy} + \frac{3}{2}v_{0,y}\partial_x^{-1}v_{0,y} + \frac{9}{2}v_0^2 v_{0,x} + \frac{7}{2}v_{0,x}v_{0,xx} + \frac{11}{4}v_0 v_{0,xxx} - \frac{1}{8}v_{0,xyy},\end{aligned} \tag{3.19}$$

where $t = t_{14}$. Its single soliton solution is

$$v_0 = \frac{1}{6(-7+3\sqrt{41})}[(57-9\sqrt{41})a^2 + (330-50\sqrt{41})k^2 - (990-150\sqrt{41})k^2 sech^2\xi], \tag{3.20}$$

where $\xi = k(\omega t + x + ay) + b$ in which

$$\omega = \frac{1}{48(-5601+869\sqrt{41})}[(-8157592+1273848\sqrt{41})k^4 + (187893-29385\sqrt{41})a^4],$$

a, b and k are constants.

• Taking the operator pair (B_2, B_3) in (3.2), we have

$$\begin{aligned}\frac{\partial v_0}{\partial t_{23}} &= v_{4,x} + 2v_{3,xx} + v_{2,xxx} + 3v_0 v_{1,xx} + 3v_1 v_{0,xx} + 3v_{0,x} v_{0,xx} \\ &+3v_2 v_{0,x} + v_0 v_{0,xxx} + 6v_{0,x} v_{1,x} + 6v_1 v_{1,x} + 3v_0 v_{2,x}.\end{aligned} \tag{3.21}$$

Substituting the expressions of $v_i, i = 1,2,3,4$ in (3.5) into (3.21), we obtain

$$\begin{aligned}\frac{\partial v_0}{\partial t} &= \frac{1}{16} v_{0,xxxxx} + \frac{1}{4}\partial_x^{-1}(v_0 v_{0,yy}) + \frac{1}{4}\partial_x^{-1}(v_{0,y})^2 + \frac{1}{16}\partial_x^{-3} v_{0,yyyy} + \frac{1}{4} v_0 \partial_x^{-1} v_{0,yy} \\ &+v_{0,y}\partial_x^{-1} v_{0,y} - 3v_0^2 v_{0,x} + \frac{9}{4} v_{0,x} v_{0,xx} + \frac{1}{2} v_0 v_{0,xxx} - \frac{1}{8} v_{0,xyy},\end{aligned} \tag{3.22}$$

where $t = t_{23}$. Its single soliton solution is

$$v_0 = \frac{a^2+10k^2}{4} - \frac{15k^2}{2} sech^2\xi, \tag{3.23}$$

where $\xi = k(\omega t + x + ay) + b$ in which $\omega = \frac{a^4-51k^4}{4}$, a, b and k are constants.

• Taking the operator pair (B_1, B_5) in (3.2), we have

$$\begin{aligned}\frac{\partial v_0}{\partial t_{15}} &= 4v_{5,x} + 10v_{4,xx} + 10v_{3,xxx} + 5v_{2,xxxx} + v_{1,xxxxx} + 20v_{3,x} v_0 + 20v_3 v_{0,x} \\ &+20v_1 v_{0,xxx} + 20v_0 v_{1,xxx} + 10(v_{0,xx})^2 + 20v_0^2 v_{0,xx} + 20v_1 v_{1,xx} + 20v_{1,x}^2 \\ &+80v_1 v_0 v_{0,x} + 5v_0 v_{0,xxxx} + 30v_{1,x} v_{0,xx} + 30v_{0,x} v_{1,xx} + 15v_{0,x} v_{0,xxx} + 40v_0 v_{0,x}^2 \\ &+40v_{2,x} v_{0,x} + 30v_0 v_{2,xx} + 10v_2 v_{0,xx} + 20v_2 v_{1,x} + 20v_1 v_{2,x} + 40v_0^2 v_{1,x}.\end{aligned} \tag{3.24}$$

Substituting the expressions of $v_i, i = 1, 2, \cdots 5$ in (3.5) into (3.24), we obtain

$$\begin{aligned}\frac{\partial v_0}{\partial t} &= 4\partial_x^{-1}(v_{0,xxy}\partial_x^{-1}v_{0,y}) + 4\partial_x^{-1}(v_{0,xx}\partial_x^{-1}v_{0,yy}) + 2(v_{0,y})^2 + \frac{3}{2}\partial_x^{-2}(v_{0,y}v_{0,yy})\\ &+6v_0v_{0,x}\partial_x^{-1}v_{0,y} + \frac{1}{8}\partial_x^{-4}v_{0,yyyyy} - \frac{3}{2}\partial_x^{-1}(v_{0,x}\partial_x^{-2}v_{0,yyy}) - \frac{1}{8}v_{0,xxxxy}\\ &-\frac{3}{2}\partial_x^{-1}(v_{0,xy}\partial_x^{-2}v_{0,yy}) + 8v_0^2v_{0,y} + 2v_{0,x}\partial_x^{-1}(v_0v_{0,y}) + \frac{5}{2}v_0\partial_x^{-2}(v_{0,yyy})\\ &+\frac{1}{2}v_{0,x}\partial_x^{-3}v_{0,yyy} - \frac{7}{2}\partial_x^{-1}(v_0v_{0,xxxy}) + 4\partial_x^{-1}(v_{0,x}v_{0,yy}) - \frac{7}{2}\partial_x^{-1}(v_{0,y}v_{0,xxx})\\ &+\frac{21}{2}v_{0,xx}v_{0,y} - 4v_{0,x}\partial_x^{-1}v_{0,yy} + \frac{11}{2}v_{0,xxx}\partial_x^{-1}v_{0,y} + \frac{11}{2}v_0v_{0,xxy} - 4v_{0,xy}\partial_x^{-1}v_{0,y}\\ &-\partial_x^{-1}(v_{0,y}\partial_x^{-1}v_{0,yy}) - \partial_x^{-1}(v_0\partial_x^{-1}v_{0,yyy}) - \frac{1}{2}v_{0,x}v_{0,xy} + \frac{1}{2}\partial_x^{-2}(v_0v_{0,yyy})\\ &+\frac{3}{2}\partial_x^{-1}v_{0,yy}\partial_x^{-1}v_{0,y} + \frac{5}{2}v_{0,y}\partial_x^{-2}v_{0,yy},\end{aligned} \tag{3.25}$$

where $t = t_{15}$. Its single soliton solution is

$$v_0 = \frac{1}{12(-1+\sqrt{7})}[(5-2\sqrt{7})a^2 + (76-28\sqrt{7})k^2 - (228-84\sqrt{7})k^2 sech^2\xi], \tag{3.26}$$

where $\xi = k(\omega t + x + ay) + b$ in which

$$\omega = \frac{1}{96(-125+47\sqrt{7})}[(-857120+323936\sqrt{7})k^4 + (2654-1001\sqrt{7})a^4],$$

a, b and k are constants.

- Taking the operator pair (B_2, B_4) in (3.2), we have

$$\begin{aligned}\frac{\partial v_0}{\partial t_{24}} &= 2v_{5,x} + 5v_{4,xx} + 4v_{3,xxx} + v_{2,xxxx} + 8v_{3,x}v_0 + 8v_3v_{0,x} + 12v_0v_{0,x}^2\\ &+4v_1v_{0,xxx} + 12v_0v_{1,xxx} + 2(v_{0,xx})^2 + 6v_0^2v_{0,xx} + 12v_1v_{1,xx} + 12v_{1,x}^2\\ &+24v_1v_0v_{0,x} + 4v_0v_{0,xxxx} + 10v_{1,x}v_{0,xx} + 18v_{0,x}v_{1,xx} + 6v_{0,x}v_{0,xxx}\\ &+18v_{2,x}v_{0,x} + 12v_0v_{2,xx} + 6v_2v_{0,xx} + 12v_2v_{1,x} + 12v_1v_{2,x} + 12v_0^2v_{1,x}.\end{aligned} \tag{3.27}$$

Substituting the expressions of $v_i, i = 1, 2, \cdots 5$ in (3.5) into (3.27), we obtain

$$\begin{aligned}\frac{\partial v_0}{\partial t} &= 2\partial_x^{-1}(v_{0,xxy}\partial_x^{-1}v_{0,y}) + 2\partial_x^{-1}(v_{0,xx}\partial_x^{-1}v_{0,yy}) + (v_{0,y})^2 + \frac{3}{4}\partial_x^{-2}(v_{0,y}v_{0,yy})\\ &-2v_0v_{0,x}\partial_x^{-1}v_{0,y} + \frac{1}{16}\partial_x^{-4}v_{0,yyyyy} - \frac{3}{4}\partial_x^{-1}(v_{0,x}\partial_x^{-2}v_{0,yyy}) + \frac{1}{16}v_{0,xxxxy}\\ &-\frac{3}{4}\partial_x^{-1}(v_{0,xy}\partial_x^{-2}v_{0,yy}) + \frac{1}{2}v_0^2v_{0,y} + v_0\partial_x^{-2}(v_{0,yyy}) - \frac{1}{8}v_{0,yyy} - \frac{7}{4}\partial_x^{-1}(v_0v_{0,xxxy})\\ &+2\partial_x^{-1}(v_{0,x}v_{0,yy}) - \frac{7}{4}\partial_x^{-1}(v_{0,y}v_{0,xxx}) + \frac{9}{2}v_{0,xx}v_{0,y} - 2v_{0,x}\partial_x^{-1}v_{0,yy} + \frac{1}{2}v_{0,xxx}\partial_x^{-1}v_{0,y}\\ &+5v_0v_{0,xxy} - 2v_{0,xy}\partial_x^{-1}v_{0,y} - \frac{1}{2}\partial_x^{-1}(v_{0,y}\partial_x^{-1}v_{0,yy}) - \frac{1}{2}\partial_x^{-1}(v_0\partial_x^{-1}v_{0,yyy})\\ &+\frac{7}{2}v_{0,x}v_{0,xy} + \frac{1}{4}\partial_x^{-2}(v_0v_{0,yyy}) + \partial_x^{-1}v_{0,yy}\partial_x^{-1}v_{0,y} + \frac{3}{2}v_{0,y}\partial_x^{-2}v_{0,yy},\end{aligned} \tag{3.28}$$

where $t = t_{24}$. Its single soliton solution is

$$v_0 = \frac{1}{6(4+\sqrt{61})}[(38+5\sqrt{61})a^2 + (736+94\sqrt{61})k^2 - (2208+282\sqrt{61})k^2 sech^2\xi], \qquad (3.29)$$

where $\xi = k(\omega t + x + ay) + b$ in which

$$\omega = \frac{1}{48(51272+6563\sqrt{61})}[-(820537408+105059032\sqrt{61})k^4 + (2478952+317383\sqrt{61})a^4],$$

a, b and k are constants.

By applying the Painlevé analysis to the equations derived in this subsection, we find that they do not pass the test. However it is interesting to note that they have the single soliton solutions. The KP hierarchy is introduced as a generalization of KdV hierarchies. The higher-order KdV equations have attracted a lot of interest from physical and mathematical points of view [12, 13, 16]. It is known that the fifth order KdV equations describe motions of long waves in shallow water under gravity. Recently the following general fifth order KdV equation has been investigated [16]:

$$\frac{\partial v}{\partial t} = \rho v_{xxxxx} + \alpha v v_{xxx} + \beta v_x v_{xx} + \gamma v^2 v_x, \qquad (3.30)$$

where ρ, α, β and γ are the arbitrary real parameters. By using the *exp* function method, the solutions of (3.30) have also been presented there.

Under the low dimensional reduction $y = 0$, we note that (3.19) and (3.22) reduce to the following fifth order KdV equations,

$$\frac{\partial v_0}{\partial t} = -\frac{1}{16}v_{0,xxxxx} + \frac{11}{4}v_0 v_{0,xxx} + \frac{7}{2}v_{0,x}v_{0,xx} + \frac{9}{2}v_0^2 v_{0,x} \qquad (3.31)$$

and

$$\frac{\partial v_0}{\partial t} = \frac{1}{16}v_{0,xxxxx} + \frac{1}{2}v_0 v_{0,xxx} + \frac{9}{4}v_{0,x}v_{0,xx} - 3v_0^2 v_{0,x}, \qquad (3.32)$$

respectively. We immediately recognize that (3.31) and (3.32) are the special cases of (3.30).

4. Summary

In this paper, in terms of the operator Nambu 3-bracket and the Lax pair (L, B_n) of the KP hierarchy, we proposed the generalized Lax equation with respect to the Lax triple (L, B_n, B_m). When one of two operators B_n and B_m in the Lax triple is unit operator, we noted that the generalized Lax equation reduces to the usual Lax equation of the KP hierarchy. Due to the Lax triple (L, B_n, B_m) involving two operators B_n and B_m, we found that there is an intrinsic equivalent relation between the different Lax triples. We derived the KP equation from the generalized Lax equation with two different Lax triples (L, B_0, B_3) and (L, B_1, B_2), respectively. For the case of the Lax triples (L, B_0, B_4) and (L, B_1, B_3), by the similar way, we derived the same integrable equation in the KP hierarchy. Moreover we investigated the generalized Lax equation with more general Lax triples and derived some evolution equations. Although these equations are no integrable, however they have the single soliton solutions. The applications of these evolution equations in physics should be of interest. It should be pointed out that we only analyze the cases of several Lax triples. More properties of the generalized Lax equation with the general Lax triple still deserve further study.

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