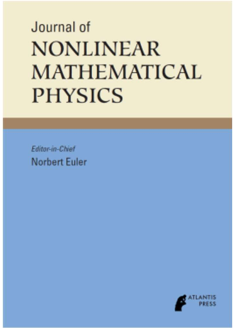

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Internal equatorial water waves in the $f-$plane

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In this paper we describe an exact, and explicit, three-dimensional nonlinear solution for geophysical internal ocean waves in the Equatorial region which incorporates a transverse-Equatorial meridional current.

Keywords: Geophysical, internal water waves, transverse current, $f-$plane, exact solutions.

2010 Mathematics Subject Classification: 76B15, 74G05, 86A05

1. Introduction

In this paper we describe an exact three-dimensional solution for nonlinear geophysical internal ocean waves in the Equatorial region which incorporates a transverse-Equatorial meridional current. The solution corresponds to the classical two-layer model describing oscillations of the thermocline (which is an interface separating two distinct vertical ocean layers of differing densities) in the equatorial region, whereby the fluid wave motion diminishes as one ascends from the thermocline towards the surface. This solution is valid for oceanic flows within a restricted meridional range of approximately 2° latitude from the Equator, a region where the $f-$plane approximation of the geophysical governing equations applies [10, 12, 27].

The solution we present is explicit in terms of Lagrangian labelling parameters, and in this, and other, respects the solution may be termed *Gerstner-like* in reference to the celebrated Gerstner's wave. Gerstner's wave is a two-dimensional wave propagating over an infinitely-deep fluid domain— cf. [3, 5, 17], and also [2, 28] for a Gerstner-like formulation of edge-waves propagating over a sloping bed. One of the Gerstner wave's primary points of significance is the fact that it is the only known explicit and exact solution of the nonlinear periodic gravity wave problem with a non-flat free-surface.

Remarkably, considering the Gerstner wave's rareness, and highly-prescribed mathematical formulation, Constantin recently presented a solution [6] to the geophysical governing equations which is Gerstner-like, in the sense that it reduces to Gerstner's solution upon ignoring Coriolis effects. However, the solution in [6] embodies a significant breakthrough since it successfully generalises to the geophysical setting, in the sense that it defines a inherently three-dimensional eastward-propagating geophysical wave which is Equatorially-trapped and whose dispersion relation is dependant on the Coriolis parameter. Subsequently a wide variety of exact and explicit solutions were derived and analysed in various papers (cf. [7–9, 15, 18–25]) with the respective solutions modelling a number of different physical and geophysical scenarios.

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Among these were some exact, explicit solutions to the $\beta-$plane governing equations corresponding to the classical two-layer model describing oscillations of the thermocline in the equatorial region [7, 8]. A feature of these particular solutions is the absence of any meridional flow, and the aim of this paper is to present a solution fitting the two-layer model describing oscillations of the thermocline which also admits transverse-Equatorial fluid flow, something which is only achievable in the $f-$plane formulation. A well-recognised [26] and advantageous characteristic of exact finite-amplitude solutions to a water wave problem (particularly if they are explicit) is the opportunity to perturb these solutions to generate more complex flows. Controlling the perturbations appropriately, it may be possible to derive detailed information about the resulting fluid motion. It is to be hoped that allowing for transverse Equatorial fluid motion is a physically useful, and mathematically interesting, extension of the $f-$plane solution presented in [22] in the sense that the additional physical complexity may be beneficial with respect to potential complex generalisations of the flow.

2. Governing equations

In a frame of reference with the origin located at a point fixed on Earth's surface and rotating with the Earth, the governing equations for geophysical ocean waves are given by [12, 27]

$$u_t + uu_x + vu_y + wu_z + 2\Omega w\cos\phi - 2\Omega v\sin\phi = -\frac{1}{\rho}P_x, \tag{2.1a}$$

$$v_t + uv_x + vv_y + wv_z + 2\Omega u\sin\phi = -\frac{1}{\rho}P_y, \tag{2.1b}$$

$$w_t + uw_x + vw_y + ww_z - 2\Omega u\cos\phi = -\frac{1}{\rho}P_z - g, \tag{2.1c}$$

together with the equation of incompressibility

$$\nabla\cdot\mathbf{U} = 0, \tag{2.2a}$$

where $\mathbf{U} = (u, v, w)$ is the velocity field of the fluid, and the equation of mass conservation

$$\rho_t + u\rho_x + v\rho_y + w\rho_z = 0, \tag{2.2b}$$

where ρ is the density of the fluid. Here the variable ϕ denotes the latitude and P is the pressure of the fluid. The Earth is taken to be a perfect sphere of radius $R = 6378km$ with constant rotational speed of $\Omega = 73\cdot 10^{-6} rad/s$, and $g = 9.8ms^{-2}$ is the gravitational acceleration at the surface of the Earth. In the Equatorial region the Coriolis terms in (2.1) are rendered more tractable by employing the small-latitude approximation

$$\sin\phi \approx \phi, \cos\phi \approx 1,$$

reducing the governing equations to the $\beta-$plane approximation form:

$$\begin{aligned}
u_t + uu_x + vu_y + wu_z + 2\Omega w - \beta yv &= -\frac{1}{\rho}P_x,\\
v_t + uv_x + vv_y + wv_z + \beta yu &= -\frac{1}{\rho}P_y,\\
w_t + uw_x + vw_y + ww_z - 2\Omega u &= -\frac{1}{\rho}P_z - g,
\end{aligned}$$

where $\beta = 2\Omega/R = 2.28 \cdot 10^{-11} m^{-1} s^{-1}$. Effectively, the Coriolis terms for the curved Earth's surface which appear in (2.1) are approximated by a planar model. If we restrict our focus to purely Equatorial waves then the governing equations are further simplified [12] giving us the f−plane approximation

$$u_t + uu_x + vu_y + wu_z + 2\Omega w = -\frac{1}{\rho} P_x, \tag{2.2ca}$$

$$v_t + uv_x + vv_y + wv_z = -\frac{1}{\rho} P_y, \tag{2.2cb}$$

$$w_t + uw_x + vw_y + ww_z - 2\Omega u = -\frac{1}{\rho} P_z - g. \tag{2.2cc}$$

In this paper we present a solution of (2.2c) that satisfies a two-layer model incorporating the thermocline, which is an interface separating two distinct vertical ocean layers of differing densities.

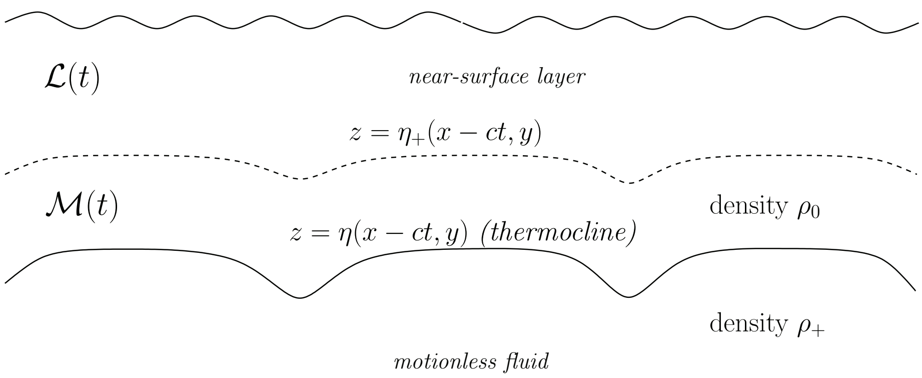

Fig. 1. Schematic of the two-layer model

This model may be described as follows. The fluid layer which lies above the thermocline is subdivided into two parts, which we denote $\mathscr{L}(t)$ and $\mathscr{M}(t)$, with both regions having a constant fluid density ρ_0. The near-surface layer, labelled $\mathscr{L}(t)$, is the region to which wind effects are confined. Typical values for the mean-depth of $\mathscr{L}(t)$ are 80m. Beneath $\mathscr{L}(t)$ is a layer where the fluid motion is entirely due to the propagation of equatorial waves, this layer is denoted $\mathscr{M}(t)$, and typical values for the mean-depth of $\mathscr{M}(t)$ are 40m, cf. [7]. The thermocline is an interface lying at the boundary of $\mathscr{M}(t)$ and the deeper, motionless layer of fluid which has density $\rho_+ > \rho_0$. The relative magnitudes of the fluid densities may be deduced from observing that the reduced gravity, defined by $\tilde{g} = g\frac{\rho_+ - \rho_0}{\rho_0}$, has a typical value of $6 \cdot 10^{-3}$m s^{-2} [13]. We denote the thermocline by $z = \eta(x - ct, y)$, while the interface separating $\mathscr{L}(t)$ and $\mathscr{M}(t)$ is denoted $z = \eta_+(x - ct, y)$, where c is the constant wave phasespeed. Beneath the thermocline the fluid is assumed motionless, and so $u \equiv v \equiv w \equiv 0$ for $z < \eta(x - ct, y)$. The stillness of fluid beneath the thermocline, coupled with (2.2c), leads naturally to the boundary condition

$$P = P_0 - \rho_+ g z \quad \text{on } z = \eta(x - ct, y), \tag{2.2d}$$

for some constant P_0. In this paper we present an exact solution of the governing equations (2.2) which describes fluid flow in the $\mathscr{M}$ region.

3. Exact solution of (2.2)

The equations of motion (2.2c) may be reformulated (where D/Dt is the material or convective derivative) as

$$\frac{Du}{Dt} + 2\Omega w = -\frac{1}{\rho}P_x, \tag{3.1a}$$

$$\frac{Dv}{Dt} = -\frac{1}{\rho}P_y, \tag{3.1b}$$

$$\frac{Dw}{Dt} - 2\Omega u = -\frac{1}{\rho}P_z - g. \tag{3.1c}$$

In the following we present an exact solution for fluid motion in the $\mathscr{M}(t)$ region, which is explicit in a Lagrangian formulation using the labelling parameters q,r,s. Here $q \in \mathbb{R}$, $s \in [-s_0, s_0]$ for $s_0 = \sqrt{c_0/\beta} \approx 250$km the equatorial radius of deformation [12], and $r \in [r_0, r_+]$, where $r = r_0$ determines the thermocline η and $r = r_+$ determines the interface η_+ which separates the $\mathscr{M}(t)$ and $\mathscr{L}(t)$ region. It is shown below that $r_+(s) > r_0(s) > 0$. In this section we show that the following system of Eulerian coordinates for the flow, defined in terms of these Lagrangian labelling variables (q,r,s) and time t, represent a solution of the governing equations (2.2):

$$x = q - \frac{1}{k}e^{-kr}\sin[k(q-ct)], \tag{3.2a}$$

$$y = s + \psi(q,r)t, \tag{3.2b}$$

$$z = r - \frac{1}{k}e^{-kr}\cos[k(q-ct)]. \tag{3.2c}$$

Denoting $\xi = -kr$, $\theta = k(q-ct)$, the Jacobian matrix of the transformation (3.2) takes the following form

$$\begin{pmatrix} \frac{\partial x}{\partial q} & \frac{\partial y}{\partial q} & \frac{\partial z}{\partial q} \\ \frac{\partial x}{\partial s} & \frac{\partial y}{\partial s} & \frac{\partial z}{\partial s} \\ \frac{\partial x}{\partial r} & \frac{\partial y}{\partial r} & \frac{\partial z}{\partial r} \end{pmatrix} = \begin{pmatrix} 1 - e^{\xi}\cos\theta & \psi_q(q,r)t & e^{\xi}\sin\theta \\ 0 & 1 & 0 \\ e^{\xi}\sin\theta & \psi_r(q,r)t & 1 + e^{\xi}\cos\theta \end{pmatrix}. \tag{3.3}$$

The Jacobian has a time independent determinant $1 - e^{2\xi}$ (which is non-zero since $r_0 > 0$) thus it follows that the flow defined by (3.2) must be volume preserving, ensuring that (2.2a) holds in the Eulerian setting, cf. [1]. We note for future reference that the inverse of the Jacobian (3.3) is given by

$$\begin{pmatrix} \frac{\partial q}{\partial x} & \frac{\partial s}{\partial x} & \frac{\partial r}{\partial x} \\ \frac{\partial q}{\partial y} & \frac{\partial s}{\partial y} & \frac{\partial r}{\partial y} \\ \frac{\partial q}{\partial z} & \frac{\partial s}{\partial z} & \frac{\partial r}{\partial z} \end{pmatrix} = \frac{1}{1-e^{2\xi}} \begin{pmatrix} 1 + e^{\xi}\cos\theta & -t[\psi_q(1+e^{\xi}\cos\theta) - \psi_r e^{\xi}\sin\theta] & -e^{\xi}\sin\theta \\ 0 & 1 - e^{2\xi} & 0 \\ -e^{\xi}\sin\theta & -t[\psi_r(1-e^{\xi}\cos\theta) - \psi_q e^{\xi}\sin\theta] & 1 - e^{\xi}\cos\theta \end{pmatrix}. \tag{3.4}$$

Calculating directly from (3.2) we get

$$u = \frac{Dx}{Dt} = ce^{\xi}\cos\theta, \tag{3.5a}$$

$$v = \frac{Dy}{Dt} = \psi(q,r), \tag{3.5b}$$

$$w = \frac{Dz}{Dt} = -ce^{\xi}\sin\theta. \tag{3.5c}$$

It is apparent from (3.5) that the solution (3.2) comprises a travelling wave-like term in the zonal direction, determined by the velocity components (3.5a) and (3.5c), moving with constant wave phasespeed c (given by dispersion relations (3.11) below) and constant wavelength L, with wavenumber $k = 2\pi/L$. The velocity component (3.5b) represents a meridional transverse current term $\psi(q,r)$ which is both latitudinally- and time-independent. It is also clear from (3.5) that the wave motion is three-dimensional if $\psi \not\equiv 0$. Furthermore, if the current term ψ is non-constant then the vorticity is also three-dimensional with a steady periodic time-dependence, since we have

$$\boldsymbol{\omega} = \nabla \times \mathbf{U} = (w_y - v_z, u_z - w_x, v_x - u_y)$$
$$= \frac{1}{1-e^{-2kr}}\left(\psi_q e^{-kr}\sin\theta + \psi_r(1-e^{-kr}\cos\theta), 2kce^{-2kr}, \psi_q(1+e^{-kr}\cos\theta) + \psi_r e^{-kr}\sin\theta\right),$$

where the vorticity component may be easily obtained from the velocity gradient tensor

$$\begin{pmatrix} \frac{\partial u}{\partial x} & \frac{\partial v}{\partial x} & \frac{\partial w}{\partial x} \\ \frac{\partial u}{\partial y} & \frac{\partial v}{\partial y} & \frac{\partial w}{\partial y} \\ \frac{\partial u}{\partial z} & \frac{\partial v}{\partial z} & \frac{\partial w}{\partial z} \end{pmatrix} = \begin{pmatrix} \frac{\partial q}{\partial x} & \frac{\partial s}{\partial x} & \frac{\partial r}{\partial x} \\ \frac{\partial q}{\partial y} & \frac{\partial s}{\partial y} & \frac{\partial r}{\partial y} \\ \frac{\partial q}{\partial z} & \frac{\partial s}{\partial z} & \frac{\partial r}{\partial z} \end{pmatrix} \begin{pmatrix} \frac{\partial u}{\partial q} & \frac{\partial v}{\partial q} & \frac{\partial w}{\partial q} \\ \frac{\partial u}{\partial s} & \frac{\partial v}{\partial s} & \frac{\partial w}{\partial s} \\ \frac{\partial u}{\partial r} & \frac{\partial v}{\partial r} & \frac{\partial w}{\partial r} \end{pmatrix}$$
$$= \frac{1}{1-e^{2\xi}}\begin{pmatrix} -cke^{\xi}\sin\theta & \psi_q(1+e^{\xi}\cos\theta)+\psi_r e^{\xi}\sin\theta & -cke^{\xi}(\cos\theta+e^{\xi}) \\ 0 & 0 & 0 \\ -cke^{\xi}(\cos\theta-e^{\xi}) & -\psi_q e^{\xi}\sin\theta - \psi_r(1-e^{\xi}\cos\theta) & cke^{\xi}\sin\theta \end{pmatrix}.$$

Indeed even if $\psi \equiv 0$ the flow is inherently rotational, an observation which may be inferred from the closed (circular) particle trajectories resulting from (3.2), which are redolent of Gerstner's wave solution— see [14]. It has recently been shown that a characteristic of irrotational flow (at least in the gravity wave setting) is the non-closed nature of particle paths, cf. [4,11,16]. To prove that (3.2) defines an exact solution of (3.1), we calculate

$$\frac{Du}{Dt} = kc^2 e^{\xi}\sin\theta, \tag{3.6a}$$
$$\frac{Dv}{Dt} = 0, \tag{3.6b}$$
$$\frac{Dw}{Dt} = kc^2 e^{\xi}\cos\theta, \tag{3.6c}$$

and inserting the terms from (3.5) and (3.6) into (3.1) gives us

$$P_x = -\rho_0(kc^2 e^{\xi}\sin\theta - 2\Omega c e^{\xi}\sin\theta), \tag{3.7a}$$
$$P_y = 0, \tag{3.7b}$$
$$P_z = -\rho_0(kc^2 e^{\xi}\cos\theta - 2\Omega c e^{\xi}\cos\theta + g). \tag{3.7c}$$

Multiplying both sides of (3.7) by the Jacobian matrix (3.3) we derive the following expression for the pressure gradient in terms of the Lagrangian variables

$$\begin{pmatrix} P_q \\ P_s \\ P_r \end{pmatrix} = -\rho_0 \begin{pmatrix} (kc^2 - 2\Omega c + g)e^{\xi}\sin\theta \\ 0 \\ (kc^2 - 2\Omega c)e^{2\xi} + (kc^2 - 2\Omega c + g)e^{\xi}\cos\theta + g \end{pmatrix}. \tag{3.8}$$

To conclude our demonstration that (3.2) is an exact solution of the governing equations (2.2), we must prescribe a suitable pressure function P which satisfies (3.8) and which satisfies suitable boundary conditions on the interface η. To this end we propose

$$\tilde{P} = \rho_0 \frac{kc^2 - 2\Omega c}{2k} e^{2\xi} - \rho_0 g r + \rho_0 \frac{kc^2 - 2\Omega c + g}{k} e^{\xi} \cos\theta + \tilde{P}_0 \tag{3.9}$$

whose gradient takes the form

$$\begin{aligned}
\tilde{P}_q &= -\rho_0 (kc^2 - 2\Omega c + g) e^{\xi} \sin\theta \\
\tilde{P}_s &= 0 \\
\tilde{P}_r &= -\rho_0 (kc^2 - 2\Omega c) e^{2\xi} - \rho_0 g - \rho_0 (kc^2 - 2\Omega c + g) e^{\xi} \cos\theta,
\end{aligned}$$

which correctly matches the right hand side of (3.8). The pressure function itself must also match the boundary condition (2.2d) at the thermocline $\eta(x-ct)$, so we work as follows. Let us first suppose that the thermocline is determined by setting $r = r_0$, for some r_0, then (2.2d) together with (3.2c) gives us

$$P|_{z=\eta(x-ct)} = P_0 - \rho_+ g r_0 + \frac{\rho_+ g}{k} e^{\xi_0} \cos\theta, \tag{3.10}$$

and comparing the time-dependent θ term in (3.10) with that in (3.9) leads us to impose the matching condition

$$\rho_0 (kc^2 - 2\Omega c + g) = \rho_+ g,$$

or $kc^2 - 2\Omega c = \tilde{g}$, a quadratic in c which we solve to get the dispersion relations

$$c = \frac{\Omega \pm \sqrt{\Omega^2 + k\tilde{g}}}{k}. \tag{3.11}$$

Choosing the plus sign in (3.11) gives $c > 0$ for which the wavelike term is eastward propagating, whereas choosing the minus sign gives $c < 0$ and so the wavelike term is then propagating westwards. This apparent freedom in propagation direction is unique to the $f-$plane formulation, as similar exact solutions in the $\beta-$plane are exclusively eastward propagating [7, 8], in the sense that westward propagating waves exist theoretically, but would exhibit an amplitude growth proportional to the meridional distance from the Equator, being thus physically unrealistic. With the wave speed c prescribed by (3.11), rematching the interface pressure condition (3.10) with (3.9) leads to the equation

$$\frac{e^{-2kr_0}}{2k} + r_0 = \frac{1}{g(\rho_+ - \rho_0)} \left(P_0 - \tilde{P}_0\right). \tag{3.12}$$

This relation implies that for a given $\tilde{P}_0$, if a unique solution r_0 exists for which (3.12) holds, then the parameter choice $r = r_0$ uniquely determines the thermocline. Since the mapping

$$r \mapsto \frac{e^{-2kr}}{2k} + r$$

is strictly increasing, and since $r > 0$, it follows that if

$$\frac{1}{g(\rho_+ - \rho_0)} \left(P_0 - \tilde{P}_0\right) > \frac{1}{2k}$$

then there is a unique value $r_0 > 0$ where (3.12) holds, and accordingly the parameter choice $r = r_0$ prescribes the thermocline. A similar analysis shows that we can determine the interface $z = \eta_+$

which separates $\mathscr{L}$ and $\mathscr{M}$ by fixing some constant

$$P^* > \frac{1}{g(\rho_+ - \rho_0)}\left(P_0 - \tilde{P}_0\right) > \frac{1}{2k},$$

and the unique solution of

$$P^* = \frac{e^{-2kr}}{2k} + r,$$

given by $r = r_+ > r_0$, will specify this interface. This shows that the solution (3.2) allows some freedom in locating both interfaces η and η_+, in the sense that we have some choice in the constants $\tilde{P}_0$ and P^*. We further remark that, reasoning along similar lines as the $\beta-$plane setting (cf. [7, 8]), these interfaces will be troichoidal (and therefore inherently nonlinear) in appearance.

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