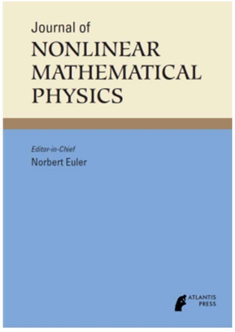

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On a supersymmetric nonlinear integrable equation in (2+1) dimensions

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A supersymmetric integrable equation in (2+1) dimensions is constructed by means of the approach of the homogenous space of the super Lie algebra, where the super Lie algebra osp(3/2) is considered. For this (2+1) dimensional integrable equation, we also derive its Bäcklund transformation.

Keywords: Integrable systems; supersymmetry; super Lie algebra.

2000 Mathematics Subject Classification: 17B80, 17A70, 35Q58

1. Introduction

The supersymmtric generalizations of the integrable systems in (1+1) dimensions have been paid more attention. Since various techniques like Painlevé test [11], Darboux and Bäcklund transformations [6, 8, 19], Hirota bilinear method [7, 13] and prolongation structure theory [15] have been extended to analysis supersymmetric integrable systems, a large number of (1+1)-dimensional integrable supersymmetric equations have been well studied, such as supersymmetric Korteweg-de Vries equation [5, 12], supersymmetric Kadomtsev-Petviashvili hierarchy [10, 18], supersymmetric nonlinear Schrödinger equation [14] and Heisenberg supermagnet model [4, 9, 21].

Not as the case of the (1+1)-dimensional supersymmetric integrable systems, the supersymmtric higher dimensional integrable systems have not been dealt with in such detail. Thus its investigation has attracted interest from physical and mathematical points of view. The Heisenberg supermagnet model [9] is an integrable supersymmetric system which can be regarded as the superextension of the Heisenberg ferromagnet model. Due to its connection with the strong electron correlated Hubbard model, much interest has arisen in the construction of its higher dimensional integrable models. Recently the (2+1)-dimensional integrable Heisenberg supermagnet models have been constructed [20]. It should be pointed out that the super Lie algebra $uspl(2/1)$ play an important role there. Saha and Chowdhury [16] presented two approaches to construct (2+1)-dimensional supersymmetric systems. In one approach, they investigated the (2+1)-dimensional supersymmetric systems based on the ideal of Fordy and Kulish [3] with the homogeneous space of the super Lie algebra. In the other approach, a different technique of extending the dimension of the system were

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used. Furthermore the fermionic covariant prolongation structure technique has been applied to the multidimensional super nonlinear evolution equation [22].

The integrable quantum field theories have attracted a lot of interest. Recently it is found that the super Lie algebra osp(m/2n) plays an important role in the integrable quantum field theories [17]. Thus one has paid more attention on the super Lie algebra osp(m/2n) [1, 2]. In this paper, we shall construct a supersymmetric integrable equation in (2+1) dimensions by means of the approach of the homogenous space of the super Lie algebra osp(3/2).

2. The integrable system of the super Lie algebra Osp(3/2)

The orthosympletic osp(3/2) can be defined as the set of all 5×5 matrices of the form

$$\begin{pmatrix} a & 0 & b & g & j \\ 0 & -a & c & h & k \\ -c & -b & 0 & i & l \\ k & j & l & d & e \\ -h & -g & -i & f & -d \end{pmatrix}, \tag{2.1}$$

where the non-zero entries are arbitrary complex numbers. The even subalgebra so(3)$\oplus$ sp(2) consists of all matrices (2.1), for which $g = h = i = j = k = l = 0$, whereas the odd subspace is obtained taking $a = b = c = d = e = f = 0$. Let $\hat{x}$ be the matrix (2.1) where we put the variable x equal to 1 and all other variables equal to zero. The super Lie algebra osp(3/2) is generated by the bosonic generators $(\hat{a},\hat{b},\hat{c},\hat{d},\hat{e},\hat{f})$ and fermionic generators $(\hat{g},\hat{h},\hat{i},\hat{j},\hat{k},\hat{l})$.

Let us denote the super Lie bracket

$$[X,Y] = X \cdot Y - (-1)^{P(X)P(Y)} X \cdot Y, \tag{2.2}$$

in which $P(X)$ and $P(Y)$ are the parities of the corresponding matrices or generators.

Substituting the bosonic and fermionic generators of the super Lie algebra osp(3/2) into the super Lie bracket (2.2), it is easy to give their (anti) commutation relations. From these (anti) commutation relations, we note that the super Lie algebra osp(3/2) can be decomposed as

$$\mathbf{g} = \mathbf{k} \oplus \mathbf{m}, \tag{2.3}$$

and

$$[\mathbf{k},\mathbf{k}] \subset \mathbf{k}, [\mathbf{k},\mathbf{m}] \subset \mathbf{m}, [\mathbf{m},\mathbf{m}] \subset \mathbf{k}, \tag{2.4}$$

where

$$\begin{aligned} \mathbf{k} &= \left\{\hat{a},\hat{d},\hat{e},\hat{f},\hat{i},\hat{l}\right\}, \\ \mathbf{m} &= \left\{\hat{b},\hat{c},\hat{g},\hat{h},\hat{j},\hat{k}\right\}. \end{aligned} \tag{2.5}$$

Let us turn to consider the following equations:

$$\begin{aligned} \phi_x &= [\lambda \cdot A + Q(x,y,t)]\,\phi \triangleq M\phi, \\ \phi_t &= \lambda^{n-1}\phi_y + \sum_{i=0}^{n-2} B_{n-i}(x,y,t)\,\lambda^i\phi \triangleq (\partial_t - \lambda^{n-1}\partial_y - N)\phi, \end{aligned} \tag{2.6}$$

where the constant matrix A belongs to the Cartan subalgebra of $\mathbf{g}$, $Q(x,y,t) \in \mathbf{m}$ and $B_{n-i}(x,y,t) \in \mathbf{g}$.

By means of the integrability condition

$$[\partial_x - M, \partial_t - \lambda^{n-1}\partial_y - N] = 0, \tag{2.7}$$

we have

$$\begin{aligned}
\partial_y Q(x,y,t) &= [A, B_2(x,y,t)], \\
\partial_x B_{n-i}(x,y,t) &= [A, B_{n-i+1}(x,y,t)] + [Q(x,y,t), B_{n-i}(x,y,t)], \\
\partial_t Q(x,y,t) &= \partial_x B_n(x,y,t) - [Q(x,y,t), B_n(x,y,t)],
\end{aligned} \tag{2.8}$$

where $i = 1, 2, \cdots n-2$.

Let us consider the case of $n = 2$ in (2.8) and take $A = \hat{a}$ and $B_{n-i} = B^m_{n-i} + B^k_{n-i}$, where $B^m_{n-i} \in \mathbf{m}$ and $B^k_{n-i} \in \mathbf{k}$. Then the equations (2.8) becomes

$$\partial_y Q = [A, B^m_2], \tag{2.9}$$

$$\partial_x B^k_2 = [Q, B^m_2], \tag{2.10}$$

$$\partial_t Q = \partial_x B^m_2 - \left[Q, B^k_2\right]. \tag{2.11}$$

Due to the property (2.4), we may assume

$$\begin{aligned}
Q &= q_1\hat{b} + q_2\hat{c} + q_3\hat{g} + q_4\hat{h} + q_5\hat{k} + q_6\hat{j}, \\
B^m_2 &= s_1\hat{b} + s_2\hat{c} + s_3\hat{g} + s_4\hat{h} + s_5\hat{k} + s_6\hat{j}, \\
B^k_2 &= M_1\hat{a} + M_2\hat{d} + M_3\hat{e} + M_4\hat{f} + M_5\hat{l} + M_6\hat{i}.
\end{aligned} \tag{2.12}$$

By means of (2.12), it is easy to derive the expression of B^m_2 from (2.9)

$$B^m_2 = q_{1y}\hat{b} - q_{2y}\hat{c} + q_{3y}\hat{g} - q_{4y}\hat{h} - q_{5y}\hat{k} + q_{6y}\hat{j}. \tag{2.13}$$

Substituting B^k_2 of (2.12) and (2.13) into (2.10), we obtain

$$\begin{aligned}
M_1 &= \partial_x^{-1}(q_1q_{2y} + q_2q_{1y} - q_3q_{5y} - q_4q_{6y} + q_5q_{3y} + q_6q_{4y}), \\
M_2 &= \partial_x^{-1}(-q_3q_{5y} + q_4q_{6y} + q_5q_{3y} - q_6q_{4y}), \\
M_3 &= \partial_x^{-1}(2q_5q_{6y} - 2q_6q_{5y}), \\
M_4 &= \partial_x^{-1}(2q_3q_{4y} - 2q_4q_{3y}), \\
M_5 &= \partial_x^{-1}(q_1q_{5y} - q_2q_{6y} + q_5q_{1y} - q_6q_{2y}), \\
M_6 &= \partial_x^{-1}(q_1q_{4y} - q_2q_{3y} - q_3q_{2y} + q_4q_{1y}).
\end{aligned} \tag{2.14}$$

Thus we obtain the explicit expressions of B^m_2 and B^k_2. Substituting these expressions into (2.11), we obtain the following desired supersymmetric integrable evolution equations:

$$\begin{aligned}
q_{1t} &= q_{1xy} + q_1M_1 - q_3M_5 + q_6M_6, \\
q_{2t} &= -q_{2xy} - q_2M_1 - q_4M_5 + q_5M_6, \\
q_{3t} &= q_{3xy} - q_1M_6 + q_3M_1 - q_3M_2 - q_6M_4, \\
q_{4t} &= -q_{4xy} - q_2M_6 - q_4M_1 - q_4M_2 - q_4M_2 - q_5M_4, \\
q_{5t} &= -q_{5xy} - q_2M_5 - q_4M_3 - q_5M_1 + q_5M_2, \\
q_{6t} &= q_{6xy} - q_1M_5 - q_3M_3 + q_6M_1 + q_6M_2.
\end{aligned} \tag{2.15}$$

3. Bäcklund transformation

In order to derive the Bäcklund transformation of (2.15), let us consider the first equation in (2.6). Suppose we have two sets of solution of the same nonlinear system given as Q and $Q^{'}$,

$$T_x^{'} = (\lambda A + Q')T',$$
$$T_x = (\lambda A + Q)T, \tag{3.1}$$

where $A = \hat{a}$, $Q = \begin{pmatrix} 0 & 0 & q_1 & q_3 & q_6 \\ 0 & 0 & q_2 & q_4 & q_5 \\ -q_2 & -q_1 & 0 & 0 & 0 \\ q_5 & q_6 & 0 & 0 & 0 \\ -q_4 & -q_3 & 0 & 0 & 0 \end{pmatrix}$ and $Q' = \begin{pmatrix} 0 & 0 & q_1' & q_3' & q_6' \\ 0 & 0 & q_2' & q_4' & q_5' \\ -q_2' & -q_1' & 0 & 0 & 0 \\ q_5' & q_6' & 0 & 0 & 0 \\ -q_4' & -q_3' & 0 & 0 & 0 \end{pmatrix}$.

Let us take the transformation $T' = DT$ with $D = \lambda D_0 + D_1$. Substituting this transformation into (3.1), we obtain

$$D_1 = A = \hat{a}, \tag{3.2}$$

$$[A, D_0] + Q'A - AQ = D_{1x}, \tag{3.3}$$

and

$$D_{0x} = Q'D_0 - D_0 Q, \tag{3.4}$$

Substituting (3.2) into (3.3), we obtain the off-diagonal part of D_0

$$D_0^{off} = \begin{pmatrix} 0 & 0 & q_1 & q_3 & q_6 \\ 0 & 0 & q_2 & q_4 & q_5 \\ -q_2' & -q_1' & 0 & 0 & 0 \\ q_5' & q_6' & 0 & 0 & 0 \\ -q_4' & -q_3' & 0 & 0 & 0 \end{pmatrix}. \tag{3.5}$$

Note that the elements D_0^{diag} can be given by the diagonal part of (3.4)

$$D_0^{diag} = \begin{pmatrix} \rho & 0 & 0 & 0 & 0 \\ 0 & \tau & 0 & 0 & 0 \\ 0 & 0 & \alpha & 0 & 0 \\ 0 & 0 & 0 & \beta & 0 \\ 0 & 0 & 0 & 0 & \gamma \end{pmatrix}. \tag{3.6}$$

where α, β and γ are constants, ρ and τ are given by

$$\rho = \partial_x^{-1}(q_1 q_2 - q_1' q_2' + q_3' q_5' - q_3 q_5 + q_6 q_4 - q_6' q_4') \tag{3.7}$$

and

$$\tau = \partial_x^{-1}(q_2 q_1 - q_2' q_1' + q_4' q_6' - q_4 q_6 + q_5 q_3 - q_5' q_3'), \tag{3.8}$$

respectively.

Substituting $D_0 = D_0^{off} + D_0^{diag}$ into (3.4) and taking the off-diagonal parts, we obtain the following Bäcklund transformation:

$$\begin{aligned}
&(q_1)^2 - (q_1')^2 = 2q_6'q_3' - 2q_6q_3,\ (q_2)^2 - (q_2')^2 = 2q_5'q_4' - 2q_5q_4,\\
&\alpha q_1' - \rho q_1 = q_{1x}, \quad \beta q_3' - \rho q_3 = q_{3x}, \quad \gamma q_6' - \rho q_6 = q_{6x},\\
&\alpha q_2' - \tau q_2 = q_{2x}, \quad \beta q_4' - \tau q_4 = q_{4x}, \quad \gamma q_5' - \tau q_5 = q_{5x},\\
&\rho q_2' - \alpha q_2 = q_{2x}', \quad \tau q_1' - \alpha q_1 = q_{1x}', \quad \rho q_5' - \beta q_5 = q_{5x}',\\
&\tau q_6' - \beta q_6 = q_{6x}', \quad \rho q_4' - \gamma q_4 = q_{4x}', \quad \tau q_3' - \gamma q_3 = q_{3x}'.
\end{aligned} \tag{3.9}$$

4. Summary

The supersymmtric higher dimensional integrable systems have attracted interest from physical and mathematical points of view. In this paper, we investigated the super Lie algebra osp(3/2). By means of the approach of the homogenous space of the super Lie algebra osp(3/2), we constructed the supersymmetric integrable equation in (2+1) dimensions. Moreover we also derived its Bäcklund transformation. Since the super Lie algebra osp(m/2n) plays an important role in the integrable quantum field theories, as a future study, it should be interesting to find out the applications of this supersymmetric (2+1) dimensional integrable equation.

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