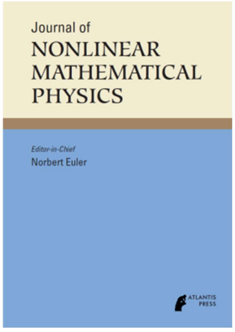

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Speed selection for coupled wave equations

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We discuss models for coupled wave equations describing interacting fields, focusing on the speed of travelling wave solutions. In particular, we propose a general mechanism for selecting and tuning the speed of the corresponding (multi-component) travelling wave solutions under certain physical conditions. A number of physical models (molecular chains, coupled Josephson junctions, propagation of kinks in chains of adsorbed atoms and domain walls) are considered as examples.

Keywords: Coupled wave equations; solitons.

2000 Mathematics Subject Classification: 35L05, 70S10, 70H33

1. Introduction

In a previous paper [4] dealing with a concrete physical model – more specifically, with the nonlinear dynamics of the DNA macromolecule [5,6] – we have observed a remarkable phenomenon. This is as follows: in the continuum limit that model reduces to two coupled nonlinear wave equations for different fields $\phi_{1,2}$; if the coupling is switched off, each of the wave equations $E_{1,2}$ obeyed by the fields $\phi_{1,2}$ is Lorentz invariant with *different* limiting speed, i.e. in particular admits travelling waves (solitons) with any speed c smaller than the limiting speed $c_{1,2}$. When the full model, including the interaction, is considered, there is no Lorentz invariance, and the travelling wave solutions admits only a given speed (see also [7]). Thus we have a selection mechanism for the speed of travelling wave (TW) solutions (the latter turn out to be also stable and thus physically relevant [6,7,9,10,16]).

In the present paper we want to study if this "speed selection mechanism" works also in a more general class of equations. We will answer to this in the positive, and actually the relevant class of equations turns out to be rather ample. An important point in this context is the fact that the speed selection mechanism of the paper mentioned above is implemented using a constraint [4]. In general the constraint will define a submanifold of the configuration space for the associated dynamical system. The crucial requirement for our speed selection mechanism to work is that the constraint is natural, i.e the associated submanifold is invariant under the dynamics. In other words, if the initial data are chosen to lie on this submanifold, the dynamics will take place entirely on it with no need to introduce external forces to enforce the constraint.

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We note that, apart from the theoretical interest, this question also has potentially very relevant practical consequences. In fact, for instance, it would point out a way to have transmission lines with physically determined speed for the travelling wave packets.

As mentioned above, we will answer in the positive to the question of applicability of the speed selection mechanism to more general equations. It turns out that, albeit we are mostly interested in nonlinear equations, a speed selection mechanism is also present in the case of linear systems. Note that for linear equations the uncoupled equations have each a well definite speed, so in this context the mechanism we study amounts to a change in the value of the n allowed speeds.

We will thus start, in Section 2, by considering simple linear systems; in this case the speed selection follows from some trivial algebraic facts and gives the possibility of tuning the speed of TW just by changing the value of an interaction parameter.

The general setting of this note will be as follows. We investigate wave equations for N fields $\phi^i(x,t)$, $i = 1 \ldots N$ in $1+1$ dimensions (one space dimension and time) described by a Lagrangian, and we are specially interested in travelling wave solutions.

The Lagrangian will be written as

$$\mathscr{L} \ = \ \sum_i \mathscr{L}_i \ + \ \mathscr{L}_{int} \ , \tag{1.1}$$

where the $\mathscr{L}_i$ is a Lagrangian for each of the fields ϕ^i, and $\mathscr{L}_{int}$ is the interaction Lagrangian.

We will make "minimal" choices for $\mathscr{L}_i$ and $\mathscr{L}_{int}$, as we want to understand the phenomenon of speed selection in the simplest possible terms. In particular, $\mathscr{L}_{int}$ will be made of a "gradient interaction term", coupling the spatial gradient ϕ^i_x of different fields – and playing an essential role in our analysis – and possibly of a potential term $V(\phi_i) = V(\phi^1, ..., \phi^N)$.

We will consider mainly Lagrangians leading to hyperbolic wave equations. In Appendix B, we will extend our considerations to parabolic, Schrödinger-like equations. Thus, we take first a Lagrangian of the form

$$\mathscr{L} \ = \ \frac{1}{2} \sum_{i=1}^N \left[\rho_i^2 \, (\partial_t \phi_i)^2 \, - \, \kappa_i^2 \, (\partial_x \phi_i)^2 \right] \ - \ \sum_{ij=1}^N \left[\gamma_{ij} (\partial_x \phi_i)(\partial_x \phi_j) \right] \ - \ V(\phi_i) \ , \tag{1.2}$$

where ρ_i, κ_i are constant, γ_{ij} are the components of a constant $(N \times N)$ matrix Γ, and the potential V (which could be zero) will be appropriately chosen below (see Sect. 2 and Sect. 3). The fields and the constants are assumed to be real; note that the speed of the waves for the fields ϕ_i we get in the decoupled case $\gamma = V = 0$ are

$$c_i \ = \ \kappa_i / \rho_i \ . \tag{1.3}$$

The matrix Γ can be taken to be symmetric, and in the present notation we can assume it has zero terms on the diagonal, as the corresponding terms are represented by the κ_i (so that Γ only represents the interaction between gradients of *different* fields).

The Euler-Lagrange equations corresponding to the Lagrangian (1.2) are[a]

$$\rho_i^2 \, \phi^i_{tt} \ - \ \kappa_i^2 \, \phi^i_{xx} \ - \ \gamma_{ij} \, \phi^j_{xx} \ = \ - \, (\partial V / \partial \phi^i) \ . \tag{1.4}$$

[a]Here and below we move the field index up and down for typographical convenience.

It is appropriate to mention immediately some physical relevant cases in which one meets Lagrangians of the type (1.2). These include e.g., beside the "composite" model of DNA dynamics [5] mentioned above and the strictly related case of long wavelength excitations in a chain of double pendulums (this also applies to polyethylene [8]), the case of coupled Josephson junctions [20, 25] and interaction between kinks in coupled chains of adsorbed atoms [21]. These cases will be discussed later on as examples of our general mechanism; see Sections 4 and 5.

Our analysis would of course also apply to Lagrangians differing from (1.2) only by boundary terms and total differentials, e.g by a term $\frac{1}{2}\sum_{ij=1}^{N}\sigma_{ij}\phi_i\partial_x\phi_j$; we focus on Lagrangians of the form (1.2) both for these make the analysis rather transparent and for their physical relevance.

Most of our discussion will be conducted in the simplest nontrivial case, $N=2$; we will then discuss the generalization to the arbitrary N case, which will be in some case rather immediate.

2. The linear case

As mentioned above, we start discussing the case of a quadratic Lagrangian, hence of linear field (Euler-Lagrange) equations.

Moreover, we will at first consider the case $N=2$; we will then write, for ease of notation, $\phi_1=\varphi$ and $\phi_2=\psi$; there is only one nontrivial term in the matrix Γ, i.e. γ_{12} (which, with a slight abuse of notation, we denote simply by γ). Thus we study the Lagrangian

$$\mathscr{L} \;=\; \frac{1}{2}\left[\left(\rho_1^2\varphi_t^2+\rho_2^2\psi_t^2\right)-\left(\kappa_1^2\varphi_x^2+\kappa_2^2\psi_x^2\right)\right]\;-\;\gamma\varphi_x\psi_x\;-\;V(\varphi,\psi)\,. \tag{2.1}$$

The Euler-Lagrange field equations are then

$$\begin{aligned}\rho_1^2\,\varphi_{tt}\;-\;\kappa_1^2\,\varphi_{xx}\;-\;\gamma\,\psi_{xx} &= -\left(\partial V/\partial\varphi\right)\,,\\ \rho_2^2\,\psi_{tt}\;-\;\kappa_2^2\,\psi_{xx}\;-\;\gamma\,\varphi_{xx} &= -\left(\partial V/\partial\psi\right)\,.\end{aligned} \tag{2.2}$$

In the simplest case, which we consider in this section, V will be a quadratic function of its arguments (including the case where V is trivial) and the equations (2.2) will hence be linear.

We will write, for the sake of definiteness,

$$V(\varphi,\psi)\;=\;\frac{1}{2}\left(\mu_1^2\,\varphi^2\;+\;2\lambda\,\varphi\,\psi\;+\;\mu_2^2\,\psi^2\right)\,, \tag{2.3}$$

where $\mu_{1,2}^2$ are the masses of the non interacting fields ($\lambda=\gamma=0$) and can hence be assumed to be positive.

Thus our Lagrangian is defined by three matrices $Q^{(t)}$, $Q^{(x)}$, $Q^{(V)}$, given by

$$Q^{(t)}\;=\;\begin{pmatrix}\rho_1^2 & 0\\ 0 & \rho_2^2\end{pmatrix}\;;\;Q^{(x)}\;=\;\begin{pmatrix}\kappa_1^2 & \gamma\\ \gamma & \kappa_2^2\end{pmatrix}\;;\;Q^{(V)}\;=\;\begin{pmatrix}\mu_1^2 & \lambda\\ \lambda & \mu_2^2\end{pmatrix}\;; \tag{2.4}$$

we have (returning for a moment to the notation with indices)

$$\mathscr{L}\;=\;\frac{1}{2}\sum_{ij=1}^{N}\left[\left(\phi_t^i\,Q_{ij}^{(t)}\,\phi_t^j\right)\;-\;\left(\varphi_x^i\,Q_{ij}^{(x)}\,\phi_x^j\right)\;-\;\left(\phi^i\,Q_{ij}^{(V)}\,\phi^j\right)\right]\,.$$

The problem – or better the source of interesting behavior – lies in that, in general, the three matrices $Q^{(t)}$, $Q^{(x)}$ and $Q^{(V)}$ do not commute with each other. More specifically, we have

$$\begin{aligned} \left[Q^{(t)},Q^{(x)}\right] &= (\rho_2^2-\rho_1^2)\,\begin{pmatrix}0 & -\gamma\\ \gamma & 0\end{pmatrix}\ ;\\ \left[Q^{(x)},Q^{(V)}\right] &= (\mu_1^2-\mu_2^2)\,\begin{pmatrix}0 & -\gamma\\ \gamma & 0\end{pmatrix}\ +\ (\kappa_2^2-\kappa_1^2)\,\begin{pmatrix}0 & -\lambda\\ \lambda & 0\end{pmatrix}\ ;\\ \left[Q^{(t)},Q^{(V)}\right] &= (\rho_2^2-\rho_1^2)\,\begin{pmatrix}0 & -\lambda\\ \lambda & 0\end{pmatrix}\ . \end{aligned} \tag{2.5}$$

This lack of commutativity between the different matrices, and in particular the fact we have $[Q^{(t)},Q^{(x)}]\neq 0$, is responsible for the breaking of the space-time Lorentz symmetry, which is one of the sources of the unusual (and interesting) features of the theory; see also the discussion in Appendix A.

2.1. *Travelling wave ansatz*

If we look for travelling wave solutions, i.e. for solutions of the form

$$\varphi(x,t)\ =\ \varphi(x\pm ct)\ =\ \varphi(z)\ ,\ \ \psi(x,t)\ =\ \psi(x\pm ct)\ =\ \psi(z)\ , \tag{2.6}$$

we have then to study the equations

$$\begin{aligned} \left(\rho_1^2c^2-\kappa_1^2\right)\varphi_{zz}\ -\ \gamma\,\psi_{zz}\ &=\ -(\mu_1^2\varphi+\lambda\,\psi)\ ,\\ \left(\rho_2^2c^2-\kappa_2^2\right)\psi_{zz}\ -\ \gamma\,\varphi_{zz}\ &=\ -(\lambda\,\varphi+\mu_2^2\psi)\ . \end{aligned} \tag{2.7}$$

These are immediately written in matrix form. Defining

$$A\ =\ \begin{pmatrix}[\kappa_1^2-\rho_1^2c^2] & \gamma\\ \gamma & [\kappa_2^2-\rho_2^2c^2]\end{pmatrix}\ ,\ \ B\ =\ \begin{pmatrix}\mu_1^2 & \lambda\\ \lambda & \mu_2^2\end{pmatrix}\ ;\ \Phi\ =\ \begin{pmatrix}\varphi\\ \psi\end{pmatrix}\ ,$$

we recast the previous equation as

$$A\,\Phi_{zz}\ =\ B\Phi\ . \tag{2.8}$$

If $\det(A)\neq 0$, i.e. if

$$c^2\ \neq\ \frac{1}{2\rho_1^2\rho_2^2}\ \left[(\kappa_1^2\rho_2^2+\kappa_2^2\rho_1^2)\ \pm(\kappa_1^2\rho_2^2-\kappa_2^2\rho_1^2)\ \sqrt{1+\frac{4\gamma^2\rho_1^2\rho_2^2}{(\kappa_1^2\rho_2^2-\kappa_2^2\rho_1^2)^2}}\ \right]\ ,$$

we can invert A and further recast (2.8) as

$$\begin{aligned} \Phi_{zz}\ &=\ M\,\Phi\ ,\\ M:=A^{-1}\,B\ &=\ \mathscr{M}\ \begin{pmatrix}\gamma\lambda-\kappa_2^2\mu_1^2+c^2\mu_1^2\rho_2^2 & -\kappa_2^2\lambda+\gamma\mu_2^2+c^2\lambda\rho_2^2\\ -\kappa_1^2\lambda+\gamma\mu_1^2+c^2\lambda\rho_1^2 & \gamma\lambda-\kappa_1^2\mu_2^2+c^2\mu_2^2\rho_1^2\end{pmatrix}\ ;\\ \mathscr{M}\ &=\ \frac{1}{\gamma^2-(\kappa_1^2-c^2\rho_1^2)(\kappa_2^2-c^2\rho_2^2)}\ ,\\ \det(M)\ &=\ \mathscr{M}\ (\lambda^2\ -\ \mu_1^2\,\mu_2^2)\ . \end{aligned}$$

One could then diagonalize the matrix M by a change of basis $\Phi=\Lambda\widehat{\Phi}$ so that in the new coordinates the equation reads $\widehat{\Phi}_{zz}=\widehat{M}\widehat{\Phi}$ with $\widehat{M}=\Lambda^{-1}M\Lambda=\mathrm{diag}(\delta_1(c),\delta_2(c))$ a diagonal matrix, and now solve easily for $\widehat{\Phi}(z)$ and hence for $\Phi(z) = \Lambda\widehat{\Phi}(z)$. As the functions $\delta_i(c)$ (which of course do also depend on the various parameters of the equations) will in general satisfy $\delta_1(c) \neq \delta_2(c)$, we will have two normal modes with frequencies (in z) $\omega_{1,2}^{(z)}$ satisfying $(\omega_1^{(z)})^2 = -\delta_1(c)$ and $(\omega_2^{(z)})^2 = -\delta_2(c)$. These relations correspond to the two branches of the dispersion relations we will find in the next section using the Fourier transform.

Note that the situation is completely different in the $V = 0$ case, i.e. for $B = 0$. In this case the equation (2.8) reduces to

$$A\,\Phi_{zz} \;=\; 0\,, \tag{2.9}$$

which admits a solution if and only if $\det(A) = 0$, which should be seen as a requirement on the speed c. More specifically, this yields with simple algebra.

$$c_\pm^2 \;=\; \frac{1}{2}\left[(c_1^2+c_2^2)\;\pm\;(c_1^2-c_2^2)\,\sqrt{P}\right]\,, \tag{2.10}$$

where we have written

$$P \;=1\;+\;4\left[\frac{\gamma}{\rho_1\rho_2(c_1^2-c_2^2)}\right]^2\;;$$

here and below $c_i = |\kappa_i/\rho_i|$ is the speed of decoupled waves defined in Eq. (1.3).

The eigenvectors of the matrix A corresponding to the eigenvalues $c_\pm$ can be also easily computed; up to a normalization factor, they are

$$\Phi_\pm \;=\; \begin{pmatrix} (c_1/c_2)\sqrt{(c_2^2-c_\pm^2)/(c_1^2-c_\pm^2)} \\ 1 \end{pmatrix}\,. \tag{2.11}$$

It is worth stressing that equations (2.10) and (2.11) describe two remarkable features of TWs in linear coupled systems (2.7)

First, we have that the linear coupling between the φ and ψ field in the wave equation (2.7) forces a synchronization of the φ- and ψ-waves: they have to propagate with the same speed, given either by c_+ or by c_-. To see this let us first diagonalize the kinetic part of the Lagrangian (2.1). It is not a priori evident that this is possible because the kinetic matrices Q^x e Q^t in Eq. (2.4) do not commute. However, one can show that a $GL(2,\mathbb{R})$ transformation S exists, which acting on the vector Φ as $\Phi = S\widetilde{\Phi}$ diagonalizes the Lagrangian (2.1) with $V = 0$:

$$\mathscr{L} \;=\; \frac{1}{2}\left[\left(\frac{\partial_t\widetilde{\varphi}}{c_+}\right)^2 \;+\; \left(\frac{\partial_t\widetilde{\psi}}{c_-}\right)^2 \;-\;(\partial_x\widetilde{\varphi})^2\;-\;(\partial_x\widetilde{\psi})^2\right]\,. \tag{2.12}$$

We see that in the $\widetilde{\Phi}$ frame we have two decoupled normal modes given by (2.11) propagating respectively at speed c_+ and c_- given by Eq. (2.10); however they represent disjoint sectors, which cannot be superimposed.

Second, we can tune the speeds of the φ and ψ waves between zero and a maximum value by changing the coupling between the two fields, which is parametrized by the (positive) ratio r,

$$0 \leq r := \left(\frac{\gamma}{\rho_1\rho_2}\right)^2 \leq c_1^2 c_2^2 := r_{max} . \tag{2.13}$$

In particular, let us start from the decoupled case; here $r = 0$ and we get $c_+ = c_1$ and $c_- = c_2$. If we now increase r, then the value of c_+ increases whereas that of c_- decreases. When $r \to r_{max}$, c_+ tends to its maximum value whereas $c_- \to 0$. Notice that for $r > r_{max}$, c_-^2 becomes negative, i.e we have an imaginary speed for the harmonic mode; this detects an instability of the system. Notice also that Eq. (2.13) defines a general bound on the coupling parameters, which can be also generalized to the case of a non vanishing potential (2.3).

2.2. *Fourier transform*

To solve the general linear case we well make use of the linear character of the field equations (2.2), and pass to consider the Fourier transforms $\widehat{f}(q,\omega)$ and $\widehat{g}(q,\omega)$ for, respectively, the fields $\varphi(x,t)$ and $\psi(x,t)$; the equations (2.2) are then recast as

$$\begin{aligned} -\left(\rho_1^2\omega^2 - \kappa_1^2 q^2\right)\widehat{f} + \gamma q^2 \widehat{g} + (\mu_1^2 \widehat{f} + \lambda \widehat{g}) &= 0 , \\ -\left(\rho_2^2\omega^2 - \kappa_2^2 q^2\right)\widehat{g} + \gamma q^2 \widehat{f} + (\lambda \widehat{f} + \mu_2^2 \widehat{g}) &= 0 . \end{aligned} \tag{2.14}$$

These are promptly rewritten: defining now

$$M = \begin{pmatrix} [\rho_1^2\omega^2 - \kappa_1^2 q^2 - \mu_1^2] & -\gamma q^2 - \lambda \\ -\gamma q^2 - \lambda & [\rho_2^2\omega^2 - \kappa_2^2 q^2 - \mu_2^2] \end{pmatrix} , \quad \widehat{F} = \begin{pmatrix} \widehat{f} \\ \widehat{g} \end{pmatrix} ,$$

equations (2.14) read

$$M\widehat{F} = 0 . \tag{2.15}$$

Again we must require the vanishing of a determinant, i.e. $\det(M) = 0$; this will now give a relation between ω^2 and q^2, i.e. we will get some *dispersion relations* (DRs).

More precisely, these read

$$\omega_\pm^2 = \frac{1}{2}\left[(c_1^2 + c_2^2)q^2 + (u_1^2 + u_2^2) \pm \sqrt{P}\,\right] , \tag{2.16}$$

where we have written

$$P = \left[(c_1^2 - c_2^2)q^2 + (u_1^2 - u_2^2)\right]^2 + 4\,\frac{(\gamma q^2 + \lambda)^2}{\rho_1^2\rho_2^2} ,$$

with $u_i = \mu_i/\rho_i$ and c_i as above.

For a general nontrivial potential, we have two different DRs, which will also determine the phase velocity $v_\pm^p = d\omega_\pm/dq$ of the wave. In this generic case the DRs have a rather involved form. They take a simpler form for some particular or limiting cases.

For a vanishing potential i.e $\mu_1 = \mu_2 = \lambda = 0$, one can easily check that the DR (2.16) becomes linear (acoustic) and that $d\omega_\pm/dq = c_\pm$ with $c_\pm$ given by (2.10). This case describes also the high energy limit $q \to \infty$ of the DR (2.16), where the potential can be neglected.

In the low-energy limit $q \to 0$ the DR (2.16) describes two optical branches, whose analytic expression can be derived expanding (2.16) near $q = 0$.

As expected, the DR becomes simple also in the decoupling limit $\gamma = \lambda = 0$. In this case, we have two branches with optical DR $\omega_{12}^2 = c_{12}^2 q^2 + u_{12}^2$. Another interesting particular case is $c_1 = c_2$, $u_1 = u_2$. We obtain also in this case two optical branches:

$$\omega_\pm^2 = \left(c_1^2 \pm \frac{\gamma}{\rho_1\rho_2}\right) q^2 + u_1^2 \pm \frac{\lambda}{\rho_1\rho_2} \tag{2.17}$$

The synchronization and tuning effect discussed in Sect. (2.1) for the TW speed c applies also to the phase speed $d\omega_\pm/dq$. Now synchronization simply means that a given phase of the φ and ψ wave must propagate at the same speed, whereas tuning means that we can change the phase velocity by acting on the coupling parameters of the two fields. In the case of acoustic and optical dispersion relations this tuning is very simple. In the acoustic case, the group and the phase speed are the same, and thus it has been already described in Sect. (2.1). In the optical case we have instead $d\omega_\pm/dq = c_\pm q/\omega_\pm$, so that the phase velocity becomes zero whenever $c_\pm = 0$ and for a given phase, grows monotonically with $c_\pm$.

2.3. *The N-fields case and Lorentz symmetry*

The discussion conducted above is readily generalized to the case of N fields. This is specially transparent in terms of the travelling wave ansatz (TWA): in this case we get again a matrix equation, but the matrix is now an $(N \times N)$ matrix. Solving the eigenvalue problem we get in general N normal modes.

In terms of the Fourier transform, we will get N branches of the dispersion relations.

Similarly to the $N = 2$ case, for a vanishing potential we will have N determinations of the allowed speeds c and we can change the values of the these speeds (for given κ_i, ρ_i and hence "uncoupled speeds" $c_i = \kappa_i/\rho_i$) by acting on the coupling parameters γ_{ij} of the model.

Let us now briefly comment on the Lorentz symmetry of our two-dimensional field theory (1.2). Space-time Lorentz symmetry is explicitly broken in the Lagrangian (1.2). This breaking has two sources. The first is the presence of several different limit speeds c_i; the breaking of the Lorentz symmetry in this case is expressed by the non commutativity of the matrices $Q^{(x)}$ and $Q^{(t)}$ in Eq. (2.4). The second source is the non covariance of the kinetic coupling term, which contains the space but not the time derivative. On the other hand, by implementing the TWA or by solving the field equations in Fourier space the relevant matrices are not the individual $Q^{(x)}$ and $Q^{(t)}$ but a linear combination of them.

In Fourier space the breaking of the Lorentz symmetry is evident from the form of the dispersion relations (2.16). They are invariant under Lorentz boosts only when $c_1 = c_2$ and $\gamma = 0$. Despite of the explicit breaking of the Lorentz symmetry, it is quite evident that some remnant of it survives in the field theory (1.2). This is in particular evident in the massless case where the dispersion relations remain linear, the Lagrangian can be diagonalized describing decoupled sectors and the presence of the coupling term γ just changes the values of the two speeds c_{12}. We will discuss this remnant Lorentz symmetry and related group theoretical aspects in Appendix A.

3. The nonlinear case

Our discussion in Section 2 makes a substantial use of the linearity of field equations; thus it cannot be extended to the nonlinear case.

In the nonlinear case, the TWA produces a system of nonlinear ODEs, and TW solutions are obtained as solutions $\phi_i(x \pm ct)$ of these nonlinear ODEs, which can be considered as describing an equivalent mechanical system; this procedure leaves (in general, see below) the TW speed completely undetermined.

This is particularly evident when we start from a Lagrangian which is Lorentz invariant: then the TW solution must be a function of the relativistic gamma factor, $\phi_i = \phi_i((x \pm ct)/\sqrt{1-c/\bar{c}})$ (where $\bar{c}$ is the limit speed). The wave speed c can be therefore arbitrarily changed in the range $[0, \bar{c}]$ by a Lorentz boost. Thus any speed selection mechanism in the nonlinear case must necessarily start from a Lagrangian in which the Lorenz invariance is explicitly broken to remove this degeneracy in c.

Any speed selection mechanism must constrain the dynamics of the equivalent mechanical system to happen in a submanifold of the configuration space in which c is fixed. Moreover, if the submanifold corresponds to a *natural constraint*, i.e. is invariant under the dynamics, the constraint can also be realized as a selection on the initial data. A simple realization of such a mechanism has been proposed in Ref. [4]. Here we discuss in detail its dynamical implications, in particular existence of invariant submanifolds and the naturalness of the constraints that can be used to select the speed of TW solutions. Our main goal is obviously the generalization of the results of Ref. [4].

We are now considering a Lagrangian of the form (1.2) with a generic (analytic) potential $V(\phi_i)$, and Euler-Lagrange equations (1.4). In the case $N = 2$ these reduce to (2.1) and (2.2) respectively.

Note that the kinetic part of the Lagrangian – and hence the second order terms in the Euler-Lagrange equations – is still characterized by the two matrices $Q^{(t)}$ and $Q^{(x)}$, which in general do not commute. This non commutativity expresses the breaking of the space-time Lorentz symmetry at the Lagrangian level. On the other hand, as already noticed in the discussion of the linear case, after using the TWA (or passing to Fourier space) the relevant matrix will be a linear combination (with coefficients depending on the speed of the TW) of $Q^{(t)}$ and $Q^{(x)}$.

With the travelling wave ansatz (2.6), we are led to consider the equations

$$(\rho_i^2 c^2 - \kappa_i^2)\, \phi^i_{zz} \;-\; \gamma_{ij}\, \phi^j_{zz} \;=\; -\,(\partial V/\partial \phi^i)\,. \tag{3.1}$$

3.1. *The two-dimensional case*

We will again start by considering the case $N = 2$; with the notation introduced above for the linear case, we are thus dealing with the equations

$$\begin{aligned} (\kappa_1^2 - \rho_1^2 c^2)\, \varphi_{zz} \;+\; \gamma\, \psi_{zz} &= (\partial V/\partial \varphi) \;:=\; f(\varphi,\psi)\;, \\ (\kappa_2^2 - \rho_2^2 c^2)\, \psi_{zz} \;+\; \gamma\, \varphi_{zz} &= (\partial V/\partial \psi) \;:=\; g(\varphi,\psi)\,. \end{aligned} \tag{3.2}$$

These are rewritten in matrix form as

$$M\, \Phi_{zz} \;=\; F\;,$$

having of course written

$$M \;=\; \begin{pmatrix} (\kappa_1^2 - \rho_1^2 c^2) & \gamma \\ \gamma & (\kappa_2^2 - \rho_2^2 c^2) \end{pmatrix} \;:=\; \begin{pmatrix} m_1 & \gamma \\ \gamma & m_2 \end{pmatrix}\;;\quad F = \begin{pmatrix} f \\ g \end{pmatrix}\;.$$

This matrix M combines the $Q^{(x)}$ and $Q^{(t)}$.

Provided $\det(M) \neq 0$, i.e. $\gamma^2 \neq m_1 m_2$, we can now consider a linear change of coordinates in the fields space,

$$\phi^i = A^i{}_j \, \eta^j ,$$

which diagonalizes (3.2).[b]

We can actually ask more, i.e. that $A^T M A = I$. This is obtained e.g. for

$$A = \begin{pmatrix} \alpha & -\sqrt{q(\alpha)}/\sqrt{m_2(m_1 m_2 - \gamma^2)} \\ -[\alpha\gamma + \sqrt{q(\alpha)}]/m_2 & [\alpha\gamma^2 - m_1 m_2 \alpha + \gamma\sqrt{q(\alpha)}]/[m_2\sqrt{m_1 m_2 - \gamma^2}] \end{pmatrix} ,$$

where α is a free parameter and

$$q(\alpha) := m_2 + (\gamma^2 - m_1 m_2)\alpha^2 .$$

In terms of the fields η, our equations (3.2) read now simply

$$\frac{d^2\eta_i}{dz^2} = -\frac{\partial W}{\partial \eta_i} ;$$

thus the dynamics of the system is described by the motion of a particle of unit mass in the effective potential $W(\eta) := V[\phi(\eta)]$.

In dealing with PDEs one should also specify a function space for the search of solutions. In view of the physical meaning of the equations, it is natural to look for solutions of finite energy. In the case of a natural Lagrangian like (1.2), this means that the solutions $\varphi(x,t)$ should go to a minimum of the potential for $x \to \pm\infty$; once we proceed to reduction to a system of ODEs via the TWA, this means that we look for solutions which go to a minimum of V (we stress the condition is on V, *not* on W) for $z \to \pm\infty$.

It is easily seen that if minima of V correspond to minima of the effective potential W, these solutions are necessarily trivial. If, on the other hand, minima of V correspond to maxima of W, nontrivial solutions with the prescribed asymptotic behavior can exist; moreover they can either be doubly asymptotic to the same local minimum of V, or connect two distinct local minima, corresponding to the same energy level.

Thus the existence of TW solutions with the relevant behavior depends on the sign of V/W, i.e. on the sign of the determinant of the Jacobian matrix A. This in turn will depend on the value of c^2. (The discussion of concrete examples will better clarify this point.)

Provided this condition is satisfied, we are thus searching for solutions to our equivalent mechanical system with prescribed limit conditions for $\eta(z)$ and $\eta_z(z)$ at $z \to \pm\infty$. In particular, we require

$$\lim_{z\to\pm\infty} \eta_z(z) = 0 ; \quad \lim_{z\to\pm\infty} \eta_z(z) = \eta_\pm$$

with $\eta_\pm$ corresponding to (necessarily degenerate) local minima of the potential V.

Such solutions are in general not unique (nor stable, even in the realm of solutions satisfying the same limit conditions), as immediately shown by the example of a doubly periodic potential, i.e. of a dynamical system on the torus [1, 2, 28].

[b] With $\widetilde{V}(\eta) = V[\phi(\eta)]$, the latter reads $MA\eta_{zz} = (A^{-1})^T(\partial\widetilde{V}/\partial\eta)$, so that we have to look for A such that $A^T M A$ is diagonal.

The situation – and hence our analysis – is greatly simplified if there are some one-dimensional invariant submanifolds connecting two degenerate minima. Needless to say, this is a non-generic feature, and in general the existence of such invariant submanifolds is related to the presence of some symmetry in the potential[c]. This could be a Noether symmetry, guaranteeing the presence of a conserved quantity and hence the reduction to a lower dimensional effective dynamics; or even a discrete symmetries – such as a reflection symmetry – in the potential. More generally it suffices to have a conserved or even a conditionally conserved quantity [29]; the existence of this is implied e.g. by the presence of a reflection symmetry.

Albeit in general a conditionally conserved quantity and hence a low-dimensional invariant manifold is not necessarily associated to a symmetry, in practice this is most often the case.

In this paper we will limit our considerations to the case in which the system has a reflection symmetry, so that we are guaranteed of the existence of a low-dimensional invariant submanifolds. We have now to distinguish between three possible cases:

1) The system is fully invariant under Lorentz symmetry, i.e. we have $c_1 = c_2 = \bar{c}$ and $\gamma = 0$ in Eq. (3.2). In this case the invariant submanifold exists for every value of the soliton speed c below the limit speed $\bar{c}$. We do not have selection of the soliton speed. The soliton speed is fully protected by the Lorentz symmetry. We will give examples of this situation in Sects. 5.2 and 5.4.

2) Lorentz symmetry is broken by $\gamma \neq 0$ but we still have a single limit speed, i.e $c_1 = c_2 = \bar{c}$. Also in this case the invariant submanifold exists for every value of the soliton speed c below a limit speed and the soliton speed is protected by the Lorentz symmetry but the limit speed is changed by the kinetic interaction term. This is fully consistent with the discussion of the linear case of Sect. 2, where we have shown that the kinetic interaction term changes the wave speed according to Eq. (2.10). This situation is realized in an example discussed in Sect. 5.2.

3) Lorentz symmetry is broken by $c_1 \neq c_2$. In this case the soliton speed is not protected by the Lorentz symmetry. This is the case in which we can have speed selection for the soliton; the invariant submanifold exists only for selected values of c. This situation is met in the examples discussed in Sections 5.1, 5.3 and 5.4.

Summarizing, a sufficient condition for the speed selection mechanism considered here to work is *the presence of a reflection symmetry defining invariant submanifolds connecting two degenerate minima of the potential W existing only for specific values of the soliton speed.* Whereas a necessary condition for this speed selection mechanism to work is *an explicit breaking of the Lorentz symmetry at the Lagrangian level trough the presence of different limiting speeds.*

The practical implementation of the speed selection mechanism requires the determination of the invariant manifold associated with the reflection symmetry. This is a rather involved problem, which we will tackle using an ansatz. Obviously the manifold we are looking for must be invariant not only under the dynamics but also under the action of the reflection transformation. Because the kinetic part of the field equations (3.2) is linear the most natural ansatz for determining the invariant manifold is a linear equation involving the fields φ and ψ. This will be our choice for the examples

[c]Strictly speaking, we are here concerned with symmetries of the *effective* potential; these will however correspond to symmetries of the original physical potential, as the transformation mapping the latter into the former is smooth (and actually linear).

described in Sects. 5.1, 5.2, 5.3. It is also possible to search for invariant manifolds using non linear invariant functions involving the fields φ and ψ. This will be our choice for the system described in Sect. 5.4.

As a final step in the practical implementation of the speed selection mechanism we need to restrict the dynamics on the invariant submanifold. This can be achieved in two different ways. A first way is through a constraint simply given by the ansatz we have used to determine the invariant manifold. Another way, making use of the fact we have a dynamically invariant manifold and hence a *natural constraint*, is to suitably select the initial conditions, i.e. choose them (position and velocity) along the chosen invariant manifold.

3.2. *The N-dimensional case*

The analysis can be conducted along the same lines also in the general N-dimensional case. Needless to say, even the preliminary step of diagonalizing the matrix M may be a substantial problem for high dimension; but we may in principles proceed along the same lines.

Note that for dimension N a Noether symmetry will reduce the dynamics to a problem in dimension $N-1$; thus we need $N-1$ Noether symmetries in involution in order to reduce our problem to a one-dimensional one. Similarly, the presence of a reflection symmetry will in general only guarantee the existence of an invariant submanifold of dimension $N-1$ (i.e. of codimension 1); thus we will need $N-1$ reflection symmetries across planes with a nontrivial intersection to be sure of the existence of an invariant one-dimensional manifold. Note this is e.g. the case if the potential $V(\phi_i)$ actually depends only on the squares ϕ_i^2 of the fields, or at least of $N-1$ among them.

Another situation which guarantees the existence of invariant one-dimensional submanifolds is that of a separable potential, $V(\phi_1, ..., \phi_N) = V_1(\phi_1) + ... + V_N(\phi_N)$, provided each $V_k(\phi_k))$ has a minimum in $\phi_k = 0$; in this case we do not have to require reflection symmetry of the V_k.

One can of course also have a combination of the two situations mentioned above, i.e. a potential which is separable in potentials depending on several groups of field variables, each of them having suitable (continuous or discrete) symmetries.

Provided the potential has a sufficient degree of symmetry or separability, we are reduced to a one-dimensional analysis and we can proceed substantially as in the $N = 2$ case above.

It is also possible that no symmetry is present, but that for specific values of the speed some one-dimensional invariant submanifold is present We expect however that this becomes increasingly unlikely – and anyway that it would be increasingly difficult to determine the allowed c in concrete terms – with increasing dimension N.

4. Examples. Linear equations

In this section we briefly apply our speed selection mechanism developed for the linear case to two examples. These describe the region of linear dynamics for some of the non linear systems to be considered in full in the next Section.

4.1. *Kinks in coupled chains of adsorbed atoms*

In the situation to be considered in Section 5.2, the linearized equations at $(\varphi,\psi)=(0,0)$ read (setting $\beta=\gamma$ to comply with our nomenclature of coupling constants for the linear case)

$$\begin{aligned} \varphi_{tt} - \varphi_{xx} &= -\varphi - \alpha(\varphi-\psi) + \gamma\psi_{xx} \\ \psi_{tt} - \psi_{xx} &= -\psi - \alpha(\psi-\varphi) + \gamma\varphi_{xx} . \end{aligned} \tag{4.1}$$

This linear system has exactly the form considered in Sect. 2, with special values of the parameters:

$$\rho_1=\rho_2=\kappa_1=\kappa_2=1, \quad \mu_1^2=\mu_2^2=1+\alpha, \quad \lambda=-\alpha \tag{4.2}$$

Here $c_1=c_2=\bar{c}=1$, $u_1=u_2$; thus Eq. (2.17) applies, giving the optical DRs

$$\omega_\pm^2 \;=\; (1\pm\gamma)\,q^2 \;+\; (1+\alpha)\mp\alpha\,,$$

from which we get the phase speeds for the two branches

$$c_-(q) \;=\; \frac{1-\gamma}{\sqrt{1+2\alpha+(1-\gamma)q^2}}\,q\,;\;\; c_+(q) \;=\; \frac{1+\gamma}{\sqrt{1+(1+\gamma)q^2}}\,q\,.$$

Note that as a consequence of Eq. (2.13) we must have $|\gamma|\le 1$ and, as expected, for $\gamma=1$ ($\gamma=-1$) only the ω_+ (ω_-) branch is present, whereas the ω_- (ω_+) branch disappears.

In the low-energy limit we get

$$c_-(q)\simeq\frac{1-\gamma}{\sqrt{1+2\alpha}}\,q\,,\;\; c_+(q)\simeq(1+\gamma)\,q\,;$$

and in the high-energy limit

$$c_-(q)\simeq\sqrt{1-\gamma}\,,\;\; c_+(q)\simeq\sqrt{1+\gamma}\,.$$

4.2. *Coupled Josephson junctions*

In the situation to be considered in Section 5.3, the linearized equations at $(\varphi,\psi)=(0,0)$ read (setting $\alpha=-\gamma$, again to comply with our general notation)

$$\begin{aligned} \varphi_{tt} - \varphi_{xx} &= -\varphi + \gamma\psi_{xx} \\ \mu^2\,\psi_{tt} - \psi_{xx} &= -\nu^2\,\psi + \gamma\varphi_{xx} . \end{aligned} \tag{4.3}$$

This linear system has the form considered in Sect. 2 with special values of the parameters:

$$\rho_1=\kappa_1=\kappa_2=1, \quad \rho_2=\mu, \quad \mu_1=1, \quad \mu_2=\nu, \quad \lambda=0 \tag{4.4}$$

The dispersion relations (2.16) give

$$\omega_\pm^2 \;=\; \frac{(\mu^2+\nu^2+(1+\mu^2)q^2)}{2\mu^2}\left[1\,\pm\,\sqrt{1-4\mu^2\frac{\nu^2+(1+\nu^2)q^2+(1-\gamma^2)q^4}{(\mu^2+\nu^2+(1+\mu^2)q^2)^2}}\right]\,.$$

The low-energy limit ($q\simeq 0$) for the speeds $c_\pm=d\omega_\pm/dq$ are given by

$$c_-(q)\;\simeq\;\frac{1}{\mu\nu}\,q\,+\,O(q^3)\,;\;\; c_+(q)\;\simeq\;q\,+\,O(q^3)\,.$$

In the high-energy limit ($q \to \infty$) both speeds go to finite limits; the explicit expressions can be computed but are rather cumbersome and thus not reported.

5. Examples. Nonlinear equations

In this section we apply our TW speed selection mechanism developed for the non linear case to several examples. For all of these examples we look for an invariant submanifold of the dynamics for the associated mechanical system.

5.1. *Double pendulums chains*

The speed selection mechanism for TW solutions was originally proposed [4] in the context of mesoscopic models for non linear DNA torsion dynamics [5–7]. In particular, this was in studying the "composite Y model" [5] of DNA, in which the state of DNA is described by two angular variables $\varphi(x,t)$ and $\vartheta(x,t)$, which can be thought as describing (in the long wavelength limit, and thus using a continuum description) a chain of double pendulums. The peculiar geometry of DNA produces rather involved equations. The model is described by the Lagrangian density

$$\begin{aligned}\mathscr{L} = \frac{1}{2}\{& I\vartheta_t^2 - \omega_t\vartheta_x^2 + r^2[m\vartheta_t^2 + \\ & -\omega_s\vartheta_x^2 + 2(m\vartheta_t(\varphi_t+\vartheta_t) - \omega_s(\varphi_x+\vartheta_x))\cos\varphi \\ & +m(\varphi_t+\vartheta_t)^2 - \omega_s(\varphi_x+\vartheta_x)^2]\} + \\ & +4r^2K_p[\cos\vartheta + \cos(\varphi+\vartheta) - (1/2)\cos\varphi - (3/2)]\ , \end{aligned} \tag{5.1}$$

and we refrain from writing the Euler-Lagrange equations for this Lagrangian; the reader is referred to [4] for the physical meaning of the various parameters appearing in $\mathscr{L}$.

Under the TWA, the Euler-Lagrange equations reduce to

$$\begin{aligned}& \mu\,\varphi_{zz} + \mu(1+\cos\varphi)\,\theta_{zz} = \\ & \qquad = -4K_p\,\sin(\varphi+\theta) - \mu\,\sin(\varphi)\,(\theta')^2 + 2K_p\,\sin(\varphi)\ ; \\ & \mu\,(1+\cos\varphi)\,\varphi_{zz} + [(J/r^2) + 2\mu(1+\cos\varphi)]\,\theta_{zz} = \\ & \qquad = -4K_p(\sin\theta + \sin(\varphi+\theta)) + \mu\,\sin(\varphi)[(\varphi_z)^2 + 2\varphi_z\theta_z]\ , \end{aligned} \tag{5.2}$$

where $\mu := (mc^2 - \omega_s)$, $J := (Ic^2 - \omega_t)$, $\omega_t = K_t\delta^2$, $\omega_s = K_s\delta^2$.

The system in invariant under the reflection symmetry $\theta \to -\theta$, $\varphi \to -\varphi$. In the Lagrangian (5.1) appear two different limit speeds, thus space-time Lorentz symmetry is broken and we expect case 3) of Sect. 3.1 to apply, i.e that the invariant submanifold associated with the reflection symmetry exists only for fixed value of c. The natural candidates as invariant submanifolds of the dynamics are therefore $\theta = 0$ and $\varphi = 0$. One can easily check that this is not the case for $\theta = 0$, whereas setting $\varphi = 0$ in Eq. (5.2), and dividing the second by a factor 4, we get

$$\mu\,\theta_{zz} = -2K_p\sin\theta\ ,\quad \left(\frac{J}{4r^2} + \mu\right)\theta_{zz} = -2K_p\sin\theta\ . \tag{5.3}$$

Thus $\varphi = 0$ is an invariant manifold if and only if $J = 0$; this fixes the speed of the TW to $c = \omega_t/I$.

5.2. *Kinks in coupled chains of adsorbed atoms*

Let us now consider chains of adsorbed atoms (also called "adatoms"), and in particular the interaction between kinks in such chains [3,21]. In our notation, these are described by the equations [21]

$$\begin{aligned} \varphi_{tt} - \varphi_{xx} &= -\sin\varphi - \alpha\sin(\varphi-\psi) + \beta\,\psi_{xx}\,, \\ \psi_{tt} - \psi_{xx} &= -\sin\psi - \alpha\sin(\psi-\varphi) + \beta\,\varphi_{xx}\,. \end{aligned} \tag{5.4}$$

(These are also studied in [22] in the case $\beta = 0$.) These correspond to the Lagrangian

$$L = \frac{1}{2}\left[(\varphi_t^2-\varphi_x^2) + (\psi_t^2-\psi_x^2)\right] - \beta\,\varphi_x\,\psi_x + [\cos(\varphi)+\cos(\psi)+\cos(\varphi-\psi)]\,. \tag{5.5}$$

The TWA produces the equations

$$\begin{aligned} (c^2-1)\varphi_{zz} &= \beta\,\psi_{zz} - \sin\varphi - \alpha\sin(\varphi-\psi) \\ (c^2-1)\psi_{zz} &= \beta\,\varphi_{zz} - \sin\psi - \alpha\sin(\psi-\varphi)\,, \end{aligned}$$

which are written in the form

$$M\,\Phi'' = F$$

by setting

$$M = \begin{pmatrix} c^2-1 & -\beta \\ -\beta & c^2-1 \end{pmatrix}, \quad F = \begin{pmatrix} -\sin\varphi - \alpha\sin(\varphi-\psi) \\ -\sin\psi - \alpha\sin(\psi-\varphi) \end{pmatrix};$$

note that $\det(M) = (c^2-1)^2 - \beta^2$. Under the condition $c^2 \neq 1\pm\beta$, the equation is rewritten as

$$\begin{aligned} \varphi_{zz} &= -\frac{(1-c^2)\sin\varphi + \alpha(1+\beta-c^2)\sin(\varphi-\psi) - \beta\sin\psi}{\beta^2 - (1-c^2)^2}\,, \\ \psi_{zz} &= -\frac{(1-c^2)\sin\psi + \alpha(1+\beta-c^2)\sin(\psi-\varphi) - \beta\sin\varphi}{\beta^2 - (1-c^2)^2}\,. \end{aligned}$$

This obviously admits two discrete symmetries:

$$(\varphi,\psi)\to(\psi,\varphi) \text{ and } (\varphi,\psi)\to(-\varphi,-\psi)\,.$$

The latter is of no use (the invariant set it identifies is just $\varphi=\psi=0$, which gives the trivial solution), while the former suggests to pass to field coordinates

$$\eta := \frac{\varphi-\psi}{2}\,, \quad \xi := \frac{\varphi+\psi}{2}\,.$$

In terms of these, the equations are rewritten as

$$\begin{aligned} \eta_{zz} &= -\frac{\alpha\sin(2\eta)+\sin(\eta)\cos(\xi)}{(c^2-1)+\beta} = -\sin(\eta)\,\frac{2\alpha\cos(\eta)+\cos(\xi)}{(c^2-1)+\beta} \\ \xi_{zz} &= -\sin(\xi)\,\frac{\cos(\eta)}{(c^2-1)-\beta}\,. \end{aligned}$$

We see immediately that both the submanifolds identified by $\xi = 0$ and by $\eta = 0$ are invariant.

In the equation of motion (5.4) appears one single limit speeds, which with the units used in (5.4) is $c_1 = c_2 = \bar{c} = 1$, and a non vanishing kinetic coupling term β, with $|\beta| \leq 1$. Thus space-time Lorentz symmetry is broken only by this coupling term, and case 2) of Sect. 3.1 applies. We therefore expect that the invariant submanifold associated with the discrete symmetry exists for every value of c below a limit value depending on β. We will now show that this is indeed the case.

For $\beta = 0$ we would have a motion in an effective potential

$$W \ = \ -\frac{1}{c^2-1} \ [\cos(\xi+\eta) \ + \ \cos(\xi-\eta) \ + \ \alpha \cos(2\eta)] \ ;$$

the analysis would be rather simple. The case $\beta = 0$ corresponds to not having the kinetic interaction term, see (5.4), (5.5).

By restricting to $\eta = 0$ and writing $h_+ = [c^2 - 1 - \beta]^{-1}$ we obtain

$$\xi_{zz} \ = \ -h_+ \ \sin(\xi) \ , \tag{5.6}$$

i.e. a standard sine-Gordon equation, which will support the standard sine-Gordon solitons [17,26].

Note however that we have $h_+ < 0$ if and only if $c^2 < 1+\beta$; thus ξ-waves are possible if and only if the speed $|c|$ is *smaller* than a critical value, $|c| < c_\xi = \sqrt{1+\beta}$.

By restricting instead to $\xi = 0$ and writing now $h_- = [(c^2-1)+\beta]^{-1}$ we get

$$\eta_{zz} \ = \ -h_- \ \sin(\eta) \ [1 + 2\alpha \cos(\eta)] \ , \tag{5.7}$$

i.e. a double sine-Gordon equation, which will also support solitons [11].

Note that here too $h_- < 0$ is not automatic: this is the case if and only if $c^2 < 1-\beta$, i.e. again η-waves are possible if and only if the speed is smaller than a limit value, which is now $|c| < c_\eta = \sqrt{1-\beta}$.

Consistently with our discussion of the linear case, in order to have a real speed c we must require $|\beta| \leq 1$ (corresponding to the condition $|\gamma| \leq 1$ used in Sect. 4).

It should also be noted that for $\beta = 0$ space-time Lorentz invariance if fully preserved, i.e it is realized the case 1) described in Sect. 3.1. The invariant submanifold exists for every value of the soliton speed smaller than $\bar{c} = 1$.

5.3. *Coupled Josephson junctions*

A weakly coupled system of two long Josephson junctions is described, in suitable units, by the equations [20]

$$\begin{aligned} \varphi_{tt} \ - \ \varphi_{xx} &= -\sin(\varphi) \ - \ \alpha\,\psi_{xx} \ - \ \sigma_1\,\varphi_t \ + \ f_1 \ , \\ \mu^2\,\psi_{tt} \ - \ \psi_{xx} &= -\nu^2 \sin(\psi) \ - \ \alpha\,\varphi_{xx} \ - \ \sigma_2\,\mu\nu\psi_t \ + \ \nu^2 f_2 \ ; \end{aligned}$$

we refer to [20] for the physical meaning of the different constants. Note that here σ_i describes dissipation effects, and f_i are external forcing needed to counterbalance dissipation. Considering the idealized case where dissipation is zero (and hence setting to zero also the external forcing terms f_i), we are reduced to

$$\begin{aligned} \varphi_{tt} \ - \ \varphi_{xx} &= -\sin(\varphi) \ - \ \alpha\,\psi_{xx} \ , \\ \mu^2\,\psi_{tt} \ - \ \psi_{xx} &= -\nu^2 \sin(\psi) \ - \ \alpha\,\varphi_{xx} \ . \end{aligned} \tag{5.8}$$

This corresponds to a potential $V = -[\cos(\varphi) + \nu^2 \cos(\psi)]$.

The TWA reduces (5.8) to

$$(c^2-1)\,\varphi_{zz} \,+\, \alpha\,\psi_{zz} = -\sin(\varphi)\,,$$
$$(\mu^2c^2-1)\,\psi_{zz} \,+\, \alpha\,\varphi_{zz} = -\nu^2\,\sin(\psi)\,; \qquad (5.9)$$

This is written as $M\Phi_{zz} = F$ if we define

$$M \;=\; \begin{pmatrix} c^2-1 & \alpha \\ \alpha & \mu^2c^2-1 \end{pmatrix}\;,\;\; \Phi = \begin{pmatrix} \varphi \\ \psi \end{pmatrix}\;,\;\; F = \begin{pmatrix} -\sin\varphi \\ -\nu^2\sin(\psi) \end{pmatrix}\;.$$

Provided $\det(M) := [(c^2-1)(\mu^2c^2-1)-\alpha^2] \neq 0$, and writing then $h = [\det(M)]^{-1}$, the equations (5.9) are rewritten as $\Phi_{zz} = M^{-1}F$, i.e. as

$$\varphi_{zz} = -h\,\left[(1-\mu^2c^2)\sin(\varphi) \,+\, \alpha\,\nu^2\sin(\psi)\right]$$
$$\psi_{zz} = -h\,\left[\nu^2(1-c^2)\sin(\psi) \,+\, \alpha\,\sin(\varphi)\right]. \qquad (5.10)$$

We would like to determine one-dimensional invariant submanifolds for the dynamics (5.10). We look for discrete symmetries of the model. The system (5.8) is invariant under the reflection symmetry $\varphi \to -\varphi$, $\psi \to -\psi$. The most natural candidates for invariant submanifolds are therefore $\varphi = 0$ and $\psi = 0$. These are, however, of no use, because correspond to trivial solutions of the field equations. Thus in general the determination of the invariant submanifold is a hard problem, but we can look for *linear* invariant submanifolds.

That is we will look for linear combinations of the fields, $\xi = d_1\varphi + d_2\psi$ and $\eta = d_3\varphi + d_4\psi$, such that the submanifolds $\xi = 0$ and $\eta = 0$ are invariant under (5.10). The inverse transformation can be written as[d]

$$\varphi = k_1\,\eta \,+\, k_2\,\xi\;,\;\; \psi \;=\; k_3\,\eta \,+\, k_4\,\xi\;;$$

in this way, and setting

$$H \;=\; \frac{h}{k_1k_3-k_2k_4} \;=\; h\,\rho\;,$$

the equations (5.10) are written as

$$\begin{aligned} \xi_{zz} = -H\,\Big[&[(\alpha k_1 - k_4(1-c^2\mu^2)]\sin(k_1\eta + k_2\xi) \\ &+ [(1-c^2)k_1 - \alpha k_4]\nu^2\sin(k_4\eta + k_3\xi)\Big] \\ \eta_{zz} = -H\,\Big[&[(1-c^2\mu^2)k_3 - \alpha k_2]\sin(k_1\eta + k_2\xi) \\ &- [(1-c^2)k_2 + \alpha k_3]\nu^2\sin(k_4\eta + k_3\xi)\Big]\;. \end{aligned}$$

On the line $\eta = 0$ these reduce to

$$\xi_{zz} = -H\,\left[[(\alpha k_1 - k_4(1-c^2\mu^2)]\sin(k_2\xi) \,+\, [(1-c^2)k_1 - \alpha k_4]\nu^2\sin(k_3\xi)\right]$$
$$\eta_{zz} = -H\,\left[[(1-c^2\mu^2)k_3 - \alpha k_2]\sin(k_2\xi) \,-\, [(1-c^2)k_2 + \alpha k_3]\nu^2\sin(k_3\xi)\right]\;;$$

thus the line $\eta = 0$ is invariant if and only if

$$(1-c^2\mu^2)k_3 - \alpha k_2 \;=\; 0 \;\text{ and }\; (1-c^2)k_2 + \alpha k_3 \;=\; 0\,. \qquad (5.11)$$

[d]Writing $\rho = (k_1k_3 - k_2k_4)^{-1}$, the relation between the d_i and the k_i coefficients is given by $d_1 = -k_4\rho$, $d_2 = k_1\rho$, $d_3 = k_3\rho$, $d_4 = -k_2\rho$.

This can be cast in the form of a matrix equation $Q_\xi K_\xi = 0$, where we have defined

$$Q_\xi \;=\; \begin{pmatrix} -\alpha & (1-c^2\mu^2) \\ (1-c^2) & \alpha \end{pmatrix} \;,\; K_\xi \;=\; \begin{pmatrix} k_2 \\ k_3 \end{pmatrix} \;.$$

The solution exists provided $\det(Q_\xi) = 0$, and this condition considered as an equation for c provides

$$c^2 \;=\; c^2_{(\xi,\pm)} \;=\; \frac{(1+\mu^2) \;\pm\; \sqrt{(1-\mu^2)^2+4\alpha^2\mu^2}}{2\,\mu^2} \;.$$

Notice that, in accordance with our results in the linear case, in order to have a real speed c we must require $|\alpha| \le 1$ (corresponding to the condition $|\gamma| \le 1$ used in Sect. 4).

Denoting by $Q_{(\xi,\pm)}$ the matrix Q_ξ with $c^2 = c^2_{(\xi,\pm)}$, the kernel of $Q_{(\xi,\pm)}$ is spanned respectively by

$$v_{(\xi,\pm)} \;=\; \begin{pmatrix} (1-\mu^2) \pm \sqrt{(1-\mu^2)^2+4\alpha^2\mu^2} \\ 2\alpha \end{pmatrix} \;.$$

One can check that with these choices for c and for k_1, k_4 the line $\eta = 0$ is invariant, and the evolution of ξ is governed by an equation of the form

$$\xi_{zz} \;=\; p_1 \sin(2\alpha\xi) \;+\; p_2 \sin(\omega_{\xi,\pm}\xi) \;,$$

where p_i are coefficients depending on (k_1, k_4), whose explicit expression can be easily computed but is not interesting here, and

$$\omega_{(\xi,\pm)} = -(1+\mu^2) \;\pm\; \sqrt{(1-\mu^2)^2+4\alpha^2\mu^2} \;.$$

One could analyze in the same way the invariance of the line $\xi = 0$. Note however that there is no physical difference between the two fields, as they are just generic linear combinations of φ and ψ; thus the $\xi = 0$ case will reproduce – with an exchange of roles between ξ and η – the same results obtained for $\eta = 0$.

Note that in the case analyzed here the Lorentz symmetry is broken by the presence of two different limit speeds in the field equations (5.8). Thus, case 3) of Sect. 3.1 applies. The invariant submanifold associated with the discrete symmetry exists only for selected values of the speed c.

5.4. *Modified Katsura model*

Our last example is a "two speeds of light" generalization of the model of Katsura [19]; this has been proposed to describe the coupling of magnetic and ferroelectric domain walls [18, 19]. The model we consider is described by the Lagrangian (for simplicity we set in the model of Ref. [19] the coupling constant $\lambda = \mu = 1$)

$$\mathscr{L} = \frac{1}{2}\left(\frac{\varphi_t^2}{c_1^2} - \varphi_x^2 + \frac{\psi_t^2}{c_2^2} - \psi_x^2\right) - V(\varphi,\psi) \;,\quad V(\varphi,\psi) = \frac{\varphi^4}{4} - \varphi\cos\beta\psi. \qquad (5.12)$$

The Lagrangian has the discrete symmetry $\psi \to -\psi$, so that the natural candidate for the invariant submanifold for the dynamical system one obtains after the TWA is $\psi = 0$. Indeed such invariant manifold exists for every value of the parameters and the reduced dynamics is described by

$\varphi_{zz} = \varphi^3 - 1$. However, this equation does not support solitonic solutions because the potential does not have the required minima structure.

The model also has another reflection symmetry, described by

$$\varphi \to -\varphi \ , \ \ \psi \to \frac{\pi}{\beta} - \psi \ .$$

If we look for analytic functions invariant under such a reflection, they are necessarily built as (algebraic) functions of the basic invariants

$$\varphi^2 \ , \ \ \cos^2(\beta\psi) \ , \ \ \varphi\cos\beta\psi \ .$$

In fact, another invariant manifold for the dynamics can be found using the results of Ref. [19], where exact kink solutions have been found for $\beta^2 = 1/2$ using the ansatz

$$\varphi = \cos\beta\psi. \tag{5.13}$$

One can easily show that Eq. (5.13) with $\beta^2 \neq 1/2$ determines an invariant manifold for the equivalent mechanical system describing the TW dynamics stemming from (5.12), if the TW speed is fixed by

$$c \ = \ c_2 c_1 \sqrt{\frac{1-2\beta^2}{c_1^2 - 2\beta^2 c_2^2}}. \tag{5.14}$$

Note that again we have two limiting speeds in the Lagrangian (5.12); thus the model falls in case 3) of section 3.1, and we do indeed have the behavior predicted there. On the other hand, in the case $c_1 = c_2$ and $\beta^2 = 1/2$, consistently with the results of Katsura [19], equation (5.13) describes an invariant manifold of the dynamics *for every value of c*; thus in this special case the system falls in case 1) of the classification given in Sect. 3.1.

6. Discussion. Field theoretical considerations

This paper focused on the description of TW dynamics in macroscopic and mesoscopic systems. This is particular evident in the examples we have chosen in Sects. 4 and 5 to elucidate our mechanism. They represent macroscopic or mesoscopic models for molecular, biological or condensed matter systems.

Nevertheless, at least some of the results of these paper have a much broader relevance and can be applied, generically, to any two-dimensional (2D) field theory. This is particularly true for the first part of paper, which concerned linear dynamics.

An important point in this context is the presence in the action for the 2D field theory (1.2) of N different "speeds of light" c_i, see Eq. (1.3). This is a natural condition for macroscopic and mesoscopic systems, where different modes propagating in a medium experience different effective physical parameters ρ_i, κ_i (elastic, optical, magnetic and so on) of the media so that at the linear level their perturbations propagate at different speeds. On the other hand a fundamental, microscopic, 2D field theory has to be Lorentz invariant, thus characterized by a single speed of light.

The case of a single limit speed is a particular case of the description of linear waves given in the first part of this paper (see e.g Eq. (2.17)). Thus, our results including those related to synchronization and tuning of the wave speed hold true for $c_1 = c_2$.

From the field theoretical point of view a particularly important case is that of a non interacting theory. When $V = \gamma = 0$ and $c_i = \bar{c}$ the theory (1.2) is a 2D conformal field theory (CFT) . It describes CFTs with N scalar fields (i.e. CFTs with central charge N), which play a fundamental role in several contexts such as string theory [27], critical points of phase transitions [12], microscopic explanation of black hole entropy [14], just to mention only few of them. The conformal invariance of the Lagrangian (1.2) is fully evident by passing to light cone coordinates $x_\pm = x \pm ct$. In this coordinates the theory is explicitly invariant under the action of 2D-diffeomorphisms (the conformal group in 2D) and the field equations read $\partial_{x_+}\partial_{x_-}\varphi = 0$, whose general solution is a generic combination of right and left moving TW $\varphi = f(x_+) + f(x_-)$. The conformal invariance is a general feature of the massless case. In fact it is also evident from Eq. (2.9), which does not fix the dependence on z of the fields $\varphi(z)$, $\psi(z)$ but just gives a linear relation between the two fields.

As a final remark we note an analogy between the synchronization mechanism for TW in the linear regime we have found in the first part of this paper, and Quantum Mechanics; this goes as follows.

Let us consider for simplicity the field equations (2.7) with vanishing potential (our considerations can be easily extended to the $V \neq 0$ case). The system can be described in two different frames of the field space φ, ψ: a frame in which the two fields decouple completely i.e. in which the kinetic matrices are diagonal; and a frame in which the two fields interact with the interaction term γ. In the first frame TW for the two fields can propagate independently, hence with different speeds c_+ and c_- given by Eq. (2.10). Owing to the interaction, in the second frame the $\varphi-$ and ψ-waves are forced to travel with the same speed (synchronization).

This relationship between TW waves in the two frames bears a strong analogy with the eigenstates of two non-commuting quantum mechanical operators, acting on a 2D Hilbert spaces and associated to two non compatible physical observables.

7. Conclusions

In previous work, dealing with a concrete model for the nonlinear dynamics of DNA [5], we observed a peculiar mechanism which fixed the speed of solitons [4]. In this paper we investigated if this mechanism could extend to a more general class of equations, answering this question in the positive. More generally, we have considered systems of coupled wave equations; when decoupled each of them is Lorentz invariant with a limit speed c_i, with possibly different limit speeds and we have studied how the speed of travelling waves is affected by the coupling.

In particular we have considered systems admitting a variational (Lagrangian) description, and in which the coupling between the different equations could involve a kinetic term, see Eq. (1.2).

We observed that the general Lagrangian under consideration here, (see Eq. (1.2)) could fall in different classes according to its properties under Lorentz transformations. That is, Lorentz invariance could be unbroken (case 1), in which case there is of course no speed selection mechanism; it could be broken, albeit the coupled equations admit the same limit speed, by coupling terms (case 2), in which case there is still no speed selection but the limit speed can be changed due to the coupling; or finally the Lorentz invariance can be broken by the presence of different limit speeds *and* kinetic coupling terms (case 3), in which case we have a full selection of the speed of travelling wave solutions, which can only take a finite – and rather small, being limited by the number of coupled equations – set of values.

In the latter – and more interesting – case, simple travelling wave solutions can be described as dynamically invariant one-dimensional manifolds for a mechanical system (parametrically dependent on the speed c of travelling waves) associated to the system of PDEs under study, joining two extremal points for an effective potential which comply with limit conditions dictated by the finite energy condition for the PDEs system. Such manifolds only exist for special values of c, and this ignites the speed selection mechanism.

We have then validated our general discussion by a number of physically significant examples; in particular, in Sects. 5.1, 5.3 and 5.4 we have shown that the speed selection mechanism here considered is present in double pendulum chains, coupled Josephson junctions, and in a modified Katsura model for the coupling of magnetic and ferroelectric domain walls.

It is interesting that the mechanism described here – and which has a partial counterpart for linear equations as well – has some points of contact with general field-theoretic questions, as discussed in Sect. 6.

In the Appendices, we give a closer look at some group-theoretical questions (Appendix A); and argue that, albeit here we worked specifically with hyperbolic PDEs, the same mechanism can be at work in some type of parabolic equations (Appendix B).

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Appendix A. Lorentz symmetry and group theoretical considerations

In Sect. 2.3 we have noted that in the Lagrangian (1.2) the usual space-time Lorentz symmetry is explicitly broken by the presence of several limiting speeds c_i. However, there is some remnant of the Lorentz invariance; this can be described as follows.

The Lorentz transformation

$$x \to \widetilde{x} \;=\; \frac{(x-vt)}{\sqrt{1-v^2/c^2}} \; , \;\; t \to \widetilde{t} \;=\; \frac{(t-(v/c^2)x)}{\sqrt{1-v^2/c^2}} \tag{A.1}$$

is generated by the vector field

$$X \;=\; -\frac{x}{c^2}\,\frac{\partial}{\partial t} \,-\, t\,\frac{\partial}{\partial x} \; .$$

The *evolutionary representative* [15, 23, 24, 30] for this is the (generalized) vector field

$$X_{ev} \;=\; \left(\frac{x}{c^2}\,u_t \,+\, t\,u_x\right)\,\frac{\partial}{\partial u} \; ,$$

where u is the dependent variable (the physical field).

Thus if we consider the Lorentz symmetry with several limit speeds c_i acting on the field ϕ^i, this is generated in the evolutionary representation by

$$X^i_{ev} \;=\; \left(\frac{x}{c_i^2}\,\phi^a_t \,+\, t\,\phi^a_x\right)\,\frac{\partial}{\partial \phi^a} \; .$$

If now we consider a generalized vector field

$$X_L \;=\; X^1_{ev} + \ldots + X^N_{ev} \; ,$$

which (unless $c_1^2 = ... = c_N^2$) will *not* be the evolutionary representative of any Lie-point vector field acting on the space of the $(x,t;\phi^1,...,\phi^N)$ variables, this will be a symmetry for the Lagrangian $\mathscr{L}_1 + ... + \mathscr{L}_N$, where we define the partial Lagrangians as

$$\mathscr{L}_i \;=\; \frac{1}{2}\,\left[\rho_i^2\,(\phi_t^i)^2 \;-\; \kappa_i^2\,(\phi_x^i)^2\right] \ ,$$

and for the corresponding field equations $\phi^i_{tt} - c_i^2 \phi^i_{xx} = 0$ as well.[e]

Note however that this will *not* leave the interaction Lagrangian, nor the corresponding interaction terms in the field equations, invariant.

The symmetry properties of the Lagrangian and field equations described above can be also understood in terms of the commutation relations (2.5) of the matrices $Q^{(t)}, Q^{(x)}$ defined in (2.4). We have a symmetry of the Lagrangian when the matrices can be diagonalized simultaneously, i.e when $\gamma = 0$ (corresponding to absence of coupling in the kinetic sector) and $c_i = \bar{c}$, i.e the same limit for the TW of different fields.

The N-field components TW will belong to the representation of the Lorentz symmetry generated by the generalized vector field X_L. More specifically they will transform via reducible representations

$$T \;=\; T_1 \oplus T_2 \oplus ... \oplus T_N \ ,$$

where each T_a is a two-dimensional representation made of the Lorentz group. The TW solutions of the model with vanishing, respectively non-vanishing, potential belong to massless, respectively massive, representations of the Lorenz group.

Appendix B. Extension to parabolic equations

We have so far considered coupled *hyperbolic* equations; on the other hand the treatment of the linear case is based on basic algebraic facts and does not involve hyperbolicity; it thus appears that our discussion can be extended to more general (time-evolution) equations, and in particular to *parabolic* ones.

Here, we will not discuss the general case but only consider, for the sake of concreteness, TW dynamics described by coupled linear Schrödinger equations; this can be thought as emerging from the linearization of the nonlinear Schrödinger equations

$$i\,\psi_t \;=\; -\frac{1}{2}\psi_{xx} \;+\; \kappa\,|\psi|^2\,\psi \ . \qquad \text{(B.1)}$$

This is known to describe, among others, optical solitons and, upon introducing a potential term $V(x)\psi$, Bose-Einstein condensates. As usual we will consider the $N = 2$ case, the generalization to the N-fields case being performed along the lines described in Sect. 2.3.

[e]Actually, here only derivatives of first (for the Lagrangian) and second (for the field equations) order matter; so we could as well consider only transformations of the field derivatives, e.g. prescribing these are undergoing a hyperbolic rotation irrespective of any transformation on the field themselves. In this way one would be led to consider "hidden symmetries" [13].

Let us consider the system of two coupled linear Schrödinger equations,

$$\begin{aligned} \frac{i}{c_1}\varphi_t + \frac{1}{2}\varphi_{xx} - \frac{\gamma}{2}\psi_{xx} &= -\frac{\partial V}{\partial \varphi} , \\ \frac{i}{c_2}\psi_t + \frac{1}{2}\psi_{xx} - \frac{\gamma}{2}\varphi_{xx} &= -\frac{\partial V}{\partial \psi} , \end{aligned} \tag{B.2}$$

where c_i, γ are some parameters and now the fields φ, ψ are generically complex. In analogy with the hyperbolic case we have introduced a quadratic potential of the form given by Eq. (2.3). As in the hyperbolic case, to solve this equations we pass to the Fourier transforms.

In the decoupled case $\gamma = \lambda = 0$ we get two dispersion relations $\omega_i(q)$, $i = 1,2$ and the related phase velocities $v_\pm^{(p)} = d\omega_\pm/dq$,

$$\omega_i = \frac{1}{2}c_i\left(q^2 - \mu_i^2\right) \ , \ v_i^{(p)} = c_i q \ ; \tag{B.3}$$

thus c_i characterizes the phase velocity of the wave packet.

In the generic coupled case $\gamma \neq 0, V \neq 0$, we get dispersion relations similar to (2.16), i.e.

$$\omega_\pm = \frac{1}{2}\left[\frac{1}{2}(c_1+c_2)q^2 - (c_1\mu_1^2 + c_2\mu_2^2) \pm \sqrt{\mathscr{P}(q^2)}\right] , \tag{B.4}$$

$$\begin{aligned} \mathscr{P}(q^2) = & \frac{1}{4}(c_1+c_2)^2 q^4 - (c_1\mu_1^2 + c_2\mu_2^2)(c_1+c_2)q^2 + \\ & + (c_1\mu_1^2 + c_2\mu_2^2)^2 - c_1c_2(q^2 - 2\mu_1^2)(q^2 - 2\mu_2^2) + c_1c_2(q^2\gamma + \lambda)^2. \end{aligned} \tag{B.5}$$

Notice also that in the case of vanishing potential we get the dispersion relations

$$\omega_\pm \ = \ \frac{1}{2}\, c_\pm q^2 \ , \tag{B.6}$$

where $c_\pm$ has exactly the same form (2.10) found in the hyperbolic case with squared-velocities replaced by the parameters c_i :

$$c_\pm \ = \ \frac{1}{2}\left[c_1 + c_2 \ \pm \ (c_1 - c_2)\sqrt{1 + \frac{4\gamma^2 c_1 c_2}{(c_1-c_2)^2}}\right] \ . \tag{B.7}$$

As in the hyperbolic case this equation tells us that we can tune the phase velocity of the φ and ψ waves just by changing the parameter $|\gamma|$ in the range $[0,1]$, (see Eq. (2.13)). In particular, the phase velocity of a given phase in ω_+ (ω_-) changes from a maximum (minimum) value when $\gamma = 0$, given by the smallest (greatest) of $c_\pm$, to zero (a maximum value) when $|\gamma| = 1$.

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