

Journal of Nonlinear Mathematical Physics

ISSN (Online): 1776-0852

ISSN (Print): 1402-9251

Journal Home Page: <https://www.atlantis-press.com/journals/jnmp>

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Lida Mendoza, Enrique G. Reyes

To cite this article: Lida Mendoza, Enrique G. Reyes (2015) Massey products, A_∞ -algebras, differential equations, and Chekanov homology, Journal of Nonlinear Mathematical Physics 22:3, 342–360, DOI: <https://doi.org/10.1080/14029251.2015.1056616>

To link to this article: <https://doi.org/10.1080/14029251.2015.1056616>

Published online: 04 January 2021

Massey products, A_∞ -algebras, differential equations, and Chekanov homology

Lida Mendoza

*Departamento de Matemática y Ciencia de la Computación,
Universidad de Santiago de Chile
Casilla 307 Correo 2, Santiago, Chile.
lida.mendoza@usach.cl*

Enrique G. Reyes

*Departamento de Matemática y Ciencia de la Computación,
Universidad de Santiago de Chile
Casilla 307 Correo 2, Santiago, Chile.
ereyes@fermat.usach.cl ; e_g_reyes@yahoo.ca*

Received 9 January 2015

Accepted 31 March 2015

We consider (classical and generalized) Massey products on the Chekanov homology of a Legendrian knot, and we prove that they are invariant under Legendrian isotopies. We also construct a minimal A_∞ -algebra structure on the Chekanov algebra of a Legendrian knot, we prove that this structure is invariant under Legendrian isotopy, and we observe that its higher multiplications allow us to find representatives for classical Massey products. Finally, we consider differential equations: we remark that the Massey product Legendrian invariants admit a “dynamical interpretation”, in the sense that they provide solutions for a Maurer-Cartan equation posed on an infinite-dimensional bigraded Lie algebra, and we show how to set up and solve a (twisted) Kadomtsev-Petviashvili hierarchy of equations starting from the Chekanov algebra of a Legendrian knot.

Keywords: A_∞ -algebras; Legendrian contact homology; Kadomtsev-Petviashvili hierarchy.

2000 Mathematics Subject Classification: 37K10, 53D12, 55S30

1. Introduction

We say that a manifold M of dimension $2n + 1$ is a *contact manifold* if it admits a maximally non-integrable distribution η ; if we write η as the kernel of a one-form α , see [11], then the non-integrability condition translates into the fact that $\alpha \wedge (d\alpha)^n$ is nowhere vanishing. *Legendrian submanifolds* are n -dimensional submanifolds of M which are everywhere tangent to the “contact distribution” η . If $\eta = \ker(\alpha)$, then Legendrian submanifolds correspond to n -dimensional integral submanifolds of the exterior differential system determined by α .

We restrict to the case $n = 1$. In this case, *compact* Legendrian submanifolds are knots. A classical problem is to *classify* Legendrian knots in a given three-dimensional contact manifold M . Because of their definition as integral submanifolds, the classification of Legendrian knots is different from the classification of topological knots: if we define a *Legendrian isotopy* as a deformation of a Legendrian knot through Legendrian knots, it is known that a single isotopy class of topological knots admits infinitely many Legendrian isotopy classes of Legendrian knots, see [11]. Thus, in order to obtain Legendrian classification results one has to go well beyond standard topological invariants such as the Alexander or Jones polynomials.

A powerful Legendrian invariant for Legendrian knots in $M = \mathbb{R}^3$ with the contact structure determined by $\eta = \ker(\alpha)$, in which $\alpha = dz + xdy$, was introduced by Chekanov in [4]. This invariant comes “categorified”, in contradistinction with, for instance, the classical Jones polynomial from topological knot theory whose categorification was achieved in [17]: the Chekanov invariant is defined via the homology of a non-commutative differential algebra determined by the contact geometry of the ambient contact manifold (M, η) . Certainly, once we have invariants determined by homology, it is natural to investigate whether other similar invariants exist.

We show in this paper that it is indeed possible to construct further Legendrian invariants simply by using classical Massey products, see [19, 21], and their generalizations considered in [2]. We also construct an A_∞ -algebra on the (co)homology ring of the Chekanov algebra of a Legendrian knot following Kadeishvili (see [13, 22, 31]), and we remark that this A_∞ -algebra is a Legendrian invariant as well. After [20], we then observe that (classical) Massey products are related to the higher multiplications of the A_∞ -algebra just constructed. There are two reasons why this observation may be of importance. First, it allows us to get some understanding of Massey products since these higher multiplications can be computed in a relatively straightforward fashion, see [20, 22]; second, it may be used to advance a “dynamical interpretation” for Massey product Legendrian invariants: a result by He, see [12], implies that the higher multiplications of the A_∞ -algebra constructed from the Chekanov algebra provide a solution to a Maurer-Cartan equation posed on an infinite-dimensional bigraded Lie algebra.

Finally, we consider a twisted Kadomtsev-Petviashvili (KP) hierarchy of equations defined with the help of the Chekanov algebra. One reason for believing that this construction may be of interest is that it provides us with instances of *noncommutative* integrable equations, such as the ones investigated in [25], arising quite naturally from a non-trivial geometric context.

Our work is organized as follows. Section 2 is an introduction to A_∞ -algebras, and Section 3 is a rather detailed review of (generalized) Massey products after [19, 21] and [2]. Since Massey products have been used recently in Mathematical Physics, see [18], it appears reasonable to discuss them carefully. In Section 4 we introduce contact manifolds and Legendrian knots, we summarize the construction of the Chekanov algebra, we explain how to construct an A_∞ -algebra which is a Legendrian invariant, and we observe that classical and generalized Massey products also determine Legendrian invariants. Finally, in Section 5 we explain in what sense the Legendrian invariants arising from classical Massey products solve a Maurer-Cartan equation and we introduce our twisted KP hierarchy.

Remark 1.1. While writing up our results we found out that previous work on Legendrian knot invariants and classical Massey products (on linearized Chekanov (co)homology) had been carried out in [5]. The existence of this interesting paper prompted us to consider generalized Massey products after [21] and [2], which do not appear in [5]. We also note that differential equations are not studied in this reference.

2. A_∞ -algebras

Let \mathbb{K} be a field, and \mathcal{A} a \mathbb{Z} -graded \mathbb{K} -vector space, $\mathcal{A} = \bigoplus_{i \in \mathbb{Z}} \mathcal{A}^i$. An A_∞ -algebra structure on \mathcal{A} is a family of graded linear maps $m_n : \mathcal{A}^{\otimes n} \rightarrow \mathcal{A}$, $n \geq 1$, such that the degree of m_n is $2 - n$ and the

identities

$$\sum_{\substack{r+s+t=n \\ r,t \geq 0; s \geq 1}} (-1)^{rs+t} m_{r+1+t} \circ (id^{\otimes r} \otimes m_s \otimes id^{\otimes t}) = 0 \quad (2.1)$$

hold for all $n \geq 1$. Here and henceforth (see for example Equations (2.3) and (2.6) below) we follow Koszul's sign rule if we evaluate on specific elements of a tensor product space. For example, $(f \otimes g)(x \otimes y) = (-1)^{\deg(g)\deg(x)} f(x) \otimes g(y)$, where f and g are homogeneous maps and x, y are homogeneous elements in the domains of f and g respectively.

We consider some specific instances of (2.1). If $n = 1$ then $r, t = 0$ and $s = 1$, so that m_1 is a degree 1 map and the identity (2.1) is simply $m_1 \circ m_1 = 0$, that is, (\mathcal{A}, m_1) is a cochain differential complex. Also, if $n = 2$, then (2.1) can be written as

$$m_1 \circ m_2 = m_2 \circ (id \otimes m_1) + m_2 \circ (m_1 \otimes id),$$

and therefore m_2 is a bilinear map which behaves as a multiplication and the differential m_1 satisfies the graded Leibnitz rule with respect to m_2 . Note that m_2 is not necessarily associative. Indeed, the third identity arising from (2.1) is

$$m_2 \circ (m_2 \otimes id) - m_2 \circ (id \otimes m_2) = m_1 \circ m_3 + m_3 \circ (id^{\otimes 2} \otimes m_1 + id \otimes m_1 \otimes id + m_1 \otimes id^{\otimes 2}), \quad (2.2)$$

so that m_2 is associative if the right hand side of this equation is identically zero. Thus, if $m_3 = 0$, then we conclude that (\mathcal{A}, m_1, m_2) is a differential graded algebra with a differential of degree 1. Conversely, every differential graded algebra is an A_∞ -algebra with $m_3 = m_4 = \dots = 0$.

Remark 2.1. Let us write $d = m_1$ and $d^3 = m_1 \otimes id^{\otimes 2} + id \otimes m_1 \otimes id + id^{\otimes 2} \otimes m_1$. It is trivial to see that $d^3 \circ d^3 = 0$, so that (\mathcal{A}, d) and $(\mathcal{A}^{\otimes 3}, d^3)$ are cochain differential complexes. In this notation, identity (2.2) becomes, simply,

$$m_2 \circ (m_2 \otimes id) - m_2 \circ (id \otimes m_2) = d \circ m_3 + m_3 \circ d^3.$$

Now, the functions $f = m_2 \circ (m_2 \otimes id)$ and $g = m_2 \circ (id \otimes m_2)$ are cochain maps of degree zero from $\mathcal{A}^{\otimes 3}$ to \mathcal{A} , and m_3 is a map of degree -1 satisfying the above equation. This says precisely that m_3 is an homotopy between the maps f and g . In other words, m_2 is associative *up to* an homotopy which is also a part of the A_∞ -algebra structure.

A_∞ -algebras first appeared in topology, more precisely in the theory of loop spaces, see [27, 28]. A short review of their properties —and a guide to earlier literature— is in [20].

Definition 2.1. Let (\mathcal{A}, m_n) and (\mathcal{B}, m'_n) be two A_∞ -algebras. An A_∞ -morphism $f : \mathcal{A} \rightarrow \mathcal{B}$ is a family of linear maps $f_n : \mathcal{A}^{\otimes n} \rightarrow \mathcal{B}$, in which $n \geq 1$, such that for each $n \geq 1$ the degree of f_n is $1 - n$ and the following Stasheff morphism identities hold:

$$\sum_{\substack{r+s+t=n \\ r,t \geq 0; s \geq 1}} (-1)^{rs+t} f_{r+1+t}(id^{\otimes r} \otimes m_s \otimes id^{\otimes t}) = \sum_{j=1}^n \sum_{i_1+\dots+i_j=n} (-1)^u m'_j(f_{i_1} \otimes f_{i_2} \otimes \dots \otimes f_{i_j}), \quad (2.3)$$

where $i_k \geq 1$ for all k and $u = (i_{j-1} - 1) + 2(i_{j-2} - 1) + \dots + (j-2)(i_2 - 1) + (j-1)(i_1 - 1)$.

Note that the first Stasheff morphism identity is simply $f_1 m_1 = m'_1 f_1$ that is, it says that f_1 is a cochain map. We say that a morphism f is a *quasi-isomorphism* if f_1 is a quasi-isomorphism of complexes, i.e. the induced map $H(f_1) : H(\mathcal{A}) \rightarrow H(\mathcal{B})$ is an isomorphism.

Now we review Merkulov's construction, see [22], of an A_∞ -algebra starting from a differential graded algebra. As we already explained, every differential graded algebra (\mathcal{A}, d) is an A_∞ -algebra, but the importance of [22] is that it allows us to construct an explicit A_∞ -algebra structure with non-zero higher multiplications. We recall that if ϕ, ψ are two graded linear maps on the differential graded algebra (\mathcal{A}, d) , the *supercommutator* of ϕ and ψ is $[\phi, \psi] = \phi\psi - (-1)^{\deg(\phi)\deg(\psi)}\psi\phi$. Merkulov's construction relies on the following *assumption*:

Let (\mathcal{A}, d) be a differential graded algebra. We assume that there exist a subcomplex W of \mathcal{A} , and a vector space homomorphism $Q : \mathcal{A} \rightarrow \mathcal{A}$ of degree -1 , such that the image of the map $Id - [d, Q] : \mathcal{A} \rightarrow \mathcal{A}$ is in fact in W . (2.4)

Note that it is not required that W is a subalgebra of \mathcal{A} . We define a sequence of linear maps $\lambda_n : \mathcal{A}^{\otimes n} \rightarrow \mathcal{A}$, where $n \geq 1$, as follows: λ_1 is determined only by the condition $Q\lambda_1 = -Id$, and for $n \geq 2$ we set, recursively,

$$\lambda_2(v \otimes w) = v \cdot w, \quad (2.5)$$

$$\lambda_n = \sum_{s+t=n; s,t \geq 1} (-1)^{s+1} \lambda_2(Q\lambda_s \otimes Q\lambda_t). \quad (2.6)$$

The following theorem holds (see [22]):

Theorem 2.1. *Let (\mathcal{A}, d) be a differential graded algebra and assume that (2.4) holds. Define linear maps $m_n : W^{\otimes n} \rightarrow W$, where $n \geq 1$, via*

$$m_1 = d, \quad (2.7)$$

$$m_n = (Id - [d, Q]) \circ \lambda_n, \quad \text{for } n \geq 2, \quad (2.8)$$

in which λ_n are the maps constructed above. The maps m_n satisfy the identities (2.1), and therefore they determine an A_∞ -algebra structure on the complex W .

Theorem 2.1 is also discussed in [31] and [20]. As pointed out for example in [20], it is an explicit realization of a very general result due to Kadeishvili [13]. Now we follow [20] in identifying an appropriate subcomplex W and a linear map Q satisfying assumption (2.4):

Let $\mathcal{A} = \bigoplus_{p \in \mathbb{Z}} A^p$ be a differential graded algebra with differential d of degree 1. We denote by B^p and Z^p the spaces of coboundaries and cocycles of A^p respectively. Then, there are subspaces H^p of Z^p and L^p of A^p such that

$$Z^p = B^p \oplus H^p \quad \text{and} \quad A^p = Z^p \oplus L^p = B^p \oplus H^p \oplus L^p. \quad (2.9)$$

We set $W = \bigoplus_{p \in \mathbb{Z}} H^p$ and we define the map Q as follows: $Q^p : A^p \rightarrow A^{p-1}$ is given by

$$Q^p|_{L^p} = Q^p|_{H^p} = 0, \quad Q^p|_{B^p} = (d^{p-1}|_{L^{p-1}})^{-1}.$$

It is easy to see that Q determines an homotopy between Id and pr , where $pr : \mathcal{A} \rightarrow \mathcal{A}$ is the projection from \mathcal{A} onto W , that is, we have $Id - pr = dQ + Qd$ and therefore Merkulov's assumption (2.4) holds with W and Q as above. We also note that $d|_{H^p} = 0$, so that in fact, the operation m_1 of Theorem 2.1 is identically zero and therefore (see Remark 1) the operation m_2 is an *associative* multiplication on W . Using the first isomorphism theorem, we identify the complex W with the

cohomology of \mathcal{A} , that is, $W = \ker(d)/\text{Im}(d)$. Hereafter we write $H\mathcal{A}$ instead of W , to remind us of this identification. Following [20] we rewrite Merkulov's result thus:

Proposition 2.1. *Consider the functions λ_n defined above, and set $m_n = \text{pr} \circ \lambda_n : H\mathcal{A}^{\otimes n} \rightarrow H\mathcal{A}$ for $n \geq 2$. Then, $(H\mathcal{A}, 0, m_2, m_3, \dots)$ is an A_∞ -algebra and $f = \{-Q\lambda_n\}_{n \geq 1}$ is a quasi-isomorphism of A_∞ -algebras between $H\mathcal{A}$ and \mathcal{A} .*

An A_∞ -algebra constructed as above is called a *Merkulov model* or a *minimal model* of the differential graded algebra \mathcal{A} , in analogy with D. Sullivan's minimal models for differential graded commutative algebras introduced in the context of rational homotopy theory [29]. We also note that in the context of A_∞ -algebras, being quasi-isomorphic is a *transitive* property, as stressed for example in [30], and therefore all Merkulov models of \mathcal{A} (which obviously depend on the choice of the subspaces H^p and L^p introduced above) are quasi-isomorphic as A_∞ -algebras.

3. Massey products

Once a minimal model $(H\mathcal{A}, 0, m_2, m_3, \dots)$ of a differential graded algebra (\mathcal{A}, d) is available, it is very natural to investigate the *associative algebra* $(H\mathcal{A}, m_2)$ and to ask about the meaning of the higher multiplications m_n , $n \geq 3$. As observed in [20], these higher multiplications are connected to classical Massey products. Since we use classical and generalized Massey products to define Legendrian isotopy invariants of Legendrian knots, we review them in some detail. We follow the sign conventions of [21]. In particular, we write $\bar{a} = (-1)^{1+\deg(a)}a$, so that $d\bar{f} = -\bar{d}f$ and $\overline{ab} = -\bar{a}\bar{b}$.

3.1. Classical Massey products

Let (\mathcal{A}, d) be a differential graded algebra with $\deg(d) = 1$. If $\alpha_1, \alpha_2 \in H\mathcal{A}$, their *length two Massey product* $\langle \alpha_1, \alpha_2 \rangle$ is the singleton $\{\alpha_1 \alpha_2\}$; we define the *length 3 Massey product* as follows:

Suppose that $\alpha_1, \alpha_2, \alpha_3 \in H\mathcal{A}$ and assume that $\alpha_1 \alpha_2 = \alpha_2 \alpha_3 = 0$. We pick representatives $a_{i-1,i} \in \mathcal{A}$ of the cohomology classes α_i . Because we are assuming that $\alpha_1 \alpha_2 = \alpha_2 \alpha_3 = 0$, there exist cochains a_{02} and a_{13} such that

$$da_{02} = \bar{a}_{01}a_{12}, \quad \text{and} \quad da_{13} = \bar{a}_{12}a_{23}. \quad (3.1)$$

With these choices we can check that $a_{03} = \bar{a}_{02}a_{23} + \bar{a}_{01}a_{13}$ satisfies $d a_{03} = 0$. The length 3 Massey product of the cocycles a_{01} , a_{12} and a_{23} is the set $MP_3(a_{01}, a_{12}, a_{23})$ of all cohomology classes of the cocycles $a_{03} = \bar{a}_{02}a_{23} + \bar{a}_{01}a_{13}$ arising from different choices of cochains a_{02} and a_{13} .

Proposition 3.1. *The length 3 Massey product $MP_3(a_{01}, a_{12}, a_{23})$ depends only on the cohomology classes of the cocycles a_{01}, a_{12}, a_{23} .*

Indeed, we can check that $MP_3(a_{01}, a_{12}, a_{23}) = MP_3(a_{01} + db, a_{12}, a_{23}) = MP_3(a_{01}, a_{12} + db, a_{23}) = MP_3(a_{01}, a_{12}, a_{23} + db)$ for any cochain b . Proposition 3.1 follows easily from this observation as, for example, it implies that $MP_3(a_{01} + db_1, a_{12} + db_2, a_{23}) = MP_3(a_{01} + db_1, a_{12}, a_{23}) = MP_3(a_{01}, a_{12}, a_{23})$. We will provide some further details of the proof in the general case of length n Massey products, to be discussed below. This result allows us to make the following definition:

Definition 3.1. Let $\alpha_1, \alpha_2, \alpha_3$ be three cohomology classes in $H\mathcal{A}$ such that $\alpha_1 \alpha_2 = \alpha_2 \alpha_3 = 0$. Their length 3 Massey product is $\langle \alpha_1, \alpha_2, \alpha_3 \rangle = MP_3(a_{01}, a_{12}, a_{23})$, in which a_{01}, a_{12}, a_{23} are arbitrary cocycle representatives of $\alpha_1, \alpha_2, \alpha_3$.

Now we consider the general case. Let (a_1, \dots, a_n) be an n -tuple of cocycles. We say that a collection of cochains (a_{ij}) , $0 \leq i < j \leq n$, $(i, j) \neq (0, n)$, is an MP_n -defining system for (a_1, \dots, a_n) if it satisfies the following conditions:

- (1) $a_{i-1,i} = a_i$ for $1 \leq i \leq n$.
- (2) $da_{ij} = \sum_{i < r < j} \bar{a}_{ir} a_{rj}$ for $0 \leq i < j \leq n$ and $1 < j - i < n$.
- (3) $\deg(a_{ir} a_{rj}) = 1 + \deg(a_{ij})$ for all $i < r < j$.

Lemma 3.1. *Property 2 of an MP_n -defining system is consistent with $d^2 = 0$. Moreover, the cochain*

$$a_{0n} = \sum_{0 < r < n} \bar{a}_{0r} a_{rn} \quad (3.2)$$

is a cocycle.

Proof. We check the first claim by induction. It is straightforward to see that

$$d \left(\sum_{i < r < j} \bar{a}_{ir} a_{rj} \right) = \sum_{i < r < j} \sum_{i < s < r} (-1)^{\deg(a_{ir}) + \deg(a_{is})} a_{is} a_{sr} a_{rj} - \sum_{i < r < j} \sum_{r < s < j} (-1)^{1 + \deg(a_{rs})} a_{ir} a_{rs} a_{sj},$$

and consideration of the signs $(-1)^{\deg(a_{ir}) + \deg(a_{is})}$ and $(-1)^{1 + \deg(a_{rs})}$ using Property 3 above allows us to conclude that the right hand side of this equation is zero. The computations needed to check that the cochain a_{0n} given by (3.2) is a cocycle are similar. \square

The length n Massey product of the cocycles a_i , $1 \leq i \leq n$, is the set $MP_n(a_1, \dots, a_n)$ of cohomology classes of the cocycles a_{0n} associated to all possible MP_n -defining systems for (a_1, \dots, a_n) . As in the length 3 case we have the following crucial observation:

Proposition 3.2. *The length n Massey product of the cocycles a_i , $1 \leq i \leq n$, depends only on the cohomology classes of these cocycles.*

Proof. Let us fix t with $1 \leq t \leq n$ and let b be a cochain with $\deg(b) + 1 = \deg(a_t)$. As explained after Proposition 3.1, it is enough to prove that

$$MP_n(a_1, \dots, a_t, \dots, a_n) = MP_n(a_1, \dots, a_t + db, \dots, a_n)$$

and, by symmetry, it is enough to check that $MP_n(a_1, \dots, a_t, \dots, a_n) \subseteq MP_n(a_1, \dots, a_t + db, \dots, a_n)$. Matters being so, let x be a cohomology class in $MP_n(a_1, \dots, a_t, \dots, a_n)$. Then, there exists an MP_n -defining system (a_{ij}) for $(a_1, \dots, a_t, \dots, a_n)$, $0 \leq i < j \leq n$ and $(i, j) \neq (0, n)$, such that x is the cohomology class of the cocycle $a_{0n} = \sum_{0 < r < n} \bar{a}_{0r} a_{rn}$. We can exhibit an MP_n -defining system (a'_{ij}) for $(a_1, \dots, a_t + db, \dots, a_n)$ such that the corresponding cocycle a'_{0n} is cohomologous to the original

cocycle a_{0n} . Indeed, we set:

$$\begin{aligned} a'_{ij} &= a_{ij} \quad \text{if } i \neq t-1 \quad \text{and } j \neq t, \\ a'_{t-1,t} &= a_{t-1,t} + db = a_t + db, \\ a'_{it} &= a_{it} - a_{i,t-1}b \quad \text{for } i < t-1, \\ a'_{t-1,j} &= a_{t-1,j} - \bar{b}a_{tj} \quad \text{for } j > t. \end{aligned}$$

It is long, but straightforward, to check that (a'_{ij}) is in fact an MP_n -defining system for $(a_1, \dots, a_t + db, \dots, a_n)$. Now we consider the cocycle a'_{0n} . Again, a rather simple calculation yields

$$\begin{aligned} a'_{0n} &= a_{0n} + (-1)^{\deg(b)} d(ba_{1n}) \quad \text{if } t = 1, \\ a'_{0n} &= a_{0n} \quad \text{if } 1 < t < n, \\ a'_{0n} &= a_{0n} - d(a_{0,n-1}b) \quad \text{if } t = n. \end{aligned}$$

Thus, the cohomology class of a'_{0n} is also the class x we started with, and we conclude that $x \in MP_n(a_1, \dots, a_t + db, \dots, a_n)$. \square

The proof of Proposition 3.2 above is modelled after Kraines' work [19]. It allows us to define the length n Massey product on cohomology classes:

Definition 3.2. Let $\alpha_1, \dots, \alpha_n$ be n cohomology classes in $H\mathcal{A}$, let a_i be a cocycle representative of α_i , $1 \leq i \leq n$, and assume that there exists an MP_n -defining system (a_{ij}) for (a_1, \dots, a_n) . Then, the length n Massey product of $\alpha_1, \dots, \alpha_n$ is $\langle \alpha_1, \dots, \alpha_n \rangle = MP_n(a_1, \dots, a_n)$.

Remark 3.1.

- (1) Let us assume that the cohomology class α_i belongs to H^{s_i} , $1 \leq i \leq n$, and that the Massey product $\langle \alpha_1, \dots, \alpha_n \rangle$ exists, so that there is an MP_n -defining system (a_{ij}) such that the cohomology class of the cocycle $a_{0n} = \sum_{0 < r < n} \bar{a}_{0r} a_{rn}$ belongs to $\langle \alpha_1, \dots, \alpha_n \rangle$. Conditions 1–3 satisfied by (a_{ij}) imply that for each $0 < r < n-1$,

$$\deg(a_{0r} a_{rn}) = s_1 + \dots + s_n - n + 2,$$

and therefore we conclude that $\langle \alpha_1, \dots, \alpha_n \rangle \subseteq H^{s_1 + \dots + s_n - n + 2}$.

- (2) We remark that, as defined, the length n Massey product is a *partial* operation, not defined on arbitrary n -tuples of cohomology classes. A necessary and sufficient condition for the product $\langle \alpha_1, \dots, \alpha_n \rangle$ to exist is that the length $(n-1)$ Massey products $\langle \alpha_1, \dots, \alpha_{n-1} \rangle$ and $\langle \alpha_2, \dots, \alpha_n \rangle$ vanish simultaneously, see [19] and [24] for further information.

The behavior of Massey products under differential algebra morphisms, discussed for instance in [19] and [21], is crucial for us:

Proposition 3.3. Let (\mathcal{A}, d) and (\mathcal{B}, d') be differential graded algebras. Massey products are natural with respect to differential algebra morphisms, that is, if $f : \mathcal{A} \rightarrow \mathcal{B}$ is a differential algebra morphism and if the Massey product $\langle \alpha_1, \dots, \alpha_n \rangle$ exists, so does $\langle f_*\alpha_1, \dots, f_*\alpha_n \rangle$, and

$$f_* \langle \alpha_1, \dots, \alpha_n \rangle \subseteq \langle f_*\alpha_1, \dots, f_*\alpha_n \rangle. \quad (3.3)$$

In particular, if f is a quasi-isomorphism, then (3.3) is an equality.

Finally, it remains the issue of *computing* Massey products. The following result connecting Massey products with A_∞ -structures (see [20], Theorem 3.1) tells us how to proceed:

Theorem 3.1. *Let (\mathcal{A}, d) be a differential graded algebra. Up to a sign, the higher multiplications on the minimal model $H\mathcal{A}$ of \mathcal{A} give Massey products: for any $n \geq 3$, if $\alpha_1, \dots, \alpha_n \in H\mathcal{A}$ are such that $\langle \alpha_1, \dots, \alpha_n \rangle$ is defined, then*

$$(-1)^b m_n(\alpha_1 \otimes \dots \otimes \alpha_n) \in \langle \alpha_1, \dots, \alpha_n \rangle ,$$

where $b = 1 + |\alpha_{n-1}| + |\alpha_{n-3}| + |\alpha_{n-5}| + \dots$.

We note that this result does not say anything about the *non-triviality* of a given Massey product. It may well be that (notation as in Theorem 3.1) $m_n(\alpha_1 \otimes \dots \otimes \alpha_n) \neq 0$ but the cohomology classes in $\langle \alpha_1, \dots, \alpha_n \rangle$ are trivial. An example appears in [5].

3.2. Generalized Massey products

We generalize the constructions of the previous subsection following Babenko and Taimanov [2]. If (\mathcal{A}, d) is a differential graded algebra over a field k , we let $M(\mathcal{A})$ be the set of all upper triangular infinite matrices with entries in \mathcal{A} such that only finitely many entries are different from zero. Addition and multiplication on $M(\mathcal{A})$ are defined in a natural way. In particular, if $A = (a_{ij})_{i,j \geq 1}$ and $B = (b_{ij})_{i,j \geq 1}$ belong to $M(\mathcal{A})$, then $AB = (\sum_{k \geq 1} a_{ik} b_{kj})_{i,j \geq 1}$. We also extend the “bar” notation from the previous section, $\bar{a} = (-1)^{1+\deg(a)} a$ for *homogeneous* elements of \mathcal{A} , to a linear map on \mathcal{A} by setting $\lambda \bar{a} + \bar{b} = \lambda \bar{a} + \bar{b}$ for $\lambda \in k$ and a, b homogeneous. This linear map extends in an obvious way to a linear map on $M(\mathcal{A})$.

We also extend the differential on \mathcal{A} to a differential on $M(\mathcal{A})$ by setting $dA = (da_{ij})_{i,j \geq 1}$ for $A \in M(\mathcal{A})$. Obviously the extended map d is linear and it satisfies $d^2 = 0$. Also, we readily check that

$$d(AB) = (dA)B + \tilde{A}dB ,$$

in which $\tilde{A} = ((-1)^{\deg(a_{ij})} a_{ij})$, so that $\tilde{A} = -(\bar{a}_{ij}) = -\bar{A}$.

Now, let $(a)_{ij}$ be the matrix in which the (i, j) -entry is equal to a and all the other entries are zero. Given $A \in M(\mathcal{A})$ we define $\text{Ker}(A)$ as the \mathcal{A} -module spanned by the matrices $(1)_{ij}$ such that $A \cdot (1)_{ij} = (1)_{ij} \cdot A = 0$. We note that if $B \in \text{Ker}(A)$, then $\bar{B} \in \text{Ker}(A)$ as well.

Definition 3.3. A matrix $A \in M(\mathcal{A})$ is called a formal connection on \mathcal{A} . We say that A is flat if it satisfies the Maurer-Cartan equation

$$dA - \bar{A}A \equiv 0 \quad \text{mod} \quad \text{Ker}(A) . \quad (3.4)$$

The matrix $\mu(A) = dA - \bar{A}A$ is called the curvature of A .

The existence of generalized Massey products is a consequence of the following result:

Proposition 3.4. *Let A be a flat formal connection on \mathcal{A} . Then, the matrix $\mu(A)$ is closed.*

Proof. We compute:

$$d(dA - \bar{A}A) = -\{(d\bar{A})A + \widetilde{\bar{A}}dA\} = -(\mu(\bar{A}) + \bar{\bar{A}}\bar{A})A + A(\mu(A) + \bar{A}A) = -\mu(\bar{A})A + A\mu(A) .$$

On the other hand,

$$-\mu(\bar{A}) = \overline{dA} - \overline{\bar{A}A} = \overline{dA} - \overline{\bar{A}}\bar{A} = \overline{\mu(A)} ,$$

and so

$$d\mu(A) = \overline{\mu(A)}A + A\mu(A) .$$

Now, since A is flat, $\mu(A)$ and $\overline{\mu(A)}$ belong to $\text{Ker}(A)$, and therefore $d\mu(A) = 0$. \square

It follows from Proposition 3.4 that the entries of the curvature matrix $\mu(A) = (\mu_{ij})_{i,j \geq 1}$ of a flat formal connection A determine a matrix of cohomology classes $([\mu_{ij}])_{i,j \geq 1}$. After Babenko and Taimanov [2] we make the following definition:

Definition 3.4. Let A be a flat formal connection on \mathcal{A} and let $\mu(A) = (\mu_{ij})_{i,j \geq 1}$ be the corresponding curvature matrix. The generalized Massey product corresponding to A is the matrix of cohomology classes $[\mu(A)] = ([\mu_{ij}])_{i,j \geq 1}$.

We note that as they stand, generalized Massey products are defined on \mathcal{A} , and not on the cohomology of \mathcal{A} . However, Babenko and Taimanov generalize Proposition 3.2 on classical Massey products in a very interesting fashion. First of all, we make the following definition.

Definition 3.5. Let $A = (a_{ij})$ be a flat formal connection on \mathcal{A} . The initial data of the Maurer-Cartan equation (3.4) is the set of all cohomology classes of entries a_{ij} of A which are cocycles of A .

It can be checked, see [2, Prop. 1], that the matrix of cohomology classes $([\mu_{ij}])_{i,j \geq 1}$ of $\mu(A)$, in which A is a flat connection, depends only on the initial data of the Maurer-Cartan equation $\mu(A) \equiv 0 \pmod{\text{Ker}(A)}$, induced by A . Thus, this product can be considered as defined in the cohomology of \mathcal{A} . In [2] is shown that it is a true generalization of the classical Massey product considered in the previous section, and also of the matrix Massey products introduced by May in [21].

The following proposition, see [2, Prop. 2], generalizes Proposition 3.3:

Proposition 3.5. Let $f : \mathcal{A} \rightarrow \mathcal{B}$ be a morphism of differential graded algebras. We induce a map $\hat{f} : M(\mathcal{A}) \rightarrow M(\mathcal{B})$ via $\hat{f}((a_{ij})_{i,j \geq 1}) = (f(a_{ij}))_{i,j \geq 1}$. This map takes flat connections in \mathcal{A} to flat connections in \mathcal{B} and therefore we obtain a map on generalized Massey products

$$f^*([\mu(A)]) = [\mu(\hat{f}(A))] .$$

Moreover, if f is a quasi-isomorphism, then f^* is one-to-one.

4. Legendrian knots and the Chekanov algebra

4.1. Contact structures

Consider a $(2n + 1)$ -dimensional manifold M together with a differential 1-form α which satisfies the condition

$$\alpha \wedge (d\alpha)^n \neq 0. \quad (4.1)$$

Such a 1-form is called a *contact form*, and the pair (M, α) is called a *contact manifold*. If we set $\eta = \ker(\alpha)$, then η is a maximally non-integrable distribution on M and we recover the definition of contact manifolds appearing in [11].

Our main example of a contact manifold is $M = \mathbb{R}^3$ with $\alpha = dz + xdy$. It is easy to see that in this case $\eta = \ker(\alpha) = \langle \partial_x, \partial_y - x\partial_z \rangle$. Figure 1 below (taken from [11]) shows the distribution η .

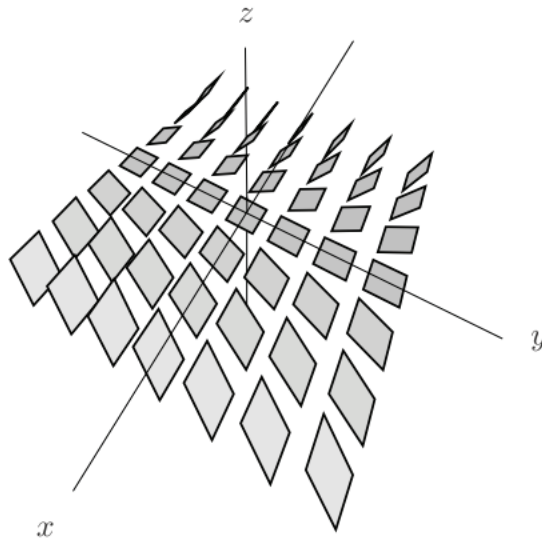


Fig. 1. The contact distribution $\langle \partial_x, \partial_y - x\partial_z \rangle$ on \mathbb{R}^3 .

Remark 4.1. It has been observed several times (see for instance [8] or the more recent [3]) that there exists a relationship between Lorentzian and contact geometry. Indeed, it is not difficult to define a Lorentzian metric on a contact manifold (M, α) so that the contact distribution is spacelike and the timelike direction is determined by the *Reeb vector field* of α (i.e., the vector field R determined by the conditions $R \in \ker(d\alpha)$ and $\alpha(R) = 1$, see [11]). In the case $(M, \alpha) = (\mathbb{R}^3, dz + xdy)$, we simply set

$$ds^2 = dx^2 + dy^2 - (dz + xdy)^2.$$

Elementary calculations yield that indeed ∂_x and $\partial_y - x\partial_z$ are spacelike and that the Reeb vector field $R = \partial_z$ is timelike. Now, an obvious question is whether this relation between contact and Lorentzian geometry may be useful for physics. We are not overly optimistic: we notice that the above metric satisfies the Einstein equations $(R_{ab} - (1/2)Rg_{ab}) + \Lambda g_{ab} = 8\pi T_{ab}$ with $\Lambda = -3/4$

and non-zero components of T_{ab} given by $T_{11} = T_{22} = -1/8\pi$. However, the very definition of contact structures implies that this spacetime model does not admit Cauchy surfaces, even locally! Thus, in this naturally constructed model is not possible to set up sensible evolution problems.

4.2. Legendrian knots

Definition 4.1. A *Legendrian knot* in a three-dimensional contact manifold (M, α) , is an embedded circle $L \subset M$ which is always tangent to the distribution $\eta = \ker(\alpha)$. In other words, a Legendrian knot is a compact one-dimensional integral submanifold of η .

Legendrian knots always exist, see [11, Theorem 3.3.1]: given an arbitrary knot $f : S^1 \rightarrow M$, there exists a Legendrian knot in M which is isotopic (in the topological sense) to f .

We also specify when two Legendrian knots K_0 and K_1 are equivalent: we say that they are *Legendrian isotopic* if there is a Legendrian isotopy between them, this is, there exists a smooth family of Legendrian knots L_t , $t \in [0, 1]$, with $L_i = K_i$, para $i = 0, 1$.

Hereafter we consider only the contact manifold (\mathbb{R}^3, α) , in which $\alpha = dz + xdy$, and η will always represent the maximally non-integrable distribution $\ker(\alpha)$ on \mathbb{R}^3 .

Definition 4.2. Consider a Legendrian knot K in (\mathbb{R}^3, α) given by $\gamma(s) = (x(s), y(s), z(s))$, $s \in S^1$.

- (1) The front projection of $\gamma(s) = (x(s), y(s), z(s))$ in (\mathbb{R}^3, η) , is the curve $\gamma_F(s) = (x(s), z(s))$ in the xz -plane. We denote this projection by $\Pi_F(K)$.
- (2) The Lagrangian projection of $\gamma(s) = (x(s), y(s), z(s))$ in (\mathbb{R}^3, α) is the curve $\gamma_L(s) = (x(s), y(s))$ in the xy -plane. We denote this projection by $\Pi_L(K)$.

Generically, the projection $\Pi_L(K)$ is an immersed curve with only double points. Explicit examples of projections appear, for example, in the treatise [11].

4.3. The Chekanov algebra

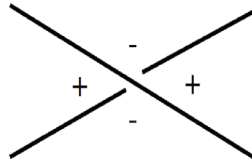
Chekanov homology is the homology of a particular differential algebra (\mathcal{A}, d) constructed using the crossings of the Lagrangian projection $\Pi_L(K)$ of a given Legendrian knot K . We denote by $e = \{a_1, \dots, a_n\}$ the double points in the Lagrangian projection $\Pi_L(K)$, and we define \mathcal{A} as the unitary tensor algebra over \mathbb{Z}_2 generated by the set e . The unit corresponds to the empty word.

First of all we describe the grading of \mathcal{A} . Suppose that we are given an immersion $\tilde{\gamma} : S^1 \rightarrow \mathbb{R}^2$; we define the *winding number* of $\tilde{\gamma}$ as $wind(\tilde{\gamma}) = \deg(d\tilde{\gamma}/ds)$. Then we can check, see [11], that $rot(K) = wind(\Pi_L(K))$ is a Legendrian invariant of K . We define a function from the set e to $\mathbb{Z}/2rot(K)$: if $a \in e$, we take a regular path $\gamma : [0, \pi] \rightarrow \Pi_L(K)$ from a to itself, starting from the upper strand (the one with bigger z -coordinate) to the lower strand (the one with smaller z -coordinate), and we define a curve $\Gamma : \mathbb{R}/2\pi\mathbb{Z} \rightarrow \mathbb{RP}^1$ by taking the projection of $d\gamma/ds(s)$, $s \in [0, \pi]$ and then clockwise rotating from $[d\gamma/ds(\pi)]$ to $[d\gamma/ds(0)]$ for $s \in [\pi, 2\pi]$. We define

$$\deg(a) = \deg(\Gamma) \mod 2rot(K),$$

and we extend to a full $\mathbb{Z}/2rot(K)$ -grading of \mathcal{A} via $\deg(a \otimes b) = \deg(a) + \deg(b)$. Chekanov explains in [4] why the grading takes place in $\mathbb{Z}/2rot(K)$ and not in \mathbb{Z} .

Now we define the differential. Let us fix a double point a in $\Pi_L(K)$. Then, there exist two lines L_1 and L_2 which locally divide the plane into four quadrants. We equip each quadrant with a sign in the following way:



Now, given a and $b_1, \dots, b_n \in e$, we define $\Delta(a; b_1, \dots, b_n)$ as the set of immersed polygons in \mathbb{R}^2 with edges in $\Pi_L(K)$ and vertices in a, b_1, \dots, b_n , and which cover an “up” quadrant near b and “down” quadrants near a_1, \dots, a_n . We denote by $\#\Delta(a; b_1, \dots, b_n)$ the cardinality of $\Delta(a; b_1, \dots, b_n) \bmod 2$. Then we define the differential d at a by

$$d(a) = \sum_{n \in \mathbb{N}} \sum_{(b_1, \dots, b_n) \in e^n} \#\Delta(a; b_1, \dots, b_n) b_1 \cdots b_n. \quad (4.2)$$

We extend d to all \mathcal{A} by linearity and the graded Leibnitz rule $d(ab) = d(a)b + (-1)^{\deg(a)}ad(b)$. The fundamental theorem proven by Chekanov in [4] is:

Theorem 4.1. *The map d is a differential on the $\mathbb{Z}/2\text{rot}(K)$ -graded algebra \mathcal{A} of degree -1 . Moreover, the homology of (\mathcal{A}, d) is unchanged under Legendrian isotopy: the homology rings of isotopic Legendrian knots are isomorphic as graded rings.*

Now, for the applications we have in mind, we need to change the $\mathbb{Z}/2\text{rot}(K)$ -grading of the Chekanov algebra (\mathcal{A}, d) for a full \mathbb{Z} -grading. We can, in fact, abelianize (\mathcal{A}, d) following [10]:

Let $\{a_1, \dots, a_n\}$ be the set of generators of the Chekanov algebra \mathcal{A} of a Legendrian knot K . Instead of \mathcal{A} we now consider the free associative algebra $\mathcal{A}_{\mathbb{Q}}$ generated by $\{a_1, \dots, a_n\}$ over $\mathbb{Q}[t, t^{-1}]$, in which t is a formal parameter. It is shown in [10] that this algebra can be equipped with a \mathbb{Z} -grading (which reduces to Chekanov’s grading if we set formally $t = 1$, see [10, Section 3.1]) and a differential ∂ which is defined in a way similar to (4.2) (see [10, Theorem 3.7]) and satisfies the conditions $\partial(\mathbb{Q}[t, t^{-1}]) = 0$ and

$$\partial(vw) = (\partial v)w + (-1)^{\deg(v)}v(\partial w). \quad (4.3)$$

The differential algebra $(\mathcal{A}_{\mathbb{Q}}, \partial)$ is made into a $(\mathbb{Z}$ -graded) commutative algebra by setting

$$wv = (-1)^{\deg(v)\deg(w)}vw.$$

It is proven in [10, Theorem 3.14] that the homology of $(\mathcal{A}_{\mathbb{Q}}, \partial)$ is an invariant of the Legendrian isotopy class of K .

4.4. The A_{∞} -algebra of a Legendrian knot and Legendrian invariants

Let $(\mathcal{A}_{\mathbb{Q}}, \partial)$ with $\mathcal{A}_{\mathbb{Q}} = \bigoplus_{i \in \mathbb{Z}} A_i$ be the abelianized Chekanov algebra of a Legendrian knot K defined above. We set $\partial_i = \partial|_{A_i} : A_i \rightarrow A_{i-1}$ and $C^{-i} = A_i$. Then, as is standard, $\mathcal{C}(K) = \bigoplus_{i \in \mathbb{Z}} C^i$ is a \mathbb{Z} -graded differential algebra with a degree 1 differential which we continue denoting by ∂ . We will call $(\mathcal{C}(K), \partial)$ the *dual Chekanov algebra* of K . The cohomology of $\mathcal{C}(K)$ will be called the *abelianized Chekanov cohomology* of K ; we will denote it by $CH_A(K)$. We apply Merkulov’s result recalled in Section 3 and we obtain:

Theorem 4.2. *Let K be a Legendrian knot. There exists an A_{∞} -algebra structure on the abelianized Chekanov cohomology $CH_A(K)$ of K : there exist higher multiplications m_n , where $n \geq 2$, on $CH_A(K)$*

such that $(CH_A(K), 0, m_2, m_3, \dots)$ satisfy the higher order associative identities (2.1). Moreover, there exists a quasi-isomorphism of A_∞ -algebras between $CH_A(K)$ and the dual Chekanov algebra $\mathcal{C}(K)$.

Let us consider two Legendrian knots K_1 and K_2 connected by a Legendrian isotopy. Then, the cohomology rings of their corresponding dual Chekanov algebras, $\mathcal{C}(K_1)$ and $\mathcal{C}(K_2)$ respectively, are isomorphic. The transitivity of quasi-isomorphisms in the context of A_∞ -algebras (see [30]) implies that the minimal models of $\mathcal{C}(K_1)$ and $\mathcal{C}(K_2)$, $CH_A(K_1)$ and $CH_A(K_2)$ respectively, are quasi-isomorphic. Now, $CH_A(K_2)$ is also a minimal model for $CH_A(K_1)$, again because of the transitivity of quasi-isomorphisms for A_∞ -algebras. It follows from the analysis carried out by Kajiura (see [13], Corollaries 5.8 and 5.10) that $CH_A(K_1)$ and $CH_A(K_2)$ are actually A_∞ -isomorphic. We have:

Corollary 4.1. *The minimal model $(CH_A(K), 0, m_2, m_3, \dots)$ of the dual Chekanov algebra $(\mathcal{C}(K), \partial)$ of a Legendrian knot K is invariant under Legendrian isotopy.*

Now, since the A_∞ -algebras $CH_A(K_1)$ and $CH_A(K_2)$ are minimal, they are not only isomorphic as A_∞ -algebras but also they are isomorphic as associative rings. The naturality of classical and generalized Massey products (Propositions 3.3 and 3.5) yields the following result:

Corollary 4.2. *The classical and generalized Massey products of the dual Chekanov algebra $(\mathcal{C}(K), \partial)$ of a Legendrian knot K are invariant under Legendrian isotopy. Moreover, in the classical case, we can find representatives for these invariants by using the A_∞ -algebra structure of $CH_A(K)$, as explained in Theorem 3.1.*

Massey product invariants are useful. Civan and his coworkers prove in [5] the existence of an A_∞ -algebra structure on a linearized complex $LC(K)$ built from the Chekanov algebra, see [4], and they show that there exists an infinite family of knots that are distinguishable from their Legendrian mirrors by using classical Massey products on the cohomology of $LC(K)$.

5. Chekanov algebra and differential equations

In this section we prove that the invariants constructed in the previous section can be provided with a dynamical interpretation, in the sense that they can be considered as determining solutions to a differential equation of Maurer-Cartan type, and we also construct nonlinear evolution equations starting from the (dual) Chekanov algebra. As we stated in Section 1, we believe these examples of differential equations are interesting because they are instances of *noncommutative* equations, such as the ones investigated in [25], which arise quite naturally from a non-trivial geometric context.

5.1. Maurer-Cartan equations

We follow [12]. Let \mathcal{A} be a \mathbb{Z} -graded associative algebra over a field \mathbb{K} . We consider the *tensor coalgebra*

$$T(\mathcal{A}) = \bigoplus_{n \geq 1} \mathcal{A}^{\otimes n}$$

equipped with the coassociative coproduct Δ uniquely determined by

$$\Delta(x) = \sum_{i=1}^{n-1} (a_1 \otimes \cdots \otimes a_i) \otimes (a_{i+1} \otimes \cdots \otimes a_n)$$

in which $x = a_1 \otimes \cdots \otimes a_n \in T(\mathcal{A})$. This coalgebra admits a natural $\mathbb{Z} \times \mathbb{N}$ bi-graduation,

$$bideg(a_1 \otimes \cdots \otimes a_n) = \left(\sum deg(a_i), n \right),$$

if $a_i \in \mathcal{A}$ are homogeneous elements of \mathcal{A} . This bi-graduation induces a bi-graduation on $Hom(T(\mathcal{A}), \mathcal{A})$ as follows: $bideg(\varphi) = (i, j)$ if and only if φ is a graded \mathbb{K} -linear map of degree i and $\varphi : \mathcal{A}^{\otimes j+1} \rightarrow \mathcal{A}$. We write $C^{i,j}(\mathcal{A}) = Hom^i(\mathcal{A}^{\otimes j+1}, \mathcal{A})$ and we define

$$L = \bigoplus_{\substack{i \in \mathbb{Z} \\ j \geq 0}} C^{i,j}(\mathcal{A}). \quad (5.1)$$

A crucial observation, see [12], is that L is a bi-graded differential Lie algebra. Its Lie bracket and differential are defined as follows. For $\varphi \in C^{i,j}(\mathcal{A})$ and $\phi \in C^{s,t}(\mathcal{A})$, we consider the map $\varphi \diamond \phi \in C^{i+s, j+t}(\mathcal{A})$ given by

$$\varphi \diamond \phi(a_1, a_2, \dots, a_{j+t+1}) = \sum_{k \leq j} (-1)^\varepsilon \varphi(a_1, \dots, a_k \phi(a_{k+1}, \dots, a_{k+t+1}), a_{(k+1)+t+1}, \dots, a_{j+t+1}),$$

where $\varepsilon = s \sum_{p=1}^k deg(a_p) + kt$, and we are identifying homogeneous elements of the form $a_1 \otimes \cdots \otimes a_r$ with (a_1, \dots, a_r) . We then define the bigraded Lie bracket $[\cdot, \cdot] : L \otimes L \rightarrow L$ as

$$[\varphi, \phi] = \varphi \diamond \phi - (-1)^{is+jt} \phi \diamond \varphi,$$

in which $\varphi \in C^{i,j}(\mathcal{A})$ and $\phi \in C^{s,t}(\mathcal{A})$, and the bigraded differential δ —of bidegree $(0, 1)$ — as $\delta = [m, \cdot]$, in which $m \in C^{0,1}(\mathcal{A}) = Hom^0(\mathcal{A} \otimes \mathcal{A}, \mathcal{A})$ denotes the multiplication operator of \mathcal{A} .

Now let us consider an A_∞ -algebra $(\mathcal{A}, m_1, m_2, \dots)$ with differential $m_1 = 0$, so that (\mathcal{A}, m_2) is an associative graded algebra. As in Section 3, we have $\mathcal{A} = \bigoplus_{i \in \mathbb{Z}} A^i$, $m_n : \mathcal{A}^{\otimes n} \rightarrow \mathcal{A}$, $n \geq 1$, and $deg(m_n) = 2 - n$. We set $\tilde{m}_n = m_{n+2}$ and $\tilde{A}^n = A^{-n}$. Then, $\tilde{m}_n : \mathcal{A}^{\otimes n+2} \rightarrow \mathcal{A}$ and $deg(\tilde{m}_n) = n$, so that $\tilde{m}_n \in Hom^n(\mathcal{A}^{\otimes(n+1)+1}, \mathcal{A})$ and $bideg(\tilde{m}_n) = (n, n+1)$. In terms of \mathcal{A} and operations \tilde{m}_n , identities (2.1) now read

$$\sum_{\substack{l+j=n \\ 0 \leq i \leq l+1 \\ l, j \geq 0}} (-1)^k \tilde{m}_l(a_1, \dots, a_i, \tilde{m}_j(a_{i+1}, \dots, a_{i+j+2}), a_{i+j+3}, \dots, a_{n+3}) = 0, \quad (5.2)$$

for $n \geq 0$, in which $0 \leq i \leq n+3$, and $k = j(deg(a_1) + \cdots + deg(a_i)) + i + j(l-i-1)$.

We apply the foregoing considerations to the minimal model $(CH_A(K), 0, m_2, m_3, \dots)$ of a Legendrian knot K constructed in Section 4.4. With notation as above, Theorem 2.2 of [12] yields:

Theorem 5.1. *Let F be the bi-graded cocommutative coalgebra determined by the conditions*

$$F = span\{f_1, f_2, \dots\}; \quad bideg(f_i) = (i, i); \quad \Delta : F \rightarrow F \otimes F; \quad \Delta(f_n) = \sum_{\substack{i+j=n \\ i, j \geq 1}} (-1)^{ij} f_i \otimes f_j.$$

Consider the \mathbb{Z} -graded vector space $\widetilde{CH_A(K)}$ equipped with multiplication operators $\{\tilde{m}_n\}_{n \geq 0}$ and construct the bi-graded differential Lie algebra L as in (5.1). Define also a linear map $\alpha : F \rightarrow L$

via $f_i \mapsto \tilde{m}_i$, $\text{bideg}(\alpha) = (0, 1)$. This map satisfies the Maurer-Cartan equation

$$\delta \circ \alpha + \frac{1}{2} \mu \circ (\alpha \oplus \alpha) \circ \Delta = 0, \quad (5.3)$$

in which μ indicates the Lie product of L .

We note that this theorem states that the higher multiplications of the *Legendrian invariant* minimal model $CH_A(K)$ provide solutions to a Maurer-Cartan equation. Moreover, since we observed in Theorem 3.1 that the functions \tilde{m}_n belong to the higher Massey products of the dual Chekanov algebra $\mathcal{C}(K)$, we can also interpret Theorem 5.1 as providing a dynamic interpretation for our higher order Legendrian invariants.

Remark 5.1. Maurer-Cartan equations, this time posed on A_∞ -algebras, have been considered by Kajiura in [14, 15]. He has made two important observations. First, he has pointed out that the field equations of motion of string field theory are the Maurer-Cartan equations on an A_∞ -algebra determined by the string theory in question. Second, he has remarked that the construction of a minimal model for an A_∞ -algebra \mathcal{A} (a generalization of Theorem 2.1, see [13, 22] and [14, 15]) amounts to constructing a solution to a Maurer-Cartan equation on \mathcal{A} .

5.2. Integrable equations

We begin with the abelianized differential graded algebra $(\mathcal{A}_Q, \partial)$ and we consider the dual Chekanov algebra $(\mathcal{C}(K), \partial)$ of a Legendrian knot K .

Now we linearize. We introduce a word-length filtration on $\mathcal{C}(K)$ as follows: $\mathcal{C}(K)^n$ is the subalgebra of $\mathcal{C}(K)$ generated, as a vector space over $\mathbb{Q}[t, t^{-1}]$, by all words in $\mathcal{C}(K)$ of length at least n . The *linearization* $\mathcal{L}\mathcal{C}(K)$ of $\mathcal{C}(K)$ is the quotient space $\mathcal{C}(K)/\mathcal{C}(K)^2$, which we can consider as being embedded into $\mathcal{C}(K)$. For each generator a_i we set $\partial_1(a_i) = \pi \circ \partial(a_i)$, in which $\pi: \mathcal{C}(K) \rightarrow \mathcal{L}\mathcal{C}(K)$ is the standard projection, and we obtain a $\mathbb{Q}[t, t^{-1}]$ -linear map $\partial_1: \mathcal{L}\mathcal{C}(K) \rightarrow \mathcal{L}\mathcal{C}(K)$. Extending this map to $\mathcal{C}(K)$ so that the extension satisfies the graded Leibnitz rule (4.3), we obtain a *graded derivation* δ on $\mathcal{C}(K)$. We consider $\mathcal{C}(K)$ as a \mathbb{Q} -algebra equipped with the derivation δ . The (graded) commutative and associative algebra $\mathcal{C}(K)$ equipped with the graded derivation δ is our basic arena for setting differential equations.

We define a $\mathbb{Q}[t, t^{-1}]$ -algebra automorphism $S: \mathcal{C}(K) \rightarrow \mathcal{C}(K)$ by setting $S(a_i) = (-1)^{\deg(a_i)} a_i$ and extending linearly. Then, the graded Leibnitz rule for δ becomes $\delta(vw) = (\delta v)w + S(v)\delta w$, and clearly, the identity $\delta \circ S = S \circ \delta$ also holds. Following Demidov [6, 7] we consider the vector space ΨDO_S of twisted pseudo-differential operators given by

$$\Psi DO_S = \left\{ \sum_{\infty < i \leq n} f_i D^i : n \in \mathbb{Z} \text{ and } f_i \in \mathcal{C}(K) \right\}.$$

We equip ΨDO_S with a multiplication determined by the rule

$$D^n \cdot f = \sum_{k=0}^{\infty} \binom{n}{k} S^{n-k}(\delta^k f) D^{n-k}$$

for any $f \in \mathcal{C}(K)$ and $n \in \mathbb{Z}$. The vector space ΨDO_S then becomes an associative (but not commutative) algebra called the algebra of *twisted pseudo-differential operators of $\mathcal{C}(K)$* .

Remark 5.2. Other explicit instances of twisted algebras of formal pseudo-differential operators have already appeared in the literature. We refer the reader for example to [16].

Now we are ready to introduce differential equations. For $L \in \Psi DO_S$, we consider

$$\frac{dL}{dt_k} = \left[(L^k)_+, L \right], \quad (5.4)$$

in which $(\cdot)_+$ indicates projection into the subalgebra of ΨDO_S consisting of *differential* operators. This is our example of a non-commutative differential equation. Indeed, (5.4) gives rise to a (twisted) Kadomtsev-Petviashvili (KP) hierarchy of partial differential equations for the coefficients of the pseudo-differential operator L .

Our main observation is that Equation (5.4) can be solved explicitly in a formal setting. We follow the classical work [23] as retold in [6, 7] and [9].

Let us equip the algebra $\mathcal{C}(K)$ with a valuation v (valuations on rings are considered for example in [1]). We let I be the valuation ideal and $\pi : \mathcal{C}(K) \rightarrow \mathcal{C}(K)/I$ the canonical projection. We assume that $v \circ S = v$ and that $\delta I \subset I$, so that in particular $S(I) \subset I$ and the derivation δ and morphism S descend to the quotient ring $\mathcal{C}(K)/I$.

Definition 5.1. The space of formal pseudo-differential and differential operators of infinite order are, respectively, $\widehat{\Psi DO_S}$ and $\widehat{\mathcal{D}_S}$, in which

$$\widehat{\Psi DO_S} = \left\{ \sum_{\alpha \in \mathbb{Z}} a_\alpha D^\alpha : a_\alpha \in \mathcal{C}(K) \text{ and } \exists C \in \mathbb{R}^+, N \in \mathbb{Z}^+ \text{ so that } \pi(a_\alpha) > C\alpha - N \forall \alpha \gg 0 \right\}$$

and

$$\widehat{\mathcal{D}_S} = \left\{ P = \sum_{\alpha \in \mathbb{Z}} a_\alpha D^\alpha : P \in \widehat{\Psi DO_S} \text{ and } a_\alpha = 0 \text{ for } \alpha < 0 \right\}.$$

We also define the Volterra group (notation as in [6, 7])

$$V_{\mathcal{C}(K)} = 1 + \left\{ P = \sum_{\alpha \in \mathbb{Z}} a_\alpha D^\alpha \in \Psi DO_S : a_\alpha = 0 \text{ for } \alpha \geq 0 \right\}.$$

We have the fundamental result

Theorem 5.2. *The sets*

$$\widehat{\Psi DO_S}^\times = \{P \in \widehat{\Psi DO_S} : \pi(P) \in V_{\mathcal{C}(K)/I}\}$$

and

$$\widehat{\mathcal{D}_S}^\times = \{X \in \widehat{\mathcal{D}_S} : \pi(P) = 1\}$$

are groups: for each P in $\widehat{\Psi DO_S}^\times$ and each X in $\widehat{\mathcal{D}_S}^\times$ there exist unique inverses given by $P^{-1} = \sum_{n \geq 0} (1 - P)^n$ and $X^{-1} = \sum_{n \geq 0} (1 - X)^n$. Moreover, for any $P \in \widehat{\Psi DO_S}^\times$ there exist unique operators

$W \in V_{\mathcal{C}(K)}$ and $Y \in \widehat{\mathcal{D}}_S^\times$ such that $P = W^{-1}Y$. In other words, the group $\widehat{\Psi DO}_S^\times$ admits the global factorization

$$\widehat{\Psi DO}_S^\times = V_{\mathcal{C}(K)} \widehat{\mathcal{D}}_S^\times.$$

A proof of Theorem 5.2 appears in [7]. This result is a twisted version of a deep theorem due to Mulase, see [23], on an analog of the loop group Birkhoff factorization [26] for infinite-dimensional formal Lie groups of formal pseudo-differential operators. A recent detailed account of Mulase's result is in [9]. The importance of Theorem 5.2 for us is that it allows us to solve (5.4). Indeed, reasoning as in [9] we have:

Theorem 5.3. *Consider the system of equations*

$$\frac{dL}{dt_k} = \left[(L^k)_+, L \right] \quad (5.5)$$

with initial condition $L(0) = L_0 \in \Psi DO_S$, and let $Y(t_k) \in \widehat{\mathcal{D}}_S^\times$ and $S(t_k) \in V_{\mathcal{C}(K)}$ be the unique solution to the factorization problem

$$\exp(t_k L_0^k) = S^{-1}(t_k) Y(t_k).$$

Then, the unique solution to Equation (5.5) with $L(0) = L_0$ is

$$L(t_k) = Y L_0 Y^{-1}. \quad (5.6)$$

6. Conclusions

In this paper we have presented Legendrian invariants for Legendrian knots using classical and generalized Massey products as defined in [19, 21] and [2] respectively. Classical (in the sense of [19]) Massey products on *linearized* Chekanov homology have been used earlier in [5] to distinguish between some Legendrian knots and their mirror images. The observation that matrix and generalized Massey products, as in [2, 21], also yield Legendrian invariants seems to be new. We leave explicit applications for another publication. We have also consider the issue of a possible “physical interpretation” of the Massey product invariants. To examine this issue, we have used the following: equip the (co)homology $CH_A(K)$ of the dual Chekanov algebra $\mathcal{C}(K)$ of a Legendrian knot K with a minimal A_∞ -algebra structure following [13–15, 20, 22]; we have: (a) if $\alpha_1, \dots, \alpha_n$ are (co)homology classes in $CH_A(K)$ and m_n is a higher multiplication in this A_∞ -algebra, then (see [20]) $m_n(\alpha_1, \dots, \alpha_n)$ belongs to the classical Massey product $\langle \alpha_1, \dots, \alpha_n \rangle$, if this product is defined; and (b) the higher multiplication operations of a minimal A_∞ -algebra solve a Maurer-Cartan equation posed on a bi-graded differential Lie algebra (see [12]). Thus, classical Massey product invariants admit a “dynamical” interpretation, in the sense that representatives of them solve a “nonlinear field equation”. Most probably this interpretation extends to the case of generalized Massey products; we will report on this elsewhere. Finally, we have presented a natural system of differential equations (the system of equations for the coefficients of the formal pseudo-differential operator L appearing in Equation (5.4)) which arises from the Chekanov algebra of a Legendrian knot, and we have showed how to solve it in an algebraic setting following [9]. We note that in contradistinction with the classical KP hierarchy case, see [9] and references therein, we do not have, as yet, a hamiltonian interpretation for (5.4). Also, as A. Eslami-Rad has pointed out to us, it is natural to believe that

(5.4) may encode geometric information on Legendrian knots, since the Chekanov algebra homology is a combinatorial translation of contact homology, see [10]. We hope that the explicit solution (5.6) will allow us to extract (at least some of) it.

Acknowledgments

Both authors have been partially supported by FONDECYT grant #1111042. L.M. also acknowledges financial support from MECESUP. Important remarks and observations by Anahita Eslami-Rad are also gratefully acknowledged. In particular, it was her who stressed the importance of solving Equation (5.4).

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