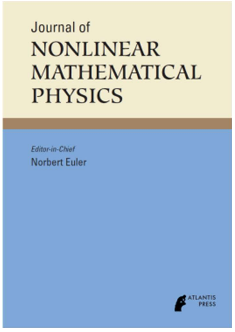

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Maohua Li, Jipeng Cheng, Jingsong He

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The compatibility of additional symmetry and gauge transformations for the constrained discrete Kadomtsev-Petviashvili hierarchy

Maohua Li

Department of Mathematics, Ningbo University,
Ningbo, 315211 Zhejiang, China
Institute for Theoretical Physics, KU Leuven,
3001 Leuven, Belgium
limaohua@nbu.edu.cn

Jipeng Cheng

Department of Mathematics, China University of Mining and Technology,
Xuzhou, 221116 Jiangsu, China
chengjp@cumt.edu.cn

Jingsong He*

Department of Mathematics, Ningbo University,
Ningbo, 315211 Zhejiang, China
hejingsong@nbu.edu.cn

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In this paper, the compatibility between the gauge transformations and the additional symmetry of the constrained discrete Kadomtsev-Petviashvili hierarchy is given, which preserves the form of the additional symmetry of the cdKP hierarchy, up to shifting of the corresponding additional flows by ordinary time flows.

Keywords: constrained discrete KP hierarchy; gauge transformation; additional symmetry.

2000 Mathematics Subject Classification: 35Q53, 37K10, 37K40

1. Introduction

The discrete Kadomtsev-Petviashvili (dKP) hierarchy [a] [15, 17, 23, 25, 29] is an attractive research object in the field of the discrete integrable systems. The dKP hierarchy is defined by means of the difference derivative Δ instead of the usual derivative ∂ with respect of x in a classical system [10, 11], and the continuous spatial variable is replaced by a discrete variable n. By using a non-uniform shift of space variable, the τ-function of KP hierarchy implies a special kind of τ-function for the dKP hierarchy [17]. With the symmetry constraint or symmetry reduction technique, which was used in the continuous KP hierarchy [6, 7, 24], the constrained discrete KP (cdKP) hierarchy is truncated dKP hierarchy by adding a constrained operator form (see (2.14)) on the Lax operator L of the dKP hierarchy [27]. And the discrete nonlinear Schrödinger equation and other equations can be derived from it.

*Corresponding author

[a]It has also been called the differential-difference hierarchy, or the semi-discrete hierarchy.

There are many methods for constructing the solutions of the integrable systems for the continuous KP hierarchy [3,4,19,31,33,34,37,43,45]. Among these methods, the gauge transformation is one of the important methods to construct the solutions of the integrable systems for the continuous KP hierarchy [3,4,19,33,34,37], the dKP hierarchy [27,30,35] and the cdKP hierarchy [27], which in fact reflects the intrinsic integrability of the KP hierarchy and dKP hierarchy. Chau *et al* [3] introduce two kinds of elementary gauge transformation operators: the differential type T_D and the integral type T_I. By now, the gauge transformations of many integrable hierarchies related to KP hierarchy have been derived, for example, the constrained KP (cKP) hierarchy [4,6,7,20,24,34,44], the constrained BKP and CKP hierarchy [21,33] (cBKP and cCKP), the dKP hierarchy [30,35], the cdKP hierarchy [27], the q-KP hierarchy [18,42] and so on. The additional symmetry [1,5,8,12,14,16,22,26,36,38–41] is a kind of symmetry depending explicitly on the space and time variables, involved in so-called string equation and the generalized Virasoro constraints in matrix models of the 2d quantum gravity (see [11,32] and references therein). Regarding the possible application of the additional symmetry flows of the KP hierarchy in physics, it is natural to ask whether these flows are compatible with the gauge transformation. It is a highly non-trivial question because the gauge transformation is only defined to be consistent with ordinary KP flows. For example, Ref. [2] has shown the compatibility between the the differential type of gauge transformation and the additional symmetry flow of cKP hierarchy separately, up to a shift of ordinary flow of cKP hierarchy. In order to construct the additional symmetry flows of the cKP hierarchy from the corresponding flows of the KP hierarchy, it is necessary to do a remarkable amendment [2] in its definition. So it is an interesting problem to show the compatibility between the gauge transformations and the additional symmetry of the cdKP hierarchy. The additional symmetry flows for the cdKP hierarchy are constructed in [9,28] through a subtle modification of the standard additional symmetry flows by adding a complicated term, which form a Virasoro type algebraic structure [28]. And the action of the Virasoro symmetry on the tau function of the cdKP hierarchy is also derived [9].

In this paper, it is shown that the additional symmetry flows for the cdKP hierarchy commute with the integral type and difference type gauge transformations preserving the form of the additional symmetry of the cdKP hierarchy, up to shifting of the corresponding additional flows by ordinary time flows, which reflects the compatibility between the two types of the gauge transformations and the additional symmetries of the cdKP hierarchy.

This paper is organized as follows. Some backgrounds on the dKP hierarchy are given in Section 2. Then the two types gauge transformation operators of the cdKP hierarchy are reviewed in Section 3. And the additional symmetry for the cdKP hierarchy are reviewed in Section 4. In Section 5, it is derived that the additional symmetry commute with the gauge transformations preserving the form of the additional symmetry of the cdKP hierarchy.

2. Background On The dKP Hierarchy

Some basic facts about the dKP hierarchy are demonstrated as follows [17]. Firstly a space F, namely

$$F = \left\{ f(n) = f(n, t_1, t_2, \cdots, t_j, \cdots); n \in \mathbb{Z}, t_i \in \mathbb{R} \right\} \tag{2.1}$$

is defined for the space of the dKP hierarchy. Λ and Δ are denote for the shift operator and the difference operator, respectively. Their actions on function $f(n)$ are defined as

$$\Lambda f(n) = f(n+1) \tag{2.2}$$

and

$$\Delta f(n) = f(n+1) - f(n) = (\Lambda - I)f(n) \tag{2.3}$$

respectively, where I is the identity operator.

For any $j \in \mathbb{Z}$, the Leibniz rule of Δ operation is,

$$\Delta^j f = \sum_{i=0}^{\infty} \binom{j}{i} (\Delta^i f)(n+j-i)\Delta^{j-i}, \quad \binom{j}{i} = \frac{j(j-1)\cdots(j-i+1)}{i!}. \tag{2.4}$$

So an associative ring $F(\Delta)$ of formal pseudo difference operators (PDO) is obtained, namely $F(\Delta) = \left\{R = \sum_{j=-\infty}^{d} f_j(n)\Delta^j, f_j(n) \in R, n \in \mathbb{Z}\right\}$. The adjoint operator to the Δ operator is given by Δ^*,

$$\Delta^* f(n) = (\Lambda^{-1} - I)f(n) = f(n-1) - f(n), \tag{2.5}$$

where $\Lambda^{-1} f(n) = f(n-1)$, and the corresponding j-times operation is

$$\Delta^{*j} f = \sum_{i=0}^{\infty} \binom{j}{i} (\Delta^{*i} f)(n+i-j)\Delta^{*j-i}. \tag{2.6}$$

Then the adjoint ring $F(\Delta^*)$ to the $F(\Delta)$ is obtained, and the formal adjoint to $R \in F(\Delta)$ is defined by $R^* \in F(\Delta^*)$ as $R^* = \sum_{j=-\infty}^{d} \Delta^{*j} f_j(n)$. The "$*$" stands for the conjugate operation which satisfies the rules as $(FG)^* = G^* F^*$, $\Delta^* = -\Delta$, $f^* = f$ for two operators F and G and $f(n)^* = f(n)$ for a function $f(n)$. Here for any (pseudo-) difference operator A and a function f, the symbol $A(f)$ will indicate the action of A on f, whereas the symbol Af (or $A \cdot f$) will denote just operator product of A and f.

The dKP hierarchy [17, 23] is a family of evolution equations depending on infinitely many variables $t = (t_1, t_2, \cdots)$

$$\frac{\partial L}{\partial t_k} = [B_k, L], \quad B_k := (L^k)_+, \tag{2.7}$$

where L is a general first-order PDO

$$L(n) = \Delta + \sum_{j=1}^{\infty} f_j(n)\Delta^{-j}. \tag{2.8}$$

$B_m = (L^m)_+ = \sum_{j=0}^{m} a_j(n)\Delta^j$, i. e. $(L^m)_+$ is the non-negative projection of L^m, and $(L^m)_- = L^m - (L^m)_+$ is the negative projection of L^m. The Lax operator in eq.(2.8) can be generated by a dressing operator

$$W(n;t) = 1 + \sum_{j=1}^{\infty} w_j(n;t)\Delta^{-j}. \tag{2.9}$$

through

$$L = W\Delta W^{-1}. \tag{2.10}$$

Further the flow equation (2.7) is equivalent to the so-called Sato equation,

$$\frac{\partial W}{\partial t_k} = -(L^k)_- W. \tag{2.11}$$

If the functions $q(t)$ and $r(t)$ satisfy

$$\frac{\partial q}{\partial t_k} = B_k(q), \qquad \frac{\partial r}{\partial t_k} = -B_k^*(r), \tag{2.12}$$

then we call them the eigenfunction and the adjoint eigenfunction respectively.

The cdKP hierarchy [28] is defined by restricting the Lax operator of the dKP hierarchy

$$\frac{\partial L}{\partial t_k} = [B_k, L], \quad B_k := (L^k)_+, \tag{2.13}$$

with the following l-constrained form:

$$L^l = L^l_+ + \sum_{i=1}^{m} q_i \Delta^{-1} r_i = \Delta^l + \sum_{j=0}^{k-2} v_j \Delta^j + \sum_{i=1}^{m} q_i \Delta^{-1} r_i, \tag{2.14}$$

where q_i and r_i are the eigenfunction and adjoint eigenfunction respectively.

3. The Two Types Gauge Transformations Of The cdKP Hierarchy

Let L be the original Lax operator of the cdKP hierarchy (2.14), and T be a pseduo-difference operator. If the transformation

$$\widetilde{L} = TLT^{-1} \tag{3.1}$$

such that

$$\frac{\partial \widetilde{L}}{\partial t_k} = [\widetilde{B}_k, \widetilde{L}], \quad \widetilde{B}_k = (\widetilde{L}^k)_+, \; k = 1,2,3,\cdots \tag{3.2}$$

still holds for transformed Lax operator $\widetilde{L}$, then T is called the gauge transformation operator of the cdKP hierarchy.

Similar to the KP hierarchy [3], there are two types of gauge transformation operators of the dKP hierarchy as [30, 35]

$$\text{Type I}: \; T_D(q) = \Lambda(q)\Delta q^{-1}, \tag{3.3}$$

$$\text{Type II}: \; T_I(r) = \Lambda^{-1}(r^{-1})\Delta^{-1} r, \tag{3.4}$$

where q and r are the eigenfunction and adjoint eigenfunction respectively. The type I transformation is called the difference type, while the type II is called the integral type.

Here we review some results about the integral type and the difference type gauge transformations of the cdKP hierarchy [27]. Under the integral type gauge transformation $T_I(r)$, the transformed Lax operator will be:

$$\widetilde{L} = T_I(r)LT_I(r)^{-1} = \widetilde{L}_+ + \widetilde{L}_-, \tag{3.5}$$
$$\widetilde{L}_+ = \Lambda^{-1}(L_+) - \Lambda^{-1}(r^{-1})\Delta^{-1}(\Delta^*(r^{-1}L_+^* r)_{\geq 1})^*(r), \tag{3.6}$$
$$\widetilde{L}_- = \widetilde{q}_0\Delta^{-1}\widetilde{r}_0 + \sum_{i=1}^{m}\widetilde{q}_i\Delta^{-1}\widetilde{r}_i, \tag{3.7}$$
$$\widetilde{q}_0 = \Lambda^{-1}(r^{-1}), \quad \widetilde{r}_0 = T_I(r)^{*-1}L^{(0)*}(r), \tag{3.8}$$
$$\widetilde{q}_i = T_I(r)(q_i), \quad \widetilde{r}_i = T_I(r)^{*-1}(r_i). \tag{3.9}$$

In order to preserve the form (2.14) of the Lax operator L, r is required to coincide with one of the original adjoint eigenfunctions of L, e.g. $r = r_1$, since $\widetilde{r}_1 = 0$ in this case.

Under the gauge transformation of $T_D(q)$, the transformed Lax operator reads as

$$\widetilde{L} = \widetilde{L}_+ + \widetilde{L}_-, \tag{3.10}$$
$$\widetilde{L}_+ = \Lambda(L_+) + \Lambda(q)\Delta(q^{-1}L_+q)_{\geq 1}\Delta^{-1}\Lambda(q^{-1}), \tag{3.11}$$
$$\widetilde{L}_- = \widetilde{q}_0\Delta^{-1}\widetilde{r}_0 + \sum_{i=1}^{m}\widetilde{q}_i\Delta^{-1}\widetilde{r}_i, \tag{3.12}$$
$$\widetilde{q}_0 = T_D(q)(L)(q), \;\; \widetilde{r}_0 = \Lambda(q^{-1}), \tag{3.13}$$
$$\widetilde{q}_i = T_D(q)q_i, \;\; \widetilde{r}_i = (T_D^{-1})^*(q)(r_i). \tag{3.14}$$

For the difference type gauge transformation $T_D(q)$, in order to preserve the form (2.14) of the Lax operator L, q is required to coincide with one of the original adjoint eigenfunctions of L, e.g. $q = q_1$, since $\widetilde{q}_1 = 0$ in this case.

In order to calculate the transformed formula of the part as $f\Delta^{-1}g$ in the Lax operator under the integral type gauge transformation, the following lemma is necessary.

Lemma 3.1.

$$T_I(r_a)\cdot M\Delta^{-1}r_a\cdot T_I^{-1}(r_a) = \Lambda^{-1}(r_a^{-1})\Delta^{-1}\left\{T_I^{*-1}(r_a)(M\Delta^{-1}r_a)^*(r_a)\right\}, \tag{3.15}$$
$$T_I(r_a)\cdot q_a\Delta^{-1}N\cdot T_I(r_a)^{-1} = \Lambda^{-1}(r_a^{-1})r_a^{-1}\Delta^{-1}\left\{T_I^{*-1}(r_a)(q_a\Delta^{-1}N)^*(r_a)\right\} + \widetilde{L}(\widetilde{q}_a)\Delta^{-1}\widetilde{N}, \tag{3.16}$$
$$T_I(r_a)\cdot M\Delta^{-1}N\cdot T_I(r_a)^{-1} = \Lambda^{-1}(r_a^{-1})\Delta^{-1}\left\{T_I^{*-1}(r_a)(M\Delta^{-1}N)^*(r_a)\right\} + \widetilde{M}\Delta^{-1}\widetilde{N}, \tag{3.17}$$
$$\widetilde{L}^{k+1}(\widetilde{q}_a) = T_I(r_a)L^k(q_a), \;\; k = 0,1,2,...., \tag{3.18}$$
$$(\widetilde{L}^*)^{k-1}(\widetilde{r}_a) = T_I(r_a)^{*-1}(L^*)^k(r_a), \;\; k = 1,2,3.... \tag{3.19}$$

where r_a is one of the adjoint eigenfunctions of the cdKP hierarchy (2.14), M and N are two functions of t, and

$$\widetilde{L} = T_I(r_a)LT_I(r_a)^{-1}, \;\; \widetilde{q}_a = \Lambda^{-1}(1/r_a), \;\; \widetilde{r}_a = T_I(r_a)^{*-1}L^*(r_a), \quad \widetilde{M} = T_I(r_a)(M), \;\; \widetilde{N} = T_I(r_a)^{*-1}(N). \tag{3.20}$$

Proof. Firstly, according to $\Delta^{-1}f\Delta^{-1}=(\Delta^{-1}f)\Delta^{-1}-\Delta^{-1}\Lambda(\Delta^{-1}f)$ and $\Delta f-\Lambda(f)\Delta=\Delta(f)$,

$$\begin{aligned}
&T_I(r_a)\cdot M\Delta^{-1}N\cdot T_I(r_a)^{-1}=\Lambda^{-1}(r_a^{-1})\Delta^{-1}r_a\cdot M\Delta^{-1}N\cdot r_a^{-1}\Delta\Lambda^{-1}(r_a)\\
&=\Lambda^{-1}(r_a^{-1})\Big(\Delta^{-1}(r_aM)\Delta^{-1}-\Delta^{-1}\Lambda(\Delta^{-1}(r_aM))\Big)Nr_a^{-1}\Delta\Lambda^{-1}(r_a)\\
&=\Lambda^{-1}(r_a^{-1})\Delta^{-1}(r_aM)\Delta^{-1}\Big(\Delta\Lambda^{-1}(Nr_a^{-1})-\Delta\Lambda^{-1}(Nr_a^{-1})\Big)\Lambda^{-1}(r_a)\\
&\quad-\Lambda^{-1}(r_a^{-1})\Delta^{-1}\Big(\Delta\Lambda^{-1}(\Lambda(\Delta^{-1}(r_aM))Nr_a^{-1})-\Delta(\Delta^{-1}(r_aM)\Lambda^{-1}(Nr_a^{-1}))\Big)\Lambda^{-1}(r_a)\\
&=\Lambda^{-1}(r_a^{-1})\Delta^{-1}(r_aM)\Lambda^{-1}(N)-\Lambda^{-1}(r_a^{-1})\Delta^{-1}(r_aM)\Delta^{-1}\Big(\Delta\Lambda^{-1}(Nr_a^{-1})\Big)\Lambda^{-1}(r_a)\\
&\quad-\Lambda^{-1}(r_a^{-1})\Delta^{-1}(r_aM)\Lambda^{-1}(N)+\Lambda^{-1}(r_a^{-1})\Delta^{-1}\Delta\Big(\Delta^{-1}(r_aM)\Lambda^{-1}(Nr_a^{-1})\Big)\Lambda^{-1}(r_a)\\
&=T_I(r_a)(M)\Delta^{-1}T_I(r_a)^{*-1}(N)+\Lambda^{-1}(r_a^{-1})\Delta^{-1}\left\{T_I(r_a)^{*-1}(M\Delta^{-1}N)^*(r_a)\right\}\\
&=\widetilde{M}\Delta^{-1}\widetilde{N}+\Lambda^{-1}(r_a^{-1})\Delta^{-1}\left\{T_I(r_a)^{*-1}(M\Delta^{-1}N)^*(r_a)\right\}.
\end{aligned}$$

So (3.17) is be proved. (3.15) can be derived from (3.17) for $N=r_a$, since $T_I(r_a)^{*-1}(r_a)=0$.

Then for (3.18),

$$\begin{aligned}
\widetilde{L}^{k+1}(\widetilde{q}_a)&=T_I(r_a)L^{k+1}T_I(r_a)^{-1}(\Lambda^{-1}(r_a^{-1}))\\
&=T_I(r_a)L^k(L_++\sum_{i=0}^{k}q_i\Delta^{-1}r_i)r_a^{-1}\Delta\Lambda^{-1}(r_a)\Lambda^{-1}(r_a^{-1})=T_I(r_a)L^k(q_a).
\end{aligned}$$

Here we let $q_i\Delta^{-1}(0)=0$ for $i\neq a$, and $q_a\Delta^{-1}(0)=q_a$. And (3.16) can be derived from (3.17) and (3.18). At last,

$$T_I(r_a)^{*-1}(L^*)^k(r_a)=T_I(r_a)^{*-1}(L^*)^{k-1}T_I(r_a)^*T_I(r_a)^{*-1}L^*(r_a)=(\widetilde{L}^*)^{k-1}(\widetilde{r}_a).$$

□

In order to calculate the transformed formula of the part as $f\Delta^{-1}g$ in the Lax operator under the difference type gauge transformation, the following lemma is necessary.

Lemma 3.2.

$$T_D(q_a)\cdot M\Delta^{-1}r_a\cdot T_D(q_a)^{-1}=T_D(q_a)(M\Delta^{-1}r_a)(q_a)\Delta^{-1}\Lambda(q_a^{-1})+\widetilde{M}\Delta^{-1}\widetilde{L}^*(\widetilde{r}_a),\qquad(3.21)$$

$$T_D(q_a)\cdot q_a\Delta^{-1}N\cdot T_D(q_a)^{-1}=T_D(q_a)(q_a\Delta^{-1}N)(q_a)\Delta^{-1}\Lambda(q_a^{-1}),\qquad(3.22)$$

$$T_D(q_a)\cdot M\Delta^{-1}N\cdot T_D(q_a)^{-1}=T_D(q_a)(M\Delta^{-1}N)(q_a)\Delta^{-1}\Lambda(q_a^{-1})+\widetilde{M}\Delta^{-1}\widetilde{N},\qquad(3.23)$$

$$\widetilde{L}^{k-1}(\widetilde{q}_a)=T_D(q_a)L^k(q_a),\ \ k=0,1,2,...,\qquad(3.24)$$

$$(\widetilde{L}^*)^k(\widetilde{r}_a)=T_D^*(q_a)^{-1}(L^*)^{k-1}(r_a),\ \ k=1,2,3,...,\qquad(3.25)$$

where r_a is one of the adjoint eigenfunctions of the cdKP hierarchy (2.14), M and N are two functions of t, and

$$\widetilde{L}=T_D(q_a)LT_D(q_a)^{-1},\ \ \widetilde{q}_a=T_D(q_a)L(q_a),\ \ \widetilde{r}_a=\Lambda(r_a^{-1}),\qquad \widetilde{M}=T_D(q_a)(M),\ \ \widetilde{N}=T_D(q_a)^{*-1}(N).\qquad(3.26)$$

Proof. Firstly, according to $\Delta^{-1}f\Delta^{-1}=(\Delta^{-1}f)\Delta^{-1}-\Delta^{-1}\Lambda(\Delta^{-1}f)$ and $\Delta f-\Lambda(f)\Delta=\Delta(f)$,

$$
\begin{aligned}
&T_D(q_a)\cdot M\Delta^{-1}N\cdot T_D(q_a)^{-1}=\Lambda(q_a)\Delta q_a^{-1}\cdot M\Delta^{-1}N\cdot q_a\Delta^{-1}\Lambda(q_a^{-1})\\
&=\Lambda(q_a)\Delta(q_a^{-1}M)\Big(\Delta^{-1}(Nq_a)\Delta^{-1}-\Delta^{-1}\Lambda(\Delta^{-1}(q_aN))\Big)\Lambda(q_a^{-1})\\
&=\Lambda(q_a)\Big(\Delta(q_a^{-1}M\Delta^{-1}(Nq_a))+\Lambda(q_a^{-1}M\Delta^{-1}(Nq_a^{-1}))\Delta\Big)\Delta^{-1}\Lambda(q_a^{-1})\\
&\quad-\Lambda(q_a)\Big(\Delta(q_a^{-1}M)+\Lambda(q_a^{-1}M)\Delta\Big)\Delta^{-1}\Lambda(\Delta^{-1}(Nq_a))\Lambda^{-1}(q_a^{-1})\\
&=\Lambda(q_a)\Delta\Big(q_a^{-1}M\Delta^{-1}(Nq_a)\Big)\Delta^{-1}\Lambda(q_a^{-1})+\Lambda(q_a)\Lambda\Big(q_a^{-1}M\Delta^{-1}(Nq_a)\Big)\Lambda(q_a^{-1})\\
&\quad-\Lambda(q_a)\Delta(q_a^{-1}M)\Delta^{-1}\Lambda\Big(\Delta^{-1}(Nq_a)\Big)\Lambda(q_a^{-1})-\Lambda(q_a)\Lambda\Big(q_a^{-1}M\Delta^{-1}(Nq_a)\Big)\Lambda(q_a^{-1})\\
&=\Lambda(q_a)\Delta\Big(q_a^{-1}M\Delta^{-1}(Nq_a)\Big)\Delta^{-1}\Lambda(q_a^{-1})-\Lambda(q_a)\Delta(q_a^{-1}M)\Delta^{-1}\cdot\Lambda\Delta^{-1}(Nq_a)\Lambda(q_a^{-1})\\
&=T_D(q_a)(M)\Delta^{-1}T_D(q_a)^{*-1}(N)+T_D(q_a)(M\Delta^{-1}N)(q_a)\Delta^{-1}\Lambda(q_a^{-1})\\
&=\widetilde{M}\Delta^{-1}\widetilde{N}+T_D(q_a)(M\Delta^{-1}N)(q_a)\Delta^{-1}\Lambda(q_a^{-1}).
\end{aligned}
$$

So (3.23) is be proved. (3.22) can be derived from (3.23) for $M=q_a$, since $T_D(q_a)(q_a)=0$.

For (3.24),

$$\widetilde{L}^{k-1}(\widetilde{q}_a)=T_D(q_a)L^{k-1}T_D(q_a)^{-1}T_D(q_a)L(q_a^{-1})=T_D(q_a)L^k(q_a).$$

Then (3.21) can be derived from (3.23) and (3.25).

At last, for (3.25),

$$T_D^*(q_a)^{-1}(L^*)^{k+1}(r_a)=T_D^*(q_a)^{-1}(L^*)^kT_D^*(q_a)T_D^*(q_a)^{-1}L^*(r_a)=(\widetilde{L}^*)^k(\widetilde{r}_a).$$

□

In order to prove the compatibility between the two types of the gauge transformation and the additional symmetry, the following operator identities are necessary.

Lemma 3.3. *Let q,r be suitable function and A be a PDO, then*

$$
\begin{aligned}
\big(\Lambda^{-1}(r^{-1})\Delta^{-1}rAr^{-1}\Delta\Lambda^{-1}(r)\big)_-&=\Lambda^{-1}(r^{-1})\Delta^{-1}rA_-r^{-1}\Delta\Lambda^{-1}(r)\\
&\quad-\Lambda^{-1}(r^{-1})\Delta^{-1}\Lambda^{-1}(r)\Delta(\Lambda^{-1}(r^{-1}A_+^*r)),\qquad(3.27)\\
\big(\Lambda(q)\Delta q^{-1}Aq\Delta^{-1}\Lambda(q^{-1})\big)_+&=\Lambda(q)\Delta q^{-1}A_+q\Delta^{-1}\Lambda(q^{-1})\\
&\quad-\Lambda(q)\Delta(q^{-1}A_+(q))\Delta^{-1}\Lambda(q^{-1}).\qquad(3.28)
\end{aligned}
$$

Proof. With $(Kq\Delta^{-1}r)_-=K(q)\Delta^{-1}r,(q\Delta^{-1}rK)_-=q\Delta^{-1}K^*(r)$ for pure-difference operator K [28],

$$
\begin{aligned}
&\big(\Lambda^{-1}(r^{-1})\Delta^{-1}rAr^{-1}\Delta\Lambda^{-1}(r)\big)_-\\
&=\big(\Lambda^{-1}(r^{-1})\Delta^{-1}rA_-r^{-1}\Delta\Lambda^{-1}(r)\big)_-+\big(\Lambda^{-1}(r^{-1})\Delta^{-1}rA_+r^{-1}\Delta\Lambda^{-1}(r)\big)_-\\
&=\Lambda^{-1}(r^{-1})\Delta^{-1}rA_-r^{-1}\Delta\Lambda^{-1}(r)-\Lambda^{-1}(r^{-1})\Delta^{-1}\Lambda^{-1}(r)\Delta(\Lambda^{-1}(r^{-1}A_+^*r)),
\end{aligned}
$$

so the (3.27) is proved. For (3.28), with $(Kq\Delta^{-1}r)_- = K(q)\Delta^{-1}r$,

$$\begin{aligned}\left(\Lambda(q)\Delta q^{-1}Aq\Delta^{-1}\Lambda(q^{-1})\right)_+ &= \left(\Lambda(q)\Delta q^{-1}A_+q\Delta^{-1}\Lambda(q^{-1})\right)_+ \\ &= \Lambda(q)\Delta q^{-1}A_+q\Delta^{-1}\Lambda(q^{-1}) - \Lambda(q)\Delta(q^{-1}A_+(q))\Delta^{-1}\Lambda(q^{-1}).\end{aligned}$$

□

Remark 1: This lemma is a difference-analogue of the corresponding identities of PDO given by [2,37].

4. Additional Symmetries Of The cdKP Hierarchy

Define

$$X_k^{(1)} = \sum_{i=1}^{m}\sum_{j=0}^{k-1}\left(j-\frac{1}{2}(k-1)\right)L^{k-1-j}(q_i)\Delta^{-1}(L^*)^j(r_i);\ \ k\geq 1, \tag{4.1}$$

which is the essential to ensure the compatibility of the additional Virasoro symmetry with the constraints (2.14) defining the cdKP hierarchy. The additional symmetry flows for the cdKP hierarchy, spanning the Virasoro algebra, are given by [28]:

$$\partial_k^* L = [-(M_\Delta L^k)_- + X_{k-1}^{(1)}, L]. \tag{4.2}$$

M_Δ is the Orlov-Schulman operator [38] defined in the dressing the "bare" $M^{(0)}$ operator:

$$M^{(0)} = \sum_{l\geq 1} l t_l \Delta^{l-k} = X_{(k)} + \sum_{l\geq 1}(l+k)t_{l+k}\Delta^l;\ \ X_{(k)} = \sum_{l=1}^{k} l t_l \Delta^{l-k} \tag{4.3}$$

that is,

$$M_\Delta = WM^{(0)}W^{-1} = WX_{(k)}W^{-1} + \sum_{l\geq 1}(l+k)t_{l+k}L^l = \sum_{l\geq 0}(l+k)t_{l+k}L_+^l + (M_\Delta)_-, \tag{4.4}$$

$$(M_\Delta)_- = WX_{(k)}W^{-1} - kt_k - \sum_{l\geq 1}(l+k)t_{l+k}\frac{\partial W}{\partial t_l}W^{-1}, \tag{4.5}$$

with (2.11) used in (4.5).

Then accordingly, the actions of the additional symmetry flows on the dressing operators and BA functions are showed that:

$$\partial_k^* W = \left(-(M_\Delta L^k)_- + X_{k-1}^{(1)}\right)W;\ \ \partial_k^*\psi(t,\lambda) = \left(-(M_\Delta L^k)_- + X_{k-1}^{(1)}\right)(\psi(t,\lambda)). \tag{4.6}$$

The corresponding actions on the eigenfunctions q_i and the adjoint eigenfunctions r_i are derived by considering $(\partial_k^* L)_-$ listed as follows [28]:

$$\partial_k^* q_i = (M_\Delta L^k)_+(q_i) + \frac{k}{2}L^{k-1}(q_i) + X_{k-1}^{(1)}(q_i), \tag{4.7}$$

$$\partial_k^* r_i = -(M_\Delta L^k)_+^*(r_i) + \frac{k}{2}(L^*)^{k-1}(r_i) - (X_{k-1}^{(1)})^*(r_i). \tag{4.8}$$

5. Additional Symmetries Versus Two Types Gauge Transformations

In this section, we will restrict to the cdKP hierarchy ((2.14) for $m = 1, l = 1$). And thus its Lax operator is given by

$$L = \Delta + q\Delta^{-1}r. \tag{5.1}$$

In order to investigate the changes of the additional symmetries under the integral type gauge transformation $T_I(r)$, some useful lemmas are needed.

Lemma 5.1.

$$T_I(r)X^{(1)}_{k-1}T_I(r)^{-1} = \widetilde{X}^{(1)}_{k-1} + \sum_{j=0}^{k-2}\widetilde{L}^{k-j-2}(\widetilde{q})\Delta^{-1}\widetilde{L}^j(\widetilde{r}) + \Lambda(r^{-1})\Delta^{-1}\left\{(T_I^*(r))^{-1}(X^{(1)}_{k-1} - \frac{k}{2}L^{k-1})^*(r)\right\}. \tag{5.2}$$

Proof. According to Lemma 3.1 and (4.1), then

$$\begin{aligned}
&T_I(r)X^{(1)}_{k-1}T_I(r)^{-1}\\
&= -\Lambda^{-1}(r^{-1})\Delta^{-1}T_D(r)(X^{(1)}_{k-1})^*(r) + \sum_{j=1}^{k-2}\left(j - \frac{1}{2}(k-2)\right)\widetilde{L}^{k-j-1}(\widetilde{q})\Delta^{-1}(\widetilde{L}^*)^{j-1}(\widetilde{r})\\
&= -\Lambda^{-1}(r^{-1})\Delta^{-1}T_D(r)(X^{(1)}_{k-1})^*(r) + \sum_{j=0}^{k-2}\left(j - \frac{1}{2}(k-2)\right)\widetilde{L}^{k-j-2}(\widetilde{q})\Delta^{-1}(\widetilde{L}^*)^{j}(\widetilde{r})\\
&\quad + \sum_{j=0}^{k-2}\widetilde{L}^{k-j-2}(\widetilde{q})\Delta^{-1}(\widetilde{L}^*)^{j}(\widetilde{r}) - \left(1 + k - 2 - \frac{1}{2}(k-2)\right)\widetilde{q}\Delta^{-1}(\widetilde{L}^*)^{k-2}(\widetilde{r})\\
&= -\Lambda^{-1}(r^{-1})\Delta^{-1}T_D(r)(X^{(1)}_{k-1})^*(r) + \widetilde{X}^{(1)}_{k-1} + \sum_{j=0}^{k-2}\widetilde{L}^{k-j-2}(\widetilde{q})\Delta^{-1}(\widetilde{L}^*)^{j}(\widetilde{r})\\
&\quad - \frac{k}{2}r^{-1}\Delta^{-1}T_I(r)^{*-1}(L^*)^{k-1}(r)\\
&= \widetilde{X}^{(1)}_{k-1} + \sum_{j=0}^{k-2}\widetilde{L}^{k-j-2}(\widetilde{q})\Delta^{-1}\widetilde{L}^j(\widetilde{r}) + r^{-1}\Delta^{-1}\left\{T_I(r)^{*-1}(X^{(1)}_{k-1} - \frac{k}{2}L^{k-1})^*(r)\right\}.
\end{aligned}$$

□

Lemma 5.2.

$$\partial_k^* T_I(r)\cdot T_I(r)^{-1} = \Lambda^{-1}(r^{-1})\Delta^{-1}\left\{T_I(r)^{*-1}\left(-(M_\Delta L^k)^*_+ + \frac{k}{2}(L^*)^{k-1} - (X^{(1)}_{k-1})^*\right)(r)\right\}. \tag{5.3}$$

Proof. By (4.8),

$$\begin{aligned}\partial_k^* T_I(r)\cdot T_I(r)^{-1} &= -T_I(r)\partial_k^*(T_I(r)^{-1})\\ &= \Lambda^{-1}(r^{-1})\Delta^{-1}r^{-1}\partial_k^*(r)\Delta\Lambda^{-1}(r) - \Lambda^{-1}(r^{-1}\partial_k^*(r))\\ &= \Lambda^{-1}(r^{-1})\Delta^{-1}\left(\Delta\Lambda^{-1}(\partial_k^*(r)r^{-1}) - \Delta\Lambda^{-1}(\partial_k^*(r)r^{-1})\right)\Lambda^{-1}(r) - \Lambda^{-1}(r^{-1}\partial_k^*(r))\\ &= -\Lambda^{-1}(r^{-1})\Delta^{-1}(\Delta\Lambda^{-1}(\partial_k^*(r)r^{-1}))\Lambda^{-1}(r)\\ &= \Lambda^{-1}(r^{-1})\Delta^{-1}T_I(r)^{*-1}\Lambda^{-1}(\partial_k^* r)\\ &= \Lambda^{-1}(r^{-1})\Delta^{-1}\left\{T_I(r)^{*-1}\left(-(M_\Delta L^k)_+^* + \frac{k}{2}(L^*)^{k-1} - (X_{k-1}^{(1)})^*\right)(r)\right\}.\end{aligned}$$

□

Theorem 5.1. *The additional symmetry flows (4.2) for the cdKP hierarchy ((2.14) for $m = 1, l = 1$) commute with the integral type transformations $T_I(r)$ preserving the form of cdKP hierarchy, up to shifting of (4.2) by ordinary time flows, that is,*

$$\partial_k^*\widetilde{L} = [-(\widetilde{M}_\Delta\widetilde{L}^k)_- + \widetilde{X}_{k-1}^{(1)}, \widetilde{L}] - \frac{\partial\widetilde{L}}{\partial t_{k-1}}. \tag{5.4}$$

Proof. Firstly, by (4.2),

$$\begin{aligned}\partial_k^*\widetilde{L} &= \partial_k^* T_I(r)\cdot LT_I(r)^{-1} + T_I(r)\partial_k^* L\cdot T_I(r)^{-1} - T_I(r)LT_I(r)^{-1}\cdot\partial_k^* T_I(r)\cdot T_I(r)^{-1}\\ &= \left[T_I(r)\left(-(M_\Delta L^k)_- + X_{k-1}^{(1)}\right)T_I(r)^{-1} + \partial_k^* T_I(r)\cdot T_I(r)^{-1}, \widetilde{L}\right]\end{aligned} \tag{5.5}$$

Then with the help of (5.2), (5.3), and the following useful formula in Lemma.3.3, we have

$$\begin{aligned}&T_I(r)\left(-(M_\Delta L^k)_- + X_{k-1}^{(1)}\right)T_I(r)^{-1} + \partial_k^* T_I(r)\cdot T_I(r)^{-1}\\ &= T_I(r)\left(-M_\Delta L^k)_-\right)T_I^{-1}(r) - \Lambda^{-1}(r^{-1})\Delta^{-1}\left\{T_I^*(r)^{-1}\left((M_\Delta L^k)_+^* - \frac{k}{2}(L^*)^{k-1} + (X_{k-1}^{(1)})^*\right)(r)\right\}\\ &\quad + \widetilde{X}_{k-1}^{(1)} + \sum_{j=0}^{k-2}\widetilde{L}^{k-j-2}(\widetilde{q})\Delta^{-1}\widetilde{L}^j(\widetilde{r}) + \Lambda^{-1}(r^{-1})\Delta^{-1}\left\{T_I(r)^{*-1}\left((X_{k-1}^{(1)})^* - \frac{k}{2}(L^*)^{k-1}\right)(r)\right\}\\ &= -(\widetilde{M}_\Delta\widetilde{L}^k)_- + \widetilde{X}_{k-1}^{(1)} + (\widetilde{L}^{k-1})_-,\end{aligned} \tag{5.6}$$

where the following relation [28] is used,

$$(\widetilde{L}^{k-1})_- = \sum_{j=0}^{k-2}\widetilde{L}^{k-j-2}(\widetilde{q})\Delta^{-1}\widetilde{L}^j(\widetilde{r}). \tag{5.7}$$

In the above process,

$$\begin{aligned}&-T_I(r)(M_\Delta L^k)_- T_I^{-1}(r) - \Lambda^{-1}(r^{-1})\Delta^{-1}T_I^*(r)^{-1}((M_\Delta L^k)_+^*(r))\\ &= -T_I(r)(M_\Delta L^k)_- T_I^{-1}(r) + \Lambda^{-1}(r^{-1})\Delta^{-1}(r(M_\Delta L^k)_+ T_I(r)^{-1})^*)\\ &= -(\widetilde{M}_\Delta\widetilde{L}^k)_-,\end{aligned}$$

which can be got by means of the identity (3.27) of Lemma.3.3.

At last, the substituting (5.6) into (5.5) gives rise to (5.4). □

For the difference type gauge transformation $T_D(q)$, there are some lemma as following.

Lemma 5.3.

$$T_D(q)X^{(1)}_{k-1}T_D^{-1}(q) = \widetilde{X}^{(1)}_{k-1} - (\widetilde{L}^{k-1})_- + \left(T_D(q)(X^{(1)}_{k-1} + \frac{k}{2}L^{k-1}(q_i))\right) q\Delta^{-1}\Lambda(q^{-1}). \qquad (5.8)$$

Proof. According to Lemma 3.2 and (4.1), then

$$\begin{aligned}
& T_D(q)X^{(1)}_{k-1}T_D(q)^{-1} \\
= \;& \Lambda(q)\Delta q^{-1}\sum_{j=0}^{k-2}\left(j-\frac{k-2}{2}\right)L^{k-j-2}(q_i)\Delta^{-1}(L^*)^j(r_i)q\Delta^{-1}\Lambda(q^{-1}) \\
\overset{(3.23)}{=\!=} \;& \sum_{j=0}^{k-2}\left(j-\frac{k-2}{2}\right)T_D(q)(L^{k-2}(q_i)\Delta^{-1}r_i)(q)\Delta^{-1}\Lambda(q^{-1}) + \widetilde{L}^{k-j-3}(\widetilde{q}_i)\Delta^{-1}(\widetilde{L}^*)^{j+1}(\widetilde{r}_i) \\
= \;& \sum_{j=0}^{k-2}\left(j-\frac{k-2}{2}\right)T_D(q)(L^{k-2}(q_i)\Delta^{-1}r_i)(q)\Delta^{-1}\Lambda(q^{-1}) + \sum_{j=1}^{k-1}\left(j-\frac{k}{2}\right)\widetilde{L}^{k-j-2}(\widetilde{q}_i)\Delta^{-1}(\widetilde{L}^*)^j(\widetilde{r}_i) \\
= \;& T_D(q)(X^{(1)}_{k-1})(q)\Delta^{-1}\Lambda(q^{-1}) + \sum_{j=1}^{k-1}\left(j-\frac{k-2}{2}\right)\widetilde{L}^{k-j-2}(\widetilde{q}_i)\Delta^{-1}(\widetilde{L}^*)^j(\widetilde{r}_i) \\
& -\sum_{j=1}^{k-1}\widetilde{L}^{k-j-2}(\widetilde{q}_i)\Delta^{-1}(\widetilde{L}^*)^j(\widetilde{r}_i) \\
= \;& \sum_{j=0}^{k-2}\left(j-\frac{k-2}{2}\right)\widetilde{L}^{k-j-2}(\widetilde{q}_i)\Delta^{-1}(\widetilde{L}^*)^j(\widetilde{r}_i) - \sum_{j=0}^{k-2}\widetilde{L}^{k-j-2}(\widetilde{q}_i)\Delta^{-1}(\widetilde{L}^*)^j(\widetilde{r}_i) \\
& +T_D(q)(X^{(1)}_{k-1})(q)\Delta^{-1}\Lambda(q^{-1}) + \frac{k}{2}T_D(q)(L^{k-2}(q_i)\Delta^{-1}(\widetilde{L}^*)^0 r_i)(q)\Delta^{-1}\Lambda(q^{-1}) \\
= \;& \widetilde{X}^{(1)}_{k-1} - (\widetilde{L}^{k-1})_- + \left(T_D(q)(X^{(1)}_{k-1} + \frac{k}{2}L^{k-1}(q_i))\right)(q)\Delta^{-1}\Lambda(q^{-1}).
\end{aligned}$$

Here we use the relation of $T_D(q_a)(q_a) = 0$. □

Lemma 5.4.

$$\partial^*_k T_D(q)\cdot T_D^{-1}(q) = -\Lambda(q)\Delta\left(q^{-1}((M_\Delta L^k)_+ + \frac{k}{2}L^{k-1} + X^{(1)}_{k-1})\right)(q)\Delta^{-1}\Lambda(q^{-1}). \qquad (5.9)$$

Proof. By (4.7),

$$\begin{aligned}
\partial^*_k T_D(q)\cdot T_D(q)^{-1} &= -T_D(q)\partial^*_k(T_D(q)^{-1}) \\
&= -\Lambda(q)\Delta q^{-1}\partial^*_k(q)\Delta^{-1}\Lambda(q^{-1}) - \Lambda(q\partial^*_k(q^{-1})) \\
&= -\Lambda(q)\left(\Lambda(q^{-1}\partial^*_k(q))\Delta + \Delta(q^{-1}\partial^*_k(q))\right)\Delta^{-1}\Lambda(q^{-1}) - \Lambda(q\partial^*_k(q^{-1})) \\
&= -\Lambda(q)\Delta(q^{-1}\partial^*_k(q))\Delta^{-1}\Lambda(q^{-1}) \\
&= -\Lambda(q)\Delta\left(q^{-1}((M_\Delta L^k)_+ + \frac{k}{2}L^{k-1} + X^{(1)}_{k-1})\right)(q)\Delta^{-1}\Lambda(q^{-1}).
\end{aligned}$$

□

Theorem 5.2. *The additional symmetry flows (4.2) for the cdKP hierarchy ((2.14) for $m=1, l=1$) commute with the difference type gauge transformation $T_D(q)$ preserving the form of cdKP , up to shifting of (4.2) by ordinary time flows, that is,*

$$\partial_k^* \widetilde{L} = \left[-(\widetilde{M}_\Delta \widetilde{L}^k)_- + \widetilde{X}_{k-1}^{(1)}, \widetilde{L}\right] + \frac{\partial \widetilde{L}}{\partial t_{k-1}}. \tag{5.10}$$

Proof. Firstly, by (4.2),

$$\begin{aligned}
\partial_k^* \widetilde{L} &= \partial_k^* \left(T_D(q) L T_D^{-1}(q)\right) \\
&= \partial_k^* T_D(q) \cdot L T_D(q)^{-1} + T_D(q) \partial_k^* L \cdot T_D^{-1}(q) - T_D(q) L T_D^{-1}(q) \cdot \partial_k^* T_D(q) \cdot T_D^{-1}(q) \\
&= \partial_k^* T_D(q) \cdot T_D^{-1}(q) \widetilde{L} + T_D(q) [-(M_\Delta L^k)_- + X_{k-1}^{(1)}, L] T_D^{-1}(q) - \widetilde{L} T_D^{-1}(q) \cdot \partial_k^* T_D(q) \\
&= \left[T_D(q) \left(-(M_\Delta L^k)_- + X_{k-1}^{(1)}\right) T_D^{-1}(q) + \partial_k^* T_D(q) \cdot T_D(q)^{-1}, \widetilde{L}\right].
\end{aligned} \tag{5.11}$$

Then with the help of (5.8), (5.9), and the following useful formula (3.9) in [27], we have

$$\begin{aligned}
& T_D(q) \left(-(M_\Delta L^k)_- + X_{k-1}^{(1)}\right) T_D^{-1}(q) + \partial_k^* T_D(q) \cdot T_D^{-1}(q) \\
&= -T_D(q)(M_\Delta L^k)_- T_D^{-1}(q) + T_D(q) X_{k-1}^{(1)} T_D^{-1}(q) + \partial_k^* T_D(q) \cdot T_D^{-1}(q) \\
&= -T_D(q)(M_\Delta L^k)_- T_D^{-1}(q) + \widetilde{X}_{k-1}^{(1)} + \left(T_D(q)(X_{k-1}^{(1)} + \frac{k}{2} L^{k-1}(q_i))\right)(q) \Delta^{-1} \Lambda(q^{-1}) \\
&\quad - (\widetilde{L}^{k-1})_- - T_D(q) \left\{(M_\Delta L^k)_+ + \frac{k}{2} L^{k-1} + X_{k-1}^{(1)}\right\} (T_D^{-1}(q)) \\
&= -(\widetilde{M}_\Delta \widetilde{L}^k)_- + \widetilde{X}_{k-1}^{(1)} - (\widetilde{L}^{k-1})_-,
\end{aligned} \tag{5.12}$$

where the following relation [28] is used,

$$(\widetilde{L}^{k-1})_- = \sum_{j=0}^{k-2} \widetilde{L}^{k-j-2}(\widetilde{q}) \Delta^{-1} \widetilde{L}^j(\widetilde{r}). \tag{5.13}$$

In the above process,

$$\begin{aligned}
& -T_D(q)(M_\Delta L^k)_- T_D^{-1}(q) - T_D(q)(M_\Delta L^k)_+ (T_D^{-1}(q)) \\
&= -T_D(q)(M_\Delta L^k) T_D^{-1}(q) + T_D(q)(M_\Delta L^k)_+ T_D^{-1}(q) - T_D(q)(M_\Delta L^k)_+ (T_D^{-1}(q)) \\
&= -(\widetilde{M}_\Delta \widetilde{L}^k) + (\widetilde{M}_\Delta \widetilde{L}^k)_+ \\
&= -(\widetilde{M}_\Delta \widetilde{L}^k)_-,
\end{aligned}$$

which can be got by means of the identities (3.28) of Lemma.3.3.

At last, the substituting (5.12) into (5.11) gives rise to

$$\begin{aligned}
\partial_k^* \widetilde{L} &= \left[-(\widetilde{M_\Delta \widetilde{L}^k})_- + \widetilde{X}_{k-1}^{(1)} - (\widetilde{L}^{k-1})_-, \widetilde{L}\right] \\
&= \left[-(\widetilde{M_\Delta \widetilde{L}^k})_- + \widetilde{X}_{k-1}^{(1)}, \widetilde{L}\right] - \left[(\widetilde{L}^{k-1})_-, \widetilde{L}\right] \\
&= \left[-(\widetilde{M_\Delta \widetilde{L}^k})_- + \widetilde{X}_{k-1}^{(1)}, \widetilde{L}\right] + \frac{\partial \widetilde{L}}{\partial t_{k-1}}.
\end{aligned}$$

□

Remark 2: When m is not 1, Theorem 5.1 and Theorem 5.2 do not hold. We make a example for the integral type transformations $T_I(r)$ in Theorem 5.1. In fact, when $m>1$, (5.2) will become into

$$T_I(r_a)X^{(1)}_{k-1}T_I(r_a)^{-1}=\widetilde{X}^{(1)}_{k-1}+\sum_{j=0}^{k-2}\widetilde{L}^{k-j-2}(\widetilde{q_a})\Delta^{-1}\widetilde{L}^j(\widetilde{r_a})$$
$$+\Lambda^{-1}(r_a^{-1})\Delta^{-1}\left\{T_I(r_a)^{*-1}(X^{(1)}_{k-1}-\frac{k}{2}L^{k-1})^*(r_a)\right\}, \qquad (5.14)$$

where r_a is one of the adjoint eigenfunctions in (2.14). The term $\sum_{j=0}^{k-2}\widetilde{L}^{k-j-2}(\widetilde{q_a})\Delta^{-1}\widetilde{L}^j(\widetilde{r_a})$ in (5.14) will cause the difficulty for the proof of Theorem 5.1. Actually, for $m>1$ [13],

$$(\widetilde{L}^{k-1})_-=\sum_{a=1}^{m}\sum_{j=0}^{k-2}\widetilde{L}^{k-j-2}(\widetilde{q_a})\Delta^{-1}\widetilde{L}^j(\widetilde{r_a}),$$

so the term $\sum_{j=0}^{k-2}\widetilde{L}^{k-j-2}(\widetilde{q_a})\Delta^{-1}\widetilde{L}^j(\widetilde{r_a})$ can not be written as $(\widetilde{L}^{k-1})_-$ like the condition of $m=1$, which is corresponding to

$$\partial_{t_{k-1}}\widetilde{L}=-[\widetilde{L}^{k-1}_-,\widetilde{L}].$$

So for $m>1$, Theorem 5.1 does not hold.
Remark 3: For simplicity, it is proved for $l=1$ in Theorem 5.1 and Theorem 5.2. But for $l\neq 1$ it is also satisfied for Theorem 5.1 and Theorem 5.2.

6. Conclusions And Discussions

After some technical identities of two types gauge transformations of the cdKP hierarchy, the interplay of the integral type gauge transformation T_I and the difference type gauge transformation T_D with the additional symmetry at the instance of the cdKP hierarchy are obtained in Theorem 5.1, Theorem 5.2 (see (5.4, 5.10)), which preserves the form of the additional symmetry of the cdKP hierarchy, up to shifting of the corresponding additional flows by ordinary time flows. Nonetheless the shifting is different from the integral type gauge transformation and the difference type gauge transformation of the cdKP hierarchy. It reflects one of the intrinsic features for the cdKP hierarchy. These results provide a mathematical background from the point of view of integrable systems of the potential applications in physics for the additional symmetry flows of the cdKP hierarchy.

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