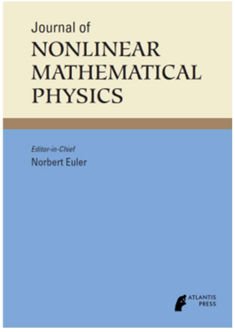

Journal of Nonlinear Mathematical Physics

ISSN (Online): 1776-0852 **ISSN (Print): 1402-9251**
Journal Home Page: https://www.atlantis-press.com/journals/jnmp

The number of independent traces and supertraces on symplectic reflection algebras

S.E. Konstein, I.V. Tyutin

To cite this article: S.E. Konstein, I.V. Tyutin (2014) The number of independent traces and supertraces on symplectic reflection algebras, Journal of Nonlinear Mathematical Physics 21:3, 308–335, DOI: https://doi.org/10.1080/14029251.2014.936755

To link to this article: https://doi.org/10.1080/14029251.2014.936755

Published online: 04 January 2021

Journal of Nonlinear Mathematical Physics, Vol. 21, No. 3 (September 2014), 308-335

The number of independent traces and supertraces on symplectic reflection algebras

S.E. Konstein

I.E. Tamm Department of Theoretical Physics, P.N. Lebedev Physical Institute, 53, Leninsky Prospect Moscow, 117924, Russia
konstein@lpi.ru

I.V. Tyutin

I.E. Tamm Department of Theoretical Physics, P.N. Lebedev Physical Institute, 53, Leninsky Prospect Moscow, 117924, Russia
tyutin@lpi.ru

Received 15 August 2013

Accepted 19 March 2014

It is shown that $A := H_{1,\eta}(G)$, the sympectic reflection algebra over $\mathbb{C}$, has T_G independent traces, where T_G is the number of conjugacy classes of elements without eigenvalue 1 belonging to the finite group $G \subset Sp(2N) \subset End(\mathbb{C}^{2N})$ generated by the system of symplectic reflections.

Simultaneously, we show that the algebra A, considered as a superalgebra with a natural parity, has S_G independent supertraces, where S_G is the number of conjugacy classes of elements without eigenvalue -1 belonging to G.

We consider also A as a Lie algebra A^L and as a Lie superalgebra A^S.

It is shown that if A is a simple associative algebra, then the supercommutant $[A^S, A^S]$ is a simple Lie superalgebra having at least S_G independent supersymmetric invariant non-degenerate bilinear forms, and the quotient $[A^L, A^L]/([A^L, A^L] \cap \mathbb{C})$ is a simple Lie algebra having at least T_G independent symmetric invariant non-degenerate bilinear forms.

Keywords: Symplectic reflection algebra; Cherednik algebra; trace; supertrace; invariant bilinear form.

2000 Mathematics Subject Classification: 17B80, 16W55

1. Introduction

In [7], it was shown that $H_{W(\mathfrak{R})}(\eta)$, the algebra of observables of the rational Calogero model based on the root system $\mathfrak{R} \subset \mathbb{R}^N$, has $T_\mathfrak{R}$ independent traces, where $T_\mathfrak{R}$ is the number of conjugacy classes of elements without eigenvalue 1 belonging to the Coxeter group $W(\mathfrak{R}) \subset End(\mathbb{R}^N)$ generated by the root system $\mathfrak{R}$, and that the algebra $H_{W(\mathfrak{R})}(\eta)$, considered as a superalgebra with a natural parity, has $S_\mathfrak{R}$ independent supertraces, where $S_\mathfrak{R}$ is the number of conjugacy classes of elements without eigenvalue -1 belonging to $W(\mathfrak{R})$.

Unlike the case of finite-dimensional associative algebras, the presence of several (super)traces on the infinite-dimensional superalgebra $H_{W(\mathfrak{R})}(\eta)$ in the case of irreducible $\mathfrak{R}$ does not necessarily imply violation of simplicity except certain particular values of parameter(s) η.

It is easy to show that $H_{W(\mathfrak{R})}(\eta) = H_{1,\eta}(W(\mathfrak{R}))$, where $H_{t,\eta}(G)$ is a symplectic reflection algebra introduced in [2] for any finite group $G \subset Sp(2N)$ generated by symplectic reflections. Here we extend the results of [7] from $H_{1,\eta}(W(\mathfrak{R}))$ to $H_{1,\eta}(G)$.

Besides, we consider Lie (super)algebras generated by $H_{1,\eta}(G)$ and invariant (super)symmetric bilinear forms on these Lie (super)algebras generated by the traces and supertraces.

2. Preliminaries

2.1. *Traces*

Let $\mathscr{A}$ be an associative superalgebra with parity π. All expressions of linear algebra are given for homogenous elements only and are supposed to be extended to inhomogeneous elements via linearity.

A linear function str on $\mathscr{A}$ is called a *supertrace* if

$$str(fg) = (-1)^{\pi(f)\pi(g)} str(gf) \text{ for all } f,g \in \mathscr{A}.$$

A linear function tr on $\mathscr{A}$ is called a *trace* if

$$tr(fg) = tr(gf) \text{ for all } f,g \in \mathscr{A}.$$

Let $\varkappa = \pm 1$. We can unify the definitions of trace and supertrace by introducing a $\varkappa$-trace. We say that a linear function[a] sp on $\mathscr{A}$ is a *$\varkappa$-trace* if

$$sp(fg) = \varkappa^{\pi(f)\pi(g)} sp(gf) \text{ for all } f,g \in \mathscr{A}. \tag{2.1}$$

A linear function L is *even* (resp. *odd*) if $L(f) = 0$ for any odd (resp. even) $f \in \mathscr{A}$.

Clearly, any linear function L can be decomposed in the sum $L = L_+ + L_-$ of even linear function L_+ and odd linear function L_-.

Observe that each odd trace is simultaneously an odd supertrace and vice versa.

Let $\mathscr{A}_1$ and $\mathscr{A}_2$ be associative superalgebras with parities π_1 and π_2, respectively.

Define their tensor product[b] $\mathscr{A} = \mathscr{A}_1 \otimes \mathscr{A}_2$ as a superalgebra with the product

$$(a_1 \otimes a_2)(b_1 \otimes b_2) = (a_1 b_1) \otimes (a_2 b_2) \text{ for any } a_1, b_1 \in \mathscr{A}_1,\ a_2, b_2 \in \mathscr{A}_2$$

and the parity π defined by the formula $\pi(a_1 \otimes a_2) = \pi_1(a_1) + \pi_2(a_2)$.

Let T_i be a trace on $\mathscr{A}_i$. Clearly, the function T such that $T(a \otimes b) = T_1(a) T_2(b)$ is a trace on $\mathscr{A}$.

Let S_i be an *even* supertrace on $\mathscr{A}_i$. Clearly, the function S such that $S(a \otimes b) = S_1(a) S_2(b)$ is an even supertrace on $\mathscr{A}$.

In what follows, we use three types of brackets:

$$\begin{aligned}
[f,g] &= fg - gf,\\
\{f,g\} &= fg + gf,\\
[f,g]_\varkappa &= fg - \varkappa^{\pi(f)\pi(g)} gf.
\end{aligned}$$

Every $\varkappa$-trace $sp(\cdot)$ on superalgebra $\mathscr{A}$ generates the following bilinear form on $\mathscr{A}$:

$$B_{sp}(f,g) := sp(fg) \text{ for any } f,g \in \mathscr{A}. \tag{2.2}$$

---

[a]From the German word *Spur*.

[b]In this paper we do not need *the supertensor product* introduced by setting

$$(a_1 \otimes a_2)(b_1 \otimes b_2) = (-1)^{\pi_1(b_1)\pi_2(a_2)} (a_1 b_1) \otimes (a_2 b_2) \text{ for any } a_1, b_1 \in \mathscr{A}_1,\ a_2, b_2 \in \mathscr{A}_2.$$

It is obvious that if such a bilinear form B_{sp} is degenerate, then the null-vectors of B_{sp} (i.e., $v \in \mathscr{A}$ such that $B(v,x) = 0$ for any $x \in \mathscr{A}$) constitute the two-sided ideal $\mathscr{I} \subset \mathscr{A}$. If the $\varkappa$-trace generating degenerate bilinear form is homogeneous (even or odd), then the corresponding ideal is a superalgebra.

If $\varkappa = -1$, the ideals of this sort are present, for example, in the superalgebras $H_{1,\eta}(A_1)$ (corresponding to the two-particle Calogero model) at $\eta = k + \frac{1}{2}$, see [13], and in the superalgebras $H_{1,\eta}(A_2)$ (corresponding to three-particle Calogero model) at $\eta = k + \frac{1}{2}$ and $\eta = k \pm \frac{1}{3}$, see [5], for every integer k. For all other values of η all supertraces on these superalgebras generate non-degenerate bilinear forms (2.2).

The general case of $H_{1,\eta}(A_{n-1})$ for arbitrary n is considered in [10]. Theorem 5.8.1 of [10] states that the associative algebra $H_{1,\eta}(A_{n-1})$ is not simple if and only if $\eta = \frac{q}{m}$, where q,m are mutually prime integers such that $1 < m \leqslant n$, and presents the structure of corresponding ideals.

The dimension of the space of supertraces on $H_{1,\eta}(A_{n-1})$ is the number of partitions of $n \geqslant 1$ into the sum of different positive integers, see [9], and the space of the traces on $H_{W(A_{n-1})}(\eta)$ is one-dimensional for $n \geqslant 2$ due to Theorem 5.2, see also [8].

So, every algebra $H_{1,\eta}(A_{n-1})$ with $\eta \neq \frac{q}{m}$, where q,m are mutually prime integers, $1 < m \leqslant n$, and $n \geq 2$, is an example of simple superalgebra with several independent supertraces (see also [6]).

Conjecture 2.1. *Each of the ideals of $H_{1,\eta}(A_{n-1})$ is the set of null-vectors of the degenerate bilinear form (2.2) for some $\varkappa$-trace sp on $H_{1,\eta}(A_{n-1})$.*

More examples of associative simple (super)algebra with several (super)traces are presented in Section 6.

2.2. *Symplectic reflection group*

Let $V = \mathbb{C}^{2N}$ be endowed with a non-degenerate anti-symmetric $Sp(2N)$-invariant bilinear form $\omega(\cdot,\cdot)$, let the vectors $a_i \in V$, where $i = 1, \dots, 2N$, constitute a basis in V.

The matrix $(\omega_{ij}) := \omega(a_i, a_j)$ is anti-symmetric and non-degenerate.

Let x^i be the coordinates of $x \in V$, i.e., $x = a_i x^i$. Then $\omega(x, y) = \omega_{ij} x^i y^j$ for any $x, y \in V$. The indices i are raised and lowered by means of the forms (ω_{ij}) and (ω^{ij}), where $\omega_{ij}\omega^{kj} = \delta_i^k$.

Definition 2.1. The element $R \in Sp(2N) \subset EndV$ is called a *symplectic reflection*, if $rk(R-1) = 2$.

Definition 2.2. Any finite subgroup G of $Sp(2N)$ generated by a set of symplectic reflections is called a *symplectic reflection group*.

We collect some elementary properties of the elements of the symplectic reflection group in the following Proposition.

Proposition 2.1. *Let G be a symplectic reflection group and $g \in G$. Then*

(1) The Jordan normal form of g is diagonal.
(2) Each eigenvalue of g is a root of unity.
(3) $\det g = 1$.
(4) If λ is an eigenvalue of g, then λ^{-1} is also an eigenvalue of g.
(5) The spectrum of g has an even number of -1 and an even number of $+1$.
(6) $g^{tr}\omega g = \omega$, where g^{tr} is the transposed of g, or, equivalently, $g_i^k\, \omega_{kl}\, g_j^l = \omega_{ij}$.

Clearly, each item of Proposition 2.1 follows either from the fact that $G \subset Sp(2N)$ or from the fact that G is a finite group. Item 6 is just the defining property of $Sp(2N)$.

In what follows, G stands for a symplectic reflection group, and $\mathscr{R}$ stands for the set of all symplectic reflections in G.

Let $R \in \mathscr{R}$. Set[c]

$$V_R := Im(R-1)\,, \tag{2.3}$$
$$Z_R := Ker(R-1)\,. \tag{2.4}$$

Clearly, V_R and Z_R are symplectically perpendicular, i.e., $\omega(V_R, Z_R) = 0$, and $V = V_R \oplus^{\perp} Z_R$. Hereafter the expression $U \oplus^{\perp} W$ denotes a direct sum whose summands are symplectically perpendicular to each other.

So, let $x = x_{V_R} + x_{Z_R}$ for any $x \in V$, where $x_{V_R} \in V_R$ and $x_{Z_R} \in Z_R$. Set

$$\omega_R(x,y) := \omega(x_{V_R}, y_{V_R}). \tag{2.5}$$

Item 5 of Proposition 2.1 allows one to introduce the following grading on $\mathbb{C}[G]$. Recall that $\varkappa = \pm 1$, and that we consider both values of $\varkappa$.

Definition 2.3. Let the grading E on $\mathbb{C}[G]$ be defined by the formula

$$E(g) := \frac{1}{2}\dim \mathscr{E}(g) \text{ for any } g \in G, \tag{2.6}$$

where

$$\mathscr{E}(g) := Ker(g - \varkappa). \tag{2.7}$$

For any $g \in G$, the number $E(g)$ is an integer such that $0 \leqslant E(g) \leqslant N$.

The following Lemma is crucial in what follows.[d]

Lemma 2.1. *Let $g \in G$, $R \in \mathscr{R}$. If there exist $c^1, c^2 \in Ker(g-\varkappa)$ such that $\omega_R(c^1, c^2) \neq 0$, then*

$$E(Rg) = E(g) - 1. \tag{2.8}$$

Besides,

$$\mathscr{E}(Rg) = Z_R \cap \mathscr{E}(g). \tag{2.9}$$

Proof. Clearly, $Rgx = \varkappa Rx = \varkappa x$ if $gx = \varkappa x$ and $Rx = x$. Hence, $Z_R \cap \mathscr{E}(g) \subset \mathscr{E}(Rg)$.

Denote

$$\mathscr{E}_R(g) := Z_R \cap \mathscr{E}(g). \tag{2.10}$$

Since $\omega_R(c^1, c^2) \neq 0$, it follows that the vectors $c^l_{V_R} \in V_R$, where $l = 1,2$, are independent, so the $c^l_{V_R}$ constitute a basis of V_R (recall that $\dim V_R = 2$).

Clearly, $\mathscr{E}_R(g) \oplus span(c^1, c^2) = \mathscr{E}(g)$, implying $\dim \mathscr{E}_R(g) + 2 = \dim \mathscr{E}(g)$.

[c] Hereafter we denote all the units in groups, algebras, etc, by 1, and $c \cdot 1$ by c for any number c.

[d] An analogous Lemma is proved in [7] for the *real orthogonal* matrices and *reflections* in $\mathbb{R}^N$.

It remains to prove that $\mathscr{E}_R(g) = \mathscr{E}(Rg)$.

Suppose that there exists a vector $u \in \mathscr{E}(Rg)$ such that $u \notin \mathscr{E}_R(g)$. Since $\dim \mathscr{E}(Rg)$ is even, our supposition implies that there exist two vectors u^1, $u^2 \in \mathscr{E}(Rg)$ such that $span(u^1, u^2) \cap \mathscr{E}_R(g) = 0$.

Let $Z_{\text{rem}} \subset Z_R$ be subspace of Z_R such that $Z_R = Z_{\text{rem}} \oplus \mathscr{E}_R(g)$, so $\dim Z_{\text{rem}} = 2s$, where $s := N - E(g)$. Then

$$V = V_R \oplus^{\perp} [Z_{\text{rem}} \oplus \mathscr{E}_R(g)]. \tag{2.11}$$

The vectors c^l ($l = 1, 2$) can be decomposed according to decomposition (2.11):

$$c^l = c^l_{V_R} + c^l_{\text{rem}} + c^l_{\mathscr{E}_R(g)}. \tag{2.12}$$

Define a linear map $\rho: \ V_R \mapsto Z_{\text{rem}}$ by the formula

$$\rho c^l_{V_R} = c^l_{\text{rem}}. \tag{2.13}$$

Clearly, $x + \rho x \in \mathscr{E}(g)$ for each $x \in V_R$.

In the decomposition (2.11) the matrices of R and g have the block forms

$$R = \begin{pmatrix} R_{2\times 2} & 0 & 0 \\ 0 & 1_{2s\times 2s} & 0 \\ 0 & 0 & 1_{(2E(g)-2)\times(2E(g)-2)} \end{pmatrix}, \ g = \begin{pmatrix} g_{11} & g'_{12} & 0 \\ g'_{21} & g'_{22} & 0 \\ g'_{31} & g'_{32} & \varkappa_{(2E(g)-2)\times(2E(g)-2)} \end{pmatrix}, \tag{2.14}$$

where the blocks of g are of the same sizes as those of R.

From previous consideration we know that g in (2.14) has 2-dimensional space of eigenvectors c with eigenvalue $\varkappa$, which can be written in the form

$$c = \begin{pmatrix} x \\ \rho x \\ 0 \end{pmatrix}, \text{ where } \begin{pmatrix} x \\ 0 \\ 0 \end{pmatrix} \in V_R, \ \begin{pmatrix} 0 \\ \rho x \\ 0 \end{pmatrix} \in Z_{\text{rem}},$$

i.e., the following relations take place

$$g_{11} = \varkappa - g'_{12}\rho, \quad g'_{21} = (\varkappa - g'_{22})\rho, \quad g'_{31} = -g'_{32}\rho.$$

Let us look for null-vectors u of $Rg - \varkappa$ on $V_R \oplus^{\perp} Z_{\text{rem}}$ in the form

$$u = \begin{pmatrix} x \\ \rho x + z \\ 0 \end{pmatrix},$$

which is, in fact, a general form of such u.

Since

$$Rg = \begin{pmatrix} R_{2\times 2}g_{11} & R_{2\times 2}g'_{12} & 0 \\ g'_{21} & g'_{22} & 0 \\ g'_{31} & g'_{32} & \varkappa \end{pmatrix},$$

the equation

$$(Rg-\varkappa)c_R = 0 \tag{2.15}$$

gives

$$\varkappa(R_{2\times 2}-1)x + R_{2\times 2}g'_{12}z = 0, \tag{2.16}$$

$$(g'_{22}-\varkappa)z = 0, \quad g'_{32}z = 0. \tag{2.17}$$

So, if eqs. (2.17) do not have any nontrivial solutions, eq. (2.15) has no nontrivial solutions either. If eqs. (2.17) have null-vectors z_0 such that $g'_{12}z_0 = 0$, then eq. (2.16) shows that $x = 0$, and we see that $\begin{pmatrix} 0 \\ z_0 \\ 0 \end{pmatrix} \in Z_{\mathrm{rem}} \cap \mathscr{E}_R(g)$, which is impossible.

So, the only opportunity for Rg to have eigenvalue $\varkappa$ with multiplicity $> 2E(g)-2$ (see eq. (2.14)), is the existence of a vector

$$u = \begin{pmatrix} 0 \\ z \\ 0 \end{pmatrix} \in Z_{\mathrm{rem}}$$

which satisfies eqs. (2.17) and $g'_{12}z \neq 0$, i.e.,

$$(g-\varkappa)u = \begin{pmatrix} g'_{12}z \\ (g'_{22}-\varkappa)z \\ g'_{32}z \end{pmatrix} = \begin{pmatrix} g'_{12}z \\ 0 \\ 0 \end{pmatrix} \in V_R.$$

Because the multiplicity of $\varkappa$ in the spectrum of Rg is even, the supposition that $Rg-\varkappa$ has null-vectors besides $\mathscr{E}_R(g)$ leads to existence of a 2-dimensional subspace $Z_0 \subset Z_{\mathrm{rem}}$ such that

$$(g-\varkappa)Z_0 = V_R. \tag{2.18}$$

Suppose that $Z_0 \neq 0$. Represent Z_{rem} in the form $Z_{\mathrm{rem}} = Z_0 \oplus Z_{\mathrm{r}}$. In the basis for decomposition $V = V_R \oplus^{\perp} (Z_0 \oplus Z_{\mathrm{r}} \oplus \mathscr{E}_R(g))$, the matrix g has the form

$$g = \begin{pmatrix} g_{11} & g_{12} & g_{13} & 0 \\ g_{21} & g_{22} & g_{23} & 0 \\ g_{31} & g_{32} & g_{33} & 0 \\ g_{41} & g_{42} & g_{43} & \varkappa \end{pmatrix},$$

where

$$\begin{pmatrix} g_{21} \\ g_{31} \end{pmatrix} = g'_{21},\ g_{41} = g'_{31},\ \begin{pmatrix} g_{12} & g_{13} \end{pmatrix} = g'_{12},$$

$$\begin{pmatrix} g_{22} & g_{23} \\ g_{32} & g_{33} \end{pmatrix} = g'_{22},\ \begin{pmatrix} g_{42} & g_{43} \end{pmatrix} = g'_{32},$$

$$g_{11} = \varkappa - g_{12}\rho_1 - g_{13}\rho_2,$$
$$g_{21} = (\varkappa - g_{22})\rho_1 - g_{23}\rho_2,\ g_{31} = (\varkappa - g_{33})\rho_2 - g_{32}\rho_1,$$
$$g_{41} = -g_{42}\rho_1 - g_{43}\rho_2,$$

and where ρ_1 and ρ_2 give the decomposition of ρ:

$$\rho x = \rho_1 x + \rho_2 x, \text{ where } \rho_1 x \in Z_0\ ,\ \ \rho_2 x \in Z_{\mathrm{r}}.$$

Due to condition (2.18), the matrix g acquires the form

$$g = \begin{pmatrix} g_{11} & g_{12} & g_{13} & 0 \\ g_{21} & \varkappa & g_{23} & 0 \\ g_{31} & 0 & g_{33} & 0 \\ g_{41} & 0 & g_{43} & \varkappa \end{pmatrix},\ \det g_{12} \neq 0, \tag{2.19}$$

and the symplectic form ω has the shape $\omega = \begin{pmatrix} \omega^R_{2\times 2} & 0 & 0 & 0 \\ 0 & \omega^{22}_{2\times 2} & \omega^{23} & \omega^{24} \\ 0 & \omega^{32} & \omega^{33} & \omega^{34} \\ 0 & \omega^{42} & \omega^{43} & \omega^{44} \end{pmatrix}$, where $\omega^R_{2\times 2}$ is non-degenerate. Due to (2.19), the equality $\omega = g^{\mathrm{tr}}\omega g$ gives for the 22-block:

$$\omega^{22}_{2\times 2} = \begin{pmatrix} * & * & * & * \\ g^{\mathrm{tr}}_{12} & \varkappa & 0 & 0 \\ * & * & * & * \\ 0 & 0 & 0 & \varkappa \end{pmatrix} \begin{pmatrix} \omega^R_{2\times 2} & 0 & 0 & 0 \\ 0 & \omega^{22}_{2\times 2} & * & * \\ 0 & * & * & * \\ 0 & * & * & * \end{pmatrix} \begin{pmatrix} * & g_{12} & * & 0 \\ * & \varkappa & * & 0 \\ * & 0 & * & 0 \\ * & 0 & * & \varkappa \end{pmatrix}\Bigg|_{22} =$$

$$= \begin{pmatrix} * & * & * & * \\ g^{\mathrm{tr}}_{12} & \varkappa & 0 & 0 \\ * & * & * & * \\ 0 & 0 & 0 & \varkappa \end{pmatrix} \begin{pmatrix} * & \omega^R_{2\times 2} g_{12} & * & * \\ * & \varkappa\omega^{22}_{2\times 2} & * & * \\ * & * & * & * \\ * & * & * & * \end{pmatrix}\Bigg|_{22} = g^{\mathrm{tr}}_{12}\omega^R_{2\times 2} g_{12} + \omega^{22}_{2\times 2}$$

or $g^{tr}_{12}\omega^R_{2\times 2} g_{12} = 0$ which contradicts the nondegeneracy of g_{12}.

So, $Z_0 = 0$ and the matrix $Rg - \varkappa$ has no null-vectors besides $\mathscr{E}_R(g)$. □

3. Symplectic reflection algebra

The superalgebra $H_{1,\eta}(G)$ is a deform of the skew product[e] of the Weyl algebra W_N and the group algebra of a finite subgroup $G \subset Sp(2N)$ generated by symplectic reflections, see Definition 3.1 below.

3.1. *Definitions*

Let $\mathbb{C}[G]$ be the *group algebra* of G, i.e., the set of all linear combinations $\sum_{g\in G}\alpha_g \bar{g}$, where $\alpha_g \in \mathbb{C}$, and we temporarily write $\bar{g}$ to distinguish g considered as an element of $G \subset End(V)$ from the same

[e]Let $\mathscr{A}$ and $\mathscr{B}$ be superalgebras, and $\mathscr{A}$ a $\mathscr{B}$-module. We say that the superalgebra $\mathscr{A} * \mathscr{B}$ is a *skew product* of $\mathscr{A}$ and $\mathscr{B}$ if $\mathscr{A} * \mathscr{B} = \mathscr{A} \otimes \mathscr{B}$ as a superspace and $(a_1 \otimes b_1) * (a_2 \otimes b_2) = a_1 b_1(a_2) \otimes b_1 b_2$. The element $b_1(a_2)$ may include a sign factor imposed by the Sign Rule.

element $\bar{g} \in \mathbb{C}[G]$ considered as an element of the group algebra. The addition in $\mathbb{C}[G]$ is defined as follows:

$$\sum_{g\in G} \alpha_g \bar{g} + \sum_{g\in G} \beta_g \bar{g} = \sum_{g\in G} (\alpha_g + \beta_g)\bar{g}$$

and the multiplication is defined by setting $\overline{g_1}\,\overline{g_2} = \overline{g_1 g_2}$.

Let η be a function on $\mathscr{R}$, i.e., a set of constants η_R with $R \in \mathscr{R}$ such that $\eta_{R_1} = \eta_{R_2}$ if R_1 and R_2 belong to one conjugacy class of G.

Definition 3.1. The algebra $H_{t,\eta}(G)$, where $t \in \mathbb{C}$, is an associative algebra with unity **1**; it is the algebra $\mathbb{C}[V]$ of polynomials in the elements of V with coefficients in the group algebra $\mathbb{C}[G]$ subject to the relations

$$gx = g(x)g \text{ for any } g \in G \text{ and } x \in V, \text{ where } g(x) = a_i g^i_j x^j \text{ for } x = a_i x^i, \tag{3.1}$$

$$[x,y] = t\omega(x,y) + \sum_{R\in\mathscr{R}} \eta_R \omega_R(x,y) R \text{ for any } x,y \in V. \tag{3.2}$$

The algebra $H_{t,\eta}(G)$ is called a *symplectic reflection algebra*, see [2].

The commutation relations (3.2) suggest to define the *parity* π by setting:

$$\pi(x) = 1,\ \pi(g) = 0 \text{ for any } x \in V, \text{ and } g \in G, \tag{3.3}$$

enabling one to consider $H_{1,\eta}(G)$ as an associative *superalgebra*.

We consider the case $t \neq 0$ only, which is equivalent to the case $t = 1$.

3.2. *Bases of eigenvectors*

We say that a polynomial $f \in \mathbb{C}[V]$ is *monomial* if it can be expressed in the form $f = u_1 u_2 ... u_k$, where $u_i \in V$.

We say that an element $h \in H_{1,\eta}(G)$ is *monomial* if it can be expressed in the form $h = u_1 u_2 ... u_k g$, where $u_i \in V$ and $g \in G$.

Due to item 1 of Proposition 2.1, for each $g \in G$, there exists a basis $\mathfrak{B}_g = \{b_1, \dots, b_{2N}\}$ of V such that (no summation here)

$$g(b_I) = \lambda_I b_I, \text{ where } I = 1,2,...,2N, \tag{3.4}$$

or, equivalently,

$$g b_I = \lambda_I b_I g. \tag{3.5}$$

We can represent any element $h \in H_{1,\eta}(G)$ in the form $h = \sum_{g\in G} h_g g$, where the polynomials h_g depend on $b_I \in \mathfrak{B}_g$.

Definition 3.2. Let $b_I \in \mathfrak{B}_g$. A monomial $b_{I_1} \dots b_{I_k} g$ is said to be *regular* if $\lambda_{I_s} \neq \varkappa$ for some s, where $1 \leq s \leq k$, and *special* if $\lambda_{I_s} = \varkappa$ for each s, where $1 \leq s \leq k$.

Set also:

$$F_{IJ} := [b_I, b_J], \tag{3.6}$$

$$\mathscr{C}_{IJ} := \omega(b_I, b_J), \tag{3.7}$$

$$f_{IJ} := [b_I, b_J] - \mathscr{C}_{IJ}. \tag{3.8}$$

Lemma 3.1. *Let $g \in G$. Let $b_I, b_J \in \mathscr{E}(g)$. Then*

$$E(f_{IJ}g) = E(g) - 1. \tag{3.9}$$

Proof. Proof follows from eq. (3.2) (recall that $t = 1$) and Lemma 2.1. □

3.3. *Partial orderings in* $H_{1,\eta}(G)$

Definition 3.3. Let $f_1, f_2 \in \mathbb{C}[V]$ be monomials either both even or both odd. Let $g_1, g_2 \in G$. We say $f_1 g_1 < f_2 g_2$ if either $deg f_1 < deg f_2$ or $deg f_1 = deg f_2$ and $E(g_1) < E(g_2)$.

It is easy to describe all *minimal elements* in $H_{1,\eta}(G)$, i.e., the elements f_{min} such that there exists $f \in H_{1,\eta}(G)$ such that $f_{min} < f$, and there are no elements $f_<$ such that $f_< < f_{min}$:

a) In the even subspace of $H_{1,\eta}(G)$ the minimal elements are $g \in G$ such that $E(g) = 0$.

b) In the odd subspace of $H_{1,\eta}(G)$, the minimal elements are the elements of the form xg, where $x \in V$, $g \in G$ and $E(g) = 0$.

Proposition 3.1. *Let $\varkappa = 1$. Then for each trace tr and for each odd minimal element xg, the following equality takes place*

$$tr(xg) = 0. \tag{3.10}$$

Proof. Since $\varkappa = 1$ and $E(g) = 0$, the element g does not have eigenvalue $+1$.

Decompose $x \in V$ in the basis $\mathfrak{B}_g$: $x = b_I x^I$, where $g(b_I) = \lambda_I b_I$ and $\lambda_I \neq 1$ for any $I = 1, \ldots, 2N$.

Further, $tr(b_I g) = tr(g b_I) = tr(g(b_I)g) = \lambda_I tr(b_I g)$, which implies $tr(b_I g) = 0$ and, as a consequence, $tr(xg) = 0$. □

Consider the defining relations (2.1) as a system of linear equations for the linear function sp. Clearly, this system is equivalent to the following two its subsystems:

$$sp\left([b_I, P(a)g]_\varkappa\right) = 0, \tag{3.11}$$

$$sp\left(\tau^{-1} P(a) g \tau\right) = sp\left(P(a)g\right) \tag{3.12}$$

for all monomials $P \in \mathbb{C}[V]$, $b_I \in \mathfrak{B}_g$, and $g, \tau \in G$.

If the $\varkappa$-trace is either even or the $\varkappa$-trace is odd and $\varkappa = 1$, then eq. (3.11) can be rewritten in the form

$$sp\left(b_I P(a) g - \varkappa P(a) g b_I\right) = 0. \tag{3.13}$$

Eq. (3.13) enables us to express a $\varkappa$-trace of any even monomial in $H_{1,\eta}(G)$ in terms of the $\varkappa$-trace of even minimal elements. Besides, it implies that each odd trace on $H_{1,\eta}(G)$ is equal to zero. Both these statements can be proved in a finite number of the following step operations.

Regular step operation. Let $b_{I_1} b_{I_2} \dots b_{I_k} g$ be a regular monomial. Up to a polynomial of lesser degree, this monomial can be expressed in a form such that $\lambda_{I_1} \neq \varkappa$.

Then

$$sp(b_{I_1} b_{I_2} \dots b_{I_k} g) = \varkappa sp(b_{I_2} \dots b_{I_k} g b_{I_1}) = \varkappa \lambda_{I_1} sp(b_{I_2} \dots b_{I_k} b_{I_1} g),$$

which implies

$$sp(b_{I_1} b_{I_2} \dots b_{I_k} g) - \varkappa \lambda_{I_1} sp(b_{I_1} b_{I_2} \dots b_{I_k} g) = \varkappa \lambda_{I_1} sp([b_{I_2} \dots b_{I_k}, b_{I_1}] g).$$

Thus,

$$sp(b_{I_1} b_{I_2} \dots b_{I_k} g) = \frac{\varkappa \lambda_{I_1}}{1 - \varkappa \lambda_{I_1}} sp([b_{I_2} \dots b_{I_k}, b_{I_1}] g). \tag{3.14}$$

This step operation expresses the $\varkappa$-trace of any regular degree k monomial in terms of the $\varkappa$-trace of degree $k-2$ polynomials.

Special step operation. Let $M := b_{I_1} b_{I_2} \dots b_{I_k} g$ be a special monomial and $E(g) = l > 0$. The monomial M can be expressed in the form

$$M = b_I^p b_J^q b_{L_1} \dots b_{L_{k-p-q}} g + \text{ a lesser-degree-polynomial},$$

where

$$\begin{aligned} &0 \leqslant p, q \leqslant k, \quad p + q \leqslant k, \\ &\lambda_I = \lambda_J = \lambda_{L_s} = \varkappa \quad \text{for } s = 1, ..., k-p-q, \\ &\mathscr{C}_{IJ} = 1, \quad \mathscr{C}_{IL_s} = 0, \quad \mathscr{C}_{JL_s} = 0 \quad \text{for } s = 1, ..., k-p-q\,. \end{aligned} \tag{3.15}$$

Let $M' := b_I^p b_J^q b_{L_1} \dots b_{L_{k-p-q}}$ and derive the equation for $sp(M'g)$. Since

$$sp(b_J b_I M' g) = \varkappa sp(b_I M' g b_J) = sp(b_I M' b_J g),$$

it follows that

$$sp([b_I M', b_J] g) = 0. \tag{3.16}$$

Since $[b_I M', b_J]$ can be expressed in the form:

$$\begin{aligned} [b_I^{p+1} b_J^q b_{L_1} \dots b_{L_{k-p-q}}, b_J] = &\sum_{t=0}^{p} b_I^t (1 + f_{IJ}) b_I^{p-t} b_J^q b_{L_1} \dots b_{L_{k-p-q}} + \\ &+ \sum_{t=1}^{k-p-q} b_I^{p+1} b_J^q b_{L_1} \dots b_{L_{t-1}} f_{L_t J} b_{L_{t+1}} \dots b_{L_{k-p-q}}\,, \end{aligned} \tag{3.17}$$

it follows that eq. (3.16) can be rewritten in the form

$$\begin{aligned} (p+1) sp(M'g) = &- sp\left(\sum_{t=0}^{p} b_I^t f_{IJ} b_I^{p-t} b_J^q b_{L_1} \dots b_{L_{k-p-q}} g + \right. \\ &\left. + \sum_{t=1}^{k-p-q} b_I^{p+1} b_J^q b_{L_1} \dots b_{L_{t-1}} f_{L_t J} b_{L_{t+1}} \dots b_{L_{k-p-q}} g \right). \end{aligned} \tag{3.18}$$

Due to Lemma 2.1 it is easy to see that eq. (3.18) can be rewritten in the form

$$sp(M'g) = \sum_{\tilde{g}\in G:\, E(\tilde{g})=E(g)-1} sp(P_{\tilde{g}}(a_i)\tilde{g}), \tag{3.19}$$

where the $P_{\tilde{g}}$ are some polynomials such that $\deg P_{\tilde{g}} = \deg M'$.

So, the special step operation expresses the $\varkappa$-trace of a special polynomial in terms of the $\varkappa$-trace of polynomials lesser in the sense of the ordering introduced by Definition 3.3.

Thus, we showed that it is possible to express the $\varkappa$-trace of any polynomial as a linear combination of the $\varkappa$-trace of minimal elements of $H_{1,\eta}(G)$ using a finite number of regular and special step operations.

Since each step operation is manifestly G-invariant, the resulting $\varkappa$-trace is also G-invariant if the $\varkappa$-trace of minimal elements of $H_{1,\eta}(G)$ is G-invariant.

Due to Proposition 3.1, each trace of any odd minimal element is zero, so each odd trace is zero. But since each odd trace is also a supertrace, we can say that each odd $\varkappa$-trace is zero.

These arguments proved the following Theorem and Proposition:

Theorem 3.1. *Each nonzero $\varkappa$-trace on $H_{1,\eta}(G)$ is even.*

Proposition 3.2. *Each $\varkappa$-trace on $H_{1,\eta}(G)$ is completely defined by its values on the minimal elements of G.*

Note that, due to G-invariance, the restriction of the $\varkappa$-trace on G is a *central function*, i.e., a function constant on the conjugacy classes.

Below we will prove that any central function on the set of minimal elements of G can be extended to a $\varkappa$-trace on $H_{1,\eta}(G)$.

4. Ground Level Conditions

Clearly, $\mathbb{C}[G]$ is a subalgebra of $H_{1,\eta}(G)$.

It is easy to describe all $\varkappa$-traces on $\mathbb{C}[G]$. Every $\varkappa$-trace on $\mathbb{C}[G]$ is completely determined by its values on G and is a central function on G due to G-invariance. Thus, the number of $\varkappa$-traces on $\mathbb{C}[G]$ is equal to the number of conjugacy classes in G.

Since $\mathbb{C}[G] \subset H_{1,\eta}(G)$, some additional restrictions on these functions follow from the definition (2.1) of $\varkappa$-trace and the defining relations (3.2) for $H_{1,\eta}(G)$. Namely, for any $g \in G$, consider elements $c_I, c_J \in \mathscr{E}(g)$ such that

$$gc_I = \varkappa c_I g, \quad gc_J = \varkappa c_J g. \tag{4.1}$$

Then, eqs. (2.1) and (4.1) imply that

$$sp\,(c_I c_J g) = \varkappa sp\,(c_J g c_I) = sp\,(c_J c_I g)\,,$$

and therefore

$$sp\,([c_I, c_J]g) = 0. \tag{4.2}$$

Since $[c_I, c_J]g \in \mathbb{C}[G]$, the conditions (4.2) single out the central functions on $\mathbb{C}[G]$, which can in principle be extended to $\varkappa$-traces on $H_{1,\eta}(G)$, and Theorem 5.3 states that each central function on $\mathbb{C}[G]$ satisfying conditions (4.2) can indeed be extended to a $\varkappa$-trace on $H_{1,\eta}(G)$. In [9], the conditions (4.2) are called *Ground Level Conditions*.

4.1. *The solutions of Ground Level Conditions*

Ground Level Conditions (4.2) is an overdetermined system of linear equations for the central functions on $\mathbb{C}[G]$.

Theorem 4.1. *The dimension of the space of solutions of Ground Level Conditions (4.2) is equal to the number of conjugacy classes in G with $E(g)=0$. Each central function on conjugacy classes in G with $E(g)=0$ can be uniquely extended to a solution of Ground Level Conditions.*

4.2. *Proof of Theorem 4.1*

Let us prove a couple of simple statements we will use below.

Proposition 4.1. *Let $h\in G$, $c\in\mathscr{E}(h)$, $x\in\mathfrak{V}_h$, $h(x)=\lambda x$, where $\lambda\neq\varkappa$. Then, for any central function f on $\mathbb{C}[G]$, we have*

$$f([c,x]h)\equiv 0. \tag{4.3}$$

Proof. Since f is a central function, we have $f([c,x]h)=f(h[c,x]hh^{-1})=f([h(c),h(x)]h)=\varkappa\lambda f([c,x]h)$. □

Proposition 4.2. *Let $h\in G$, $c\in\mathscr{E}(h)$, and Ground Level Conditions (4.2) be satisfied. Then*

$$sp([c,x]h)\equiv 0 \textit{ for any } x\in V. \tag{4.4}$$

Proof. Let $x=\sum_{\lambda\neq\varkappa}x_\lambda+x_\varkappa$, where $h(x_\lambda)=\lambda x_\lambda$. Since sp is a central function, Proposition 4.1 gives $sp([c,x]h)=sp([c,x_\varkappa]h)$, and eq. (4.2) gives $sp([c,x_\varkappa]h)\equiv 0$. □

We prove Theorem 4.1 by induction on $E(g)$.

The first step is simple: if $E(g)=0$, then $sp(g)$ is an arbitrary central function. The next step is also simple: if $E(g)=1$, then there exists a pair of elements $c_1,c_2\in\mathscr{E}(g)$ such that $\omega(c_1,c_2)\neq 0$. Since $([c_1,c_2]-\omega(c_1,c_2))g\in\mathbb{C}[G]$ and $E(([c_1,c_2]-\omega(c_1,c_2))g)=0$ due to Lemma 2.1, then

$$sp(g)=-\frac{1}{\omega(c_1,c_2)}sp(([c_1,c_2]-\omega(c_1,c_2))g) \tag{4.5}$$

is the only possible value of $sp(g)$ for any $g\in G$ with $E(g)=1$. Clearly, the right-hand side of eq. (4.5) does not depend on the choice of basis vectors c_1,c_2 in $\mathscr{E}(g)$.

Suppose that the Ground Level Conditions (4.2) considered for all g with $E(g)\leqslant l$ and for all $c_I,c_J\in\mathscr{E}(g)$ have Q_l independent solutions.

Proposition 4.3. *The value Q_l does not depend on l.*

Proof. It was shown above that $Q_1=Q_0$. Let $l\geqslant 1$.

Suppose that Q_k does not depend on k for $k\leqslant l$. Consider $g\in G$ with $E(g)=l+1$. Let $c_I\in\mathscr{E}(g)$, where $I=1,2,\ldots,2E(g)$, be a basis in $\mathscr{E}(g)$ such that the symplectic form $\mathscr{C}_{IJ}=\omega(c_I,c_J)$ has a normal shape:

$$\mathscr{C} = \begin{pmatrix} 0 & 1 & 0 & 0 & \dots & 0 & 0 \\ -1 & 0 & 0 & 0 & \dots & 0 & 0 \\ 0 & 0 & 0 & 1 & \dots & 0 & 0 \\ 0 & 0 & -1 & 0 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \dots & 0 & 1 \\ 0 & 0 & 0 & 0 & \dots & -1 & 0 \end{pmatrix}.$$

We will show that for a fixed $g \in G$, all the Ground Level Conditions

$$\mathscr{C}_{IJ} sp(g) = -sp(([c_I, c_J] - \mathscr{C}_{IJ})g) \text{ for } I,J = 1, \dots, 2E(g) \tag{4.6}$$

follow from the inductive hypothesis and just one of them, e.g.,

$$sp(g) = -sp(([c_1, c_2] - 1)g). \tag{4.7}$$

For this purpose, it clearly suffices to consider eq. (4.6) only for $I,J = 1,...,4$:

$$sp(g) = -sp(([c_1, c_2] - 1)g), \tag{4.8}$$
$$sp(g) = -sp(([c_3, c_4] - 1)g), \tag{4.9}$$
$$0 = -sp([c_1, c_3]g), \tag{4.10}$$
$$0 = -sp([c_1, c_4]g), \tag{4.11}$$
$$0 = -sp([c_2, c_3]g), \tag{4.12}$$
$$0 = -sp([c_2, c_4]g). \tag{4.13}$$

Below we prove that eqs. (4.8) and (4.9) are equivalent, namely,

$$sp(([c_1, c_2] - 1)g) \equiv sp(([c_3, c_4] - 1)g) \tag{4.14}$$

and eqs. (4.10) – (4.13) follow from eq. (4.14).

Note that due to the inductive hypothesis both sides of eq. (4.14) are well defined, because

$$E(([c_1, c_2] - 1)g) = E(([c_3, c_4] - 1)g) = l.$$

Represent the left-hand side of eq. (4.14) as follows:

$$sp(([c_1, c_2] - 1)g) = sp(A_{12}) + sp(B_{12}), \text{ where} \tag{4.15}$$
$$A_{12} := \sum_{R \in \mathscr{R}:\ \omega_R(c_3, c_4) = 0} \eta_R \omega_R(c_1, c_2) Rg, \tag{4.16}$$
$$B_{12} := \sum_{R \in \mathscr{R}:\ \omega_R(c_3, c_4) \neq 0} \eta_R \omega_R(c_1, c_2) Rg. \tag{4.17}$$

Analogously,

$$sp(([c_3, c_4] - 1)g) = sp(A_{34}) + sp(B_{34}), \text{ where} \tag{4.18}$$
$$A_{34} := \sum_{R \in \mathscr{R}:\ \omega_R(c_1, c_2) = 0} \eta_R \omega_R(c_3, c_4) Rg, \tag{4.19}$$
$$B_{34} := \sum_{R \in \mathscr{R}:\ \omega_R(c_1, c_2) \neq 0} \eta_R \omega_R(c_3, c_4) Rg. \tag{4.20}$$

It is clear from eqs. (4.16) – (4.20) and Lemma 2.1 that

$$E(A_{12}) = E(B_{12}) = E(A_{34}) = E(B_{34}) = l.$$

Consider $R \in \mathscr{R}$ such that $\omega_R(c_1, c_2) \neq 0$. Then there exists a 2×2 matrix $(U^R_{\alpha i})$, where $\alpha = 3,4$ and $i = 1,2$, such that

$$c^R_3 := c_3 - U^R_{31} c_1 - U^R_{32} c_2 \in Z_R, \tag{4.21}$$

$$c^R_4 := c_4 - U^R_{41} c_1 - U^R_{42} c_2 \in Z_R. \tag{4.22}$$

This matrix defines the decomposition of c_{3V_R}, c_{4V_R} in V_R with respect to the basis c_{1V_R}, c_{2V_R}.

Clearly, due to Lemma 2.1 we have

$$c^R_3,\, c^R_4 \in \mathscr{E}(Rg). \tag{4.23}$$

If $\omega_R(c_3, c_4) \neq 0$, then $\det U^R \neq 0$.

If $\omega_R(c_3, c_4) = 0$, then $\det U^R = 0$. Since

$$\omega(c^R_3,\, c^R_4) = \omega(c_3,\, c_4) + \det U^R \omega(c_1,\, c_2) = 1 + \det U^R,$$

it follows that if $\det U^R = 0$, then $\omega(c^R_3,\, c^R_4) = 1$. So, if $\omega_R(c_3, c_4) = 0$ and $\omega_R(c_1, c_2) \neq 0$, then

$$sp(Rg) = -sp(([c^R_3,\, c^R_4] - 1)Rg), \tag{4.24}$$

and

$$E(([c^R_3,\, c^R_4] - 1)Rg) = l - 1. \tag{4.25}$$

Now, let us express $sp(A_{12})$ by means of the $\varkappa$-trace of elements of G with grading $l-1$:

$$\begin{aligned} sp(A_{12}) = &\sum_{R \in \mathscr{R}:\ \omega_R(c_3, c_4) = 0} \eta_R \omega_R(c_1,\, c_2) sp(Rg) = \\ &= -\sum_{R \in \mathscr{R}:\ \omega_R(c_3, c_4) = 0} \eta_R \omega_R(c_1,\, c_2) sp(([c^R_3,\, c^R_4] - 1)Rg). \end{aligned} \tag{4.26}$$

Since

$$sp([c^R_3, x]Rg) \equiv sp([c^R_4, x]Rg) \equiv 0 \text{ for any } x \in V \tag{4.27}$$

due to Proposition 4.2, and since $\det U^R = 0$ for any summand in eq. (4.26), we have

$$sp(([c^R_3,\, c^R_4] - 1)Rg) = sp(([c_3,\, c_4] - 1)Rg), \tag{4.28}$$

and as a result

$$sp(A_{12}) = -\sum_{R \in \mathscr{R}:\ \omega_R(c_3, c_4) = 0} \eta_R \omega_R(c_1,\, c_2) sp(([c_3,\, c_4] - 1)Rg) = -sp(([c_3,\, c_4] - 1)A_{12}). \tag{4.29}$$

As $A_{12} = ([c_1,\, c_2] - 1)g - B_{12}$, eq. (4.29) gives

$$\begin{aligned} sp(A_{12} + B_{12}) &= sp(-([c_3,\, c_4] - 1)(([c_1,\, c_2] - 1)g - B_{12}) + B_{12}) = \\ &= sp(-([c_3,\, c_4] - 1)([c_1,\, c_2] - 1)g + [c_3,\, c_4]B_{12}). \end{aligned} \tag{4.30}$$

Analogously,

$$sp(A_{34}+B_{34}) = sp(-([c_1, c_2]-1)([c_3, c_4]-1)g+[c_1, c_2]B_{34}). \tag{4.31}$$

Proposition 4.4. *Let $g \in G$, $c_1, c_2 \in \mathscr{E}(g)$, $\omega(c_1, c_2) = 1$, and f a central function on $\mathbb{C}[G]$. Then*

$$f(([c_1, c_2]-1)([c_3, c_4]-1)g) = f(([c_3, c_4]-1)([c_1, c_2]-1)g). \tag{4.32}$$

Proof. Since $[c_1, c_2]-1 = \sum_{R\in\mathscr{R}} \eta_R \omega_R(c_1, c_2)R$, we have

$$\begin{aligned} f(([c_1, c_2]-1)([c_3, c_4]-1)g) &= \sum_{R\in\mathscr{R}} \eta_R \omega_R(c_1, c_2) f((R[c_3, c_4]-1)g) = \\ &= \sum_{R\in\mathscr{R}} f(\eta_R \omega_R(c_1, c_2)([c_3, c_4]-1)gR) = \\ &= f(([c_3, c_4]-1)g([c_1, c_2]-1)). \end{aligned}$$

Clearly, $g([c_1, c_2]-1) = ([c_1, c_2]-1)g$ because $c_1, c_2 \in \mathscr{E}(g)$ and $\varkappa^2 = 1$. □

Due to Proposition 4.4, eqs. (4.30) and (4.31) imply

$$sp((A_{12}+B_{12})-(A_{34}+B_{34})) = sp([c_3, c_4]B_{12}-[c_1, c_2]B_{34}) \tag{4.33}$$

or

$$\begin{aligned} &sp((A_{12}+B_{12})-(A_{34}+B_{34})) = \\ &= sp(\sum_{R\in\mathscr{R}:\ \omega_R(c_1,c_2)\neq 0,\ \omega_R(c_3,c_4)\neq 0} \eta_R([c_3, c_4]\omega_R(c_1, c_2)-[c_1, c_2]\omega_R(c_3, c_4))Rg). \end{aligned} \tag{4.34}$$

Consider one summand in eq. (4.34)

$$I_R := ([c_3, c_4]\omega_R(c_1, c_2)-[c_1, c_2]\omega_R(c_3, c_4))Rg. \tag{4.35}$$

Rewrite I_R using transformation defined in eqs. (4.21) – (4.22):

$$c_3 = c_3^R + U_{31}^R c_1 + U_{32}^R c_2, \quad \text{where } c_3^R \in Z_R, \tag{4.36}$$
$$c_4 = c_4^R + U_{41}^R c_1 + U_{42}^R c_2, \quad \text{where } c_4^R \in Z_R. \tag{4.37}$$

Note that now $\det U^R \neq 0$ since $\omega_R(c_3, c_4) \neq 0$.

Express all the terms in the right-hand side of eq.(4.35) by means of c_1, c_2, c_3^R and c_4^R:

$$\begin{aligned} &[c_3, c_4]\omega_R(c_1, c_2)Rg = \\ &\qquad = [c_3^R, c_4^R]\omega_R(c_1, c_2)Rg + \\ &\qquad + [c_3^R, U_{41}^R c_1 + U_{42}^R c_2]\omega_R(c_1, c_2)Rg + \\ &\qquad + [U_{31}^R c_1 + U_{32}^R c_2, c_4^R]\omega_R(c_1, c_2)Rg + \\ &\qquad + (U_{31}^R U_{42}^R - U_{32}^R U_{41}^R)[c_1, c_2]\omega_R(c_1, c_2)Rg \end{aligned} \tag{4.38}$$

$$\begin{aligned} &[c_1, c_2]\omega_R(c_3, c_4)Rg = \\ &\qquad = (U_{31}^R U_{42}^R - U_{32}^R U_{41}^R)[c_1, c_2]\omega_R(c_1, c_2)Rg. \end{aligned} \tag{4.39}$$

Since $c_3^R, c_4^R \in \mathscr{E}(Rg)$, Proposition 4.2 shows that

$$sp(([c_3^R, c_4^R] + [c_3^R, U_{41}^R c_1 + U_{42}^R c_2] + [U_{31}^R c_1 + U_{32}^R c_2, c_4^R])\omega_R(c_1, c_2)Rg) \equiv 0. \tag{4.40}$$

So, from eqs. (4.38) – (4.39) it follows that

$$sp(I_R) \equiv 0 \tag{4.41}$$

and eq. (4.14) is proven.

Consider another four elements of $\mathscr{E}(g)$:

$$\begin{aligned} c_1' &= \frac{1}{\sqrt{2}}(\mu c_1 + \nu c_3), \quad c_2' = \frac{1}{\sqrt{2}}(\frac{1}{\mu}c_2 + \frac{1}{\nu}c_4), \\ c_3' &= \frac{1}{\sqrt{2}}(\mu c_1 - \nu c_3), \quad c_4' = \frac{1}{\sqrt{2}}(\frac{1}{\mu}c_2 - \frac{1}{\nu}c_4). \end{aligned} \tag{4.42}$$

Clearly, $\omega(c_i', c_j')$ is in a normal form and the relation (4.14) holds for c_i replaced by c_i':

$$sp(([c_1', c_2'] - 1)g) \equiv sp(([c_3', c_4'] - 1)g), \tag{4.43}$$

which implies, when eq. (4.14) is taken in account,

$$\frac{\mu}{\nu} sp([c_1, c_4]g) + \frac{\nu}{\mu} sp([c_3, c_2]g) \equiv 0 \text{ for arbitrary nonzero } \mu, \nu \in \mathbb{C}. \tag{4.44}$$

So

$$sp([c_1, c_4]g) \equiv str([c_2, c_3]g) \equiv 0. \tag{4.45}$$

Analogously, considering

$$\begin{aligned} c_1'' &= \frac{1}{\sqrt{2}}(\mu c_1 + \nu c_4), \quad c_2'' = \frac{1}{\sqrt{2}}(\frac{1}{\mu}c_2 - \frac{1}{\nu}c_3), \\ c_3'' &= \frac{1}{\sqrt{2}}(\mu c_1 - \nu c_4), \quad c_4'' = \frac{1}{\sqrt{2}}(\frac{1}{\mu}c_2 + \frac{1}{\nu}c_3) \end{aligned} \tag{4.46}$$

we see that

$$sp([c_1, c_3]g) \equiv str([c_2, c_4]g) \equiv 0. \tag{4.47}$$

This finishes the proof of Proposition 4.3 and Theorem 4.1. □

5. The number of independent $\varkappa$-traces on $H_{1,\eta}(G)$

5.1. *Main theorems*

Theorem 5.1. *The dimension of the space of $\varkappa$-traces on the superalgebra $H_{1,\eta}(G)$ is equal to the number of conjugacy classes of elements without eigenvalue $\varkappa$ belonging to the symplectic reflection group $G \subset End(V)$. Each central function on conjugacy classes of elements without eigenvalue $\varkappa$ belonging to the symplectic reflection group $G \subset End(V)$ can be uniquely extended to a $\varkappa$-trace on $H_{1,\eta}(G)$.*

Proof. This Theorem follows from Theorem 5.3 (see below), Theorem 3.1, and Theorem 4.1. □

Clearly, Theorem 5.1 is equivalent to the following theorem.

Theorem 5.2. *Let the symplectic reflection group $G \subset End(V)$ have T_G conjugacy classes without eigenvalue* 1 *and S_G conjugacy classes without eigenvalue* -1.

Then the superalgebra $H_{1,\eta}(G)$ possesses T_G independent traces and S_G independent supertraces.

Theorem 5.3. *Every $\varkappa$-trace on the algebra $\mathbb{C}[G]$ satisfying the equation*

$$sp([c_1, c_2]g) = 0 \qquad \text{for any } g \in G \text{ with } E(g) \neq 0 \text{ and any } c_1, c_2 \in \mathscr{E}(g), \tag{5.1}$$

can be uniquely extended to an even $\varkappa$-trace on $H_{1,\eta}(G)$.

For proof of Theorem 5.3, see the rest of this section and Appendices.

The proof of Theorem 5.3 was published in [9] for the case of supertraces (i.e., $\varkappa = -1$) on the superalgebra of observables of Calogero model (i.e., $G = A_n$) and in [7] for the case $H_{1,\eta}(G)$, where the group G is a finite group generated by a root system in $\mathbb{R}^N$.

Here we chose definitions of symbols such that the rest of this section and Appendices coincide almost literally with analogous parts of [7] (and of [9], if we change $\sigma \in S_N$ to $g \in G \subset Sp(2N)$).

5.2. *The $\varkappa$-trace of General Elements*

Proposition 3.2 does not prove Theorem 5.3 because the resulting values of $\varkappa$-traces may a priori depend on the sequence of step operations used and may in principle impose additional constraints on the values of $\varkappa$-trace on $\mathbb{C}[G]$.

Below we prove that the value of $\varkappa$-trace does not depend on the sequence of step operations used. We use the following inductive procedure:

($\star$) *Let $F := P(b_I)g \in H_{1,\eta}(G)$, where P is an even monomial such that* $\deg P = 2k$, *$b_I \in \mathfrak{B}_g$ and $g \in G$. Assuming that a $\varkappa$-trace is well defined for all elements of $H_{1,\eta}(G)$ lesser than F relative to the ordering from Definition 3.3, we prove that $sp(F)$ is defined also without imposing any additional constraint on the solution of the Ground Level Conditions.*

The central point of the proof is consistency conditions (5.17), (5.18) and (5.34) proved in Appendices A.1 and A.2.

Assume that the Ground Level Conditions hold. The proof of Theorem 5.3 will be given in a constructive way by the following double induction procedure, equivalent to ($\star$):

(i) Assume that

$$sp\left([b_I, P_p(a)g]_\varkappa\right) = 0 \quad \text{for any } P_p(a),\ g \text{ and } I \text{ provided } b_I \in \mathfrak{B}_g$$

and

$$\begin{aligned} &\lambda(I) \neq \varkappa;\ p \leqslant k \text{ or} \\ &\lambda(I) = \varkappa,\ E(g) \leqslant l,\ p \leqslant k \text{ or} \\ &\lambda(I) = \varkappa;\ p \leqslant k-2, \end{aligned}$$

where $P_p(a)$ is an arbitrary degree p polynomial in a_i and p is odd. This implies that there exists a unique extension of the $\varkappa$-trace such that the same is true for l replaced with $l+1$.

(ii) Assuming that $sp\left(b_I P_p(a)g - \varkappa P_p(a)g b_I\right) = 0$ for any $P_p(a)$, g and $b_I \in \mathfrak{B}_g$, where $p \leqslant k$, one proves that there exists a unique extension of the $\varkappa$-trace such that the assumption **(i)** is true for k replaced with $k+2$ and $l = 0$.

As a result, this inductive procedure uniquely extends any solution of the Ground Level Conditions to a $\varkappa$-trace on the whole $H_G(\eta)$. (Recall that the $\varkappa$-trace of any odd element of $H_G(\eta)$ vanishes because the $\varkappa$-trace is even.)

It is convenient to work with the exponential generating functions

$$\Psi_g(\mu) = sp\left(e^S g\right), \text{ where } S = \sum_{L=1}^{2N} (\mu^L b_L), \tag{5.2}$$

where g is a fixed element of G, $b_L \in \mathfrak{B}_g$, and $\mu^L \in \mathbb{C}$ are independent parameters.

The indices I, J are raised and lowered with the help of the symplectic forms $\mathscr{C}^{IJ}$ and $\mathscr{C}_{IJ}$ (see eq. (3.7)):

$$\mu_I = \sum_J \mathscr{C}_{IJ}\mu^J, \qquad \mu^I = \sum_J \mu_J \mathscr{C}^{JI}; \qquad \sum_M \mathscr{C}_{IM}\mathscr{C}^{MJ} = -\delta_I^J. \tag{5.3}$$

By differentiating eq. (5.2) n times with respect to μ^L at $\mu = 0$ one obtains a $\varkappa$-trace of an arbitrary polynomial of n-th degree in b_L as a coefficient of g, up to polynomials of lesser degrees. In these terms, the induction on the degree of polynomials is equivalent to the induction on the homogeneity degree in μ of the power series expansions of $\Psi_g(\mu)$.

As a consequence of general properties of the $\varkappa$-trace, the generating functions $\Psi_g(\mu)$ must be G-covariant:

$$\Psi_{\tau g \tau^{-1}}(\mu) = \Psi_g(\tilde{\mu}), \tag{5.4}$$

where the G-transformed parameters are of the form

$$\tilde{\mu}^I = \left(\mathfrak{M}(\tau g \tau^{-1})\mathfrak{M}^{-1}(\tau)\Lambda^{-1}(\tau)\mathfrak{M}(\tau)\mathfrak{M}^{-1}(g)\right)^I_J \mu^J \tag{5.5}$$

and matrices $\mathfrak{M}(g)$ and $\Lambda(g)$ are defined below by eqs. (5.6) and (5.7).

Let $\mathfrak{M}(g)$ be the matrix of the map $\mathfrak{B}_\mathbf{1} \longrightarrow \mathfrak{B}_g$, such that

$$b_I = \sum_i \mathfrak{M}_I^i(g)\, a_i. \tag{5.6}$$

Obviously, this map is invertible. Using the matrix notation one can rewrite (3.4) as

$$g(b_I) = \sum_{J=1}^{2N} \Lambda_I^J(g)\, b_J, \tag{5.7}$$

where the matrix (Λ_I^J) is diagonal, namely, $\Lambda_I^J = \delta_I^J \lambda_I$.

The necessary and sufficient conditions for the existence of an even $\varkappa$-trace are the G-covariance conditions (5.4) and the condition

$$sp\left([b_L, e^S g]_\varkappa\right) = 0 \qquad \text{for any } g \text{ and } L, \tag{5.8}$$

or, equivalently, taking in account that linear function sp is an even $\varkappa$-trace,

$$sp\left(b_L e^S g - \varkappa e^S g b_L\right) = 0 \qquad \text{for any } g \text{ and } L. \tag{5.9}$$

5.3. *General relations*

To transform eq. (5.9) to a form convenient for the proof, we use the following two general relations true for arbitrary operators X and Y and parameter $\mu \in \mathbb{C}$:

$$X\exp(Y+\mu X) = \frac{\partial}{\partial\mu}\exp(Y+\mu X) + \int t_2 \exp(t_1(Y+\mu X))[X,Y]\exp(t_2(Y+\mu X))D^1t, \quad (5.10)$$

$$\exp(Y+\mu X)X = \frac{\partial}{\partial\mu}\exp(Y+\mu X) - \int t_1 \exp(t_1(Y+\mu X))[X,Y]\exp(t_2(Y+\mu X))D^1t \quad (5.11)$$

with the convention that

$$D^{n-1}t = \delta(t_1+\ldots+t_n-1)\theta(t_1)\ldots\theta(t_n)dt_1\ldots dt_n\,. \quad (5.12)$$

The relations (5.10) and (5.11) can be derived with the help of partial integration (e.g., over t_1) and the following formula

$$\frac{\partial}{\partial\mu}\exp(Y+\mu X) = \int \exp(t_1(Y+\mu X))X\exp(t_2(Y+\mu X))D^1t \quad (5.13)$$

which can be proven by expanding in power series. The well-known formula

$$[X,\exp(Y)] = \int \exp(t_1Y)[X,Y]\exp(t_2Y)D^1t \quad (5.14)$$

is a consequence of eqs. (5.10) and (5.11).

With the help of eqs. (5.10), (5.11) and (3.5) one rewrites eq. (5.9) as

$$(1-\varkappa\lambda_L)\frac{\partial}{\partial\mu^L}\Psi_g(\mu) = \int(-\varkappa\lambda_Lt_1-t_2)sp\Big(\exp(t_1S)[b_L,S]\exp(t_2S)g\Big)D^1t\,. \quad (5.15)$$

This condition should be true for any g and L and plays the central role in the analysis in this section. Eq. (5.15) is an overdetermined system of linear equations for sp; below we show that it has the only solution extending any fixed solution of the Ground Level Conditions.

There are two essentially distinct cases, $\lambda_L \neq \varkappa$ and $\lambda_L = \varkappa$. In the latter case, the eq. (5.15) takes the form

$$0 = \int sp\Big(\exp(t_1S)[b_L,S]\,exp(t_2S)g\Big)D^1t\,, \qquad \lambda_L = \varkappa\,. \quad (5.16)$$

In Appendix A.1 we prove by induction that eqs. (5.15) and (5.16) are consistent in the following sense:

$$(1-\varkappa\lambda_K)\frac{\partial}{\partial\mu^K}\int(-\varkappa\lambda_Lt_1-t_2)sp\Big(\exp(t_1S)[b_L,S]\exp(t_2S)g\Big)D^1t - (L\leftrightarrow K) \equiv 0 \quad (5.17)$$
$$\text{for } \lambda_L \neq \varkappa,\ \lambda_K \neq \varkappa$$

and

$$(1-\varkappa\lambda_K)\frac{\partial}{\partial\mu^K}\int sp\Big(\exp(t_1S)[b_L,S]\exp(t_2S)g\Big)D^1t \equiv 0 \text{ for } \lambda_L = \varkappa. \quad (5.18)$$

Note that this part of the proof is quite general and does not depend on a concrete form of the commutation relations in eq. (3.2).

By expanding the exponential e^S in eq. (5.2) into power series in μ^K (equivalently b_K) we conclude that eq. (5.15) uniquely reconstructs the $\varkappa$-trace of monomials containing b_K with $\lambda_K \neq \varkappa$ (i.e., *regular monomials*) in terms of $\varkappa$-traces of some lower degree polynomials. Then the consistency conditions (5.17) and (5.18) guarantee that eq. (5.15) does not impose any additional conditions on the $\varkappa$-traces of lower degree polynomials and allow one to represent the generating function in the form

$$\Psi_g = \Phi_g(\mu) + \\ + \sum_{L:\,\lambda_L \neq \varkappa} \int_0^1 \frac{\mu_L d\tau}{1-\varkappa\lambda_L} \int D^1 t\,(-\varkappa\lambda_L t_1 - t_2) sp\left(e^{t_1(\tau S'' + S')}[b_L, (\tau S'' + S')]\, e^{t_2(\tau S'' + S')} g\right), \tag{5.19}$$

where we introduced the generating functions Φ_g for the $\varkappa$-trace of *special polynomials*, i.e., the polynomials depending only on b_L with $\lambda_L = \varkappa$, i.e., $b_L \in \mathscr{E}(g)$:

$$\Phi_g(\mu) := sp\left(e^{S'} g\right) = \Psi_g(\mu)\Big|_{(\mu^I = 0 \text{ if } \lambda_I \neq \varkappa)} \tag{5.20}$$

and

$$S' = \sum_{L:\, b_L \in \mathfrak{B}_g,\, \lambda_L = \varkappa} (\mu^L b_L); \qquad S'' = S - S'. \tag{5.21}$$

The relation (5.19) successively expresses the $\varkappa$-trace of higher degree regular polynomials via the $\varkappa$-traces of lower degree polynomials.

One can see that the arguments above prove the inductive hypotheses **(i)** and **(ii)** for the particular case where the polynomials $P_p(a)$ are regular and/or $\lambda_I \neq \varkappa$. Note that for this case the induction **(i)** on the grading E is trivial: one simply proves that the degree of the polynomial can be increased by two.

Let us now turn to a less trivial case of the special polynomials:

$$sp\left(b_I e^{S'} g - \varkappa e^{S'} g b_I\right) = 0\,, \text{ where } \lambda_I = \varkappa\,. \tag{5.22}$$

This equation implies

$$sp\left([b_I, e^{S'}] g\right) = 0\,, \text{ where } \lambda_I = \varkappa\,. \tag{5.23}$$

Consider the part of $sp\left([b_I, \exp S']g\right)$ which is of degree k in μ and let $E(g) = l+1$. By eq. (5.16) the conditions (5.23) give

$$0 = \int sp\left(\exp(t_1 S')[b_I, S'] \exp(t_2 S') g\right) D^1 t\,. \tag{5.24}$$

Substituting $[b_I, S'] = \mu_I + \sum_M f_{IM}\mu^M$, where the quantities f_{IJ} and μ_I are defined in eqs. (3.7), (3.8) and (5.3), one can rewrite eq. (5.24) in the form

$$\mu_I \Phi_g(\mu) = -\int sp\left(\exp(t_1 S') \sum_M f_{IM}\mu^M \exp(t_2 S') g\right) D^1 t\,. \tag{5.25}$$

Now we use the inductive hypothesis **(i)**. The integrand in eq. (5.25) is a $\varkappa$-trace of a polynomial of degree $\leqslant k-1$ in the $a_{\alpha i}$ in the sector of degree k polynomials in μ, and $E(f_{IM} g) = l$. Therefore one can use the inductive hypothesis **(i)** to obtain the equality

$$\int sp\Big(\exp(t_1S')\sum_M f_{IM}\mu^M \exp(t_2S')g\Big)D^1t = \int sp\Big(\exp(t_2S')\exp(t_1S')\sum_M f_{IM}\mu^M g\Big)D^1t,$$

where we used that $sp(S'Fg) = \varkappa sp(FgS')= sp(FS'g)$ by definition of S'.

As a result, the inductive hypothesis allows one to transform eq. (5.22) to the form:

$$X_I = 0, \text{ where } X_I := \mu_I\Phi_g(\mu) + sp\Big(\exp(S')\sum_M f_{IM}\mu^M g\Big). \tag{5.26}$$

By differentiating this equation with respect to μ^J one obtains after symmetrization

$$\frac{\partial}{\partial\mu^J}(\mu_I\Phi_g(\mu)) + (I\leftrightarrow J) = -\int sp\Big(e^{t_1S'}b_Je^{t_2S'}\sum_M f_{IM}\mu^M g\Big)D^1t + (I\leftrightarrow J). \tag{5.27}$$

An important point is that the system of equations (5.27) is equivalent to the original equations (5.26) except for the ground level part $\Phi_g(0)$. This can be easily seen from the simple fact that the general solution of the system of equations for entire functions $X_I(\mu)$

$$\frac{\partial}{\partial\mu^J}X_I(\mu) + \frac{\partial}{\partial\mu^I}X_J(\mu) = 0$$

is of the form

$$X_I(\mu) = X_I(0) + \sum_J c_{IJ}\mu^J$$

where $X_I(0)$ and $c_{IJ}=-c_{JI}$ are some constants.

The part of eq. (5.26) linear in μ is however equivalent to the Ground Level Conditions analyzed in Section 4. Thus, eq. (5.27) contains all information on eq. (4.2) additional to the Ground Level Conditions. For this reason, we will from now on analyze equation (5.27).

Using again the inductive hypothesis we move b_I to the left and to the right of the right hand side of eq. (5.27) with weights equal to $\frac{1}{2}$ each to get

$$\begin{aligned}\frac{\partial}{\partial\mu^J}\mu_I\Phi_g(\mu) + (I\leftrightarrow J) &= -\frac{1}{2}\sum_M sp\Big(\exp(S')\{b_J, f_{IM}\}\mu^M g\Big) - \\ &-\frac{1}{2}\int\sum_{L,M}(t_1-t_2)sp\Big(\exp(t_1S')F_{JL}\mu^L\exp(t_2S')f_{IM}\mu^M g\Big)D^1t + (I\leftrightarrow J).\end{aligned} \tag{5.28}$$

The terms with the factor $t_1 - t_2$ vanish as is not difficult to show, so eq. (5.28) reduces to

$$L_{IJ}\Phi_g(\mu) = -\frac{1}{2}R_{IJ}(\mu), \tag{5.29}$$

where

$$R_{IJ}(\mu) = \sum_M sp\Big(\exp(S')\{b_J, f_{IM}\}\mu^M g\Big) + (I\leftrightarrow J) \tag{5.30}$$

and

$$L_{IJ} = \frac{\partial}{\partial\mu^J}\mu_I + \frac{\partial}{\partial\mu^I}\mu_J, \tag{5.31}$$

or, equivalently,

$$L_{IJ} = \mu_I \frac{\partial}{\partial \mu^J} + \mu_J \frac{\partial}{\partial \mu^I} . \tag{5.32}$$

The differential operators L_{IJ} satisfy the standard commutation relations of the Lie algebra $\mathfrak{sp}(2E(g))$

$$[L_{IJ}, L_{KL}] = -(\mathscr{C}_{IK} L_{JL} + \mathscr{C}_{IL} L_{JK} + \mathscr{C}_{JK} L_{IL} + \mathscr{C}_{JL} L_{IK}) . \tag{5.33}$$

In Appendix A.2 we show by induction that this Lie algebra $\mathfrak{sp}(2E(g))$ realized by differential operators is consistent with the right-hand side of the basic relation (5.29), i.e., that

$$[L_{IJ}, R_{KL}] - [L_{KL}, R_{IJ}] = -(\mathscr{C}_{IK} R_{JL} + \mathscr{C}_{JL} R_{IK} + \mathscr{C}_{JK} R_{IL} + \mathscr{C}_{IL} R_{JK}) . \tag{5.34}$$

Generally, these consistency conditions guarantee that eqs. (5.29) express $\Phi_g(\mu)$ in terms of R^{IJ} in the following way

$$\Phi_g(\mu) = \Phi_g(0) + \frac{1}{8E(g)} \sum_{I,J=1}^{2E(g)} \int_0^1 \frac{dt}{t} (1 - t^{2E(g)}) (L_{IJ} R^{IJ})(t\mu) , \tag{5.35}$$

provided

$$R^{IJ}(0) = 0 . \tag{5.36}$$

The latter condition must hold for a consistency of eqs. (5.29) since its left hand side vanishes at $\mu^I = 0$. In the expression (5.35) it guarantees that the integral over t converges. In the case under consideration the condition (5.36) is met as follows from definition (5.30).

Taking Lemma 2.1 and the explicit form (5.30) of R_{IJ} into account one concludes that eq. (5.35) uniquely expresses the $\varkappa$-trace of special polynomials in terms of the $\varkappa$-traces of polynomials of lower degrees or in terms of the $\varkappa$-traces of special polynomials of the same degree multiplied by elements of G with a smaller value of E provided that the μ-independent term $\Phi_g(0)$ is an arbitrary solution of the Ground Level Conditions. This completes the proof of Theorem 5.3. □

6. Non-deformed skew product $H_{1,0}(G)$ of the Weyl superalgebra and a finite symplectic reflection group

Consider $H_{1,0}(G)$. It has the same number of traces and supertraces as $H_{1,\eta}(G)$ for an arbitrary η and the generating functions of these traces and supertraces are written below explicitly. The algebra $H_{1,0}(G)$ is the skew product $W_N * G$ of the Weyl superalgebra W_N and the group algebra $\mathbb{C}[G]$ of the finite group $G \subset Sp(2N)$ generated by a system $\mathscr{R} \subset G$ of symplectic reflections. Algebras of this type, and their generalizations, were considered in [12].

Because the Weyl superalgebra W_N is simple, the algebras $H_{1,0}(G) = W_N * G$ are also simple (see [12], p. 48, Exercise 6). This is a way to augment the stock of known simple associative (super)algebras with several (super)traces.

It is easy to find the general solution of eqs. (4.6), (5.15) and (5.16) for the generating function of $\varkappa$-traces in the case $\eta = 0$:

(1) If $g \in G$ and $E(g) \neq 0$, then $sp(P(a_i)g) = 0$ for any polynomial P.

(2) If $g \in G$ and $E(g) = 0$, then $sp(g)$ is an arbitrary central function on G.

(3) Let $E(g)=0$. Let $S=\sum_i \mu^i a_i$, $\Psi(g,\mu,t):=sp(e^{tS}g)$, $\Psi(g,\mu)=sp(e^S g)=\Psi(g,\mu,1)$. Then

$$sp\left([a_i,e^{tS}g]_\varkappa\right)=sp\left(t\omega_{ij}\mu^j e^{tS}g-e^{tS}a_j g p_i^j\right),\ \text{ where } p_i^j=(1-\varkappa g)_i^j. \tag{6.1}$$

Since $E(g)=0$, the matrix (p_i^j) is invertible, so eq. (6.1) gives

$$\frac{d}{dt}\Psi(g,\mu,t)=-\mu^j\omega_{ij}q_k^i\mu^k\Psi(g,\mu,t),\ \text{where } q_k^i=\left(\frac{1}{1-\varkappa g}\right)_k^i=\frac{1}{2}\left(\frac{\varkappa+g}{\varkappa-g}\right)_k^i+\frac{1}{2}\delta_k^i.$$

So

$$\frac{d}{dt}\Psi(g,\mu,t)=-Q(\mu)\Psi(g,\mu,t),\ \text{where } Q=\frac{1}{2}\mu^i\mu^j\tilde{\omega}_{ij},\ \text{and } \tilde{\omega}_{ij}=\omega_{ki}\left(\frac{\varkappa+g}{\varkappa-g}\right)_j^k$$

and finally

$$\Psi(g,\mu)=\exp\left(-\frac{1}{2}\mu^i\mu^j\omega_{ki}\left(\frac{\varkappa+g}{\varkappa-g}\right)_j^k\right)sp(g).$$

It is easy to check that the form $\tilde{\omega}_{ij}$ is symmetric.

7. Lie algebras $H_{1,\eta}(G)^L$ and Lie superalgebras $H_{1,\eta}(G)^S$

We can consider the space of associative algebra $H_{1,\eta}(G)$ as a Lie algebra $H_{1,\eta}(G)^L$ with the brackets[f] $[f,g]_{+1}=fg-gf$ for all $f,g\in H_{1,\eta}(G)^L$.

We can also consider the space of associative algebra $H_{1,\eta}(G)$ as a Lie superalgebra $H_{1,\eta}(G)^S$ with the brackets $[f,g]_{-1}=fg-(-1)^{\pi(f)\pi(g)}gf$ for all $f,g\in H_{1,\eta}(G)^S$.

S. Montgomery in [11] showed that it is possible to construct simple Lie superalgebra A^L from simple associative superalgebra A if the supercenter of A satisfies some conditions. In particular, if the supercenter is $\mathbb{C}$, then these conditions are satisfied.

Let $\mathscr{Z}^L$ be the center of $H_{1,\eta}(G)$, i.e., $fz-zf=0$ for all $z\in\mathscr{Z}^L$ and for all $f\in H_{1,\eta}(G)$.

Let $\mathscr{Z}^S$ be the supercenter of $H_{1,\eta}(G)$, i.e., $fz-(-1)^{\pi(z)\pi(f)}zf=0$ for all $z\in\mathscr{Z}^S$ and for all $f\in H_{1,\eta}(G)$. Clearly, $\mathscr{Z}^S=\mathscr{Z}_0^S\oplus\mathscr{Z}_1^S$, where $\pi(\mathscr{Z}_0^S)=0$ and $\pi(\mathscr{Z}_1^S)=1$. Evidently, $\mathscr{Z}_0^S\subset\mathscr{Z}^L$.

Theorem 7.1. $\mathscr{Z}^L=\mathscr{Z}^S=\mathbb{C}$.

Proof. The first part of this Theorem, $\mathscr{Z}^L=\mathbb{C}$, is proven in [1]. Further, $\mathscr{Z}_0^S=\mathbb{C}$, and it remains to prove that $\mathscr{Z}_1^S=0$.

Suppose that there exists $z\in\mathscr{Z}_1^S$. Then $z=\sum_{g\in G}P_g g$. Consider $[z,b_g]_{-1}=zb_g+b_g z$ for all $b_g\in\mathfrak{B}_g$ for all $g\in G$. One can see that $deg[z,b_g]_{-1}>degz$ unless there exists the element $K=-1$ in $G\subset Sp(2N)$ and $z=P_K K$.

If such element K does not exist[g] then $z=0$ otherwise $zK\in\mathscr{Z}^L$ which also implies $z=0$ due to $\pi(zK)=1$. □

[f]Recall that $[f,g]_\varkappa:=fg-\varkappa^{\pi(f)\pi(g)}gf$.

[g]Clearly, $Kf=(-1)^{\pi(f)}fK$ for all $f\in H_{1,\eta}(G)$, $\pi(K)=0$ and $K^2=1$. We call such element of $H_{1,\eta}(G)$ *Klein operator*. If Klein operator K exists, then it defines the isomorphism of the spaces of the traces and the supertraces on $H_{1,\eta}(G)$ (see [8]).

Since $1 \in G$, it follows from Theorem 5.1 that there exists a supertrace str_1 such that $str_1(1) \neq 0$. So, $[H_{1,\eta}(G)^S, H_{1,\eta}(G)^S] \cap \mathscr{Z}^S = 0$.

Definition 7.1. Set

$$L_{1,\eta}(G) := [H_{1,\eta}(G)^L, H_{1,\eta}(G)^L]_{+1} \,/\, \big([H_{1,\eta}(G)^L, H_{1,\eta}(G)^L]_{+1} \cap \mathscr{Z}^L\big)\,; \tag{7.1}$$
$$S_{1,\eta}(G) := [H_{1,\eta}(G)^S, H_{1,\eta}(G)^S]_{-1}. \tag{7.2}$$

Now one can apply Theorem 3.8 of [11] (which generalizes the results of I.N.Herstein (see [3], [4]) to formulate the following statement

Theorem 7.2. *If $H_{1,\eta}(G)$ is a simple associative algebra, then*
1) $L_{1,\eta}(G)$ is a simple Lie algebra,
2) $S_{1,\eta}(G)$ is a simple Lie superalgebra.

If there exists a trace tr_1 on $H_{1,\eta}(G)$ such that $tr_1(1) \neq 0$, then

$$[H_{1,\eta}(G)^L, H_{1,\eta}(G)^L]_{+1} \cap \mathscr{Z}^L = 0.$$

If $tr(1) = 0$ for any trace tr, then

$$tr((f+\alpha)(g+\beta)) = tr(fg) \text{ for any } f,g \in [H_{1,\eta}(G)^L, H_{1,\eta}(G)^L]_{+1} \text{ and any } \alpha,\beta \in \mathbb{C}.$$

So, it is possible to define a bilinear symmetric invariant form B_{tr} on $L_{1,\eta}(G)$.

Definition 7.2. Let tr be a trace on $H_{1,\eta}(G)$. Let

$$\rho : [H_{1,\eta}(G)^L, H_{1,\eta}(G)^L]_{+1} \mapsto [H_{1,\eta}(G)^L, H_{1,\eta}(G)^L]_{+1} \,/\, [H_{1,\eta}(G)^L, H_{1,\eta}(G)^L]_{+1} \cap \mathscr{Z}^L$$

be the natural projection. Then

$$B_{tr}(\rho(f), \rho(g)) := tr(fg) \text{ for any } f,g \in [H_{1,\eta}(G)^L, H_{1,\eta}(G)^L]_{+1} \tag{7.3}$$

is a well defined bilinear form on $L_{1,\eta}(G)$.

Define also a bilinear symmetric invariant form B_{str} on $S_{1,\eta}(G)$.

Definition 7.3. Let str be a supertrace on $H_{1,\eta}(G)$. Set

$$B_{str}(f, g) := str(fg) \text{ for any } f,g \in S_{1,\eta}(G). \tag{7.4}$$

To finish this section, let us show that if $H_{1,\eta}(G)$ is a simple associative algebra, then maps Eqs. (7.3) and (7.4) sending the (super)traces into the spaces of bilinear invariant (super)symmetric forms are injections.

Indeed, suppose that $B_{str} \equiv 0$ for some supertrace str, i.e., $str([a,b]_{-1}[c,d]_{-1}) = 0$ for any $a,b,c,d \in H_{1,\eta}(G)$. Hence, $str([[a,b]_{-1},c]_{-1}d) = 0$ for any $a,b,c,d \in H_{1,\eta}(G)$. Since $H_{1,\eta}(G)$ is simple, we have $[[a,b]_{-1},c]_{-1} = 0$ for any $a,b,c \in S_{1,\eta}(G)$, which contradicts to the simplicity of $S_{1,\eta}(G)$.

The proof for the traces is analogous.

Acknowledgements

This work was supported by the Russian Foundation for Basic Research, Grant No. 11-02-00685.

Appendix A. Proof of consistency conditions.

A.1. *Proof of consistency condition (5.17) for $\lambda \neq \varkappa$.*

Let parameters $\mu_1 := \mu^{K_1}$ and $\mu_2 := \mu^{K_2}$ be such that $\lambda_1 \neq \varkappa$ and $\lambda_2 \neq \varkappa$, where $\lambda_1 := \lambda_{K_1}$ and $\lambda_2 := \lambda_{K_2}$. Let $b^1 := b_{K_1}$ and $b^2 := b_{K_2}$. Let us prove by induction that conditions (5.17) hold. To implement induction, we select the part of degree k in μ from eq. (5.15) and observe that this part contains a degree $k+1$ polynomial in b_M in the left-hand side of eq. (5.15) while the part on the right hand side of the differential version (5.15) of eq. (5.8), which is of the same degree in μ, has a degree $k-1$ as polynomial in b_M.

This happens because of the presence of the commutator $[b_L, S]$ which is a zero degree polynomial due to the basic relations (3.2). As a result, the inductive hypothesis allows us to use the properties of the $\varkappa$-trace provided that the commutator $[b_L, S]$ is always handled as the right hand side of eq. (3.2), i.e., we are not allowed to represent it again as a difference of the second-degree polynomials.

Direct differentiation of Eq. (5.15) with the help of eq. (5.13) gives

$$
\begin{aligned}
&(1-\varkappa\lambda_2)\frac{\partial}{\partial\mu_2}\int(-\varkappa\lambda_1 t_1 - t_2)sp\Big(e^{t_1 S}[b^1,S]\,e^{t_2 S}g\Big)D^1 t - \Big(1\leftrightarrow 2\Big) = \\
&= \left(\int (1-\varkappa\lambda_2)(-\varkappa\lambda_1 t_1 - t_2)sp\left(e^{t_1 S}[b^1,b^2]\,e^{t_2 S}g\right)D^1 t - \Big(1\leftrightarrow 2\Big)\right) + \\
&+\left(\int(1-\varkappa\lambda_2)(-\varkappa\lambda_1(t_1+t_2)-t_3)sp\Big(e^{t_1 S}b^2 e^{t_2 S}[b^1,S]\,e^{t_3 S}\Big)D^2 t - \Big(1\leftrightarrow 2\Big)\right) + \\
&+\left(\int(1-\varkappa\lambda_2)(-\varkappa\lambda_1 t_1 - t_2 - t_3)sp\Big(e^{t_1 S}[b^1,S]\,e^{t_2 S}b^2 e^{t_3 S}g\Big)D^2 t - \Big(1\leftrightarrow 2\Big)\right).
\end{aligned}
\tag{A.1}
$$

We have to show that the right hand side of eq. (A.1) vanishes. Let us first transform the second and the third terms on the right-hand side of eq. (A.1). The idea is to move the operators b^2 through the exponentials towards the commutator $[b^1, S]$ in order to use then the Jacobi identity for the double commutators. This can be done in two different ways inside the $\varkappa$-trace so that one has to fix appropriate weight factors for each of these processes. Let the notation $\overrightarrow{A}$ and $\overleftarrow{A}$ mean that the operator A has to be moved from its position to the right and to the left, respectively.

The correct weights turn out to be

$$
\begin{aligned}
&D^2 t(-\varkappa\lambda_1(t_1+t_2)-t_3)b^2 \equiv D^2 t(-\varkappa\lambda_1 - t_3(1-\varkappa\lambda_1))b^2 = \\
&= D^2 t\left(\left(\frac{\lambda_1\lambda_2}{1-\varkappa\lambda_2} - t_3(1-\varkappa\lambda_1)\right)\overrightarrow{b^2} + \frac{-\varkappa\lambda_1}{1-\varkappa\lambda_2}\overleftarrow{b^2}\right)
\end{aligned}
\tag{A.2}
$$

and

$$
\begin{aligned}
&D^2 t(-\varkappa\lambda_1 t_1 - t_2 - t_3)b^2 \equiv D^2 t((-\varkappa\lambda_1+1)t_1 - 1)b^2 = \\
&= D^2 t\left(\left(t_1(1-\varkappa\lambda_1) - \frac{1}{1-\varkappa\lambda_2}\right)\overleftarrow{b^2} - \frac{-\varkappa\lambda_2}{1-\varkappa\lambda_2}\overrightarrow{b^2}\right)
\end{aligned}
\tag{A.3}
$$

for the second and third terms in the right hand side of eq. (A.1), respectively. Using eq. (5.14) along with the simple formula

$$\int \phi(t_3,\dots t_{n+1})D^n t = \int t_1 \phi(t_2,\dots t_n)D^{n-1}t \tag{A.4}$$

we find that all terms which involve both $[b^1,S]$ and $[b^2,S]$ pairwise cancel after antisymmetrization $1 \leftrightarrow 2$.

As a result, one is left with some terms involving double commutators which, thanks to the Jacobi identities and antisymmetrization, are all reduced to

$$\int \Big(\lambda_1\lambda_2 t_1 + t_2 - t_1 t_2(1-\varkappa\lambda_1)(1-\varkappa\lambda_2)\Big) sp\Big(\exp(t_1 S)[S,[b^1,b^2]]\exp(t_2 S)g\Big)D^1 t\,. \tag{A.5}$$

Finally, we observe that this expression can be equivalently rewritten in the form

$$\int \Big(\lambda_1\lambda_2 t_1 + t_2 - t_1 t_2(1-\varkappa\lambda_1)(1-\varkappa\lambda_2)\Big)\left(\frac{\partial}{\partial t_1}-\frac{\partial}{\partial t_2}\right) sp\Big(\exp(t_1 S)[b^1,b^2]\exp(t_2 S)g\Big)D^1 t \tag{A.6}$$

and after integration by parts cancel the first term on the right-hand side of eq. (A.1). Thus, we showed that eqs. (5.15) are compatible for the case $\lambda_{1,2} \neq \varkappa$.

Analogously, we can show that eqs. (5.15) are compatible with eq. (5.16). Indeed, let $\lambda_1 = \varkappa$, $\lambda_2 \neq \varkappa$. Let us prove that

$$\frac{\partial}{\partial \mu_2} sp\Big([b^1,\exp(S)]g\Big) = 0 \tag{A.7}$$

provided the $\varkappa$-trace is well-defined for the lower degree polynomials. The explicit differentiation gives

$$\begin{aligned}\frac{\partial}{\partial \mu_2} sp\Big([b^1,\exp(S)]g\Big) &= \int sp\Big([b^1,\exp(t_1 S)b^2\exp(t_2 S)]g\Big)D^1 t = \\ &= (1-\varkappa\lambda_2)^{-1} sp\Big([b^1,(b^2\exp(S)-\varkappa\lambda_2\exp(S)b^2)]g\Big)+\dots\end{aligned} \tag{A.8}$$

where dots denote some terms of the form $sp\Big([b^1,B]g\Big)$ involving more commutators inside B, which therefore amount to some lower degree polynomials and vanish by the inductive hypothesis. As a result, we find that

$$\begin{aligned}\frac{\partial}{\partial \mu_2} sp\Big([b^1,\exp(S)]g\Big) &= (1-\varkappa\lambda_2)^{-1} sp\Big((b^2[b^1,\exp(S)]-\varkappa\lambda_2[b^1,\exp(S)]b^2)g\Big)+ \\ &+ (1-\varkappa\lambda_2)^{-1} sp\Big(([b^1,b^2]\exp(S)-\varkappa\lambda_2\exp(S)[b^1,b^2])g\Big)\,.\end{aligned} \tag{A.9}$$

This expression vanishes by the inductive hypothesis, too.

A.2. *The proof of consistency conditions (5.34) (the case of special polynomials)*

In order to prove eq. (5.34) we use the inductive hypothesis **(i)**. In this Appendix we use the convention that any expression with the coinciding upper or lower indices are automatically symmetrized, e.g., $U^{II} := \frac{1}{2}(U^{I_1I_2} + U^{I_2I_1})$. In this Appendix, all the eigenvectors b_I of g belong to $\mathscr{E}(g)$. The identity

$$0 = \sum_M sp\Big(\Big[\exp(S')\{b_I, f_{IM}\}\mu^M, b_J b_J\Big] g\Big) - (I \leftrightarrow J) \qquad \text{(A.10)}$$

holds due to Lemma 3.1 for all terms of degree $k-1$ in μ with $E(g) \leqslant l+1$ and for all lower degree polynomials in μ, because one can always move f_{IJ} to g in eq. (A.10) combining $f_{IJ}g$ into a combination of elements of G analyzed in Lemma 3.1.

Straightforward calculation of the commutator in the right-hand-side of eq. (A.10) gives $0 = X_1 + X_2 + X_3$, where

$$\begin{aligned} X_1 &= -\sum_{M,L} \int sp\left(\exp(t_1 S')\{b_J, F_{JL}\}\mu^L \exp(t_2 S')\{b_I, f_{IM}\}\mu^M g\right) D^1 t - (I \leftrightarrow J)\,, \\ X_2 &= \sum_M sp\left(\exp(S')\Big\{\{b_J, F_{IJ}\}, f_{IM}\Big\}\mu^M g\right) - (I \leftrightarrow J)\,, \\ X_3 &= \sum_M sp\left(\exp(S')\Big\{b_I, \{b_J, [f_{IM}, b_J]\}\Big\}\mu^M g\right) - (I \leftrightarrow J)\,. \end{aligned} \qquad \text{(A.11)}$$

The terms of X_1 bilinear in f cancel due to the antisymmetrization $(I \leftrightarrow J)$ and the inductive hypothesis **(i)**. As a result, one can transform X_1 to the form

$$X_1 = \left(-\frac{1}{2}[L_{JJ}, R_{II}] + 2sp\left(e^{S'}\{b_I, f_{IJ}\}\mu_J g\right)\right) - (I \leftrightarrow J). \qquad \text{(A.12)}$$

Substituting $F_{IJ} = \mathscr{C}_{IJ} + f_{IJ}$ and $f_{IM} = [b_I, b_M] - \mathscr{C}_{IM}$ one transforms X_2 to the form

$$X_2 = 2\mathscr{C}_{IJ} R_{IJ} - 2\left(sp\left(e^{S'}\{b_J, f_{IJ}\}\mu_I g\right) - (I \leftrightarrow J)\right) + Y, \qquad \text{(A.13)}$$

where

$$Y = sp\left(e^{S'}\Big\{\{b_J, f_{IJ}\}, [b_I, S']\Big\} g\right) - (I \leftrightarrow J)\,. \qquad \text{(A.14)}$$

Using that

$$sp\left(\exp(S')\left[P f_{IJ} Q, S'\right] g\right) = 0 \qquad \text{(A.15)}$$

provided the inductive hypothesis can be used, one transforms Y to the form

$$Y = sp\left(e^{S'}\left(-[f_{IJ}, (b_I S' b_J + b_J S' b_I)] - b_I[f_{IJ}, S'] b_J - b_J [f_{IJ}, S'] b_I + [f_{IJ}, \{b_I, b_J\}] S'\right) g\right). \qquad \text{(A.16)}$$

Let us rewrite X_3 in the form $X_3 = X_3^s + X_3^a$, where

$$\begin{aligned} X_3^s &= \frac{1}{2}\sum_M sp\left(e^{S'}\left(\Big\{b_I, \{b_J, [f_{IM}, b_J]\}\Big\} + \Big\{b_J, \{b_I, [f_{IM}, b_J]\}\Big\}\right)\mu^M g\right) - (I \leftrightarrow J)\,, \\ X_3^a &= \frac{1}{2}\sum_M sp\left(e^{S'}\left(\Big\{b_I, \{b_J, [f_{IM}, b_J]\}\Big\} - \Big\{b_J, \{b_I, [f_{IM}, b_J]\}\Big\}\right)\mu^M g\right) - (I \leftrightarrow J)\,. \end{aligned}$$

With the help of the Jacobi identity $[f_{IM}, b_J] - [f_{JM}, b_I] = [f_{IJ}, b_M]$ one expresses X_3^s in the form

$$X_3^s = \frac{1}{2} sp \left(e^{S'} \left(\{b_I, b_J\}[f_{IJ}, S'] + [f_{IJ}, S']\{b_I, b_J\} + 2b_I[f_{IJ}, S']b_J + 2b_J[f_{IJ}, S']b_I \right) g \right).$$

Let us transform this expression for X_3^a to the form

$$X_3^a = \frac{1}{2} \sum_M sp \left(e^{S'} [F_{IJ}, [f_{IM}, b_J]] \mu^M g \right) - (I \leftrightarrow J). \tag{A.17}$$

Substitute $F_{IJ} = \mathscr{C}_{IJ} + f_{IJ}$ and $f_{IM} = [b_I, b_M] - \mathscr{C}_{IM}$ in eq. (A.17). After simple transformations we find that $Y + X_3 = 0$. From eqs. (A.12) and (A.13) it follows that the right hand side of eq. (A.10) is equal to

$$\frac{1}{2}([L_{II}, R_{JJ}] - [L_{JJ}, R_{II}]) + 2\mathscr{C}_{IJ} R_{IJ}.$$

This completes the proof of the consistency conditions (5.34).

References

[1] K.A. Brown, I. Gordon, “Poisson orders, symplectic reflection algebras and representation theory”, J. Reine Angew. Math. 559 (2003), 193 – 216; arXiv:math/0201042v2 [math.RT].

[2] P. Etingof and V. Ginzburg, “Symplectic reflection algebras, Calogero–Moser space, and deformed Harish–Chandra homomorphism”, Inv. Math. 147 (2002), 243 – 348.

[3] I.N. Herstein, “On the Lie and Jordan rings of a simple associative ring”, Amer. J. Math 77 (1955), 279-285.

[4] I.N. Herstein, “Topics in Ring Theory”, Chicago Lecture Notes in Math, University of Chicago Press, 1969.

[5] S.E. Konstein, “3-particle Calogero Model: Supertraces and Ideals on the Algebra of Observables”, Theor.Math.Phys. 116 (1998) 836 – 845; arXiv:hep-th/9803213.

[6] S.E. Konstein, “An example of simple Lie superalgebra with several invariant bilinear forms”, Resenhas IME-USP 2004; Vol. 6, No. 2/3, 249-255; arXiv:math-ph/0112063.

[7] S.E. Konstein and I.V. Tyutin, “Traces on the Superalgebra of Observables of Rational Calogero Model based on the Root System”, Journal of Nonlinear Mathematical Physics, 20:2 (2013), 271 – 294; arXiv:1211.6600; arXiv:math-ph/9904032.

[8] S.E. Konstein and R. Stekolshchik, “Klein operator and the Number of Traces and Supertraces on the Superalgebra of Observables of Rational Calogero Model based on the Root System”, Journal of Nonlinear Mathematical Physics, Vol. 20:2 (2013), 295 – 308; arXiv:0811.2487; arXiv:1212.0508.

[9] S.E. Konstein and M.A. Vasiliev, “Supertraces on the Algebras of Observables of the Rational Calogero Model with Harmonic Potential”, J. Math. Phys. **37** (1996) 2872.

[10] I. Losev, “Completions of symplectic reflection algebras”, arXiv:1001.0239v4.

[11] S. Montgomery, “Constructing simple Lie superalgebras from associative graded algebras”, J. of Algebra, **195** (1997) 558.

[12] D.S. Passman, Infinite Crossed Products, Pure and Applied Math vol. 135, Academic Press, San Diego, 1989.

[13] M.A. Vasil’ev, “Quantization on sphere and high-spin superalgebras”, JETP Letters, **50** (1989) 344 – 347; M.A. Vasiliev, “Higher spin algebras and quantization on the sphere and hyperboloid”, Int. J. Mod. Phys. **A6** (1991) 1115.