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Symmetric solutions to the four dimensional degenerate Painlevé type equation NY^{A_4}

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Abstract: We have classified symmetric solutions around the origin to the four dimensional degenerate Painlevé type equation NY^{A_4} with generic values of parameters. We obtained sixteen meromorphic solutions, which are transformed each other by the Bäcklund transformation. We calculated the linear monodromy for one of them, explicitly.

Keywords: the four dimensional degenerate Painlevé type equation, monodromy data, Bäcklund transformation.

2000 Mathematics Subject Classification: Primary 34M55; Secondary 33C15.

1. Introduction

The Painlevé equations $P_J (J = I, II, \dots, VI)$ have the phase space of dimension two, while many higher dimensional Painlevé type equations are obtained recently as the extension of the sixth Painlevé equation.

For the four dimensional case, T. Oshima showed that any irreducible Fuchsian equation with four accessory parameters can be reduced to thirteen types of equations [20], among which H. Sakai found that only four types of equation have the isomonodromic deformation equation and these are the source equations of all of the four dimensional Painlevé type equations. They are the well-known Garnier system with two variables [8], the Fuji-Suzuki system [7], the Sasano system [22] and the matrix Painlevé system. The last system is the new one which is found by H. Sakai by the monodromy preserving deformation of the Fuchsian differential equation [21].

The four dimensional Painlevé type equations have the degeneration diagram similar to the Painlevé equations [14], which is from the extension of the sixth Painlevé equation, step by step, to the extension of the second Painlevé equation. The extension of the first Painlevé equation is not yet found. The four dimensional degenerate Painlevé type equation NY^{A_4} was proposed by V.E. Adler [1] and Noumi and Yamada [18] independently. In [14], NY^{A_4} is obtained by degenerating the Fuji-Suzuki system as the extension of the fourth Painlevé equation P_{IV} , which corresponds to the well-known Noumi-Yamada system [18]. The equation NY^{A_4} is also derived by the isomonodromic deformation of the third kind, non Fuchsian ordinary differential equation, which has one regular singularity and one irregular singularity of Poincaré rank 2 on the Riemann sphere (See eq.(2.3)).

It has been important to have the relation between each Painlevé function and the monodromy data of the associated linear equation (we call the linear monodromy). By calculating the linear

monodromy, we can obtain the characteristics of each Painlevé function. R. Fuchs calculated first the linear monodromy for the Picard's solution [6], which is a special solution to the sixth Painlevé equation with special values of parameters $\alpha = \beta = \gamma = \frac{1}{2} - \delta = 0$. For generic value of parameter, A. V. Kitaev calculated first the linear monodromy explicitly for the symmetric solutions by taking examples of the first and second Painlevé equations [16]. We call them the symmetric solutions, which are invariant under the symmetric transformations (see Remark 2.1). Based on A.V. Kitaev's idea, we have studied special solutions with generic values of parameters for the fourth, fifth, sixth and third Painlevé equations, for which the linear monodromy can be calculated explicitly [10–13]. For NY^{A_4} , it is not easy to determine the linear monodromy for generic solutions, but we can determine the linear monodromy for the symmetric solutions under the transformation in Remark 2.1. We remark that P. Appell [2] also studied the symmetric solutions to the first and second Painlevé equations, but he did not study the linear monodromy problems.

There are few research for the special solutions to the four dimensional Painlevé type equations compared with the Painlevé equations. We will study the four dimensional degenerate Painlevé type equations by applying the same method to NY^{A_4} first, which we have used for the Painlevé equations above. Some new discovery is expected by viewing Painlevé equations from the four dimensional Painlevé type equations.

The aim of this paper is to give the symmetric solutions with generic values of parameters to NY^{A_4} , for which the linear monodromy $\{M_0 = S_1 S_2 S_3 S_4 e^{2\pi i T_0}, C, M_\infty\}$ can be calculated explicitly. We obtained the sixteen symmetric solutions with generic values of parameters around the origin, which are transformed each other by the Bäcklund transformations (see subsection 4.2 and Fig.1). Similar calculations were made by N. Tahara [23] and K. Matsuda [17] by using the Noumi-Yamada system [18]. N. Tahara gave the formal solutions with a pole of order one at any t and K. Matsuda completely classified the rational solutions to NY^{A_4} . Both papers treat neither the symmetric solutions nor the linear monodromy. For NY^{A_4} , the linear monodromy can be calculated only for the symmetric solutions (see Section 5). We calculated the linear monodromy for one of the obtained symmetric solutions, explicitly. For the other solutions, we can obtain the linear monodromy by using the Bäcklund transformations.

In Appendix, we show the Noumi-Yamada system and the transformation formulae to the Hamiltonian system $\mathcal{H}_{NY}^{A_4}$.

2. The four dimensional degenerate Painlevé type equation NY^{A_4}

In this section, we write down the deformation equation and monodromy preserving deformation and Hamiltonian system $\mathcal{H}_{NY}^{A_4}$ [14].

2.1. The system of the deformation equation

The system of the deformation equation for $\mathcal{H}_{NY}^{A_4}$ is given as follows:

$$\frac{\partial Y}{\partial x} = \left(\frac{A_0^{(-2)}}{x^3} + \frac{A_0^{(-1)}}{x^2} + \frac{A_0^{(0)}}{x} \right) Y, \tag{2.1}$$

$$\frac{\partial Y}{\partial t} = \frac{A_0^{(-2)}}{x} Y, \tag{2.2}$$

where

$$\begin{aligned}
 A_0^{(j)} &= U^{-1}P^{-1}\widehat{A}_0^{(j)}PU, \quad j \in \{-2, -1, 0\}, \quad U = \text{diag}(1, u, v), \\
 \frac{1}{u} \frac{du}{dt} &= -p_1 - 2p_2 + q_2 + t, \quad \frac{1}{v} \frac{dv}{dt} = q_1 - 2p_1 - 2p_2 + t, \\
 P &= \begin{pmatrix} 1 & 0 & 0 \\ \frac{a}{\theta_1^\infty - \theta_2^\infty} & 1 & 0 \\ \frac{1}{\theta_1^\infty - \theta_3^\infty} \left(b + \frac{ac}{\theta_1^\infty - \theta_2^\infty} \right) & \frac{c}{\theta_2^\infty - \theta_3^\infty} & 1 \end{pmatrix}, \quad \widehat{A}_0^{(-2)} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} (1 \ 1 \ 1), \\
 \widehat{A}_0^{(-1)} &= \begin{pmatrix} p_1 + p_2 - t & q_2 & q_1 \\ -p_2 & -p_2 & -p_2 \\ -p_1 & -p_1 & -p_1 \end{pmatrix}, \quad \widehat{A}_0^{(0)} = \begin{pmatrix} -\theta_1^\infty & 0 & 0 \\ -a & -\theta_2^\infty & 0 \\ -b & -c & -\theta_3^\infty \end{pmatrix}, \\
 a &= p_2(p_2 - q_2 - t) + p_1p_2 + \theta_1^0 + \theta_2^\infty, \quad b = p_1(p_1 - q_1 - t) + p_1p_2 + \theta_3^\infty, \\
 c &= p_1(q_2 - q_1) + \theta_3^\infty.
 \end{aligned}$$

The Riemann scheme of (2.1) is

$$P \begin{pmatrix} \overbrace{x=0}^{x=0} & x=\infty \\ 0 & 0 & 0 & \theta_1^\infty \\ 0 & 0 & \theta_1^0 & \theta_2^\infty \\ 1 & -t & \theta_2^0 & \theta_3^\infty \end{pmatrix}, \quad \theta_1^0 + \theta_2^0 + \theta_1^\infty + \theta_2^\infty + \theta_3^\infty = 0. \quad (2.3)$$

The formal solution of (2.1) around $x = 0$ has the form

$$Y^{(0)}(x) = \exp \left[\frac{1}{x^2} \text{diag}(0, 0, 1) + \frac{1}{x} \text{diag}(0, 0, t) \right] x^{T_0} \left(I_3 + \sum_{k=1}^{\infty} \widehat{Y}_k^{(0)} x^k \right),$$

where $T_0 = \text{diag}(0, \theta_1^0, \theta_2^0)$. The series $\widehat{Y}^{(0)} = I_3 + \sum_{k=1}^{\infty} \widehat{Y}_k^{(0)} x^k$ is divergent since $x = 0$ is an irregular singularity of Poincaré rank two.

The local solution of (2.1) around $x = \infty$ has the form

$$Y^{(\infty)}(x) = x^{-T_\infty} \left(I_3 + \sum_{k=1}^{\infty} Y_k^{(\infty)} x^k \right),$$

where $T_\infty = \text{diag}(\theta_1^\infty, \theta_2^\infty, \theta_3^\infty)$. This series is convergent, since $x = \infty$ is a regular singular point. Since $Y^{(\infty)}(x)$ is multi-valued functions, we take a branch of $Y_\infty(x)$ on $\{x \mid |\arg x| < \pi, |x| > R\}$.

The Stokes regions \mathcal{S}_j around $x = 0$ are given by

$$\mathcal{S}_j = \left\{ x \mid -\varepsilon + \frac{(j-1)\pi}{2} < \arg x < \frac{j\pi}{2} + \varepsilon, |x| < r \right\},$$

where ε and r are sufficiently small. There exist a holomorphic function $\bar{Y}_j(x)$ of (2.1) on \mathcal{S}_j such that

$$\bar{Y}_j(x) \sim \widehat{Y}^{(0)} \quad x \rightarrow 0$$

and

$$Y_j(x) = \exp \left[\frac{1}{x^2} \text{diag}(0, 0, 1) + \frac{1}{x} \text{diag}(0, 0, t) \right] x^{T_0} \bar{Y}_j(x)$$

is a solution of (2.1) on \mathcal{S}_j . The Stokes matrix S_j is defined by

$$Y_{j+1}(x) = Y_j(x)S_j.$$

We notice that $Y_5(x) = Y_1(xe^{-2\pi i})e^{-2\pi iT_0}$.

The connection matrix C between $x = 0$ and $x = \infty$ is given by $Y^{(\infty)}(x) = Y^{(0)}C$.

The linear equation (2.1) has s data of the linear monodromy

$$\{M_\infty, C, S_1, S_2, S_3, S_4, e^{2\pi iT_0}\},$$

where $M_\infty = e^{2\pi iT_\infty}$ is the local monodromy around $x = \infty$ and $e^{2\pi iT_0}$ is the formal monodromy around $x = 0$.

The linear monodromy are represented as follows:

$$M_\infty = \begin{pmatrix} e^{2\pi i\theta_1^\infty} & 0 & 0 \\ 0 & e^{2\pi i\theta_2^\infty} & 0 \\ 0 & 0 & e^{2\pi i\theta_3^\infty} \end{pmatrix}, \quad e^{2\pi iT_0} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & e^{2\pi i\theta_1^0} & 0 \\ 0 & 0 & e^{2\pi i\theta_2^0} \end{pmatrix}, \quad (2.4)$$

$$S_{2n-1} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ s_{2n-1}^{(1)} & s_{2n-1}^{(2)} & 1 \end{pmatrix}, \quad S_{2n} = \begin{pmatrix} 1 & 0 & s_{2n}^{(1)} \\ 0 & 1 & s_{2n}^{(2)} \\ 0 & 0 & 1 \end{pmatrix}, \quad C = (c_{ij})_{i,j=1,2,3},$$

$$CM_\infty C^{-1}S_1S_2S_3S_4e^{2\pi iT_0} = I_3. \quad (2.5)$$

2.2. Monodromy preserving deformation and Hamiltonian system

When the linear monodromy of (2.1) is independent of the parameter t , $Y = Y(x, t)$ satisfies both of (2.1) and (2.2). The monodromy preserving deformation equation is given by the compatibility condition of (2.1) and (2.2). This equation is expressed by the Hamiltonian system using the two Hamiltonians H_{IV} of the fourth Painlevé equation with different parameters and the coupled term:

$$H_{NY}^{A_4}(\alpha, \beta, ;, t; q_1, p_1) = H_{IV}(\beta, \alpha; t; q_1, p_1) + H_{IV}(\tilde{\beta}, \tilde{\alpha}; t; q_2, p_2) + 2q_1p_1p_2,$$

where

$$H_{IV}(\beta, \alpha; t; q, p) = qp(p - q - t) + \alpha p + \beta q,$$

$$\alpha = \theta_1^\infty - \theta_3^\infty - 1, \quad \beta = \theta_3^\infty, \quad \tilde{\alpha} = \theta_1^\infty - \theta_2^\infty - \theta_3^\infty - 1, \quad \tilde{\beta} = \theta_1^0 + \theta_2^\infty.$$

The Hamiltonian system $\mathcal{H}_{NY}^{A_4}$ is expressed as follows:

$$\frac{dq_1}{dt} = \frac{\partial H_{NY}^{A_4}}{\partial p_1} = q_1(2p_1 - q_1 - t) + \alpha + 2q_1p_2,$$

$$-\frac{dp_1}{dt} = \frac{\partial H_{NY}^{A_4}}{\partial q_1} = p_1(p_1 - t - 2q_1) + \beta + 2p_1p_2,$$

$$\frac{dq_2}{dt} = \frac{\partial H_{NY}^{A_4}}{\partial p_2} = q_2(2p_2 - q_2 - t) + \tilde{\alpha} + 2q_1p_1,$$

$$-\frac{dp_2}{dt} = \frac{\partial H_{NY}^{A_4}}{\partial q_2} = p_2(p_2 - t - 2q_2) + \tilde{\beta}.$$

Remark 2.1. The Hamiltonian system $\mathcal{H}_{NY}^{A_4}$ is invariant under the following symmetric transformations:

$$q_1 \longrightarrow -q_1, p_1 \longrightarrow -p_1, q_2 \longrightarrow -q_2, p_2 \longrightarrow -p_2, t \longrightarrow -t.$$

q_1, p_1, q_2, p_2 are all odd functions.

By using this property, we can calculate the symmetric solutions to the Hamiltonian system $\mathcal{H}_{NY}^{A_4}$.

The system $\mathcal{H}_{NY}^{A_4}$ is the monodromy preserving deformation for (2.1). Therefore, there exists almost one-to-one correspondence between the monodromy data and the solutions of $\mathcal{H}_{NY}^{A_4}$. In this sense, we call the corresponding monodromy data *the linear monodromy* of the solution of $\mathcal{H}_{NY}^{A_4}$. It is not easy to determine the linear monodromy for generic solutions, but we can determine the linear monodromy for the symmetric solutions under the transformation in Remark 2.1.

3. Symmetric solutions around the origin

In this section, we give the obtained symmetric solutions around $t = 0$, which are satisfied with the Hamiltonian system $\mathcal{H}_{NY}^{A_4}$.

When q_i and p_i ($i = 1, 2$) are meromorphic, we can show that they have at most a simple pole around $t = 0$.

Since $\mathcal{H}_{NY}^{A_4}$ is the fourth order differential equations, the solution space of $\mathcal{H}_{NY}^{A_4}$ is for dimensional. But we can show that the number of solutions which are invariant under the action in Remark 2.1 is finite.

Theorem 3.1. For generic values of parameters, the Hamiltonian system $\mathcal{H}_{NY}^{A_4}$ has the following sixteen symmetric solutions (1), (2), \dots , (16) around the origin:

$$\begin{aligned} q_1 &= \frac{a_{-1}}{t} + \sum_{k=1}^{\infty} a_{2k-1} t^{2k-1}, & p_1 &= \frac{\tilde{a}_{-1}}{t} + \sum_{k=1}^{\infty} \tilde{a}_{2k-1} t^{2k-1}, \\ q_2 &= \frac{b_{-1}}{t} + \sum_{k=1}^{\infty} b_{2k-1} t^{2k-1}, & p_2 &= \frac{\tilde{b}_{-1}}{t} + \sum_{k=1}^{\infty} \tilde{b}_{2k-1} t^{2k-1}, \end{aligned}$$

• One holomorphic solution (1):

$$\begin{aligned} (1) : \quad a_{-1} &= 0, a_1 = \alpha, \quad a_3 = \frac{-\alpha}{3}(\alpha + 2\beta + 1 + 2\tilde{\beta}), \dots, \\ \tilde{a}_{-1} &= 0, \tilde{a}_1 = -\beta, \tilde{a}_3 = \frac{-\beta}{3}(2\alpha + \beta + 1 + 2\tilde{\beta}), \dots, \\ b_{-1} &= 0, b_1 = \tilde{\alpha}, \quad b_3 = \frac{-1}{3}[\tilde{\alpha}(\tilde{\alpha} + 2\tilde{\beta} + 1) + 2\alpha\beta], \dots, \\ \tilde{b}_{-1} &= 0, \tilde{b}_1 = -\tilde{\beta}, \tilde{b}_3 = \frac{-\tilde{\beta}}{3}(2\tilde{\alpha} + \tilde{\beta} + 1), \dots. \end{aligned}$$

• Fifteen meromorphic solutions (2), (3), \dots , (16) with a pole of order one:

	a_{-1}	\tilde{a}_{-1}	b_{-1}	\tilde{b}_{-1}	a_1	\tilde{a}_1	b_1	\tilde{b}_1
(2)	-3	1	-2	-3
(3)	1	-3	1	3
(4)	0	3	-1	-1	$-\alpha/3$
(5)	-1	0	-1	-1	...	$-\beta$
(6)	-1	-1	-1	0	$-\tilde{\beta}/3$
(7)	-1	-1	2	0	$\tilde{\beta}/3$
(8)	0	0	-1	-1	$\alpha/3$	β
(9)	0	-1	0	1	α	...	$-(\tilde{\alpha} - 2\alpha)$...
(10)	0	1	1	0	$-\alpha$	$\tilde{\beta}$
(11)	3	0	0	1	...	$\beta/3$	$-\tilde{\alpha} - 2\beta$...
(12)	1	0	1	0	...	β	...	$\tilde{\beta}$
(13)	0	0	0	1	$-\alpha$	$-\beta/3$	$-\tilde{\alpha}$...
(14)	0	0	1	0	α	$-\beta$...	$\tilde{\beta}$
(15)	0	1	0	0	$-\alpha$...	$\tilde{\alpha} - 2\alpha$	$-\tilde{\beta}$
(16)	1	0	0	0	...	β	$\tilde{\alpha} + 2\beta$	$-\tilde{\beta}$

Remark 3.1. (1) Higher order expansion of these solutions are uniquely determined recursively by the Hamiltonian system $\mathcal{H}_{NY}^{A_4}$ and do not contain any other parameter than the parameters $\{\alpha, \beta, \tilde{\alpha}, \tilde{\beta}\}$.

(2) These solutions are convergent by Briot-Bouquet's theorem [3].

(3) The values of parameters are generic.

4. Bäcklund transformation

In this section we show the Bäcklund transformation for the NY^{A_4} given by M. Noumi and Y. Yamada [18] and the operated results to the sixteen symmetric solutions.

4.1. the Bäcklund transformation for the NY^{A_4}

The Bäcklund transformations are given as follows:

x	$s_0(x)$	$s_1(x)$	$s_2(x)$	$s_3(x)$	$s_4(x)$	$\pi(x)$
q_1	$q_1 - \frac{\alpha_0}{q_2 - p_1 - p_2 + t}$	q_1	$q_1 + \frac{\alpha_2}{p_1}$	q_1	q_1	$-p_1$
p_1	p_1	$p_1 - \frac{\alpha_1}{q_1}$	p_1	$p_1 - \frac{\alpha_3}{q_1 - q_2}$	p_1	$q_1 - q_2$
q_2	$q_2 - \frac{\alpha_0}{q_2 - p_1 - p_2 + t}$	q_2	q_2	q_2	$q_2 + \frac{\alpha_4}{p_2}$	$-p_1 - p_2$
p_2	$p_2 - \frac{\alpha_0}{q_2 - p_1 - p_2 + t}$	p_2	p_2	$p_2 + \frac{\alpha_3}{q_1 - q_2}$	p_2	$q_2 - p_1 - p_2 + t$
t	t	t	t	t	t	t
α_0	$-\alpha_0$	$\alpha_0 + \alpha_1$	α_0	α_0	$\alpha_0 + \alpha_4$	α_1
α_1	$\alpha_1 + \alpha_0$	$-\alpha_1$	$\alpha_1 + \alpha_2$	α_1	α_1	α_2
α_2	α_2	$\alpha_2 + \alpha_1$	$-\alpha_2$	$\alpha_2 + \alpha_3$	α_2	α_3
α_3	α_3	α_3	$\alpha_3 + \alpha_2$	$-\alpha_3$	$\alpha_3 + \alpha_4$	α_4
α_4	$\alpha_4 + \alpha_0$	α_4	α_4	$\alpha_4 + \alpha_3$	$-\alpha_4$	α_0

$$s_i^2 = 1, \quad \pi^5 = 1, \quad \alpha = -\alpha_1, \quad \beta = -\alpha_2, \quad \tilde{\alpha} = -\alpha_1 - \alpha_3, \quad \tilde{\beta} = -\alpha_4.$$

4.2. The operated results to the sixteen symmetric solutions

The operated results to the sixteen symmetric solutions are shown in Fig 1.

The sixteen symmetric solutions are arranged on the vertices of the concentric three pentagons and the center; the fifteen meromorphic solutions with a pole of order one are on the vertices and the holomorphic solution is on the center.

The fifteen solutions arranged on the vertices are transformed each other by the reflection $s_i (i = 0, 1, 2, 3, 4)$ in the radial direction and by the rotation π in the counter clockwise direction along the edges of the pentagons. Besides these, there are five transformations by the reflection s_i from every vertex of the inner first pentagon to one of the vertices of the second pentagon.

These fifteen solutions are closed by the Bäcklund transformation $\{s_i (i = 0, 1, 2, 3, 4), \pi\}$ but the holomorphic solution (1) arranged on the center jumps out from these three pentagons by the rotation π and the transformed holomorphic solution is not satisfied the Hamiltonian system $\mathcal{H}_{NY}^{A_4}$.

We can see that the holomorphic solution (1) arranged on the center is the key solution. We show the calculated results of the linear monodromy for the solution (1) in the next section.

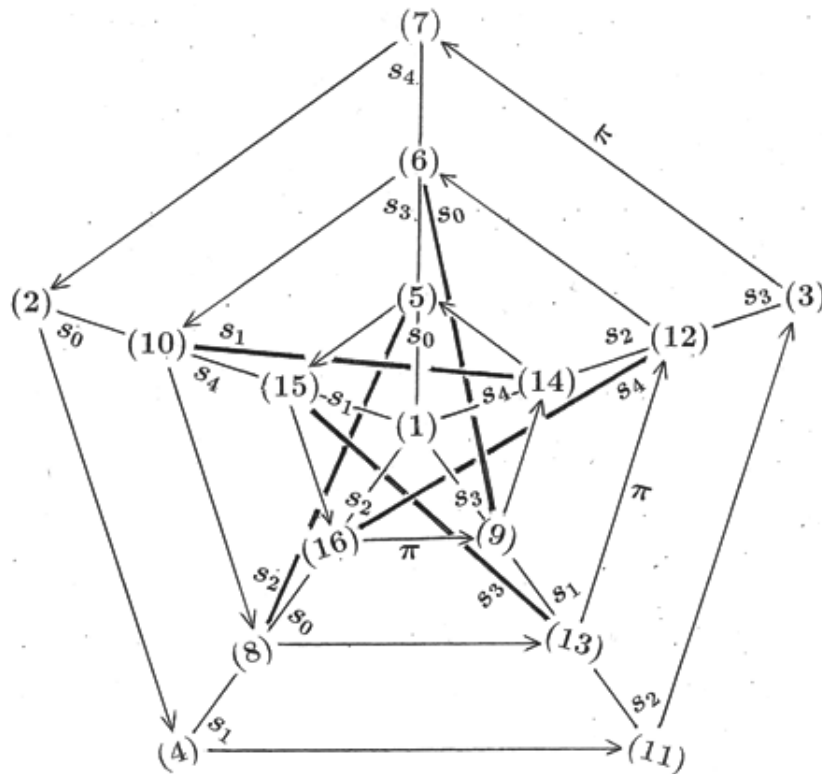


Fig. 1. NY^{A_4} 16 solutions and Bäcklund transformations

Remark 4.1. The fourth Painlevé equation P_{IV} has one holomorphic solution (1) and three meromorphic solutions with a pole of order one (2), (3) and (4), which are transformed each other by the Bäcklund transformation $\{s_i (i = 0, 1, 2), \pi | \pi^3 = 1\}$ as shown in Fig 2.

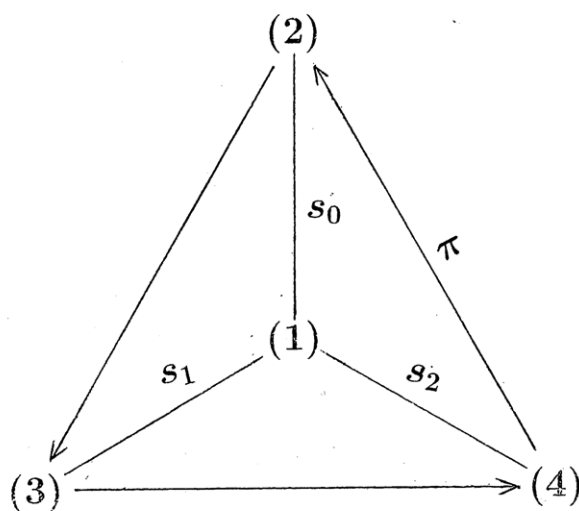


Fig. 2. P_{IV} : 4 solutions and Bäcklund transformations

Conjecture 4.1. For the $2n$ dimensional case, we will have $4n^2$ meromorphic solutions around the origin, which will be arranged on the vertices of $2n + 1$ concentric $2n - 1$ polygons and the center.

5. The linear monodromy for the symmetric solution (1)

In this section, we show the calculation results on the connection problem of the confluent hypergeometric equations ${}_2F_2$ and the linear monodromy for the symmetric solution (1).

We can put $t = 0$ after substituting the symmetric solution (1) into the linear equation (2.1), since the monodromy preserving deformation theory tells the linear monodromy is invariant under any value change of the deformation parameter t . Only when putting $t = 0$ after substituting the symmetric solution (1) into the linear equation (2.1), the equation (2.1) can be reduced to the confluent hypergeometric equations ${}_2F_2$ and we can calculate the linear monodromy. For ${}_2F_2$, A. Duval and C. Mitchi calculated the connection matrix and the Stokes matrices in [4, 5].

In subsection 5.1, we review the results by Duval and Mitchi. In subsection 5.2, we determine the linear monodromy for the symmetric solution (1).

5.1. The confluent hypergeometric equation ${}_2F_2$

The confluent hypergeometric equation ${}_2F_2$ is

$$\eta^2 \frac{d^3 \phi}{d\eta^3} + (1 + \beta_1 + \beta_2 - \eta) \eta \frac{d^2 \phi}{d\eta^2} + (\beta_1 \beta_2 - (1 + \alpha_1 + \alpha_2) \eta) \frac{d\phi}{d\eta} - \alpha_1 \alpha_2 \phi = 0.$$

The local solutions (w_1^0, w_2^0, w_3^0) around $\eta = 0$ are given by the confluent hypergeometric series ${}_2F_2$:

$$w_1^0 = {}_2F_2 \left(\begin{matrix} \alpha_1, \alpha_2 \\ \beta_1, \beta_2 \end{matrix} ; \eta \right) = \sum_{k=0}^{\infty} \frac{(\alpha_1)_k (\alpha_2)_k \eta^k}{(\beta_1)_k (\beta_2)_k k!},$$

$$w_2^0 = (-\eta)^{1-\beta_1} {}_2F_2 \left(\begin{matrix} \alpha_1 - \beta_1 + 1, \alpha_2 - \beta_1 + 1 \\ 2 - \beta_1, \beta_2 - \beta_1 + 1 \end{matrix} ; \eta \right),$$

$$w_3^0 = (-\eta)^{1-\beta_2} {}_2F_2 \left(\begin{matrix} \alpha_1 - \beta_2 + 1, \alpha_2 - \beta_2 + 1 \\ 2 - \beta_2, \beta_1 - \beta_2 + 1 \end{matrix}; \eta \right).$$

Here $(\alpha)_k = \alpha(\alpha + 1)(\alpha + 2) \cdots (\alpha + k - 1)$. Since $\eta = \infty$ is an irregular singular point of the Poincaré rank one, we have formal solutions $(\tilde{w}_1^\infty, \tilde{w}_2^\infty, \tilde{w}_3^\infty)$ around $\eta = \infty$:

$$\begin{aligned} \tilde{w}_1^\infty &= \left(\frac{1}{-\eta} \right)^{\alpha_1} {}_3F_1 \left(\begin{matrix} \alpha_1, \alpha_1 - \beta_1 + 1, \alpha_1 - \beta_2 + 1 \\ \alpha_1 - \alpha_2 + 1 \end{matrix}; \frac{-1}{\eta} \right), \\ \tilde{w}_2^\infty &= \left(\frac{1}{-\eta} \right)^{\alpha_2} {}_3F_1 \left(\begin{matrix} \alpha_2, \alpha_2 - \beta_1 + 1, \alpha_2 - \beta_2 + 1 \\ \alpha_2 - \alpha_1 + 1 \end{matrix}; \frac{-1}{\eta} \right), \\ \tilde{w}_3^\infty &= e^\eta \left(\frac{1}{-\eta} \right)^{\beta_1 + \beta_2 - \alpha_1 - \alpha_2} \left[1 + \frac{1}{\eta} \{ (\beta_1 + \beta_2 - \alpha_1 - \alpha_2)(1 - \alpha_1 - \alpha_2) + \beta_1 \beta_2 - \alpha_1 \alpha_2 \} + O\left(\frac{1}{\eta^2}\right) \right]. \end{aligned}$$

The third solution around $\eta = \infty$: \tilde{w}_3^∞ cannot be represented by the confluent hypergeometric series.

We take a branch of the solutions (w_1^0, w_2^0, w_3^0) on $\{x \mid |\arg x| < \pi\}$ since the ${}_2F_2(x)$ converges on the entire plane. The Stokes regions $\tilde{\mathcal{S}}_j$ around $x = \infty$ are given by

$$\tilde{\mathcal{S}}_j = \{x \mid -\varepsilon + (j - 1)\pi < \arg x < j\pi + \varepsilon, |x| > R\},$$

where ε is sufficiently small and R is sufficiently large.

There exist holomorphic solutions $(w_{1,j}^\infty, w_{2,j}^\infty, w_{3,j}^\infty)$ on $\tilde{\mathcal{S}}_j$, which is asymptotic to divergent series $(\tilde{w}_1^\infty, \tilde{w}_2^\infty, \tilde{w}_3^\infty)$. The Stokes matrices Σ_j are given by

$$(w_{1,j+1}^\infty, w_{2,j+1}^\infty, w_{3,j+1}^\infty) = (w_{1,j}^\infty, w_{2,j}^\infty, w_{3,j}^\infty) \Sigma_j.$$

And the connection matrix D is given by

$$(w_{1,1}^\infty, w_{2,1}^\infty, w_{3,1}^\infty) = (w_1^0, w_2^0, w_3^0) D.$$

The Stokes matrices have the following form:

$$\Sigma_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ \sigma_1^{(1)} & \sigma_1^{(2)} & 1 \end{pmatrix}, \quad \Sigma_2 = \begin{pmatrix} 1 & 0 & \sigma_2^{(1)} \\ 0 & 1 & \sigma_2^{(2)} \\ 0 & 0 & 1 \end{pmatrix}.$$

We set $\tilde{T}_0 = \text{diag}(0, 1 - \beta_1, 1 - \beta_2)$, $\tilde{T}_\infty = \text{diag}(\alpha_1, \alpha_2, \alpha_0)$ and $\alpha_0 = \beta_1 + \beta_2 - \alpha_1 - \alpha_2$. Then $\tilde{M}_0 = \exp(2\pi i \tilde{T}_0)$ is a monodromy matrix around $\eta = 0$ and we have a relation

$$D \tilde{M}_0 D^{-1} \Sigma_1 \Sigma_2 e^{2\pi i \tilde{T}_\infty} = I_3. \tag{5.1}$$

The following Proposition has been obtained in A. Duval and C. Mitchi [4, 5], but our form is slightly different from them:

Proposition 5.1. *The monodromy data $(D = (d_{ij}), \Sigma_1, \Sigma_2)$ of ${}_2F_2$ is as follows:*

$$d_{11} = \frac{\Gamma(1 + \alpha_1 - \alpha_2) \Gamma(1 - \beta_1) \Gamma(1 - \beta_2)}{\Gamma(1 - \alpha_2) \Gamma(1 + \alpha_1 - \beta_1) \Gamma(1 + \alpha_1 - \beta_2)} e^{\pi i \alpha_1},$$

$$\begin{aligned}
 d_{21} &= \frac{\Gamma(1 + \alpha_1 - \alpha_2)\Gamma(\beta_1 - 1)\Gamma(\beta_1 - \beta_2)}{\Gamma(\alpha_1)\Gamma(1 + \alpha_1 - \beta_2)\Gamma(\beta_1 - \alpha_2)} e^{\pi i(1 + \alpha_1 - \beta_1)}, \\
 d_{31} &= \frac{\Gamma(1 + \alpha_1 - \alpha_2)\Gamma(\beta_2 - 1)\Gamma(\beta_2 - \beta_1)}{\Gamma(\alpha_1)\Gamma(1 + \alpha_1 - \beta_1)\Gamma(\beta_2 - \alpha_2)} e^{\pi i(1 + \alpha_1 - \beta_2)}, \\
 d_{12} &= \frac{\Gamma(1 + \alpha_2 - \alpha_1)\Gamma(1 - \beta_1)\Gamma(1 - \beta_2)}{\Gamma(1 - \alpha_1)\Gamma(1 + \alpha_2 - \beta_1)\Gamma(1 + \alpha_2 - \beta_2)} e^{\pi i\alpha_2}, \\
 d_{22} &= \frac{\Gamma(1 + \alpha_2 - \alpha_1)\Gamma(\beta_1 - 1)\Gamma(\beta_1 - \beta_2)}{\Gamma(\alpha_2)\Gamma(1 + \alpha_2 - \beta_2)\Gamma(\beta_1 - \alpha_1)} e^{\pi i(1 + \alpha_2 - \beta_1)}, \\
 d_{32} &= \frac{\Gamma(1 + \alpha_2 - \alpha_1)\Gamma(\beta_2 - 1)\Gamma(\beta_2 - \beta_1)}{\Gamma(\alpha_2)\Gamma(1 + \alpha_2 - \beta_1)\Gamma(\beta_2 - \alpha_1)} e^{\pi i(1 + \alpha_2 - \beta_2)}, \\
 d_{13} &= \frac{\Gamma(1 - \beta_1)\Gamma(1 - \beta_2)}{\Gamma(1 - \alpha_1)\Gamma(1 - \alpha_2)}, \quad d_{23} = \frac{\Gamma(\beta_1 - \beta_2)\Gamma(\beta_1 - 1)}{\Gamma(\beta_1 - \alpha_1)\Gamma(\beta_1 - \alpha_2)}, \quad d_{33} = \frac{\Gamma(\beta_2 - \beta_1)\Gamma(\beta_2 - 1)}{\Gamma(\beta_2 - \alpha_1)\Gamma(\beta_2 - \alpha_2)}, \\
 \sigma_2^{(1)} &= \frac{-2\pi i\Gamma(\alpha_2 - \alpha_1)}{\Gamma(1 - \alpha_1)\Gamma(\beta_1 - \alpha_1)\Gamma(\beta_2 - \alpha_1)} e^{\pi i(\alpha_1 - \alpha_0)}, \\
 \sigma_2^{(2)} &= \frac{-2\pi i\Gamma(\alpha_1 - \alpha_2)}{\Gamma(1 - \alpha_2)\Gamma(\beta_1 - \alpha_2)\Gamma(\beta_2 - \alpha_2)} e^{\pi i(\alpha_2 - \alpha_0)}, \\
 \sigma_1^{(1)} &= \frac{-2\pi i\Gamma(1 - \alpha_2 + \alpha_1)}{\Gamma(\alpha_1)\Gamma(1 - \beta_1 + \alpha_1)\Gamma(1 - \beta_2 + \alpha_1)}, \\
 \sigma_1^{(2)} &= \frac{-2\pi i\Gamma(1 - \alpha_1 + \alpha_2)}{\Gamma(\alpha_2)\Gamma(1 - \beta_1 + \alpha_2)\Gamma(1 - \beta_2 + \alpha_2)}.
 \end{aligned}$$

Remark 5.1. (1) Every d_{ij} ($i, j = 1, 2, 3$) coincides with A. Duval and C. Mitschi's calculation results except for $d_{2j} \longleftrightarrow d_{3j}$.
 (2) Every $\sigma_i^{(j)}$ ($i, j = 1, 2$) is the negative sign of A. Duval and C. Mitschi's calculation results, since they use Σ^{-1} in our sense as the Stokes matrix.

5.2. Linear monodromy of the symmetric solution (I)

Putting $t = 0$ after substituting the solution (1) into the linear equation (2.1), we have

$$\begin{aligned}
 \frac{d}{dx} \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} &= \begin{pmatrix} \frac{\tilde{a}}{x^3} - \frac{\theta_1^\infty}{x} & \frac{\tilde{b}}{x^3}u & \frac{1}{x^3}v \\ \tilde{p}_{21} \frac{\tilde{a}}{x^3} \frac{1}{u} & \tilde{p}_{21} \frac{\tilde{b}}{x^3} - \frac{\theta_2^\infty}{x} & \tilde{p}_{21} \frac{1}{x^3} \frac{v}{u} \\ \tilde{p}_{31} \frac{\tilde{a}}{x^3} \frac{1}{v} & \tilde{p}_{31} \frac{\tilde{b}}{x^3} \frac{u}{v} & \tilde{p}_{31} \frac{1}{x^3} - \frac{\theta_3^\infty}{x} \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix}, \\
 {}^t(y_1, y_2, y_3) &= Y, \quad \tilde{a} = \frac{\theta_1^\infty(\theta_1^0 + \theta_1^\infty)}{(\theta_1^\infty - \theta_2^\infty)(\theta_1^\infty - \theta_3^\infty)}, \quad \tilde{b} = \frac{\theta_2^\infty}{\theta_2^\infty - \theta_3^\infty}, \\
 \tilde{p}_{21} &= \frac{-(\theta_1^0 + \theta_2^\infty)}{\theta_1^\infty - \theta_2^\infty}, \quad \tilde{p}_{31} = \frac{\theta_3^\infty(\theta_1^0 + \theta_3^\infty)}{(\theta_1^\infty - \theta_3^\infty)(\theta_2^\infty - \theta_3^\infty)}.
 \end{aligned} \tag{5.2}$$

The system of equations (5.2) can be reduced to the following confluent hypergeometric equations ${}_2F_2$ with different parameters:

$$\eta^2 \frac{d^3 \phi_i}{d\eta^3} + (1 + \beta_1^{(i)} + \beta_2^{(i)} - \eta) \eta \frac{d^2 \phi_i}{d\eta^2} + (\beta_1^{(i)} \beta_2^{(i)} - (1 + \alpha_1^{(i)} + \alpha_2^{(i)}) \eta) \frac{d\phi_i}{d\eta} - \alpha_1^{(i)} \alpha_2^{(i)} \phi_i = 0, \quad (5.3)$$

$$y_i = x^{-\theta_i^\infty} \phi_i (i = 1, 2, 3), \quad x = (-2\eta)^{-\frac{1}{2}}, \quad (5.4)$$

$$\begin{aligned} \alpha_1^{(1)} &= \frac{\theta_1^\infty}{2}, & \alpha_2^{(1)} &= \frac{\theta_1^\infty + \theta_1^0}{2}, & \beta_1^{(1)} &= \frac{\theta_1^\infty - \theta_2^\infty}{2}, & \beta_2^{(1)} &= \frac{\theta_1^\infty - \theta_3^\infty}{2}, \\ \alpha_1^{(2)} &= \frac{\theta_2^\infty}{2}, & \alpha_2^{(2)} &= \frac{\theta_2^\infty + \theta_1^0}{2}, & \beta_1^{(2)} &= \frac{\theta_2^\infty - \theta_3^\infty}{2}, & \beta_2^{(2)} &= \frac{\theta_2^\infty - \theta_1^\infty}{2}, \\ \alpha_1^{(3)} &= \frac{\theta_3^\infty}{2}, & \alpha_2^{(3)} &= \frac{\theta_3^\infty + \theta_1^0}{2}, & \beta_1^{(3)} &= \frac{\theta_3^\infty - \theta_1^\infty}{2}, & \beta_2^{(3)} &= \frac{\theta_3^\infty - \theta_2^\infty}{2}, \end{aligned}$$

whose Riemann scheme is

$$P \begin{pmatrix} \eta = 0 & \eta = \infty \\ 0 & \alpha_1^{(i)} \\ 1 - \beta_1^{(i)} & 0 & \alpha_2^{(i)} \\ 1 - \beta_2^{(i)} & 1 & \beta_1^{(i)} + \beta_2^{(i)} - \alpha_1^{(i)} - \alpha_2^{(i)} \end{pmatrix}.$$

A fundamental solution matrix is

$$\begin{pmatrix} F^{11} & (-\eta)^{1-\beta_1^{(1)}} F^{12} & (-\eta)^{1-\beta_2^{(1)}} F^{13} \\ k_1 (-\eta)^{1-\beta_2^{(2)}} F^{23} & k_2 F^{21} & k_3 (-\eta)^{1-\beta_1^{(2)}} F^{22} \\ k_4 (-\eta)^{1-\beta_1^{(3)}} F^{32} & k_5 (-\eta)^{1-\beta_2^{(3)}} F^{33} & k_6 F^{31} \end{pmatrix},$$

where

$$\begin{aligned} F^{i1} &= {}_2F_2 \left(\begin{matrix} \alpha_1^{(i)}, \alpha_2^{(i)} \\ \beta_1^{(i)}, \beta_2^{(i)} \end{matrix}; \eta \right), & F^{i2} &= {}_2F_2 \left(\begin{matrix} \alpha_1^{(i)} - \beta_1^{(i)} + 1, \alpha_2^{(i)} - \beta_1^{(i)} + 1 \\ 2 - \beta_1^{(i)}, \beta_2^{(i)} - \beta_1^{(i)} + 1 \end{matrix}; \eta \right), \\ F^{i3} &= {}_2F_2 \left(\begin{matrix} \alpha_1^{(i)} - \beta_2^{(i)} + 1, \alpha_2^{(i)} - \beta_2^{(i)} + 1 \\ 2 - \beta_2^{(i)}, \beta_1^{(i)} - \beta_2^{(i)} + 1 \end{matrix}; \eta \right), & (i = 1, 2, 3), \\ k_1 &= \frac{2\beta_1^{(1)}}{u} \frac{\alpha_1^{(1)} \alpha_2^{(1)} (\beta_1^{(1)} - \alpha_2^{(1)})}{(\beta_1^{(1)})^2 \beta_2^{(1)} (\beta_1^{(1)} + 1)}, & k_2 &= \frac{2\beta_1^{(1)}}{u} \frac{(\beta_2^{(1)} - \beta_1^{(1)})(1 - \beta_1^{(1)})}{\alpha_1^{(1)} - \beta_1^{(1)}}, \\ k_3 &= -\frac{2\beta_1^{(1)}}{u} \frac{\alpha_2^{(2)} (1 - \beta_2^{(1)})}{\beta_1^{(1)} (\beta_1^{(1)} - \beta_2^{(1)} + 1)}, & k_4 &= \frac{2\beta_2^{(1)}}{v} \frac{\alpha_1^{(1)} \alpha_2^{(1)} (\beta_2^{(1)} - \alpha_1^{(1)})(\beta_2^{(1)} - \alpha_2^{(1)})}{\beta_1^{(1)} (\beta_2^{(1)})^2 (\beta_2^{(1)} + 1)(\beta_2^{(1)} - \beta_1^{(1)})}, \\ k_5 &= \frac{2\beta_2^{(1)}}{v} \frac{(\alpha_1^{(1)} - \beta_2^{(1)})(\alpha_2^{(1)} - \beta_2^{(1)})(1 - \beta_1^{(1)})}{\beta_1^{(3)} (\beta_2^{(1)} - \beta_1^{(1)} + 1)(\beta_1^{(1)} - \beta_2^{(1)})}, & k_6 &= \frac{2\beta_2^{(1)}}{v} (1 - \beta_2^{(1)}). \end{aligned}$$

We obtain the linear monodromy of (5.2) from Proposition 5.1:

$$M_\infty = \begin{pmatrix} e^{2\pi i\theta_1^\infty} & 0 & 0 \\ 0 & e^{2\pi i\theta_2^\infty} & 0 \\ 0 & 0 & e^{2\pi i\theta_3^\infty} \end{pmatrix} = \begin{pmatrix} e^{4\pi i\alpha_1} & 0 & 0 \\ 0 & e^{4\pi i(\alpha_1-\beta_1)} & 0 \\ 0 & 0 & e^{4\pi i(\alpha_1-\beta_2)} \end{pmatrix} = \left(e^{2\pi i\alpha_1} \tilde{M}_0 \right)^2,$$

$$e^{2\pi iT_0} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & e^{2\pi i\theta_1^0} & 0 \\ 0 & 0 & e^{2\pi i\theta_2^0} \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & e^{4\pi i(\alpha_2-\alpha_1)} & 0 \\ 0 & 0 & e^{4\pi i(\alpha_0-\alpha_1)} \end{pmatrix} = \left(e^{-2\pi i\alpha_1} e^{2\pi i\tilde{T}_\infty} \right)^2,$$

$$CM_\infty C^{-1}S_1S_2S_3S_4e^{2\pi iT_0} = I_3. \tag{5.5}$$

$$C \left(e^{2\pi i\alpha_1} \tilde{M}_0 \right)^2 C^{-1}S_1S_2S_3S_4 \left(e^{-2\pi i\alpha_1} e^{2\pi i\tilde{T}_\infty} \right)^2 = I_3. \tag{5.6}$$

Comparing (5.1) and (5.6), we have

$$s_3^{(1)} = s_1^{(1)} e^{2\pi i(\alpha_0-\alpha_1)}, \quad s_3^{(2)} = s_1^{(2)} e^{2\pi i(\alpha_0-\alpha_2)},$$

$$s_4^{(1)} = s_2^{(1)} e^{2\pi i(\alpha_1-\alpha_0)}, \quad s_4^{(2)} = s_2^{(2)} e^{2\pi i(\alpha_2-\alpha_0)}.$$

Summarizing the above calculation, we have the linear monodromy.

Theorem 5.1. *The linear equation (2.1) has the linear monodromy $\{M_\infty, C, S_1, S_2, S_3, S_4, e^{2\pi iT_0}\}$ for the symmetric solution (1) as follows:*

$$M_\infty = \begin{pmatrix} e^{2\pi i\theta_1^\infty} & 0 & 0 \\ 0 & e^{2\pi i\theta_2^\infty} & 0 \\ 0 & 0 & e^{2\pi i\theta_3^\infty} \end{pmatrix}, \quad e^{2\pi iT_0} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & e^{2\pi i\theta_1^0} & 0 \\ 0 & 0 & e^{2\pi i\theta_2^0} \end{pmatrix},$$

$$S_{2n-1} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ s_{2n-1}^{(1)} & s_{2n-1}^{(2)} & 1 \end{pmatrix}, \quad S_{2n} = \begin{pmatrix} 1 & 0 & s_{2n}^{(1)} \\ 0 & 1 & s_{2n}^{(2)} \\ 0 & 0 & 1 \end{pmatrix}, \quad C = (c_{ij})_{i,j=1,2,3},$$

$$s_1^{(1)} = \frac{-2\pi i\Gamma(1-\frac{\theta_1^0}{2})}{\Gamma(\frac{\theta_1^\infty}{2})\Gamma(1+\frac{\theta_2^\infty}{2})\Gamma(1+\frac{\theta_3^\infty}{2})}, \quad s_1^{(2)} = \frac{-2\pi i\Gamma(1+\frac{\theta_1^0}{2})}{\Gamma(\frac{\theta_1^0+\theta_1^\infty}{2})\Gamma(1+\frac{\theta_1^0+\theta_2^\infty}{2})\Gamma(1+\frac{\theta_1^0+\theta_3^\infty}{2})},$$

$$s_2^{(1)} = \frac{-2\pi i e^{\pi i(-\frac{\theta_2^0}{2})}\Gamma(\frac{\theta_3^0}{2})}{\Gamma(1-\frac{\theta_1^\infty}{2})\Gamma(-\frac{\theta_2^\infty}{2})\Gamma(-\frac{\theta_3^\infty}{2})}, \quad s_2^{(2)} = \frac{-2\pi i e^{\pi i(\frac{\theta_1^0-\theta_2^0}{2})}\Gamma(-\frac{\theta_1^0}{2})}{\Gamma(1-\frac{\theta_1^0+\theta_1^\infty}{2})\Gamma(-\frac{\theta_1^0+\theta_2^\infty}{2})\Gamma(-\frac{\theta_1^0+\theta_3^\infty}{2})},$$

$$s_3^{(1)} = \frac{-2\pi i e^{\pi i\theta_2^0}\Gamma(1-\frac{\theta_1^0}{2})}{\Gamma(\frac{\theta_1^\infty}{2})\Gamma(1+\frac{\theta_2^\infty}{2})\Gamma(1+\frac{\theta_3^\infty}{2})}, \quad s_3^{(2)} = \frac{-2\pi i e^{\pi i(\theta_2^0-\theta_1^0)}\Gamma(1+\frac{\theta_1^0}{2})}{\Gamma(\frac{\theta_1^0+\theta_1^\infty}{2})\Gamma(1+\frac{\theta_1^0+\theta_2^\infty}{2})\Gamma(1+\frac{\theta_1^0+\theta_3^\infty}{2})},$$

$$s_4^{(1)} = \frac{-2\pi i e^{\pi i(-\frac{3\theta_2^0}{2})}\Gamma(\frac{\theta_3^0}{2})}{\Gamma(1-\frac{\theta_1^\infty}{2})\Gamma(-\frac{\theta_2^\infty}{2})\Gamma(-\frac{\theta_3^\infty}{2})}, \quad s_4^{(2)} = \frac{-2\pi i e^{3\pi i(\frac{\theta_1^0-\theta_2^0}{2})}\Gamma(-\frac{\theta_1^0}{2})}{\Gamma(1-\frac{\theta_1^0+\theta_1^\infty}{2})\Gamma(-\frac{\theta_1^0+\theta_2^\infty}{2})\Gamma(-\frac{\theta_1^0+\theta_3^\infty}{2})}.$$

$$c_{11} = \frac{\Gamma(1-\frac{\theta_1^0}{2})\Gamma(1-\frac{\theta_1^\infty-\theta_2^\infty}{2})\Gamma(1-\frac{\theta_1^\infty-\theta_3^\infty}{2})e^{\pi i\frac{\theta_1^0}{2}}}{\Gamma(1+\frac{\theta_2^\infty}{2})\Gamma(1+\frac{\theta_3^\infty}{2})\Gamma(1-\frac{\theta_1^0+\theta_1^\infty}{2})},$$

$$\begin{aligned}
c_{12} &= \frac{\Gamma(1+\frac{\theta_1^0}{2})\Gamma(1-\frac{\theta_1^\infty-\theta_2^\infty}{2})\Gamma(1-\frac{\theta_1^\infty-\theta_3^\infty}{2})e^{\pi i\frac{\theta_1^0+\theta_1^\infty}{2}}}{\Gamma(1+\frac{\theta_1^0+\theta_2^\infty}{2})\Gamma(1+\frac{\theta_1^0+\theta_3^\infty}{2})\Gamma(1-\frac{\theta_1^\infty}{2})}, & c_{13} &= \frac{\Gamma(1-\frac{\theta_1^\infty-\theta_2^\infty}{2})\Gamma(1-\frac{\theta_1^\infty-\theta_3^\infty}{2})}{\Gamma(1-\frac{\theta_1^0+\theta_1^\infty}{2})\Gamma(1-\frac{\theta_1^\infty}{2})}, \\
c_{21} &= \frac{\Gamma(1-\frac{\theta_1^0}{2})\Gamma(\frac{\theta_1^\infty-\theta_2^\infty}{2}-1)\Gamma(\frac{\theta_3^\infty-\theta_2^\infty}{2})e^{\pi i(1+\frac{\theta_2^\infty}{2})}}{\Gamma(1+\frac{\theta_3^\infty}{2})\Gamma(\frac{-\theta_1^0-\theta_2^\infty}{2})\Gamma(\frac{\theta_1^\infty}{2})}, \\
c_{22} &= \frac{\Gamma(1+\frac{\theta_1^0}{2})\Gamma(\frac{\theta_1^\infty-\theta_2^\infty}{2}-1)\Gamma(\frac{\theta_3^\infty-\theta_2^\infty}{2})e^{\pi i(1+\frac{\theta_1^0+\theta_2^\infty}{2})}}{\Gamma(1+\frac{\theta_1^0+\theta_3^\infty}{2})\Gamma(\frac{-\theta_2^\infty}{2})\Gamma(\frac{\theta_1^0+\theta_1^\infty}{2})}, & c_{23} &= \frac{\Gamma(\frac{\theta_1^\infty-\theta_2^\infty}{2}-1)\Gamma(\frac{\theta_3^\infty-\theta_2^\infty}{2})}{\Gamma(\frac{-\theta_1^0-\theta_2^\infty}{2})\Gamma(\frac{-\theta_2^\infty}{2})}, \\
c_{31} &= \frac{\Gamma(1-\frac{\theta_1^0}{2})\Gamma(\frac{\theta_1^\infty-\theta_3^\infty}{2}-1)\Gamma(\frac{\theta_2^\infty-\theta_3^\infty}{2})e^{\pi i(1+\frac{\theta_3^\infty}{2})}}{\Gamma(1+\frac{\theta_2^\infty}{2})\Gamma(\frac{-\theta_1^0-\theta_3^\infty}{2})\Gamma(\frac{\theta_1^\infty}{2})}, \\
c_{32} &= \frac{\Gamma(1+\frac{\theta_1^0}{2})\Gamma(\frac{\theta_1^\infty-\theta_3^\infty}{2}-1)\Gamma(\frac{\theta_2^\infty-\theta_3^\infty}{2})e^{\pi i(1+\frac{\theta_1^0+\theta_3^\infty}{2})}}{\Gamma(1+\frac{\theta_1^0+\theta_2^\infty}{2})\Gamma(\frac{-\theta_3^\infty}{2})\Gamma(\frac{\theta_1^0+\theta_1^\infty}{2})}, & c_{33} &= \frac{\Gamma(\frac{\theta_1^\infty-\theta_3^\infty}{2}-1)\Gamma(\frac{\theta_2^\infty-\theta_3^\infty}{2})}{\Gamma(\frac{-\theta_1^0-\theta_3^\infty}{2})\Gamma(\frac{-\theta_3^\infty}{2})}.
\end{aligned}$$

$$CM_\infty C^{-1}S_1S_2S_3S_4e^{2\pi iT_0} = I_3.$$

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6. Appendix

In this section, we write down the Noumi-Yamada system and the transformation formulae to the Hamiltonian system $\mathcal{H}_{NY}^{A_4}$.

6.1. The Noumi-Yamada system

The Noumi-Yamada system is expressed as follows:

$$\begin{aligned} \frac{df_0}{dt} &= f_0(f_1 - f_2 + f_3 - f_4) + \alpha_0, \\ \frac{df_1}{dt} &= f_1(f_2 - f_3 + f_4 - f_0) + \alpha_1, \\ \frac{df_2}{dt} &= f_2(f_3 - f_4 + f_0 - f_1) + \alpha_2, \\ \frac{df_3}{dt} &= f_3(f_4 - f_0 + f_1 - f_2) + \alpha_3, \\ \frac{df_4}{dt} &= f_4(f_0 - f_1 + f_2 - f_3) + \alpha_4, \\ f_0 + f_1 + f_2 + f_3 + f_4 &= t, \quad \alpha_0 + \alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 = 1. \end{aligned}$$

6.2. The transformation formulae to the Hamiltonian system $\mathcal{H}_{NY}^{A_4}$

We have the Hamiltonian system $\mathcal{H}_{NY}^{A_4}$ by putting as follows:

$$\begin{aligned} q_1 &= -f_1, & p_1 &= f_2, & q_2 &= -f_1 - f_3, & p_2 &= f_4, \\ \alpha &= -\alpha_1, & \beta &= -\alpha_2, & \tilde{\alpha} &= -\alpha_1 - \alpha_3, & \tilde{\beta} &= -\alpha_4. \end{aligned}$$