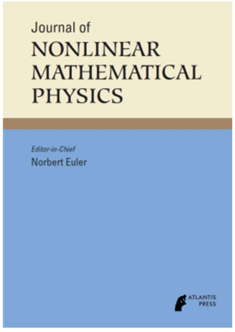

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Integrable Substructure in a Korteweg Capillarity Model. A Kármán-Tsien Type Constitutive Relation

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A classical Korteweg capillarity system with a Kármán-Tsien type (κ,ρ) constitutive relation is shown, via a Madelung transformation and use of invariants of motion, to admit integrable Hamiltonian subsystems.

1. Introduction

The Kármán-Tsien pressure-density $(p,\ \rho)$ model law

$$p = \mathbb{A} + \frac{\mathbb{C}}{\rho} \ , \tag{1.1}$$

where $\mathbb{A}$, $\mathbb{C}$ are appropriate parameters has been widely used historically to approximate real gas behaviour in both subsonic and supersonic gasdynamics (see e.g. Tsien [48], von Kármán [15], von Mises [27] and Coburn [6]). In particular, the classical hodograph equations of planar gasdynamics may be reduced to Cauchy-Riemann canonical form under such a law. This important reduction may be set in the broader context of work by Loewner [23,24] whereby reduction of the hodograph equations to appropriate canonical form is obtained via the systemic application of Bäcklund transformations [31]. The class of infinitesimal Bäcklund transformations applied in subsonic régimes in [24], under re-interpretation and extension results in a linear representation for a 2+1-dimensional master soliton system [16–18].

In classical Korteweg capillarity theory [19] the free energy adopts the form

$$\mathscr{E} = \frac{1}{2}\ \kappa(\rho)|\nabla\rho|^2 + \frac{\mathbb{K}}{\rho} \ , \tag{1.2}$$

where ρ is the liquid density and $\mathbb{K}$ is an arbitrary constant. In recent work, interest has arisen in model capillarity laws of the type [4,5]

$$\kappa = \mathbb{A} + \frac{\mathbb{C}}{\rho} \ . \tag{1.3}$$

In analogy with (1.1), we here term these Kármán-Tsien capillarity laws. It turns out that, as in the gasdynamic case, such Kármán-Tsien laws are privileged in that, in the capillarity context they are here shown to lead to integrable reductions of the Korteweg system.

2. The Capillarity System

The classical Korteweg inviscid capillarity system [19]

$$\begin{gathered}\frac{\partial \rho}{\partial t}+\operatorname{div}(\rho \mathbf{v})=0\ , \\ \frac{\partial \mathbf{v}}{\partial t}+\mathbf{v}\cdot\nabla\mathbf{v}-\nabla[\,\Pi(\rho)+\kappa(\rho)\nabla^2\rho+\frac{1}{2}\,\kappa'(\rho)|\nabla\rho|^2\,]=0\ ,\end{gathered} \tag{2.1}$$

has been the subject of recent research interest (see e.g. [4,5,20,21,26] and references cited therein). It is remarked that the system (2.1) may be set in the more general context of the isothermal, inviscid capillarity system derived *ab initio* by Antanovskii [1], namely

$$\begin{gathered}\frac{\partial \rho}{\partial t}+\operatorname{div}(\rho \mathbf{v})=0\ , \\ \frac{\partial \mathbf{v}}{\partial t}+\mathbf{v}\cdot\nabla\mathbf{v}+\nabla\,[\,\frac{\delta(\rho\,\mathscr{E})}{\delta\rho}-\Pi\,]=\mathbf{0}\ ,\end{gathered} \tag{2.2}$$

where $\mathbf{v}$ is the velocity and $\mathscr{E}(\rho,\alpha)$ with $\alpha=\frac{1}{2}\,|\nabla\rho|^2$ is the specific free energy. Here,

$$\frac{\delta\Phi}{\delta\rho}:=\frac{\partial\Phi}{\partial\rho}-\nabla\,[\,\frac{\partial\Phi}{\partial\alpha}\,\nabla\rho\,] \tag{2.3}$$

and Π denotes an external potential. The quantity

$$\zeta=\frac{\delta}{\delta\rho}\,[\,\rho\,\mathscr{E}\,] \tag{2.4}$$

is the chemical potential of the system.

The Korteweg system (2.1), by virtue of (1.2), arises as the specialisation

$$\mathscr{E}(\alpha,\rho)=\kappa(\rho)\frac{\alpha}{\rho}+\frac{\mathbb{K}}{\rho}\ , \tag{2.5}$$

in the capillarity system (2.2). The canonical Boussinesq capillarity system, in turn, is obtained as the particular case with κ constant in the Korteweg model. An analogous system arises *mutatis mutandis* in plasma physics [3].

If the irrotational constraint $\mathbf{v}=\nabla\Phi$ is imposed (c.f. Antanovskii [1]) then the momentum equation $(2.2)_2$ admits the Bernoulli integral

$$\frac{\partial\Phi}{\partial t}+\frac{1}{2}\,|\nabla\Phi|^2+\frac{\delta}{\delta\rho}\,(\rho\,\mathscr{E})-\Pi=\mathbb{B}(t) \tag{2.6}$$

where $\mathbb{B}(t)$ may be absorbed into the potential Φ without loss of generality, while the equation of continuity becomes

$$\frac{\partial\rho}{\partial t}+\nabla\rho\cdot\nabla\Phi+\rho\,\nabla^2\Phi=0\ . \tag{2.7}$$

On introduction of the Madelung transformation [25]

$$\Psi=\rho^{1/2}\exp\left(\frac{i\Phi}{2}\right)\ , \tag{2.8}$$

it is seen that the capillarity system (2.6)−(2.7) may be embodied in the NLS-type equation

$$i\,\frac{\partial\Psi}{\partial t}+\nabla^2\Psi+\left[\,\frac{-\nabla^2|\Psi|}{|\Psi|}-\frac{1}{2}\,\frac{\delta(\rho\,\mathscr{E})}{\delta\rho}+\frac{\Pi}{2}\,\right]\Psi=0 \tag{2.9}$$

which incorporates a de Broglie-Bohm potential type term $\dfrac{\nabla^2|\Psi|}{|\Psi|}$.

In previous work, integrable 1+1-dimensional reductions of (2.9) for particular capillarity models have been obtained both to the standard cubic nonlinear Schrödinger (NLS) equation in [2] and to its 'resonant' counterpart in [35]. In 2+1-dimensions, with appropriately 'driven' Π coupled to the momentum equation $(2.2)_2$, integrable capillarity models may be isolated which lead to the Davey-Stewartson I system [36] or a novel 'resonant' Davey-Stewartson system originally introduced in [37] and subsequently investigated in [22, 41].

Here, our concern is with the Korteweg capillarity system (2.1) with associated NLS-type equation under the Madelung transformation (2.8), given by

$$i\,\frac{\partial\Psi}{\partial t}+\nabla^2\Psi+\left[\frac{-\nabla^2|\Psi|}{|\Psi|}+\frac{1}{2}\left(\,\kappa(\rho)\nabla^2\rho+\frac{1}{2}\,\kappa'(\rho)(\nabla\rho)^2\,\right)+\frac{\Pi}{2}\right]\Psi=0\ , \tag{2.10}$$

that is,

$$i\,\frac{\partial\Psi}{\partial t}+\nabla^2\Psi+\left[-(1+\kappa'(\rho)\rho^2)\,\frac{\nabla^2|\Psi|}{|\Psi|}+\frac{1}{2}\left(\,\kappa(\rho)+\kappa'(\rho)\rho\,\right)\nabla^2|\Psi|^2+\frac{\Pi}{2}\right]\Psi=0\ , \tag{2.11}$$

on use of the relation

$$\frac{\nabla^2\rho^{1/2}}{\rho^{1/2}}=\frac{1}{2\rho}\,\nabla^2\rho-\frac{1}{4}\left(\frac{\nabla\rho}{\rho}\right)^2 \tag{2.12}$$

with $\rho=|\Psi|^2$. The nonlinear NLS equation (2.11) is not integrable in general. However, it will be seen for a plane wave packet ansatz to admit an integrable Hamiltonian reduction if a 'Kármán-Tsien' type constitutive law of the type (1.3) is adopted. Moreover, in 2+1-dimensions, an elliptic vortex ansatz leads to a remarkable reduction to an integrable Ermakov-Ray-Reid subsystem.

3. A Plane Wave Integrable Hamiltonian Reduction

Here, a plane wave ansatz

$$\Psi=[\,\phi\,(\,\mathbf{k}\cdot\mathbf{r}-\mu t\,)+i\,\psi\,(\,\mathbf{k}\cdot\mathbf{r}-\mu t\,)\,]\exp\,i\,(\,\mathbf{h}\cdot\mathbf{r}-\lambda t\,) \tag{3.1}$$

is introduced where $\mathbf{k} = (k_1, k_2, k_3)$, $\mathbf{h} = (h_1, h_2, h_3)$ are constant vectors and $\mathbf{r} = (x, y, z)$. Substitution into the NLS-type equation (2.11) associated with the Korteweg capillarity system yields:

$$\begin{aligned}
&|\mathbf{k}|^2\ddot{\phi} + (\ \lambda - |\mathbf{h}|^2\ )\ \phi - (\ 2\mathbf{k}\cdot\mathbf{h} - \mu\ )\ \dot{\psi} + \Bigg[-|\mathbf{k}|^2(\ 1+\kappa'(\rho)\rho^2\ )\left\{ \frac{(\phi\dot{\phi}+\psi\dot{\psi}\ )^\bullet}{\phi^2+\psi^2} \right.\\
&\qquad \left. - \left(\frac{\phi\dot{\phi}+\psi\dot{\psi}}{\phi^2+\psi^2}\right)^2 \right\} + |\mathbf{k}|^2(\kappa(\rho)+\kappa'(\rho)\rho\ )(\ \phi\dot{\phi}+\psi\dot{\psi}\ )^\bullet + \frac{\Pi}{2}\Bigg]\ \phi = 0\,,\\
&|\mathbf{k}|^2\ddot{\psi} + (\ \lambda - |\mathbf{h}|^2\ )\ \psi + (\ 2\mathbf{k}\cdot\mathbf{h} - \mu\ )\ \dot{\phi} + \Bigg[-|\mathbf{k}|^2(\ 1+\kappa'(\rho)\rho^2\ )\left\{ \frac{(\phi\dot{\phi}+\psi\dot{\psi}\ )^\bullet}{\phi^2+\psi^2} \right.\\
&\qquad \left. - \left(\frac{\phi\dot{\phi}+\psi\dot{\psi}}{\phi^2+\psi^2}\right)^2 \right\} + |\mathbf{k}|^2(\kappa(\rho)+\kappa'(\rho)\rho\ )(\ \phi\dot{\phi}+\psi\dot{\psi}\ )^\bullet + \frac{\Pi}{2}\Bigg]\ \psi = 0
\end{aligned} \tag{3.2}$$

where all dots indicate a derivative with respect to the argument $\mathbf{k}\cdot\mathbf{r} - \mu t$. The nonlinear coupled system (3.2) is readily seen to admit the dynamical invariant

$$\dot{\phi}\psi - \dot{\psi}\phi - \frac{1}{2|\mathbf{k}|^2}\ (\ 2\mathbf{k}\cdot\mathbf{h} - \mu\ )(\ \phi^2+\psi^2\ ) = \mathscr{I}\ . \tag{3.3}$$

This integral of motion for the system (3.3) may be contrasted with that characteristic of Ermakov-Ray-Reid systems [29, 30, 33, 34, 38–40, 42–44, 47] which admit an invariant of the type

$$\frac{1}{2}\ (\ \dot{\phi}\psi - \dot{\psi}\phi\ )^2 + \bar{\mathscr{E}}(\ \phi/\psi\ ) = \mathscr{I}\ . \tag{3.4}$$

Extended Ermakov-Ray-Reid systems with characteristic invariants which subsume both of the types (3.3) and (3.4) have recently been introduced in [45]. Here, in addition,

$$\begin{aligned}
&|\mathbf{k}|^2(\ \ddot{\phi}\dot{\phi} + \ddot{\psi}\dot{\psi}\ ) + \frac{1}{2}\ [\ \lambda - |\mathbf{h}|^2 + \frac{\Pi}{2}\ ]\ \dot{\Sigma}\\
&\qquad -|\mathbf{k}|^2(\ 1+\kappa'(\rho)\rho^2\ )[\ \frac{1}{4}\frac{\ddot{\Sigma}\dot{\Sigma}}{\Sigma} - \frac{1}{8}\frac{\dot{\Sigma}^3}{\Sigma^2}\ ] + \frac{|\mathbf{k}|^2}{4}\ (\ \kappa(\rho)+\kappa'(\rho)\rho\ )\ddot{\Sigma}\dot{\Sigma} = 0\ ,
\end{aligned} \tag{3.5}$$

whence, subject to the Kármán-Tsien type relation (1.3), a second dynamical invariant is obtained, namely the Hamiltonian

$$\begin{aligned}
\mathscr{H} = \frac{1}{2}\ [\ \dot{\phi}^2 + \dot{\psi}^2 + \frac{1}{|\mathbf{k}|^2}\ (\ \lambda - |\mathbf{h}|^2\ )\Sigma + \frac{1}{2|\mathbf{k}|^2}\int \Pi(\Sigma)\,d\Sigma\\
-\frac{1}{4}\ (\ 1-\mathbb{C}\ )\ \frac{\dot{\Sigma}^2}{\Sigma} + \frac{\mathbb{A}}{4}\ \dot{\Sigma}^2\ ]
\end{aligned} \tag{3.6}$$

where $\Sigma = \phi^2 + \psi^2$ is the squared wave amplitude and here it is assumed that $\Pi = \Pi(\Sigma)$.

The pair of invariants (3.3) and (3.6) now allow reduction of the nonlinear coupled system (3.2) for ϕ, ψ to quadrature. Thus, use of the identity

$$(\ \phi^2+\psi^2\ )(\ \dot{\phi}^2+\dot{\psi}^2\ ) - (\ \dot{\phi}\psi - \dot{\psi}\phi\ )^2 \equiv (\ \phi\dot{\phi} + \psi\dot{\psi}\ )^2 \tag{3.7}$$

shows that

$$\begin{aligned}&\Sigma\,[\,2\mathscr{H}-\frac{1}{|\mathbf{k}|^2}\,(\lambda-|\mathbf{h}|^2)\,\Sigma-\frac{1}{2|\mathbf{k}|^2}\int\Pi(\Sigma)\,d\Sigma+\frac{1}{4}\,(\,1-\mathbb{C}\,)\,\frac{\dot{\Sigma}^2}{\Sigma}-\frac{\mathbb{A}}{4}\,\dot{\Sigma}^2\,]\\&-[\,\mathscr{I}+\frac{1}{2|\mathbf{k}|^2}\,(\,2\mathbf{k}\,.\,\mathbf{h}-\mu\,)\,\Sigma\,]^2=\frac{1}{4}\,\dot{\Sigma}^2\ .\end{aligned}\tag{3.8}$$

Hence, if $\Pi = -\rho\, g$, corresponding to gravitational action, the squared amplitude Σ is determined by

$$\begin{aligned}&\Sigma\,[\,2\mathscr{H}-\frac{1}{|\mathbf{k}|^2}\,(\lambda-|\mathbf{h}|^2)\,\Sigma+\frac{g}{4\,|\mathbf{k}|^2}\,\Sigma^2\,]\\&-[\,\mathscr{I}+\frac{1}{2\,|\mathbf{k}|^2}\,(\,2\mathbf{k}\,.\,\mathbf{h}-\mu\,)\,\Sigma\,]^2=\frac{1}{4}\,\dot{\Sigma}^2\,[\,\mathbb{A}\Sigma+\mathbb{C}\,]\ .\end{aligned}\tag{3.9}$$

If we now set $\Sigma^* \equiv \mathbb{A}\Sigma+\mathbb{C}$, $(\mathbb{A}\neq 0)$ then (3.9) reduces to the form

$$\dot{\Sigma}^{*2}\,\Sigma^* = \alpha\,\Sigma^{*3}+\beta\,\Sigma^{*2}+\gamma\Sigma^*+\delta\ ,\quad \alpha\neq 0 \tag{3.10}$$

whence

$$\int\sqrt{\frac{\Sigma^*}{\alpha\,\Sigma^{*3}+\beta\,\Sigma^{*2}+\gamma\Sigma^*+\delta}}\,d\Sigma^* = t\ . \tag{3.11}$$

With $u = \Sigma^{*-1}$, (3.11) yields

$$-\int\frac{du}{\sqrt{\delta u^5+\gamma u^4+\beta u^3+\alpha u^2}} = t \tag{3.12}$$

and the integral therein, can be evaluated in terms of the canonical elliptic integrals of the 1st and 2nd kind, namely

$$F\,(\,\varphi,k\,) = \int_0^{\varphi}\frac{d\,\alpha}{\sqrt{1-k^2\,\sin^2\,\alpha}} = sn^{-1}(\sin\varphi,k) \tag{3.13}$$

and

$$E\,(\,\varphi,k\,) = \int_0^{\varphi}\sqrt{1-k^2\,\sin^2\,\alpha}\,d\,\alpha = \int_0^{\varphi} dn^2\,v\,dv \tag{3.14}$$

(see Gradshtyn and Ryzhnik [12]). The nature of the solution in terms of these elliptic integrals depends critically on the parametric values of $\alpha,\ \beta,\ \gamma,\ \delta$.

The variable Δ is now introduced according to

$$\Delta = \frac{\phi}{\psi} \tag{3.15}$$

whence, the Wronskian relation (3.3) shows that

$$\frac{d}{dt}\,[\,\tan^{-1}\Delta\,] = \frac{1}{\Sigma}\,[\,\mathscr{I}+\frac{1}{2\,|\mathbf{k}|^2}\,(\,2\mathbf{k}\,.\,\mathbf{h}-\mu\,)\,\Sigma\,]$$

so that

$$\Delta = \tan[\, \frac{1}{2\,|\mathbf{k}|^2}\,(\,2\mathbf{k}\,.\,\mathbf{h} - \mu\,)\,t + \mathscr{I}\int \frac{1}{\Sigma}\,dt\,]\ . \tag{3.16}$$

The original quantities ϕ, ψ are now given by

$$\phi = \pm\Delta\sqrt{\frac{\Sigma}{1+\Delta^2}}\ ,\quad \psi = \pm\sqrt{\frac{\Sigma}{1+\Delta^2}}\ . \tag{3.17}$$

4. An NLS Reduction under the Kármán Tsien Type Law

With the Kármán-Tsien type constitutive relation (1.3), the NLS equation (2.11) becomes

$$i\,\frac{\partial\Psi}{\partial t} + \nabla^2\Psi + [\,-(1-\mathbb{C})\,\frac{\nabla^2|\Psi|}{|\Psi|} + \frac{\mathbb{A}}{2}\,\nabla^2|\Psi|^2 + \frac{\Pi}{2}\,]\,\Psi = 0\ , \tag{4.1}$$

and, on setting $\Psi = e^{R-iS}$ where $R,\ S$ are real, this decomposes into the coupled nonlinear system

$$\begin{aligned} &S_t + \mathbb{C}\,[\,\nabla^2 R + (\nabla R)^2\,] - (\nabla S)^2 + \mathbb{A}[\,\nabla^2 R + 2(\nabla R)^2\,] - \frac{1}{2}\,\Pi = 0\ , \\ &R_t - \nabla^2 S - 2(\nabla R)\,.\,(\nabla S) = 0\ . \end{aligned} \tag{4.2}$$

If $\mathbb{C} > 0$, then on introduction of new variables according to

$$\begin{aligned} \tilde{t} = \mathbb{C}^{1/2}\,t\ ,\quad \tilde{S} = \mathbb{C}^{-1/2}\,S\ , \\ \tilde{R} = R \end{aligned} \tag{4.3}$$

it is seen that the system (4.2) is reduced to the canonical form

$$\tilde{S}_{\tilde{t}} + \nabla^2\tilde{R} + (\nabla\tilde{R})^2 - (\nabla\tilde{S})^2 + \frac{\mathbb{A}}{\mathbb{C}}\,[\,\nabla^2\tilde{R} + 2(\nabla\tilde{R})^2\,] - \frac{1}{2\mathbb{C}}\,\Pi = 0 \tag{4.4}$$

associated with a reduced NLS-type equation with the de-Broglie Bohm potential term removed, namely

$$i\,\frac{\partial\tilde{\Psi}}{\partial\tilde{t}} + \nabla^2\tilde{\Psi} + [\,\frac{\tilde{\mathbb{A}}}{2}\,\nabla^2|\tilde{\Psi}|^2 + \frac{\tilde{\Pi}}{2}\,]\,\tilde{\Psi} = 0 \tag{4.5}$$

where $\tilde{\Psi} = e^{\tilde{R}-i\tilde{S}}$ and $\tilde{\mathbb{A}} = \mathbb{A}/\mathbb{C}$, $\tilde{\Pi} = \Pi/\mathbb{C}$. The preceding reduction extends to 3+1-dimensions a result originally obtained in the 1+1-dimensional case by Pashaev and Lee [28]. It is noted that, in recent work Carles *et al* [5] investigated the system

$$\begin{aligned} &\frac{\partial\rho}{\partial t} + \mathrm{div}(\rho\,\mathbf{v}) = 0\ , \\ &\frac{\partial\mathbf{v}}{\partial t} + \mathbf{v}\cdot\nabla\mathbf{v} - \nabla\Pi(\rho) = \frac{\varepsilon^2}{2}\,\nabla\left(\frac{\nabla^2\sqrt{\rho}}{\sqrt{\rho}}\right) \end{aligned} \tag{4.6}$$

corresponding to the special case of the Kármán-Tsien law with

$$\kappa = \frac{\varepsilon^2}{4\rho} \tag{4.7}$$

so that $\mathbb{C} = \dfrac{\varepsilon^2}{4} > 0$ and accordingly the de Broglie-Bohm potential term in the associated NLS equation obtained via the Madelung transformation, may be removed via the transformation (4.3).

5. An Elliptic Vortex Reduction to an Integrable Ermakov-Ray-Reid System

Here, we consider the Korteweg system (2.11) with the Kármán-Tsien type capillarity law [5]

$$\kappa = \frac{\mathbb{C}}{\rho} \tag{5.1}$$

so that the governing equations become

$$\begin{aligned} &\frac{\partial \rho}{\partial t} + \operatorname{div}(\rho\,\mathbf{v}) = 0 \ , \\ &\frac{\partial \mathbf{v}}{\partial t} + \mathbf{v}\cdot\nabla\mathbf{v} - \nabla[\Pi(\rho) + 2\mathbb{C}\frac{\nabla^2\rho^{1/2}}{\rho^{1/2}}] = \mathbf{0} \ . \end{aligned} \tag{5.2}$$

Thus, it is seen that the presence of capillarity here just contributes to the incorporation of a de Broglie-Bohm type potential in the momentum equation. It is remarked that the capillarity system (5.2) with such a Kármán-Tsien type law has recently been investigated by Benzoni-Gavage [4] in a recent study of the propagation of planar capillary waves.

Integrable nonlinear dynamical subsystems of (5.2) are sought corresponding to 2+1-dimensional elliptic vortex motions with

$$\begin{aligned} &\mathbf{v} = \mathbf{L}(t)\mathbf{x} + \mathbf{M}(t) \ , \\ &\rho = \sigma(t)\exp[\,\mathbf{x}^T\mathbf{E}(t)\mathbf{x}\,] \end{aligned} \tag{5.3}$$

where

$$\mathbf{x} = \begin{pmatrix} x - q(t) \\ y - p(t) \end{pmatrix} \tag{5.4}$$

and

$$\mathbf{L}(t) = \begin{pmatrix} u_1(t) & u_2(t) \\ v_1(t) & v_2(t) \end{pmatrix} \ , \quad \mathbf{E}(t) = \begin{pmatrix} a(t) & b(t) \\ b(t) & c(t) \end{pmatrix} \ , \quad \mathbf{M} = \begin{pmatrix} \dot{q}(t) \\ \dot{p}(t) \end{pmatrix} \ . \tag{5.5}$$

Here, we proceed with the pure capillarity case in the absence of an external loading term Π such as that due to gravity.

Insertion of the ansatz (5.1) into the capillarity system (5.2) produces the nonlinear coupled 2×2 matrix subsystem

$$\dot{\mathbf{E}} + \mathbf{E}\mathbf{L} + \mathbf{L}^T\mathbf{E} = \mathbf{0} \ , \tag{5.6}$$

$$\dot{\mathbf{L}} + \mathbf{L}^2 - 4\mathbb{C}\mathbf{E}^2 = \mathbf{0} \tag{5.7}$$

together with

$$\dot{\sigma} + (\operatorname{tr}\mathbf{L})\sigma = 0 \tag{5.8}$$

and

$$\ddot{p} = \ddot{q} = 0 \ . \tag{5.9}$$

where here, and in what follows, a dot indicates a derivative with respect to time t.

In the sequel, it is convenient to introduce new variables according to

$$
\begin{aligned}
&G = u_1 + v_2 \ , \quad G_R = \frac{1}{2}(v_1 - u_2) \ , \\
&G_S = \frac{1}{2}(v_1 + u_2) \ , \quad G_N = \frac{1}{2}(u_1 - v_2) \ , \\
&B = a + c \ , \quad B_S = b \ , \quad B_N = \frac{1}{2}(a - c) \ ,
\end{aligned}
\tag{5.10}
$$

whence, the system (5.6)−(5.7) yields

$$
\begin{aligned}
&\dot{B} + BG + 4(B_N G_N + B_S G_S) = 0 \ , \\
&\dot{B}_S + B_S G + G_S B - 2B_N G_R = 0 \ , \\
&\dot{B}_N + B_N G + BG_N + 2B_S G_R = 0 \ , \\
&\dot{G} + \frac{1}{2}G^2 + 2(G_S^2 + G_N^2 - G_R^2) - 8\mathbb{C}(\frac{B^2}{4} + B_N^2 + B_S^2) = 0 \ , \\
&\dot{G}_R + GG_R = 0 \ , \\
&\dot{G}_S + GG_S - 4\mathbb{C}BB_S = 0 \ , \\
&\dot{G}_N + GG_N - 4\mathbb{C}BB_N = 0 \ ,
\end{aligned}
\tag{5.11}
$$

while (5.8) gives that

$$
\dot{\sigma} + G\sigma = 0 \ . \tag{5.12}
$$

On introduction of Ω via

$$
G = \frac{2\dot{\Omega}}{\Omega} \ , \tag{5.13}
$$

$(5.11)_5$ and (5.12) in turn, yield

$$
G_R = c_0 \Omega^{-2} \ , \quad \sigma = c_{\mathrm{I}} \Omega^{-2} \tag{5.14}
$$

where c_0, c_{I} are arbitrary constants of integration.

In terms of new modulated variables

$$
\begin{aligned}
&\bar{B} = \Omega^2 B \ , \quad \bar{B}_S = \Omega^2 B_S \ , \quad \bar{B}_N = \Omega^2 B_N \ , \\
&\bar{G}_S = \Omega^2 G_S \ , \quad \bar{G}_N = \Omega^2 G_N
\end{aligned}
\tag{5.15}
$$

the residual six equations of the nonlinear system (5.11) reduce to

$$\begin{aligned}
&\dot{\bar{B}} + 4(\bar{B}_N\bar{G}_N + \bar{B}_S\bar{G}_S)/\Omega^2 = 0 \ , \\
&\dot{\bar{B}}_S + (\bar{B}G_S - 2c_0\bar{B}_N)/\Omega^2 = 0 \ , \\
&\dot{\bar{B}}_N + (\bar{B}G_N + 2c_0\bar{B}_S)/\Omega^2 = 0 \ , \\
&\dot{\bar{G}}_S - 4\mathbb{C}\bar{B}\bar{B}_S/\Omega^2 = 0 \ , \\
&\dot{\bar{G}}_N - 4\mathbb{C}\bar{B}\bar{B}_N/\Omega^2 = 0
\end{aligned} \tag{5.16}$$

together with

$$\Omega^3\ddot{\Omega} - c_0^2 + \bar{G}_S^2 + \bar{G}_N^2 - \mathbb{C}(\bar{B}^2 + 4\bar{B}_S^2 + 4\bar{B}_N^2) = 0 \ . \tag{5.17}$$

Combination of $(5.16)_1$–$(5.16)_3$ and integration produces the invariant relation

$$\bar{B}_S^2 + \bar{B}_N^2 - \frac{\bar{B}^2}{4} = c_{\rm II} \tag{5.18}$$

while, similarly, $(5.16)_4$ and $(5.16)_5$ together produce the integral of motion

$$\bar{G}_S^2 + \bar{G}_N^2 + \mathbb{C}\bar{B}^2 = c_{\rm III} \tag{5.19}$$

In what follows, we proceed with the particular parametrisation of the relations (5.18) and (5.19) with

$$\begin{aligned}
&\bar{B}_S = -\sqrt{c_{\rm II} + \bar{B}^2/4}\cos\phi(t) \ , \quad \bar{B}_N = -\sqrt{c_{\rm II} + \bar{B}^2/4}\sin\phi(t) \ , \\
&\bar{G}_S = -\sqrt{c_{\rm III} - \mathbb{C}\bar{B}^2}\sin\theta(t) \ , \quad \bar{G}_N = \sqrt{c_{\rm III} - \mathbb{C}\bar{B}^2}\cos\theta(t) \ .
\end{aligned} \tag{5.20}$$

Thus,

$$\bar{B}_S\bar{G}_S + \bar{B}_N\bar{G}_N = \sqrt{c_{\rm II} + \bar{B}^2/4}\sqrt{c_{\rm III} - \mathbb{C}\bar{B}^2}\,\sin(\theta - \phi) \ , \tag{5.21}$$

so that $(5.16)_1$ yields

$$\dot{\bar{B}} + \frac{4}{\Omega^2}\sqrt{c_{\rm II} + \bar{B}^2/4}\,\sqrt{c_{\rm III} - \mathbb{C}\bar{B}^2}\,\sin(\theta - \phi) = 0 \ . \tag{5.22}$$

Conditions $(5.16)_{2,3}$ reduce to a single requirement, namely

$$\sqrt{c_{\rm II} + \bar{B}^2/4}\,(\dot{\phi} + \frac{2c_0}{\Omega^2}) - \frac{\bar{B}}{\Omega^2}\sqrt{c_{\rm III} - \mathbb{C}\bar{B}^2}\cos(\theta - \phi) = 0 \ , \tag{5.23}$$

while $(5.16)_{4,5}$ reduce to the single relation

$$\dot{\theta}\sqrt{c_{\rm III} - \mathbb{C}\bar{B}} - \frac{4\mathbb{C}\bar{B}}{\Omega^2}\sqrt{c_{\rm II} + \bar{B}^2/4}\cos(\theta - \phi) = 0 \ . \tag{5.24}$$

At this stage, we state the following result which is readily validated by symbolic computation:

Theorem 5.1.

$$\dot{M} = -2GM \tag{5.25}$$

and

$$\dot{N} = -2GN \tag{5.26}$$

where

$$M = au_2 + b(v_2 - u_1) - cv_1 \tag{5.27}$$

and

$$N = -a(u_2^2 + v_2^2) + 2b(u_1u_2 + v_1v_2) - c(u_1^2 + v_1^2) + 4\Delta - 4\mathbb{C}\Delta(a+c) \tag{5.28}$$

with

$$\Delta = ac - b^2 = -\frac{1}{\Omega^4}\,[\bar{B}_S^2 + \bar{B}_N^2 - \frac{1}{4}\,\bar{B}^2\,] = -\frac{c_{\mathrm{II}}}{\Omega^4}\ . \tag{5.29}$$

The relation (5.25) may now be employed to obtain the explicit solution of (5.22). Thus, (5.25) together with (5.13) show that, on integration

$$M = [-c_0\bar{B} + 2(\bar{B}_N\bar{G}_S - \bar{B}_S\bar{G}_N)]\Omega^{-4} = c_{\mathrm{IV}}\Omega^{-4} \tag{5.30}$$

whence, since

$$\bar{B}_N\bar{G}_S - \bar{B}_S\bar{G}_N = \sqrt{c_{\mathrm{II}} + \bar{B}^2/4}\,\sqrt{c_{\mathrm{III}} - \mathbb{C}\bar{B}^2}\,\cos(\theta - \phi)\ , \tag{5.31}$$

it is seen that

$$c_0\bar{B} = -c_{\mathrm{IV}} + 2\sqrt{c_{\mathrm{II}} + \bar{B}^2/4}\,\sqrt{c_{\mathrm{III}} - \mathbb{C}\bar{B}^2}\,\cos(\theta - \phi)\ . \tag{5.32}$$

Elimination of $\cos(\theta - \phi)$ in (5.23) and (5.24) by means of (5.32) now shows that

$$\dot{\phi} = \frac{2}{\Omega^2}\left[\frac{\bar{B}\,(c_0\bar{B} + c_{\mathrm{IV}})}{\bar{B}^2 + 4c_{\mathrm{II}}} - c_0\right] \tag{5.33}$$

and

$$\dot{\theta} = \left(\frac{2\mathbb{C}\bar{B}}{\Omega^2}\right)\frac{c_0\bar{B} + c_{\mathrm{IV}}}{c_{\mathrm{III}} - \mathbb{C}\bar{B}^2}\ . \tag{5.34}$$

A second key result stated below is embodied in:

Theorem 5.2.

$$(\Omega^2\ddot{\bar{B}}) = -2(N - 4\Delta)\Omega^4\ . \tag{5.35}$$

The relation (5.35) in the above is readily validated by symbolic computation and has an analog in the theory of the evolution of elliptical ocean warm-core eddies ([8, 14, 32]).

Substitution of (5.13) into (5.26) and integration shows that

$$N = c_{\mathrm{V}}\Omega^{-4} \tag{5.36}$$

whence (5.35) becomes, in view of (5.29),

$$(\ddot{\Omega^2 \bar{B}}) + 2(c_{\mathrm{V}} + 4c_{\mathrm{II}}) = 0 \ . \tag{5.37}$$

On elimination of $\theta - \phi$ in (5.22) via the relation (5.32) and use of (5.37) is seen that, $\bar{B}$ is given by the elliptic integral relation

$$\int_{c_{\mathrm{VIII}}}^{\bar{B}} \frac{d\bar{\sigma}}{\bar{\sigma}\sqrt{(\bar{\sigma}^2+4c_{\mathrm{II}})(c_{\mathrm{III}}-\mathbb{C}\bar{\sigma}^2)-(c_0\bar{\sigma}+c_{\mathrm{IV}})^2}} = \pm 2\int_0^t \frac{d\tau}{-(c_{\mathrm{V}}+4c_{\mathrm{II}})\tau^2+c_{\mathrm{VI}}\tau+c_{\mathrm{VII}}} \ . \tag{5.38}$$

Here, $\bar{B}|_{t=0} = c_{\mathrm{VIII}}$ and $\bar{B}$ is assumed non-constant. Once $\bar{B}$ has been obtained, Ω is given by the relation

$$\Omega^2 = [-(c_{\mathrm{V}}+4c_{\mathrm{II}})t^2 + c_{\mathrm{VI}}t + c_{\mathrm{VII}}]/\bar{B} \tag{5.39}$$

while $\phi(t)$ and $\theta(t)$ are obtained by integration, in turn, of (5.33) and (5.34).

The matrices $\mathbf{L}(t)$ and $\mathbf{M}(t)$ is the original ansatz (5.3) are now given by

$$\mathbf{L} = \frac{1}{\Omega^2}\begin{pmatrix} \Omega\dot{\Omega} + \sqrt{c_{\mathrm{III}}-\mathbb{C}\bar{B}^2}\cos\theta & -(c_0+\sqrt{c_{\mathrm{III}}-\mathbb{C}\bar{B}^2})\sin\theta \\ c_0 - \sqrt{c_{\mathrm{III}}-\mathbb{C}\bar{B}^2}\sin\theta & \Omega\dot{\Omega} - \sqrt{c_{\mathrm{III}}-\mathbb{C}\bar{B}^2}\cos\theta \end{pmatrix} \tag{5.40}$$

and

$$\mathbf{E} = \frac{1}{\Omega^2}\begin{pmatrix} \frac{\bar{B}}{2} - \sqrt{\frac{1}{4}\bar{B}^2 + c_{\mathrm{II}}}\sin\phi & -\sqrt{c_{\mathrm{II}} + \frac{1}{4}\bar{B}^2}\cos\phi \\ -\sqrt{c_{\mathrm{II}} + \frac{1}{4}\bar{B}^2}\cos\phi & \frac{\bar{B}}{2} + \sqrt{\frac{1}{4}\bar{B}^2 + c_{\mathrm{II}}}\sin\phi \end{pmatrix} \tag{5.41}$$

while σ is given by $(5.14)_2$ and p, q by (5.9).

It is noted that an analogous parametrization procedure may be employed *mutatis mutandis* when $\bar{B} = const$.

6. An Ermakov-Ray-Reid Connection

Here, it is established that the governing nonlinear system (5.6)−(5.7) has a remarkable Ermakov-Ray-Reid connection. Moreover, the system is shown to be Hamiltonian.

It proves convenient to introduce the quantities

$$\Phi = \Omega\sqrt{\frac{\sigma}{(B_N^2+B_S^2)^{1/2} - \frac{B}{2}}} = \Omega\sqrt{\frac{c_{\mathrm{I}}}{(c_{\mathrm{II}}+\frac{\bar{B}^2}{4})^{1/2} - \frac{\bar{B}}{2}}} \tag{6.1}$$

and

$$\Psi = \Omega\sqrt{\frac{\sigma}{-(B_N^2+B_S^2)^{1/2} - \frac{B}{2}}} = \Omega\sqrt{\frac{c_{\mathrm{I}}}{-(c_{\mathrm{II}}+\frac{\bar{B}^2}{4})^{1/2} - \frac{\bar{B}}{2}}} \tag{6.2}$$

where it is assumed that $\bar{B} < 0$ and $c_{\rm I} > 0$. Thus,

$$\frac{\Phi}{\Psi} = \frac{-1}{\sqrt{-c_{\rm II}}}\left[\frac{\bar{B}}{2} + \sqrt{c_{\rm II} + \frac{\bar{B}^2}{4}}\right] \tag{6.3}$$

whence,

$$\bar{B} = -\sqrt{-c_{\rm II}}\left(\frac{\Phi}{\Psi} + \frac{\Psi}{\Phi}\right) \ . \tag{6.4}$$

The relation (6.3) shows that

$$\Psi\dot{\Phi} - \dot{\Psi}\Phi = \frac{c_{\rm I}\Omega^2\dot{\bar{B}}}{2\sqrt{-c_{\rm II}(\bar{B}^2 + c_{\rm II})}} \equiv Z(\Phi/\Psi) \tag{6.5}$$

where it is noted that, by virtue of the relation (5.32), $\theta - \phi$ is dependent only on $\bar{B}$. Moreover,

$$\Phi^2 + \Phi^2 = \left(\frac{c_{\rm I}}{c_{\rm II}}\right)\Omega^2\bar{B} = \left(\frac{c_{\rm I}}{c_{\rm II}}\right)[-(c_{\rm V} + 4c_{\rm II})t^2 + c_{\rm VI}t + c_{\rm VII}] \equiv T(t) \ , \tag{6.6}$$

so that, on use of the identity

$$(\dot{\Phi}^2 + \dot{\Psi}^2)(\Phi^2 + \Psi^2) - (\Psi\dot{\Phi} - \dot{\Psi}\Phi)^2 \equiv (\Phi\dot{\Phi} + \Psi\dot{\Psi})^2 \tag{6.7}$$

it is seen that

$$\Phi\ddot{\Phi} + \Psi\ddot{\Psi} = \frac{\ddot{T}}{2} - \frac{1}{\Phi^2 + \Psi^2}[Z^2(\Phi/\Psi) + \frac{\dot{T}^2}{4}] \tag{6.8}$$

while (6.6) yields

$$\Psi\ddot{\Phi} - \ddot{\Psi}\Phi = \frac{1}{\Psi^2}Z'(\Phi/\Psi)Z(\Phi/\Psi) \ . \tag{6.9}$$

The relations (6.8) and (6.9) together show that Φ, Ψ are governed by

$$\begin{aligned}\ddot{\Phi} + [\frac{\dot{T}^2}{4(\Phi^2+\Psi^2)^2} - \frac{\ddot{T}}{2(\Phi^2+\Psi^2)}]\Phi &= \frac{1}{\Phi^2\Psi}\left[\frac{Z(\Phi/\Psi)Z'(\Phi/\Psi)}{[1+(\Psi/\Phi)^2]} - \left(\frac{\Psi}{\Phi}\right)\frac{Z^2(\Phi/\Psi)}{[1+(\Psi/\Phi)^2]^2}\right] \ , \\ \ddot{\Psi} + [\frac{\dot{T}^2}{4(\Phi^2+\Psi^2)^2} - \frac{\ddot{T}}{2(\Phi^2+\Psi^2)}]\Psi &= \frac{1}{\Psi^2\Phi}\left[-\left(\frac{\Phi}{\Psi}\right)\frac{Z^2(\Phi/\Psi)}{[1+(\Phi/\Psi)^2]^2} - \frac{Z(\Phi/\Psi)Z'(\Phi/\Psi)}{[1+(\Psi/\Phi)^2]}\right]\end{aligned} \tag{6.10}$$

which, on use of (6.6), reduces to the Ermakov-Ray-Reid system

$$\begin{aligned}\ddot{\Phi} &= \frac{1}{\Phi^2\Psi}\left[\frac{Z(\Phi/\Psi)Z'(\Phi/\Psi)}{[1+(\Psi/\Phi)^2]} - \frac{\Psi}{\Phi}\frac{(Z^2(\Phi/\Psi)+\mathbb{C})}{[1+(\Psi/\Phi)^2]^2}\right] = \frac{1}{\Phi^2\Psi}F(\Phi/\Psi) \ , \\ \ddot{\Psi} &= \frac{1}{\Psi^2\Phi}\left[-\frac{\Phi}{\Psi}\frac{(Z^2(\Phi/\Psi)+\mathbb{C})}{[1+(\Phi/\Psi)^2]^2} - \frac{Z(\Phi/\Psi)Z'(\Phi/\Psi)}{[1+(\Psi/\Phi)^2]}\right] = \frac{1}{\Psi^2\Phi}G(\Psi/\Phi)\end{aligned} \tag{6.11}$$

where

$$\mathbb{C} = \frac{1}{4}\left(\frac{c_{\rm I}}{c_{\rm II}}\right)^2[c_{\rm VI}^2 + 4(c_{\rm V} + 4c_{\rm II})c_{\rm VII}] \ . \tag{6.12}$$

Moreover, the relations (6.5), (6.6) and the identity (6.7) together show that

$$\dot{\Phi}^2 + \dot{\Psi}^2 = \frac{1}{\Phi^2 + \Psi^2}\,[Z^2(\Phi/\Psi) + \frac{\dot{T}^2}{4}] \ , \tag{6.13}$$

whence it is seen that the Ermakov-Ray-Reid system (6.11) is Hamiltonian with

$$\dot{\Phi}^2 + \dot{\Psi}^2 - \frac{1}{\Phi^2 + \Psi^2}\,[Z^2(\Phi/\Psi) + \mathbb{C}] = \mathscr{H} \ , \tag{6.14}$$

where

$$\mathscr{H} = \left(\frac{c_{\mathrm{I}}}{c_{\mathrm{II}}}\right)(c_{\mathrm{V}} + 4\,c_{\mathrm{II}}) \ . \tag{6.15}$$

Ermakov-Ray-Reid systems with underlying Hamiltonian structure are readily integrated in the manner described in [38]. It is remarked that such Ermakov-Ray-Reid systems are ubiquitous in nonlinear optics ([7, 9–11, 13, 14, 49]).

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