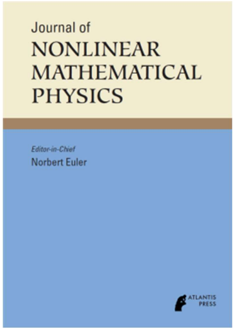

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Jipeng Cheng, Jinzheng Wang, Xingyong Zhang

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The gauge transformation of the q-deformed modified KP hierarchy

Jipeng Cheng
Department of Mathematics, China University of Mining and Technology
Xuzhou, Jiangsu 221116, P. R. China
chengjp@cumt.edu.cn and chengjp@mail.ustc.edu.cn

Jinzheng Wang
School of Mathematical Sciences, Capital Normal University
Beijing 100048, P. R. China

Xingyong Zhang*
Department of Mathematics, China University of Mining and Technology
Xuzhou, Jiangsu 221116, P. R. China
zhangxingyong163@163.com

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In this paper, we mainly study three types of gauge transformation operators for the q-mKP hierarchy. The successive applications of these gauge transformation operators are derived. And the corresponding communities between them are also investigated.

Keywords: q-KP hierarchy; q-mKP hierarchy; gauge transformation.

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1. Introduction

Recently, the q-deformed integrable system, has attracted much attention in theoretical physics and mathematics [2, 7, 9, 12–15, 17, 22–27], which is defined by means of q-derivative ∂_q (see (2.1)) instead of usual derivative ∂ with respect to x in the classical system. When $q \to 1$, the q-deformed integrable system can reduce to the classical system. There are several kinds of q-deformed integrable systems, for example: the q-deformed Kadomtsev-Petviashvili (q-KP) hierarchy [2, 9, 12, 23–25], the q-modified Kadomtsev-Petviashvili (q-mKP) hierarchy [15, 22, 26], the q-KP hierarchy and the q-mKP hierarchy with self-consistent sources [14, 15], and so on. In this paper, we will mainly study the q-mKP hierarchy.

The gauge transformation [5, 19, 21], also called the Darboux transformation, provides a simple way to construct solutions for integrable hierarchies. By now, the gauge transformations of many integrable hierarchies have been studied, for example, the KP hierarchy [3–5, 10, 19, 21, 28], the BKP and CKP hierarchies [6, 8, 11, 18], the discrete KP hierarchy [16, 20], the q-KP hierarchy [9, 25] and so on.

*Corresponding author

However, to the best of our knowledge, there are few researches on the gauge transformation for the q-mKP hierarchy in the literature. In this paper, three types of gauge transformation operators for the q-mKP hierarchy are constructed, that is, T_1, T_2 and T_3 (see (3.4), (3.5) and (3.8)), where T_3 is the composition of T_1 and T_2. Then the successive applications of these three gauge transformation operators are obtained, which will be helpful in the study of the gauge transformation for the q-mKP hierarchy. At last, the communities of these gauge transformation operators are investigated.

This paper is organized in the following way. In Section 2, we briefly review some facts about the q-mKP hierarchy. Then three types of the gauge transformations for the q-mKP hierarchy are constructed in Section 3. In Section 4, successive applications of these three types of gauge transformation operators are investigated. And in Section 5, the communities of these gauge transformation operators are also studied. At last, some conclusions and discussions are given in Section 6.

2. The q-KP hierarchy and the q-mKP hierarchy

Firstly, let's introduce some useful formulas. The q-derivative ∂_q and the q-shift operator are defined by their actions on a function $f(x)$

$$\partial_q(f(x)) = \frac{f(qx)-f(x)}{x(q-1)}, \quad \theta(f(x)) = f(qx). \tag{2.1}$$

In this paper, $A(f)$ is used to denote the action of the operator A on the function $f(x)$, while Af or $A\cdot f$ denotes the multiplication of A and f. Let ∂_q^{-1} denote the formal inverse of ∂_q. We should note that θ does not commute with ∂_q,

$$\partial_q(\theta^k(f)) = q^k\theta^k(\partial_q(f)), \quad k\in\mathbb{Z}. \tag{2.2}$$

In general, the following q-deformed Leibnitz rule holds

$$\partial_q^n f = \sum_{k\geq 0}\binom{n}{k}_q \theta^{n-k}(\partial_q^k(f))\partial_q^{n-k}, \tag{2.3}$$

where the q-number and the q-binomial are defined by

$$(n)_q = \frac{q^n-1}{q-1}, \quad \binom{n}{k}_q = \frac{(n)_q(n-1)_q\cdots(n-k+1)_q}{(1)_q(2)_q\cdots(k)_q}, \quad \binom{n}{0}_q = 1. \tag{2.4}$$

Next, we consider the algebra

$$g = \{\sum_{i\ll\infty} p_i\partial_q^i\} = \{\sum_{i\geq k} p_i\partial_q^i\}\oplus\{\sum_{i\leq k} p_i\partial_q^i\} = g_{\geq k}\oplus g_{<k} \tag{2.5}$$

of q-pseduo-differential operators (q-PDO). When $k=0,1$, $g_{\geq k}$ and $g_{<k}$ are sub-Lie algebras of g: $[g_{\geq k}, g_{\geq k}]\subset g_{\geq k}$ and $[g_{<k}, g_{<k}]\subset g_{<k}$. The projections of $A=\sum_i p_i\partial_q^i$ are

$$A_{\geq k} = \sum_{i\geq k} p_i\partial_q^i, \quad A_{<k} = \sum_{i<k} p_i\partial_q^i. \tag{2.6}$$

Then according to the famous Adler-Kostant-Symes scheme [1], the following commuting Lax equations on g can be constructed,

$$L_{t_n} = [(L^n)_{\geq k}, L], \quad n=1,2,3,\cdots, \tag{2.7}$$

where $k=0,1$ are corresponding to q-KP and q-mKP hierarchies respectively, with the Lax operator L given by

$$L=\begin{cases}\partial_q+u_0+u_{-1}\partial_q^{-1}+\cdots, & k=0;\\ v_1\partial_q+v_0+v_{-1}\partial_q^{-1}+\cdots, & k=1,\end{cases} \tag{2.8}$$

where $u_i=u_i(x,t)=u_i(x,t_1,t_2,\cdots)$ and $v_i=v_i(x,t)=v_i(x,t_1,t_2,\cdots)$.

For $k=0,1$, if the function $\Phi=\Phi(x,t)$ satisfying the linear equations

$$\Phi_{t_n}=(L^n)_{\geq k}(\Phi), \quad n=1,2,3,\cdots \tag{2.9}$$

then we will call Φ the **eigenfunction** of the hierarchy of the Lax equations $L_{t_n}=[(L^n)_{\geq k},L]$.

The q-KP hierarchy and the q-mKP hierarchy are linked by the gauge transformation, which is showed in the following proposition.

Proposition 2.1. *Let $L\in g$ be the Lax operator of the q-KP hierarchy: $L_{t_n}=[(L^n)_{\geq 0},L]$. $\Phi\neq 0$ and Ψ are the eigenfunctions of the q-KP hierarchy. Then $\tilde{L}=\Phi^{-1}L\Phi$ satisfies the q-mKP hierarchy: $\tilde{L}_{t_n}=[(\tilde{L}^n)_{\geq 1},\tilde{L}]$, and $\tilde{\Psi}=\Phi^{-1}\Psi$ is an eigenfunction of $\tilde{L}$, that is $\tilde{\Psi}_{t_n}=\tilde{L}^n_{\geq 1}(\tilde{\Psi})$.*

Proof. Firstly given any q-PDO: $A\in g$,

$$\begin{aligned}(\Phi^{-1}A\Phi)_{\geq 1}&=(\Phi^{-1}A_{\geq 0}\Phi)_{\geq 1}=(\Phi^{-1}A_{\geq 0}\Phi)_{\geq 0}-(\Phi^{-1}A_{\geq 0}\Phi)_{[0]}\\ &=\Phi^{-1}A_{\geq 0}\Phi-\Phi^{-1}A_{\geq 0}(\Phi),\end{aligned} \tag{2.10}$$

where $A_{[0]}=p_0$ if $A=\sum_i p_i\partial_q^i$.

Then for $\tilde{L}=\Phi^{-1}L\Phi$ and $\tilde{\Psi}=\Phi^{-1}\Psi$, by noting the fact that $L_{t_n}=[(L^n)_{\geq 0},L]$, $\Phi_{t_n}=(L^n)_{\geq 0}(\Phi)$ and $\Psi_{t_n}=(L^n)_{\geq 0}(\Psi)$, one has

$$\tilde{L}_{t_n}-[(\tilde{L}^n)_{\geq 1},\tilde{L}]=\Phi^{-1}(L_{t_n}-[(L^n)_{\geq 0},L])\Phi-[\Phi^{-1}\Big(\Phi_{t_n}-(L^n)_{\geq 0}(\Phi)\Big),L]=0, \tag{2.11}$$

$$\tilde{\Psi}_{t_n}-(\tilde{L}^n)_{\geq 1}(\tilde{\Psi})=-\Phi^{-2}\Psi\Big(\Phi_{t_n}-(L^n)_{\geq 0}(\Phi)\Big)+\Phi^{-1}\Big(\Psi_{t_n}-(L^n)_{\geq 0}(\Psi)\Big)=0. \tag{2.12}$$

□

Thus if we apply the gauge transformation in Proposition 2.1 to the Lax operator of the q-KP hierarchy $L=\partial_q+u_0+u_{-1}\partial_q^{-1}+u_{-2}\partial_q^{-2}+u_{-3}\partial_q^{-3}\cdots$, that is

$$\tilde{L}=\Phi^{-1}L\Phi=v_1\partial_q+v_0+v_{-1}\partial_q^{-1}+v_{-2}\partial_q^{-2}+v_{-3}\partial_q^{-3}\cdots, \tag{2.13}$$

where $\Phi\neq 0$ is the eigenfunction of the q-KP hierarchy, then

$$\begin{aligned}&v_1=\Phi^{-1}\theta(\Phi),\ \ v_0=\Phi^{-1}\partial_q(\Phi)+u_0,\\ &v_{-1}=u_{-1}\Phi^{-1}\theta^{-1}(\Phi),\ \ v_{-2}=u_{-2}\Phi^{-1}\theta^{-2}(\Phi)-q^{-1}u_{-1}\Phi^{-1}\theta^{-2}(\partial_q(\Phi)).\end{aligned} \tag{2.14}$$

3. Gauge transformation

In this section, we will investigate the gauge transformation of the q-mKP hierarchy. For the q-mKP hierarchy (see (2.7) and (2.8) with $k = 1$), suppose $T \in g$ is a q-PDO, and

$$L^{(1)} = TLT^{-1}, \tag{3.1}$$

such that

$$L^{(1)}_{t_n} = [\left(L^{(1)}\right)^n_{\geq 1}, L^{(1)}] \tag{3.2}$$

still holds for the transformed Lax operator $L^{(1)}$, then T is called a **gauge transformation operator** of the q-mKP hierarchy. According to (3.2), one can obtain the lemma below.

Lemma 3.1. *If $T \in g$ satisfies*

$$(TL^nT^{-1})_{\geq 1} = T(L^n)_{\geq 1}T^{-1} + T_{t_n}T^{-1}, \tag{3.3}$$

then T is a gauge transformation operator of the q-mKP hierarchy.

Before the construction of the gauge transformation, the following basic identities on q-PDO are needed.

Lemma 3.2. *For $A \in g$ and arbitrary function Φ, one has the following operator identities:*

(1) $(\Phi^{-1}A\Phi)_{\geq 1} = \Phi^{-1}A_{\geq 1}\Phi - \Phi^{-1}A_{\geq 1}(\Phi)$

(2) $\left((\partial_q(\Phi))^{-1} \cdot \partial_q A \partial_q^{-1} \cdot \partial_q(\Phi)\right)_{\geq 1} = (\partial_q(\Phi))^{-1} \cdot \partial_q A_{\geq 1} \partial_q^{-1} \cdot \partial_q(\Phi) - (\partial_q(\Phi))^{-1} \cdot (\partial_q A_{\geq 1})(\Phi)$

Proof. The first identity can be proved in the way below,

$$\begin{aligned}
(\Phi^{-1}A\Phi)_{\geq 1} &= (\Phi^{-1}A_{\geq 1}\Phi)_{\geq 1} = (\Phi^{-1}A_{\geq 1}\Phi)_{\geq 0} - (\Phi^{-1}A_{\geq 1}\Phi)_{[0]} \\
&= \Phi^{-1}A_{\geq 1}\Phi - \Phi^{-1}A_{\geq 1}(\Phi).
\end{aligned}$$

The derivation of the second identity is showed in the following

$$\begin{aligned}
&\left((\partial_q(\Phi))^{-1} \cdot \partial_q A \partial_q^{-1} \cdot \partial_q(\Phi)\right)_{\geq 1} = \left((\partial_q(\Phi))^{-1} \cdot \partial_q A_{\geq 1} \partial_q^{-1} \cdot \partial_q(\Phi)\right)_{\geq 1} \\
&= \left((\partial_q(\Phi))^{-1} \cdot \partial_q A_{\geq 1} \partial_q^{-1} \cdot \partial_q(\Phi)\right)_{\geq 0} - \left((\partial_q(\Phi))^{-1} \cdot \partial_q A_{\geq 1} \partial_q^{-1} \cdot \partial_q(\Phi)\right)_{[0]} \\
&= \left(\partial_q(\Phi)\right)^{-1} \cdot \partial_q A_{\geq 1} \partial_q^{-1} \cdot \partial_q(\Phi) - (\partial_q(\Phi))^{-1} \cdot (\partial_q A_{\geq 1})(\Phi).
\end{aligned}$$

□

After the preparation above, the corresponding gauge transformation operators are constructed in the following proposition below.

Proposition 3.1. *There are two elementary gauge transformation operators for the q-mKP hierarchy, that is,*

$$T_1(\Phi) = \Phi^{-1}, \tag{3.4}$$

$$T_2(\Phi) = \left(\partial_q(\Phi)\right)^{-1} \partial_q, \tag{3.5}$$

where $\Phi \neq 0$ is the eigenfunction of the q-mKP hierarchy (see (2.9) for $k = 1$), and in particular Φ is not a constant in (3.5).

Proof. From (2.9) for $k=1$, we have

$$\begin{aligned}
T_1(\Phi)_{t_n}T_1(\Phi)^{-1} &= -\Phi^{-1}\Phi_{t_n} = -\Phi^{-1}(L^n)_{\geq 1}(\Phi),\\
T_2(\Phi)_{t_n}T_2(\Phi)^{-1} &= -\Big(\partial_q(\Phi)\Big)^{-2}\Big(\partial_q(\Phi_{t_n})\Big)\partial_q\cdot\partial_q^{-1}\Big(\partial_q(\Phi)\Big)\\
&= -(\partial_q(\Phi))^{-1}\cdot(\partial_q(L^n)_{\geq 1})(\Phi).
\end{aligned}$$

Thus if setting $A=L^n$ in Lemma 3.2, then one can find that $T_1(\Phi)$ and $T_2(\Phi)$ satisfy (3.3), which leads to Proposition 3.1. □

Proposition 3.2. *Assume $\Phi\neq 0$ (Φ is nonconstant for T_2) and Ψ be the eigenfunctions of the q-mKP hierarchy, then under the gauge transformation operator $T_1(\Phi)$ and $T_2(\Phi)$,*

$$\Psi \xrightarrow{T_1(\Phi)} \Psi^{(1)} = T_1(\Phi)\Psi = \Phi^{-1}\Psi, \tag{3.6}$$

$$\Psi \xrightarrow{T_2(\Phi)} \Psi^{(1)} = T_2(\Phi)\Psi = \Big(\partial_q(\Phi)\Big)^{-1}\partial_q(\Psi). \tag{3.7}$$

$\Psi^{(1)}$ will be the corresponding eigenfunction of the transformed q-mKP hierarchy.

Proof. For (3.6), with the help of the first identity in Lemma 3.2

$$\begin{aligned}
\Psi^{(1)}_{t_n} - \Big(L^{(1)}\Big)^n_{\geq 1}(\Psi^{(1)}) &= -\Phi^{-2}\Phi_{t_n}\Psi + \Phi^{-1}\Psi_{t_n} - \Phi^{-1}L^n_{\geq 1}(\Psi) + \Phi^{-2}\Psi L^n_{\geq 1}(\Phi)\\
&= -\Phi^{-2}\Psi(\Phi_{t_n} - L^n_{\geq 1}(\Phi)) + \Phi^{-1}(\Psi_{t_n} - L^n_{\geq 1}(\Psi)) = 0.
\end{aligned}$$

By using the second identity in Lemma 3.2, one can find for (3.7),

$$\begin{aligned}
&\Psi^{(1)}_{t_n} - \Big(L^{(1)}\Big)^n_{\geq 1}(\Psi^{(1)})\\
&= -\Big(\partial_q(\Phi)\Big)^{-2}\partial_q(\Psi)\Big(\partial_q(\Phi_{t_n} - L^n_{\geq 1}(\Phi))\Big) + \Big(\partial_q(\Phi)\Big)^{-1}\partial_q(\Psi_{t_n} - L^n_{\geq 1}(\Psi)) = 0.
\end{aligned}$$

□

Since 1 is also an eigenfunction of the q-mKP hierarchy, one can get a new gauge transformation operator

$$T_3(\Phi) = T_2(1^{(1)})T_1(\Phi) = \Big(\partial_q(\Phi^{-1})\Big)^{-1}\cdot\partial_q\cdot\Phi^{-1}, \tag{3.8}$$

where Φ is nonconstant and $1^{(1)} = T_1(\Phi)(1) = \Phi^{-1}$. Then under $T_3(\Phi)$, the arbitrary eigenfunction Ψ of the q-mKP hierarchy will become into

$$\Psi \xrightarrow{T_3(\Phi)} \Psi^{(1)} = T_3(\Phi)(\Psi) = \Big(\partial_q(\Phi^{-1})\Big)^{-1}\partial_q(\Phi^{-1}\Psi). \tag{3.9}$$

4. Successive applications of gauge transformations

In this section, we discuss the successive applications of the three types of gauge transformation operators mentioned in Section 3. Given n nonzero independent eigenfunctions of the q-mKP hierarchy $\Phi_1, \Phi_2, \cdots, \Phi_n$ [a], consider the following chain of the gauge transformation operators,

$$L \xrightarrow{T_i(\Phi_1)} L^{(1)} \xrightarrow{T_i(\Phi_2^{(1)})} L^{(2)} \xrightarrow{T_i(\Phi_3^{(2)})} L^{(3)} \to \cdots \to L^{(n-1)} \xrightarrow{T_i(\Phi_n^{(n-1)})} L^{(n)}, \quad i = 1,2,3, \tag{4.1}$$

where $\Phi_j^{(k)}$ is the transformed eigenfunction by k-steps gauge transformations from Φ_j, that is,

$$\Phi_j^{(k)} = T_i(\Phi_k^{(k-1)}) \cdots T_i(\Phi_3^{(2)}) T_i(\Phi_2^{(1)}) T_i(\Phi_1)(\Phi_j), \tag{4.2}$$

and $L^{(k)}$ is transformed by k-steps gauge transformations from the initial Lax operator L. The successive applications of gauge transformation operator in (4.1) can be represented by

$$T_i^{(n)} \triangleq T_i^{(n)}(\Phi_1, \cdots, \Phi_n) \triangleq T_i(\Phi_n^{(n-1)}) \cdots T_i(\Phi_3^{(2)}) T_i(\Phi_2^{(1)}) T_i(\Phi_1), \quad i = 1,2,3. \tag{4.3}$$

Case $i = 1$,

$$\begin{aligned} T_1^{(n)} &= T_1(\Phi_n^{(n-1)}) \cdots T_1(\Phi_3^{(2)}) T_1(\Phi_2^{(1)}) T_1(\Phi_1) \\ &= (\Phi_1 \Phi_2^{(1)} \cdots \Phi_{n-2}^{(n-3)} \Phi_{n-1}^{(n-2)} \Phi_n^{(n-1)})^{-1} \\ &= (\Phi_1 \Phi_2^{(1)} \cdots \Phi_{n-2}^{(n-3)} \Phi_n^{(n-2)})^{-1} \\ &= (\Phi_1 \Phi_2^{(1)} \cdots \Phi_n^{(n-3)})^{-1} \\ &\quad \cdots \\ &= \Phi_n^{-1}, \end{aligned} \tag{4.4}$$

where we have used the following fact derived from (3.6),

$$\Phi_k^{(k-1)} \Phi_j^{(k)} = \Phi_j^{(k-1)}, \tag{4.5}$$

with $\Phi_j^{(k)} = T_1(\Phi_k^{(k-1)})(\Phi_j^{(k-1)})$.

Case $i = 2$. Assume $T_2^{(n)} = a_1\partial_q + a_2\partial_q^2 + \cdots + a_n\partial_q^n$. Since $T_2^{(n)}(\Phi_i) = 0, i = 1,2,\cdots,n-1$ and $T_2^{(n)}(\Phi_n) = 1$, one can obtain

$$\begin{cases} a_1\partial_q(\Phi_1) + a_2\partial_q^2(\Phi_1) + \cdots + a_n\partial_q^n(\Phi_1) = 0 \\ a_1\partial_q(\Phi_2) + a_2\partial_q^2(\Phi_2) + \cdots + a_n\partial_q^n(\Phi_2) = 0 \\ \vdots \\ a_1\partial_q(\Phi_{n-1}) + a_2\partial_q^2(\Phi_{n-1}) + \cdots + a_n\partial_q^n(\Phi_{n-1}) = 0 \\ a_1\partial_q(\Phi_n) + a_2\partial_q^2(\Phi_n) + \cdots + a_n\partial_q^n(\Phi_n) = 1. \end{cases} \tag{4.6}$$

[a] Φ_j is nonconstant for the cases $i = 2,3$.

Thus

$$a_i = (-1)^{i+n} \frac{\begin{vmatrix} \partial_q(\Phi_1) & \cdots & \partial_q(\Phi_{n-1}) \\ \vdots & \vdots & \vdots \\ \partial_q^{i-1}(\Phi_1) & \cdots & \partial_q^{i-1}(\Phi_{n-1}) \\ \partial_q^{i+1}(\Phi_1) & \cdots & \partial_q^{i+1}(\Phi_{n-1}) \\ \vdots & \vdots & \vdots \\ \partial_q^{n}(\Phi_1) & \cdots & \partial_q^{n}(\Phi_{n-1}) \end{vmatrix}}{\begin{vmatrix} \partial_q(\Phi_1) & \cdots & \partial_q(\Phi_n) \\ \vdots & \vdots & \vdots \\ \partial_q^{n}(\Phi_1) & \cdots & \partial_q^{n}(\Phi_n) \end{vmatrix}}. \tag{4.7}$$

Further

$$T_2^{(n)} = \frac{\begin{vmatrix} \partial_q(\Phi_1) & \cdots & \partial_q(\Phi_{n-1}) & \partial_q \\ \vdots & \vdots & \vdots & \vdots \\ \partial_q^{i}(\Phi_1) & \cdots & \partial_q^{i}(\Phi_{n-1}) & \partial_q^{i} \\ \vdots & \vdots & \vdots & \vdots \\ \partial_q^{n}(\Phi_1) & \cdots & \partial_q^{n}(\Phi_{n-1}) & \partial_q^{n} \end{vmatrix}}{\begin{vmatrix} \partial_q(\Phi_1) & \cdots & \partial_q(\Phi_n) \\ \vdots & \vdots & \vdots \\ \partial_q^{n}(\Phi_1) & \cdots & \partial_q^{n}(\Phi_n) \end{vmatrix}}. \tag{4.8}$$

Case $i = 3$. Since $T_3(\Phi) = \theta(\Phi^{-1})\Big(\partial_q(\Phi^{-1})\Big)^{-1} \cdot \partial_q + 1 = -\Phi\Big(\partial_q(\Phi)\Big)^{-1} \cdot \partial_q + 1$, thus one can let $T_3^{(n)} = 1 + b_1\partial_q + \cdots + b_n\partial_q^n$. By considering $T_3^{(n)}(\Phi_i) = 0, i = 1,2,\cdots,n$,

$$\begin{cases} b_1\partial_q(\Phi_1) + b_2\partial_q^2(\Phi_1) + \cdots + b_n\partial_q^n(\Phi_1) = -\Phi_1 \\ b_1\partial_q(\Phi_2) + b_2\partial_q^2(\Phi_2) + \cdots + b_n\partial_q^n(\Phi_2) = -\Phi_2 \\ \vdots \\ b_1\partial_q(\Phi_n) + b_2\partial_q^2(\Phi_n) + \cdots + b_n\partial_q^n(\Phi_n) = -\Phi_n. \end{cases} \tag{4.9}$$

Through solving the linear equation group,

$$b_i = (-1)^{i} \frac{\begin{vmatrix} \Phi_1 & \cdots & \Phi_n \\ \partial_q(\Phi_1) & \cdots & \partial_q(\Phi_n) \\ \vdots & \vdots & \vdots \\ \partial_q^{i-1}(\Phi_1) & \cdots & \partial_q^{i-1}(\Phi_n) \\ \partial_q^{i+1}(\Phi_1) & \cdots & \partial_q^{i+1}(\Phi_n) \\ \vdots & \vdots & \vdots \\ \partial_q^{n}(\Phi_1) & \cdots & \partial_q^{n}(\Phi_n) \end{vmatrix}}{\begin{vmatrix} \partial_q(\Phi_1) & \cdots & \partial_q(\Phi_n) \\ \vdots & \vdots & \vdots \\ \partial_q^{n}(\Phi_1) & \cdots & \partial_q^{n}(\Phi_n) \end{vmatrix}}. \tag{4.10}$$

Therefore

$$T_3^{(n)} = (-1)^n \frac{\begin{vmatrix} \Phi_1 & \cdots & \Phi_n & 1 \\ \partial_q(\Phi_1) & \cdots & \partial_q(\Phi_n) & \partial_q \\ \vdots & \vdots & \vdots & \vdots \\ \partial_q^i(\Phi_1) & \cdots & \partial_q^i(\Phi_n) & \partial_q^i \\ \vdots & \vdots & \vdots & \vdots \\ \partial_q^n(\Phi_1) & \cdots & \partial_q^n(\Phi_n) & \partial_q^n \end{vmatrix}}{\begin{vmatrix} \partial_q(\Phi_1) & \cdots & \partial_q(\Phi_n) \\ \vdots & \vdots & \vdots \\ \partial_q^n(\Phi_1) & \cdots & \partial_q^n(\Phi_n) \end{vmatrix}}. \tag{4.11}$$

5. The commutativity of the Bianchi diagram

Assume L be the Lax operator of the q-mKP hierarchy, and Φ_1 and Φ_2 be two nonzero independent eigenfunctions, we consider the following diagram

$$\begin{array}{ccc} L, \Phi_1, \Phi_2 & \xrightarrow{T_i(\Phi_2)} & \bar{L}, \bar{\Phi}_1 \\ {\scriptstyle T_i(\Phi_1)} \downarrow & & \downarrow {\scriptstyle T_i(\bar{\Phi}_1)} \\ \hat{L}, \hat{\Phi}_2 & \xrightarrow{T_i(\hat{\Phi}_2)} & \hat{\bar{L}} = \bar{\hat{L}}? \end{array}$$

where

$$\begin{aligned} &\hat{L} = T_i(\Phi_1) L T_i(\Phi_1)^{-1}, \;\; \hat{\Phi}_2 = T_i(\Phi_1)(\Phi_2), \\ &\bar{L} = T_i(\Phi_2) L T_i(\Phi_2)^{-1}, \;\; \bar{\Phi}_1 = T_i(\Phi_2)(\Phi_1), \\ &\hat{\bar{L}} = T_i(\bar{\Phi}_1) \bar{L} T_i(\bar{\Phi}_1)^{-1}, \;\; \bar{\hat{L}} = T_i(\hat{\Phi}_2) \hat{L} T_i(\hat{\Phi}_2)^{-1}. \end{aligned}$$

The question is whether this diagram will commute, that is, $T_i^{(2)}(\Phi_1, \Phi_2) = T_i^{(2)}(\Phi_2, \Phi_1)$, $i = 1, 2, 3$.

By using (4.4), (4.8) and (4.11), one can find

$$T_1^{(2)}(\Phi_1, \Phi_2) - T_1^{(2)}(\Phi_2, \Phi_1) = \Phi_2^{-1} - \Phi_1^{-1} \neq 0, \tag{5.1}$$

$$T_2^{(2)}(\Phi_1, \Phi_2) - T_2^{(2)}(\Phi_2, \Phi_1) = \frac{\begin{vmatrix} \partial_q(\Phi_1 + \Phi_2) & \partial_q \\ \partial_q^2(\Phi_1 + \Phi_2) & \partial_q^2 \end{vmatrix}}{\begin{vmatrix} \partial_q(\Phi_1) & \partial_q(\Phi_2) \\ \partial_q^2(\Phi_1) & \partial_q^2(\Phi_2) \end{vmatrix}} \neq 0, \tag{5.2}$$

$$T_3^{(2)}(\Phi_1, \Phi_2) - T_3^{(2)}(\Phi_2, \Phi_1) = \frac{\begin{vmatrix} \Phi_1 & \Phi_2 & 1 \\ \partial_q(\Phi_1) & \partial_q(\Phi_2) & \partial_q \\ \partial_q^2(\Phi_1) & \partial_q^2(\Phi_2) & \partial_q^2 \end{vmatrix}}{\begin{vmatrix} \partial_q(\Phi_1) & \partial_q(\Phi_2) \\ \partial_q^2(\Phi_1) & \partial_q^2(\Phi_2) \end{vmatrix}} - \frac{\begin{vmatrix} \Phi_2 & \Phi_1 & 1 \\ \partial_q(\Phi_2) & \partial_q(\Phi_1) & \partial_q \\ \partial_q^2(\Phi_2) & \partial_q^2(\Phi_1) & \partial_q^2 \end{vmatrix}}{\begin{vmatrix} \partial_q(\Phi_2) & \partial_q(\Phi_1) \\ \partial_q^2(\Phi_2) & \partial_q^2(\Phi_1) \end{vmatrix}} = 0. \tag{5.3}$$

Thus only in the third gauge transformation, the diagram commutes.

6. Conclusions and Discussions

Three types of the gauge transformation operators for the q-mKP hierarchy are constructed in this paper, that is T_1 (see (3.4)), T_2 (see (3.5)) and T_3 (see (3.8)), where T_3 is the composition of T_1 and T_2. The successive applications of these three gauge transformation operators are shown in (4.4), (4.8) and (4.11) respectively. From (5.1), (5.2) and (5.3), one can find that only the third gauge transformation operator T_3 can commute itself. Thus T_3 will be the most convenient gauge transformation operator for the q-mKP hierarchy. These results will be helpful for the further study of the q-mKP hierarchy. And we will consider whether there is the connection between the gauge transformations of the q-KP and q-mKP hierarchies in the future.

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