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Characteristic integrals in 3D and linear degeneracy

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Conservation laws vanishing along characteristic directions of a given system of PDEs are known as characteristic conservation laws, or characteristic integrals. In 2D, they play an important role in the theory of Darboux-integrable equations. In this paper we discuss characteristic integrals in 3D and demonstrate that, for a class of second order linearly degenerate dispersionless integrable PDEs, the corresponding characteristic integrals are parametrised by points on the Veronese variety.

Keywords: Characteristic integrals, principal symbol, linear degeneracy, dispersionless integrability, Veronese variety

2000 Mathematics Subject Classification: 35A30, 37K10

1. Introduction

Let Σ be a partial differential equation (PDE) in n independent variables x_1, \dots, x_n . A conservation law is an $(n-1)$ -form Ω which is closed on the solutions of Σ : $d\Omega = 0 \bmod \Sigma$. Since any $(n-1)$ -form in n variables possesses a unique annihilating direction, there exists a vector field F such that $\Omega(F) = 0$. We say that Ω is a characteristic integral (conservation law) if F is a characteristic direction of Σ .^a If a conservation law is represented in conventional form,

$$(F_1)_{x_1} + \dots + (F_n)_{x_n} = 0 \bmod \Sigma,$$

the corresponding vector field is $F = (F_1, \dots, F_n)$. The characteristic condition becomes particularly simple for scalar second order PDEs, in which case F can be interpreted as a null vector of the conformal structure defined by the principal symbol of the equation.

In two dimensions, characteristic integrals arise in the context of Darboux integrability, see e.g. [27, 31] and references therein. Having in mind a generalisation of Darboux integrability to higher dimensions (see [2, 3, 12, 13, 23] for some steps in this direction), in this paper we develop the concept of characteristic integrals in 3D. We believe it to be of interest in its own.

Let us begin with illustrating examples.

Example 1.1. Consider the 2 + 1 dimensional wave equation,

$$u_{tt} = u_{xx} + u_{yy}. \tag{1.1}$$

It possesses four first order conservation laws,

$$(u_x)_x + (u_y)_y - (u_t)_t = 0,$$

^aThe set of characteristic directions is projectively dual to the more conventional variety of characteristic covectors determined by the principal symbol of the equation.

$$(u_x^2 + u_t^2 - u_y^2)_x + (2u_x u_y)_y - (2u_x u_t)_t = 0,$$

$$(2u_y u_x)_x - (u_x^2 - u_y^2 - u_t^2)_y - (2u_y u_t)_t = 0,$$

$$(2u_t u_x)_x + (2u_t u_y)_y - (u_x^2 + u_y^2 + u_t^2)_t = 0.$$

Let us denote them

$$(f_i)_x + (g_i)_y + (h_i)_t = 0,$$

$i = 1, \dots, 4$. Taking their linear combination with constant coefficients J_1, \dots, J_4 , and adding trivial conservation laws, we obtain the expression $(F_1)_x + (F_2)_y + (F_3)_t = 0$ where

$$F_1 = J_1 f_1 - J_5 u_y + J_6 u_t + J_8, \quad F_2 = J_2 g_1 + J_5 u_x - J_7 u_t + J_9, \quad F_3 = J_3 h_1 - J_6 u_x + J_7 u_y + J_{10},$$

(summation over $i = 1, \dots, 4$ is assumed). Here the constants J_5, J_6, J_7 correspond to trivial conservation laws of the form $(u_x)_y - (u_y)_x = 0$, etc., and J_8, J_9, J_{10} are three extra arbitrary constants. Although the constants $J_5 - J_{10}$ correspond to trivial conservation laws, they effect non-trivially the characteristic condition, $F g^{-1} F^t = 0$, where g is the 3×3 symmetric matrix of the corresponding principal symbol,

$$g = \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

(in this particular example g coincides with g^{-1}). The characteristic condition imposes a system of quadratic constraints for J_1, \dots, J_{10} , which specify a Veronese threefold $V^3 \subset \mathbb{P}^9$ with parametric equations

$$J_1 = \frac{\sqrt{2}}{4}(\gamma\alpha + \gamma\beta + \delta\alpha - \delta\beta), \quad J_2 = -\alpha\beta, \quad J_3 = \frac{\alpha^2 - \beta^2}{2}, \quad J_4 = -\frac{\alpha^2 + \beta^2}{2},$$

$$J_5 = \frac{\sqrt{2}}{2}(\gamma\alpha - \gamma\beta - \delta\alpha - \delta\beta), \quad J_6 = -\frac{\sqrt{2}}{2}(\gamma\alpha + \gamma\beta - \delta\alpha + \delta\beta),$$

$$J_7 = \frac{\sqrt{2}}{2}(\gamma\alpha - \gamma\beta + \delta\alpha + \delta\beta), \quad J_8 = \frac{\delta^2 - \gamma^2}{2}, \quad J_9 = \delta\gamma, \quad J_{10} = \frac{\gamma^2 + \delta^2}{2}.$$

We use $\alpha, \beta, \gamma, \delta$ for homogeneous coordinates in \mathbb{P}^3 , and J_1, \dots, J_{10} for homogeneous coordinates in \mathbb{P}^9 . Recall that the Veronese threefold V^3 is the image of the projective imbedding of \mathbb{P}^3 into \mathbb{P}^9 defined by a complete system of quadrics. Thus, we have a whole V^3 -worth of characteristic integrals. It turns out that this example is not isolated, and similar phenomena take place for other classes of 3D linearly degenerate dispersionless integrable PDEs.

Example 1.2. Let us consider the equation

$$\mu u_t u_{xy} + \nu u_y u_{xt} + \eta u_x u_{yt} = 0, \tag{1.2}$$

$\mu + \nu + \eta = 0$, which appeared in the context of Veronese webs in 3D [29], as well as in the theory of Einstein-Weyl geometries of hyper-CR type [6]. This equation possesses four first order

conservation laws,

$$\eta(u_y u_t)_x + \nu(u_x u_t)_y + \mu(u_x u_y)_t = 0,$$

$$\nu \left(\frac{u_y}{u_t} \right)_x + \eta \left(\frac{u_x}{u_t} \right)_y = 0,$$

$$\mu \left(\frac{u_t}{u_y} \right)_x + \eta \left(\frac{u_x}{u_y} \right)_t = 0,$$

$$\mu \left(\frac{u_t}{u_x} \right)_y + \nu \left(\frac{u_y}{u_x} \right)_t = 0.$$

Let us denote them

$$(f_i)_x + (g_i)_y + (h_i)_t = 0,$$

$i = 1, \dots, 4$. Taking their linear combination with coefficients J_1, \dots, J_4 , and adding trivial conservation laws, we obtain the expression $(F_1)_x + (F_2)_y + (F_3)_t = 0$ where

$$F_1 = J_1 f_1 - J_5 u_y + J_6 u_t + J_8, \quad F_2 = J_2 g_2 + J_5 u_x - J_7 u_t + J_9, \quad F_3 = J_3 h_3 - J_6 u_x + J_7 u_y + J_{10}.$$

As in Example 1.1, the constants $J_5 - J_{10}$ correspond to trivial conservation laws. The characteristic condition takes the form $F g^{-1} F^t = 0$ where g is the 3×3 symmetric matrix of the corresponding principal symbol:

$$g = \begin{pmatrix} 0 & \mu u_t & \nu u_y \\ \mu u_t & 0 & \eta u_x \\ \nu u_y & \eta u_x & 0 \end{pmatrix}.$$

The characteristic condition imposes a system of quadratic constraints for J_1, \dots, J_{10} , which specify a Veronese threefold $V^3 \subset \mathbb{P}^9$ with parametric equations

$$J_1 = \alpha^2, \quad J_2 = \frac{1}{4\nu\eta}\beta^2, \quad J_3 = \frac{1}{4\eta\mu}\delta^2, \quad J_4 = \frac{1}{4\nu\mu}\gamma^2,$$

$$J_5 = \alpha\beta, \quad J_6 = \alpha\delta, \quad J_7 = \alpha\gamma, \quad J_8 = -\frac{1}{2\eta}\beta\delta, \quad J_9 = -\frac{1}{2\nu}\beta\gamma, \quad J_{10} = -\frac{1}{2\mu}\delta\gamma.$$

Further examples of this type are given in the Appendix.

The structure of the paper is as follows:

In Sect. 2 we briefly review the case of 2D hydrodynamic type systems, where the existence of characteristic integrals is known to imply linear degeneracy.

In Sect. 3 we consider characteristic integrals of second order quasilinear PDEs in 3D. We prove that the requirement of the existence of ‘sufficiently many’ integrals of this type implies linear degeneracy. For all linearly degenerate integrable cases, we obtain V^3 -worth of characteristic integrals.

We hope that the results of this paper may be useful for the theory of multi-dimensional Darboux-integrable equations (yet to be constructed).

2. Characteristic integrals in 2D and linear degeneracy

For definiteness we restrict the discussion to systems of hydrodynamic type,

$$u_t^i = v_j^i(\mathbf{u})u_x^j, \tag{2.1}$$

where $\mathbf{u} = (u^1, \dots, u^n)$ denotes dependent variables, and $v = v_j^i$ is an $n \times n$ matrix. Let λ^i be the eigenvalues (characteristic speeds) of v , and let ξ^i be the corresponding eigenvectors, so that $v\xi^i = \lambda^i\xi^i$. Characteristic directions are defined as $dx + \lambda^i dt = 0$, and the characteristic integral in i -th direction is a 1-form $h(\mathbf{u})(dx + \lambda^i dt)$ which is closed on solutions of (2.1). We will assume that the density h depends on \mathbf{u} only, although, in principle, nontrivial dependence on higher-order x -derivatives of \mathbf{u} may also be allowed. Recall that the i -th characteristic direction is called linearly degenerate if the Lie derivative of λ^i in the direction of the corresponding eigenvector ξ^i vanishes, $L_{\xi^i}\lambda^i = 0$. The following result is well-known:

Proposition 2.1. *If there exists a characteristic integral in the i -th direction, then the corresponding characteristic speed λ^i must be linearly degenerate.*

Proof. The closedness of $h(\mathbf{u})(dx + \lambda^i dt)$ is equivalent to $h_t = (\lambda^i h)_x$. This implies

$$(\nabla h)v = h\nabla\lambda^i + \lambda^i\nabla h,$$

where $\nabla = (\partial_{u^1}, \dots, \partial_{u^n})$ denotes the gradient. Evaluating both sides of this identity (which are 1-forms) on the vector ξ^i , and using $v\xi^i = \lambda^i\xi^i$, one can see that the left hand side cancels with the second term on the right hand side, leading to $\xi^i\nabla\lambda^i = L_{\xi^i}\lambda^i = 0$. This finishes the proof. \square

The requirement of the existence of characteristic integrals for all characteristic directions implies that all characteristic speeds must be linearly degenerate. Such systems are known as (totally) linearly degenerate, they have been thoroughly investigated in the literature, see e.g. [14, 24–26]. For linearly degenerate systems the gradient catastrophe, which is typical for genuinely nonlinear systems, does not occur, and one has global existence results for an open set of initial data.

There exist systems which possess infinitely many characteristic integrals.

Example 2.1. The 2-component linearly degenerate system,

$$v_t = wv_x, \quad w_t = vw_x,$$

possesses functionally many characteristic integrals in both characteristic directions:

$$\frac{\phi(v)}{w-v}(dx + wdt), \quad \frac{\psi(w)}{v-w}(dx + vdt),$$

here ϕ and ψ are arbitrary functions of v and w respectively.

3. Characteristic integrals of second order PDEs in 3D

In this section we consider quasilinear wave-type equations of the form

$$f_{11}u_{xx} + f_{22}u_{yy} + f_{33}u_{tt} + 2f_{12}u_{xy} + 2f_{13}u_{xt} + 2f_{23}u_{yt} = 0, \tag{3.1}$$

where $u(x, y, t)$ is a function of three independent variables, and the coefficients f_{ij} depend on the first order derivatives u_x, u_y, u_t only. Equations of this form generalise Examples 1.1 and 1.2 from

the introduction. Their integrability by the method of hydrodynamic reductions [7] was investigated in [4]. We recall that, for broad classes of dispersionless PDEs in 3D including (3.1), the following conditions are equivalent:

- The equation is integrable by the method of hydrodynamic reductions [7, 11].
- The equation possesses a dispersionless Lax pair [20, 30] or, equivalently, a Lax pair in vector fields [16, 17].
- For every solution of the equation, the symbol of formal linearisation defines an Einstein-Weyl structure [10].

Either of these conditions leads to one and the same system of differential constraints for the coefficients f_{ij} , which was first derived in [4]. It was shown in [4] that any integrable equation of the form (3.1) possesses exactly four conservation laws

$$(f_i)_x + (g_i)_y + (h_i)_t = 0,$$

$i = 1, \dots, 4$, where f_i, g_i, h_i are functions of u_x, u_y, u_t only. Taking their linear combination with constant coefficients J_1, \dots, J_4 , and adding trivial conservation laws, we obtain the expression $(F_1)_x + (F_2)_y + (F_3)_t = 0$ where

$$F_1 = J_1 f_i - J_5 u_y + J_6 u_t + J_8, \quad F_2 = J_1 g_i + J_5 u_x - J_7 u_t + J_9, \quad F_3 = J_1 h_i - J_6 u_x + J_7 u_y + J_{10}.$$

Although the constants $J_5 - J_{10}$ give trivial contribution to conservation laws, they do effect non-trivially the characteristic condition, $Fg^{-1}F^t = 0$, where $g = f_{ij}$ is the 3×3 symmetric matrix of the corresponding principal symbol. The characteristic condition imposes a system of quadratic constraints for the coefficients J_1, \dots, J_{10} which, in linearly degenerate integrable cases, specify a Veronese threefold $V^3 \subset \mathbb{P}^9$. For 3D equations of the form (3.1), the concept of linear degeneracy can be defined as follows. Looking for travelling wave solutions in the form $u(x, y, t) = u(\xi, \eta) + \zeta$ where ξ, η, ζ are arbitrary linear forms in the variables x, y, t , we obtain a second-order PDE for $u(\xi, \eta)$,

$$au_{\xi\xi} + 2bu_{\xi\eta} + cu_{\eta\eta} = 0,$$

where the coefficients a, b, c are certain functions of u_ξ and u_η . Setting $v = u_\xi$, $w = u_\eta$, one can rewrite this PDE as a two-component system of hydrodynamic type. We say that Equation (3.1) is linearly degenerate if all its travelling wave reductions are linearly degenerate in the sense of Sect. 2. The condition of linear degeneracy is equivalent to the identity (set $u_x, u_y, u_t = p_1, p_2, p_3$ and consider f_{ij} as functions of p_1, p_2, p_3):

$$f_{(ij,k)} = c_{(k} f_{ij)}, \tag{3.2}$$

here $f_{ij,k} = \partial_{p_k} f_{ij}$, c_k is a covector, and brackets denote complete symmetrisation in i, j, k . Linearly degenerate integrable PDEs were classified in [9]:

Theorem 3.1. *The following examples constitute a complete list of linearly degenerate integrable PDEs of the form (3.1):*

$$\mu u_t u_{xy} + \nu u_y u_{xt} + \eta u_x u_{yt} = 0, \quad \mu + \nu + \eta = 0,$$

$$u_{xx} + u_x u_{yt} - u_y u_{xt} = 0,$$

$$u_{xy} + u_y u_{xt} - u_x u_{yt} = 0,$$

$$u_{yy} + u_{xt} + u_y u_{tt} - u_t u_{yt} = 0,$$

$$u_{xt} + u_x u_{yy} - u_y u_{xy} = 0,$$

$$u_{tt} - u_{xx} - u_{yy} = 0.$$

In different contexts, the canonical forms of Theorem 3.1 have appeared in [1, 5, 16–22, 29]. The main result of this section is the following.

Theorem 3.2.

- (i) *If a 3D quasilinear PDE of the form (3.1) possesses ‘sufficiently many’ characteristic integrals, then it must be linearly degenerate. Here ‘sufficiently many’ means that the corresponding vector F satisfies no extra algebraic constraints other than the characteristic condition itself, $F g^{-1} F^t = 0$.*
- (ii) *Any linearly degenerate integrable PDE (3.1) possesses V^3 -worth of characteristic integrals.*

Proof. To demonstrate (i) we recall the result of [4] according to which the functions F_i defining a conservation law must satisfy the identity $F_{(i,j)} = s f_{ij}$, where $F_{i,j} = \partial_{p_j} F_i$, brackets denote symmetrisation in i, j , and s is a coefficient of proportionality (all entries are viewed as functions of p 's). The characteristic constraint takes the form

$$(f^{-1})^{ij} F_i F_j = 0,$$

which can be rewritten as $f_{ij} F^i F^j = 0$ where we use the notation $F_i = f_{ij} F^j$. Differentiating the characteristic condition by p_k we obtain

$$-(f^{-1})^{ip} f_{pq,k} (f^{-1})^{qj} F_i F_j + 2(f^{-1})^{ij} F_{i,k} F_j = 0,$$

which can be rewritten as

$$f_{pq,k} F^p F^q = 2F_{i,k} F^i.$$

Contracting this identity with F^k , using the condition $F_{(i,j)} = s f_{ij}$ and the characteristic constraint $f_{ij} F^i F^j = 0$ we obtain the additional algebraic condition

$$f_{ij,k} F^i F^j F^k = 0. \tag{3.3}$$

The requirement that this condition is satisfied identically modulo the characteristic constraint, $f_{ij} F^i F^j = 0$, is equivalent to saying that the cubic (3.3) is divisible by the quadric $f_{ij} F^i F^j = 0$,

$$f_{ij,k} F^i F^j F^k = (c_i F^i) (f_{ij} F^i F^j),$$

for some linear form $c_i F^i$. Symmetrisation of this identity implies the condition of linear degeneracy (3.2).

Finally, the proof of (ii) is a case-by-case calculation. Details are included in the Appendix. This finishes the proof of Theorem 3.2. \square

Note that linearly non-degenerate or non-integrable equations may also possess characteristic integrals, although not ‘as many’ as linearly degenerate integrable ones.

Example 3.1. The following integrable (linearly non-degenerate) equation,

$$u_t u_{xy} + u_y u_{xt} + u_x u_{yt} = 0,$$

admits only three characteristic conservation laws:

$$(u_x^2 u_t)_y + (u_x^2 u_y)_t = 0,$$

$$(u_y^2 u_t)_x + (u_y^2 u_x)_t = 0,$$

$$(u_t^2 u_y)_x + (u_t^2 u_x)_y = 0.$$

4. Concluding remarks

We have demonstrated that linearly degenerate second-order quasilinear integrable PDEs in 3D possess characteristic integrals parametrised by points on the Veronese variety V^3 . Our calculations suggest that similar phenomena take place for other classes of linearly degenerate dispersionless integrable systems: all of them possess nontrivial characteristic integrals, what may change is the dimension of the corresponding Veronese variety.

Note that to be Darboux-integrable, the equation is required to possess sufficiently many higher-order characteristic integrals. The existence of such integrals in multi-dimensions is still an open problem.

5. Appendix: characteristic integrals for equations from Theorem 3.1

Here we present characteristic conservation laws for all examples from Theorem 3.1. For each of the canonical forms, we present four nontrivial conservation laws (which are found by direct calculation, recall that their existence follows from the general result of [4]), and parametric expressions for the corresponding constants J_1, \dots, J_{10} as defined in Sect. 3. In each case these parametric equations are readily seen to specify a Veronese threefold.

Equation 1 (discussed in the Introduction):

$$\mu u_t u_{xy} + \nu u_y u_{xt} + \eta u_x u_{yt} = 0.$$

Four conservation laws:

$$\eta (u_y u_t)_1 + \nu (u_x u_t)_y + \mu (u_x u_y)_t = 0,$$

$$\nu \left(\frac{u_y}{u_t} \right)_1 + \eta \left(\frac{u_x}{u_t} \right)_y = 0,$$

$$\mu \left(\frac{u_t}{u_y} \right)_x + \eta \left(\frac{u_x}{u_y} \right)_t = 0,$$

$$\mu \left(\frac{u_y}{u_x} \right)_y + \nu \left(\frac{u_y}{u_x} \right)_t = 0.$$

Coefficients of characteristic integrals:

$$J_1 = \alpha^2, \quad J_2 = \frac{1}{4\nu\eta}\beta^2, \quad J_3 = \frac{1}{4\eta\mu}\delta^2, \quad J_4 = \frac{1}{4\nu\mu}\gamma^2,$$

$$J_5 = \alpha\beta, \quad J_6 = \alpha\delta, \quad J_7 = \alpha\gamma, \quad J_8 = -\frac{1}{2\eta}\beta\delta, \quad J_9 = -\frac{1}{2\nu}\beta\gamma, \quad J_{10} = -\frac{1}{2\mu}\delta\gamma.$$

Equation 2.

$$u_{xx} + u_x u_{yt} - u_y u_{xt} = 0.$$

Four conservation laws:

$$\left(\frac{u_y}{2u_x^2} \right)_x + \left(\frac{1}{2u_x} \right)_y - \left(\frac{u_y^2}{2u_x^2} \right)_t = 0,$$

$$(u_x - u_y u_t)_x + (u_x u_t)_y = 0,$$

$$(2u_x u_t - u_y u_t^2)_x + (u_t^2 u_x)_y - (u_x^2)_t = 0,$$

$$-\left(\frac{1}{u_x} \right)_x + \left(\frac{u_y}{u_x} \right)_t = 0.$$

Coefficients of characteristic integrals:

$$J_1 = \alpha^2, \quad J_2 = -\delta\beta, \quad J_3 = \frac{1}{2}\beta^2, \quad J_4 = \alpha\gamma,$$

$$J_5 = \frac{1}{2}\delta^2, \quad J_6 = \beta\gamma, \quad J_7 = \alpha\beta, \quad J_8 = -\alpha\beta - \gamma\delta, \quad J_9 = \alpha\delta, \quad J_{10} = -\frac{1}{2}\gamma^2.$$

Equation 3.

$$u_{xy} + u_y u_{xt} - u_x u_{yt} = 0.$$

Four conservation laws:

$$(u_y u_t)_x + (u_x - u_x u_t)_y = 0,$$

$$(u_y u_t^2)_x + (2u_x u_t - u_x u_t^2 - u_x)_y - (u_x u_y)_t = 0,$$

$$\begin{aligned} \left(\frac{1}{u_y}\right)_x - \left(\frac{u_x}{u_y}\right)_t &= 0, \\ -\left(\frac{1}{u_x}\right)_y - \left(\frac{u_y}{u_x}\right)_t &= 0. \end{aligned}$$

Coefficients of characteristic integrals:

$$\begin{aligned} J_1 = \alpha\beta, \quad J_2 = \alpha^2, \quad J_3 = \frac{1}{4}\delta^2, \quad J_4 = \frac{1}{4}\gamma^2, \\ J_5 = -\frac{1}{4}\beta^2, \quad J_6 = -\alpha\delta, \quad J_7 = \alpha\gamma, \quad J_8 = -\frac{1}{2}\beta\delta, \quad J_9 = \alpha\gamma - \frac{1}{2}\beta\gamma, \quad J_{10} = -\frac{1}{2}\delta\gamma. \end{aligned}$$

Equation 4.

$$u_{yy} + u_{xt} + u_y u_{tt} - u_t u_{yt} = 0.$$

Four conservation laws:

$$\begin{aligned} (u_y - u_t^2)_y + (u_x + u_y u_t)_t &= 0, \\ \left(\frac{1}{2}u_t^2\right)_x + (u_y u_t - \frac{1}{2}u_t^3)_y + \left(-u_y u_t^2 - \frac{1}{2}u_y^2 + \frac{3}{2}u_t^2 u_y\right)_t &= 0, \\ (u_y u_t - u_t^3)_x + (-u_x u_t + u_y^2 - 3u_y u_t^2 + u_t^4)_y + (u_x u_y + 2u_t u_y^2 - u_y u_t^3)_t &= 0, \\ (u_y^2 - 2u_y u_t^2 + u_t^4)_x + (-2u_x u_y + 2u_x u_t^2 - 3u_y^2 u_t + 4u_y u_t^3 - u_t^5)_y + (-2u_x u_y u_t - u_x^2 - 3u_y^2 u_t^2 + u_y u_t^4 + u_y^3)_t &= 0. \end{aligned}$$

Coefficients of characteristic integrals:

$$\begin{aligned} J_1 = -\alpha\delta - \frac{1}{2}\beta\gamma, \quad J_2 = 2\alpha\gamma + \frac{1}{2}\delta^2, \quad J_3 = \frac{1}{2}\delta\gamma, \quad J_4 = \frac{1}{4}\gamma^2, \\ J_5 = \alpha\gamma, \quad J_6 = -\alpha\delta, \quad J_7 = \alpha^2 + \frac{1}{2}\beta\delta, \quad J_8 = \alpha^2, \quad J_9 = \alpha\beta, \quad J_{10} = -\frac{1}{4}\beta^2. \end{aligned}$$

Equation 5.

$$u_{xt} + u_x u_{yy} - u_y u_{xy} = 0.$$

Four conservation laws:

$$\begin{aligned} \left(\frac{u_y}{u_x}\right)_y - \left(\frac{1}{u_x}\right)_t &= 0, \\ (-u_y^2)_x + (u_x u_y)_y + (u_x)_t &= 0, \\ (-u_t^2 + 2u_t u_y^2 - u_y^4)_x + (-2u_x u_y u_t + u_x u_y^3)_y + (u_x u_y^2)_t &= 0, \end{aligned}$$

$$(u_t u_y - u_y^3)_x + (-u_x u_t + u_x u_y^2)_y + (u_x u_y)_t = 0.$$

Coefficients of characteristic integrals:

$$J_1 = \alpha^2, \quad J_2 = -\frac{1}{2}\beta\gamma - \frac{1}{4}\delta^2, \quad J_3 = -\frac{1}{4}\beta^2, \quad J_4 = -\frac{1}{2}\beta\delta,$$

$$J_5 = -\frac{1}{2}\gamma\delta, \quad J_6 = -\frac{1}{2}\beta\gamma, \quad J_7 = \beta\alpha, \quad J_8 = \frac{1}{4}\gamma^2, \quad J_9 = \gamma\alpha, \quad J_{10} = \delta\alpha.$$

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