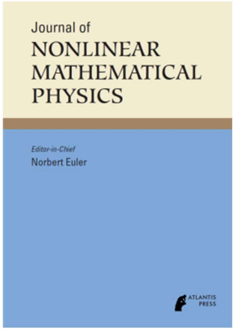

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On characteristic integrals of Toda field theories

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Characteristic integrals of Toda field theories associated to general simple Lie algebras are constructed using systematic techniques, and complete mathematical proofs are provided. Plenty of examples illustrating the results are presented in explicit forms.

1. Introduction

First consider the famous Liouville equation, for which we take the following version:

$$u_{xy} = -e^{2u}, \tag{1.1}$$

where x and y are the independent variables, and $u = u(x,y)$ is an unknown function. Let

$$I = u_{xx} - u_x^2. \tag{1.2}$$

Then it is easily checked that $I_y = \frac{\partial}{\partial y} I = 0$ for a solution $u(x,y)$ to (1.1). I is thus called a *characteristic integral* of (1.1).

Toda field theories are generalizations of the Liouville equation (1.1). Let $\mathfrak{g}$ be a simple Lie algebra of rank n with Cartan matrix $A = (a_{ij})_{i,j=1}^n$. For $1 \le i \le n$, let $u^i = u^i(x,y)$ be n unknown functions of the independent variables x and y. The Toda field theory associated to $\mathfrak{g}$ is the system of nonlinear PDEs:

$$u^i_{xy} = -e^{\rho_i} := -\exp\left(\sum_{j=1}^n a_{ij} u^j\right), \quad 1 \le i \le n. \tag{1.3}$$

The Liouville equation (1.1) is the Toda field theory associated to $A_1 = \mathfrak{sl}_2$.

Toda field theories are fundamental integrable systems with rich properties and important applications in mathematics and physics. They have been extensively studied, and we refer to the two books [2, 16] for surveys on them. In this paper, we are concerned with the explicit forms of their characteristic integrals.

Definition 1.1. A differential polynomial in the $u^i(x,y)$ for $1 \le i \le n$ is a polynomial in the u^i and their partial derivatives with respect to x of various orders $u^i_x, u^i_{xx}, \cdots$.

A *characteristic integral* of the Toda field theory (1.3) is a differential polynomial I in the $u^i(x,y)$ for $1 \le i \le n$ such that $I_y = 0$ for the solutions $u^i(x,y)$ to (1.3).

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By the symmetry between x and y in the Toda field theory (1.3), there are differential polynomials $\tilde{I}$ in the u^i and their y-partial derivatives such that $\tilde{I}_x = 0$ but they clearly follow the same patterns as those in the above definition.

Since a differential polynomial in characteristic integrals is another such integral, the characteristic integrals form a differential algebra. Shabat and Yamilov [19] showed that the characteristic integrals of the Toda field theory (1.3) form a *polynomial* differential algebra generated by *n primitive characteristic integrals*, and that the system (1.3) has such a complete set of characteristic integrals if and only if the matrix $A = (a_{ij})$ is equivalent to the Cartan matrix of a simple Lie algebra.

For a differential mononomial in the u^i, we call by its *degree* the sum of the orders of differentiation multiplied by the algebraic degrees of the corresponding factors. Therefore the I in (1.2) has a homogeneous degree 2. The primitive characteristic integrals can be chosen to be homogeneous and their degrees are equal to the degrees of the Lie algebra $\mathfrak{g}$. For a proof of this structure theorem, see [18] or [8, Theorem 2.4.10 and Proposition 2.4.7]. Recall that the algebra of adjoint-invariant functions on $\mathfrak{g}$ is a polynomial algebra on n homogenous generators, whose degrees we call the degrees of $\mathfrak{g}$ [12]. This paper is concerned with constructing these primitive characteristic integrals.

Many works [3, 5, 7, 16] have been devoted to the characteristic integrals, very often under different names such as *local conservation laws, chiral currents, intermediate integrals* or *W-algebras* and from different viewpoints. In particular, the works of Guryeva and Zhiber [10, 11] have shown that the Toda field theories (1.3) for classical and exceptional Lie algebras are of Liouville type, and have obtained explicit formulas for their generalized Laplace invariants, which can be used to construct the characteristic integrals and higher symmetries.

However to this author, the construction results about characteristic integrals for the Toda field theories (1.3) are not explicit enough, in terms of both the formulas and the proofs. Therefore we would like to present here systematic and concrete techniques for constructing the primitive characteristic integrals together with new and self-contained proofs.

Furthermore, many works [14, 16, 21] are concerned with using the characteristic integrals to obtain explicit solutions to the original Toda field theories (1.3) by the method of Darboux. Recently, Anderson, Fels and Vassiliou [1] developed a very general approach to the study of Darboux integrable systems.

By demonstrating enough characteristic integrals, this paper explicitly shows that the Toda field theories (1.3) are Darboux integrable. In sequels to this work, Anderson and the author will apply the characteristic integrals found in this paper and the structure theory in [1] to realize the Toda field theory (1.3) as the quotient of two standard differential systems [20] depending on x and y separately. We will also apply the method and result of this paper to the setting of differential invariants for parabolic geometry [4] with the standard differential system, generalizing works done by Mari Beffa [17].

This paper is organized as follows. In Section 2, we present our main results. Our main theorem is Theorem 2.1, which is applicable to all simple Lie algebras. It employs the zero curvature representation [15] of the Toda field theories (1.3) under a Drinfeld-Sokolov gauge [6]. When the Lie algebra $\mathfrak{g}$ has a non-branching representation (see (2.9)), which is the case for Lie algebras of types A, B, C and $\mathfrak{g}_2$, we present a more concrete formula in Theorem 2.2, which is easier to use. We stress that in both cases, we provide complete and novel mathematical proofs.

In Sections 3, we demonstrate the more direct method in Theorem 2.2 in all the applicable cases. In Section 4, we illustrate the general method in Theorem 2.1 by the example of the characteristic integral related to the Pfaffian of D_4. We stress that our constructions are easily implemented using mathematical softwares such as Maple, which this author has done. Maple programs and formulas for the characteristic integrals will be made available at the DifferentialGeometry Software Project website at the Digital Commons of the Utah State Univeristy (http://digitalcommons.usu.edu/dg/). Therefore our results are readily usable for various purposes.

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2. Main results

Let us first introduce some terminology. Let $\mathfrak{h} \subset \mathfrak{g}$ be a Cartan subalgebra, and we denote the corresponding set of roots of $\mathfrak{g}$ by Δ, the sets of positive/negative roots by $\Delta_\pm$, and the set of positive simple roots by $\pi = \{\alpha_i\}_{i=1}^n$. Let $\mathfrak{g} = \mathfrak{h} \oplus \bigoplus_{\alpha\in\Delta} \mathfrak{g}_\alpha$ be the root space decomposition.

For a root α, define its height by $\mathrm{ht}(\alpha) = \sum_{i=1}^n c_i$ if $\alpha = \sum_{i=1}^n c_i\alpha_i$. Also define the principal height gradation

$$\mathfrak{g} = \bigoplus_{k=-p}^{p} \mathfrak{g}_k, \quad \mathfrak{g}_k = \bigoplus_{\mathrm{ht}(\alpha)=k} \mathfrak{g}_\alpha,\ \mathfrak{g}_0 = \mathfrak{h}, \tag{2.1}$$

where p is the maximal height of the roots. We also denote by $\mathfrak{n} = \bigoplus_{\alpha\in\Delta_+} \mathfrak{g}_\alpha = \bigoplus_{k>0} \mathfrak{g}_k$ the maximal nilpotent subalgebra, and by N the corresponding unipotent group.

For $\alpha \in \Delta_+$, let e_α and $e_{-\alpha}$ be root vectors in the root spaces $\mathfrak{g}_\alpha$ and $\mathfrak{g}_{-\alpha}$ such that for $H_\alpha = [e_\alpha, e_{-\alpha}] \in \mathfrak{h}$, we have $\alpha(H_\alpha) = 2$. Then the Cartan matrix $A = (a_{ij})_{i,j=1}^n$ of $\mathfrak{g}$ is defined by $a_{ij} = \alpha_i(H_{\alpha_j})$.

Let us recall the zero curvature representation of (1.3) following [15]. Let

$$\mathbf{u} = \sum_{i=1}^n u_x^i H_{\alpha_i}, \quad \varepsilon = \sum_{i=1}^n e_{-\alpha_i}, \quad Y = \sum_{i=1}^n e^{\rho_i} e_{\alpha_i}, \tag{2.2}$$

where as in (1.3) $\rho_i = \sum_{j=1}^n a_{ij}u^j$. Then the Toda field theory (1.3) is equivalent to the following zero curvature equation

$$[-\partial_x + \varepsilon + \mathbf{u}, \partial_y + Y] = 0. \tag{2.3}$$

Now let us recall the definition of a Kostant slice $\mathfrak{s} \subset \mathfrak{g}$ [13], which is used in a Drinfeld-Sokolov gauge [6]. Let $\mathfrak{s}$ be a complement of $[\varepsilon, \mathfrak{g}]$ in $\mathfrak{g}$, that is,

$$\mathfrak{g} \cong \mathfrak{s} \oplus [\varepsilon, \mathfrak{g}]. \tag{2.4}$$

Then by [13], $\mathfrak{s} \subset \mathfrak{n}$, and $\dim(\mathfrak{s}) = n$ is equal to the rank. We call $\mathfrak{s}$ a Kostant slice, and let $\{s_j\}_{j=1}^n$ be a homogeneous basis of $\mathfrak{s}$ with respect to the height gradation (2.1).

By [6], we can bring the first element in (2.3) into its Drinfeld-Sokolov gauge. More precisely, there exists an element $g \in N$ (whose components are differential polynomials of the u^i) such that

$$g(-\partial_x + \varepsilon + \mathbf{u})g^{-1} = -\partial_x + \varepsilon + \mathbf{I}, \quad \mathbf{I} = \sum_{j=1}^{n} I_j s_j \in \mathfrak{s}, \tag{2.5}$$

where the components I_j are differential polynomials of the u^i.

Theorem 2.1. *For the solutions u^i to* (1.3)*, the differential polynomials I_j for $1 \le j \le n$ defined in* (2.5) *through a Drinfeld-Sokolov gauge are primitive characteristic integrals, that is, $\partial_y I_j = 0$ and the I_j are the generators for the differential polynomial algebra of characteristic integrals.*

Proof. Suppose that for the $g \in N$ in (2.5), we have

$$g(\partial_y + Y)g^{-1} = \partial_y + \tilde{Y}.$$

Since $g \in N$ and $Y \in \mathfrak{g}_1$ by (2.2), we have $\tilde{Y} \in \mathfrak{n}$. Suppose $\tilde{Y} = \sum_{i=1}^{p} Y_i$ by the height decomposition (2.1).

By the invariance of the zero-curvature equation (2.3) under the adjoint action by g, from (2.5) we get

$$\left[-\partial_x + \varepsilon + \sum_{j=1}^{n} I_j s_j, \partial_y + \sum_{i=1}^{p} Y_i \right] = 0. \tag{2.6}$$

We will prove by induction that all the $Y_i = 0$. Clearly this then implies that $\partial_y I_j = 0$ for $1 \le j \le n$.

First recall the basic result of Kostant [12] that $\ker(\mathrm{ad}_\varepsilon) \cap (\mathfrak{h} \oplus \mathfrak{n}) = 0$. The term on the left of (2.6) with height zero is $[\varepsilon, Y_1] = 0$, which then implies that $Y_1 = 0$.

Now assume $i \ge 2$ and that $Y_j = 0$ for $j \le i-1$. Then since the $s_j \in \mathfrak{n}$, the term on the left of (2.6) with height $i-1$ is

$$[\varepsilon, Y_i] - \sum_{\mathrm{ht}(s_j)=i-1} (\partial_y I_j) s_j = 0.$$

By the decomposition (2.4), we get $[\varepsilon, Y_i] = 0$ which then implies that $Y_i = 0$.

These I_j in (2.5) are the generators for the characteristic integrals since without the differentiation operator $-\partial_x$, we obtain the generators for the algebraic invariant functions of $\varepsilon + \mathbf{u}$ by [13]. See also [3, Section V]. □

Remark 2.1. The calculation of $g \in N$ and the I_j in (2.5) can be done inductively using the decomposition (2.4) as in [6,7]. For the reader's convenience, we provide some details. Then formula (2.5) is

$$\partial_x g \cdot g^{-1} + g(\varepsilon + \mathbf{u})g^{-1} = \varepsilon + \mathbf{I}.$$

Consider

$$g = e^{a_1} \cdots e^{a_m}, \quad a_i \in \mathfrak{g}_i,\ i = 1, \ldots, m. \tag{2.7}$$

Inductively for $i \ge 1$, assume that the a_j for $1 \le j \le i-1$ have been determined. Consider $g_{i-1} = e^{a_1} \cdots e^{a_{i-1}}$. Then the component of $\partial_x g_{i-1} \cdot g_{i-1}^{-1} + g_{i-1}(\varepsilon + \mathbf{u})g_{i-1}^{-1}$ with grade $i-1$ can be uniquely written as $[\varepsilon, a_i] + J_{i-1}$ with $J_{i-1} \in \mathfrak{s} \cap \mathfrak{g}_{i-1}$ by (2.4). Then using exactly this a_i in (2.7) will do the job of the Drinfeld-Sokolov gauging at grade $i-1$. The algorithm is easily implemented on a mathematical software such as Maple, as long as we know the structure equations of the Lie algebra. This author has worked out the characteristic integrals of the E_6 Toda field theory by this method. In Section 4, we will see the example of D_4.

Furthermore very often there are more direct formulas for the characteristic integrals as we explain now.

Take an irreducible representation $\phi : \mathfrak{g} \to \mathrm{End}\, V$. Let the $\beta_k \in \mathfrak{h}^*$ for $1 \le k \le m$ be the weights of ϕ, and

$$V = \bigoplus_{k=1}^{m} V_{\beta_k}, \tag{2.8}$$

the weight space decomposition. We assume that $\dim V_{\beta_k} = 1$ and also that the representation ϕ does not branch. That is, for each weight β_k there is one unique negative simple root $-\alpha_{i_k}$ such that $\beta_k - \alpha_{i_k}$ is another weight of ϕ. Order our weights such that $\beta_{k+1} = \beta_k - \alpha_{i_k}$ for $1 \le k \le m-1$ and we draw the following weight diagram

$$\beta_m \overset{-\alpha_{i_{m-1}}}{\longleftarrow} \cdots \overset{-\alpha_{i_2}}{\longleftarrow} \beta_2 \overset{-\alpha_{i_1}}{\longleftarrow} \beta_1. \tag{2.9}$$

Such non-branching representations occur for the first fundamental representations of the Lie algebras A_n, B_n, C_n and $\mathfrak{g}_2$.

Theorem 2.2. *If a non-branching representation ϕ as above exists with the weight diagram* (2.9)*, then we have*

$$[(\partial_x - \beta_1(\mathbf{u}))(\partial_x - \beta_2(\mathbf{u}))\cdots(\partial_x - \beta_m(\mathbf{u})), \partial_y] = 0, \tag{2.10}$$

for a $\mathbf{u}$ (2.2) *satisfying* (2.3) *or equivalently the Toda filed theory* (1.3)*. The product is in the sense of composition for operators on functions of x and y, and we strictly follow the order. If by the Leibniz rule, we expand*

$$(\partial_x - \beta_1(\mathbf{u}))(\partial_x - \beta_2(\mathbf{u}))\cdots(\partial_x - \beta_m(\mathbf{u})) = \partial_x^m + \sum_{j=1}^{m} I_j \partial_x^{m-j}, \tag{2.11}$$

then (2.10) *implies that*

$$\partial_y I_j = 0, \quad 1 \le j \le m. \tag{2.12}$$

Furthermore, if the d_j for $1 \le j \le n$ are the degrees of the simple Lie algebra $\mathfrak{g}$, then the I_{d_j} for $1 \le j \le n$ are the generators of the differential polynomial algebra of characteristic integrals.

Proof. In the weight decomposition (2.8), we let $v_{\beta_1} \in V_{\beta_1}$ be a weight vector for the highest weight, and inductively we define the other weight vectors by $v_{\beta_k} = \phi(e_{-\alpha_{i_{k-1}}})v_{\beta_{k-1}}$ for $2 \le k \le m$ by (2.9). Therefore by (2.2)

$$\phi(\varepsilon)v_{\beta_{k-1}} = v_{\beta_k}, \quad 2 \le k \le m. \tag{2.13}$$

The zero curvature equation (2.3) is the compatibility condition for the following system of equations. Let $\psi(x,y) = \sum_{k=1}^{m} \psi_k(x,y)v_{\beta_k}$ be a function of x and y with values in V. Then (2.3) implies that the following system of equations has solutions

$$\begin{cases} (-\partial_x + \phi(\mathbf{u}+\varepsilon))\psi = 0 \\ (\partial_y + \phi(Y))\psi = 0. \end{cases} \tag{2.14}$$

Then by (2.13) and $\mathbf{u} \in \mathfrak{h}$ (2.2), the first equation, at the weight vector v_{β_k}, means that

$$(\partial_x - \beta_k(\mathbf{u}))\psi_k = \psi_{k-1}, \quad 2 \le k \le m. \tag{2.15}$$

When $k = 1$, we actually have

$$(\partial_x - \beta_1(\mathbf{u}))\psi_1 = 0,$$

since β_1 is the highest weight. Therefore combining them, we have

$$(\partial_x - \beta_1(\mathbf{u}))(\partial_x - \beta_2(\mathbf{u}))\cdots(\partial_x - \beta_m(\mathbf{u}))\psi_m = 0.$$

On the other hand, the second equation in (2.14), at the lowest weight vector v_{β_m}, quickly gives that

$$\partial_y \psi_m = 0,$$

since $Y \in \mathfrak{n}$ (2.2). Therefore the above two equations have a solution $\psi_m = \psi_m(x, y)$, which is general. The implied compatibility condition is exactly (2.10), which quickly gives (2.12).

In the expansion (2.11), if we replace the differential operator ∂_x by an algebraic indeterminant λ then the corresponding I_{d_j}, corresponding to the degrees d_j of $\mathfrak{g}$ for $1 \le j \le n$, are well-known to be the generators for the algebraic invariant functions of $\mathbf{u}$. Therefore the current I_{d_j} in (2.11) with the differential operator are independent and are the generators for the characteristic integrals. □

3. Examples for Theorem 2.2

The first fundamental representations of the Lie algebras A_n, B_n, C_n and $\mathfrak{g}_2$ are non-branching representations (2.9). Therefore Theorem 2.2 applies to these situations and we obtain their primitive characteristic integrals as corollaries.

Keeping our spirit of being as explicit as possible, we present the formulas in these cases. In this section we follow [9] for notation and choices of root vectors. The Cartan subalgebras $\mathfrak{h}$ always consist of diagonal matrices. We let $L_i \in \mathfrak{h}^*$ denote the linear function of taking the ith element on the diagonal. We also let E_{ij} denote the matrix with a 1 at the (i, j)-position and zero everywhere else. For simplicity, we write ∂ for ∂_x.

Example 3.1 (A_n). The simple roots for A_n are $\alpha_i = L_i - L_{i+1}$ and the $H_{\alpha_i} = E_{i,i} - E_{i+1,i+1}$, $1 \le i \le n$. The Cartan matrix is

$$\begin{pmatrix} 2 & -1 & & & \\ -1 & 2 & -1 & & \\ & \ddots & \ddots & \ddots & \\ & & -1 & 2 & -1 \\ & & & -1 & 2 \end{pmatrix}$$

The degrees of A_n (that is, the degrees of primitive adjoint-invariant functions) are $2, 3, \cdots, n+1$. The weight diagram for the first fundamental representation is

$$L_{n+1} \xleftarrow{-\alpha_n} \cdots \xleftarrow{-\alpha_2} L_2 \xleftarrow{-\alpha_1} L_1.$$

The $\mathbf{u}$ in (2.2) is

$$\mathbf{u} = \mathrm{Diag}(u_x^1, u_x^2 - u_x^1, \cdots, u_x^n - u_x^{n-1}, -u_x^n)$$

By Theorem 2.2, consider the expansion

$$\begin{aligned}&(\partial - u_x^1)(\partial + u_x^1 - u_x^2)\cdots(\partial + u_x^{n-1} - u_x^n)(\partial + u_x^n)\\ &= \partial^{n+1} + \sum_{j=1}^{n} I_j \partial^{n-j}.\end{aligned} \tag{3.1}$$

Then the I_j for $1 \le j \le n$ are primitive characteristic integrals of the A_n Toda field theory (1.3).

Example 3.2 ($C_n, n \ge 2$). The simple roots for C_n are $\alpha_i = L_i - L_{i+1}$ for $1 \le i \le n-1$ and $\alpha_n = 2L_n$. Also $H_{\alpha_i} = E_{i,i} - E_{i+1,i+1} - E_{n+i,n+i} + E_{n+i+1,n+i+1}$ for $1 \le i \le n-1$ and $H_{\alpha_n} = E_{n,n} - E_{2n,2n}$. The Cartan matrix is

$$\begin{pmatrix} 2 & -1 & & & \\ -1 & 2 & -1 & & \\ & \ddots & \ddots & \ddots & \\ & & -1 & 2 & -1 \\ & & & -2 & 2 \end{pmatrix}$$

The degrees of C_n are $2, 4, \cdots, 2n$. The weight diagram for the first fundamental representation is

$$-L_1 \xleftarrow{-\alpha_1} \cdots \xleftarrow{-\alpha_{n-1}} -L_n \xleftarrow{-\alpha_n} L_n \xleftarrow{-\alpha_{n-1}} \cdots \xleftarrow{-\alpha_2} L_2 \xleftarrow{-\alpha_1} L_1.$$

The $\mathbf{u}$ in (2.2) is

$$\mathbf{u} = \mathrm{Diag}(u_x^1, u_x^2 - u_x^1, \cdots, u_x^n - u_x^{n-1}, -u_x^1, -u_x^2 + u_x^1, \cdots, -u_x^n + u_x^{n-1}).$$

By Theorem 2.2, consider the expansion

$$\begin{aligned}&(\partial - u_x^1)(\partial + u_x^1 - u_x^2)\cdots(\partial + u_x^{n-1} - u_x^n)\\ &\quad(\partial + u_x^n - u_x^{n-1})\cdots(\partial + u_x^2 - u_x^1)(\partial + u_x^1)\\ &= \partial^{2n} + \sum_{j=1}^{n} I_j \partial^{2n-2j} + \sum_{j=1}^{n-1} J_j \partial^{2n-2j-1}.\end{aligned} \tag{3.2}$$

Then the I_j for $1 \le j \le n$ are primitive characteristic integrals of the C_n Toda field theory, and the J_j are some differential polynomials in them.

Example 3.3 ($B_n, n \ge 2$). The simple roots for B_n are $\alpha_i = L_i - L_{i+1}$ for $1 \le i \le n-1$ and $\alpha_n = L_n$. Also $H_{\alpha_i} = E_{i,i} - E_{i+1,i+1} - E_{n+i,n+i} + E_{n+i+1,n+i+1}$ for $1 \le i \le n-1$ and $H_{\alpha_n} = 2E_{n,n} - 2E_{2n,2n}$. The

Cartan matrix is

$$\begin{pmatrix} 2 & -1 & & & \\ -1 & 2 & -1 & & \\ & \ddots & \ddots & \ddots & \\ & & -1 & 2 & -2 \\ & & & -1 & 2 \end{pmatrix}$$

The degrees of B_n are $2, 4, \cdots, 2n$. The weight diagram for the first fundamental representation is

$$-L_1 \xleftarrow{-\alpha_1} \cdots \xleftarrow{-\alpha_{n-1}} -L_n \xleftarrow{-\alpha_n} 0 \xleftarrow{-\alpha_n} L_n \xleftarrow{-\alpha_{n-1}} \cdots \xleftarrow{-\alpha_2} L_2 \xleftarrow{-\alpha_1} L_1. \tag{3.3}$$

The $\mathbf{u}$ in (2.2) is

$$\mathbf{u} = \mathrm{Diag}(u_x^1, u_x^2 - u_x^1, \cdots, 2u_x^n - u_x^{n-1}, -u_x^1, -u_x^2 + u_x^1, \cdots, -2u_x^n + u_x^{n-1}, 0).$$

By Theorem 2.2, consider the expansion

$$\begin{aligned} &(\partial - u_x^1)(\partial + u_x^1 - u_x^2) \cdots (\partial + u_x^{n-1} - 2u_x^n) \\ &\quad \partial(\partial + 2u_x^n - u_x^{n-1}) \cdots (\partial + u_x^2 - u_x^1)(\partial + u_x^1) \\ &= \partial^{2n+1} + \sum_{j=1}^{n} I_j \partial^{2n-2j+1} + \sum_{j=1}^{n} J_j \partial^{2n-2j}. \end{aligned} \tag{3.4}$$

Then the I_j for $1 \le j \le n$ are primitive characteristic integrals of the B_n Toda field theory, and the J_j are some differential polynomials in them.

Example 3.4 ($\mathfrak{g}_2$). The Cartan matrix is

$$\begin{pmatrix} 2 & -1 \\ -3 & 2 \end{pmatrix}$$

For simplicity we call the unknown functions u^1, u^2 by u and v. The Toda system is

$$\begin{cases} u_{xy} = -e^{2u-v} \\ v_{xy} = -e^{-3u+2v} \end{cases}$$

Although we can work abstractly with just the above Cartan matrix, we choose to use the embedding of $\mathfrak{g}_2 \subset \mathfrak{so}_7 = B_3$ such that the two roots are

$$\alpha_1 = L_1 - L_2, \quad \alpha_2 := L_2 - L_3,$$

and

$$\begin{aligned} H_{\alpha_1} &= \mathrm{Diag}(1, -1, 2, -1, 1, -2, 0) \\ H_{\alpha_2} &= \mathrm{Diag}(0, 1, -1, 0, -1, 1, 0). \end{aligned}$$

The $\mathbf{u}$ in (2.2) is

$$\mathbf{u} = \mathrm{Diag}(u_x, -u_x + v_x, 2u_x - v_x, -u_x, u_x - v_x, -2u_x + v_x, 0).$$

The first fundamental representation of $\mathfrak{g}_2$ is the restriction of that of B_3. Therefore we follow the weight diagram (3.3). The degrees of $\mathfrak{g}_2$ are 2 and 6.

By Theorem 2.2, consider the expansion

$$\begin{aligned}(\partial-u_x)(\partial+u_x-v_x)(\partial-2u_x+v_x)\\ \partial(\partial+2u_x-v_x)(\partial-u_x+v_x)(\partial+u_x)\\ =\partial^7+I_1\partial^5+I_2\partial+\sum_{j=1}^{3}J_j\partial^{5-j}+J_4.\end{aligned} \tag{3.5}$$

Then the I_1 and I_2 are primitive characteristic integrals of the $\mathfrak{g}_2$ Toda field theory, and the J_j are some differential polynomials in them.

4. An example of Theorem 2.1

It can be said that Theorem 2.2 is a quick way to compute a Drinfeld-Sokolov gauge (2.5) when there is a non-branching representation, and this is the viewpoint in [3]. When such a simple representation does not exist, a Drinfeld-Sokolov gauge can still be computed through an inductive procedure by Remark 2.1 so as to yield the characteristic integrals by Theorem 2.1. We stress that the Drinfeld-Sokolov gauging can be achieved intrinsically using just the adjoint representation, as this author has done for the E_6 Toda field theory.

In this section, we show as an example how to construct the characteristic integrals for D_4, where an explicit matrix presentation of the Lie algebra is easy to write down which makes the calculation more explicit. That we indeed obtain the characteristic integrals in this example is a corollary of Theorem 2.1.

First recall that the Cartan matrix of D_n is

$$\begin{pmatrix} 2 & -1 & & & & \\ -1 & 2 & -1 & & & \\ & \ddots & \ddots & \ddots & & \\ & & -1 & 2 & -1 & -1 \\ & & & -1 & 2 & \\ & & & -1 & & 2 \end{pmatrix},$$

and its first fundamental representation branches with the following weight diagram

$$\begin{array}{ccccccc} & & & L_n & & & \\ & & \swarrow_{-\alpha_n} & & \nwarrow^{-\alpha_{n-1}} & & \\ -L_1 \xleftarrow{-\alpha_1} -L_2 \xleftarrow{-\alpha_2} \cdots \xleftarrow{-\alpha_{n-2}} -L_{n-1} & & & & & & L_{n-1} \xleftarrow{-\alpha_{n-2}} \cdots \xleftarrow{-\alpha_2} L_2 \xleftarrow{-\alpha_1} L_1. \\ & & \nwarrow_{-\alpha_{n-1}} & & \swarrow^{-\alpha_n} & & \\ & & & -L_n & & & \end{array} \tag{4.1}$$

Therefore Theorem 2.2 can not be applied to the Lie algebras D_n, but Theorem 2.1 can as we will do below to the example of D_4.

Example 4.1 **(D_4).** For simplicity, we write u, v, w, z for the unknown functions $u^1, \cdots, u^4$. The Toda field theory (1.3) for D_4 is

$$\begin{cases} u_{xy} = -e^{2u-v} \\ v_{xy} = -e^{-u+2v-w-z} \\ w_{xy} = -e^{-v+2w} \\ z_{xy} = -e^{-v+2z}. \end{cases}$$

The degrees of D_4 are $2,4,4,6$, with the Pfaffian, the square root of the determinant, being an extra adjoint-invariant function of degree 4.

In this example, we use the convention that the non-degenerate symmetric matrix preserved by the orthogonal matrices in $SO(8)$ is the anti-diagonal matrix $(\delta_{i,9-j})_{i,j=1}^{8}$. Then the matrices in $D_4 = \mathfrak{so}_8$ are skew-symmetric with respect to the anti-diagonal. For simplicity, we use the notation $u_1 = u_x,\ u_2 = u_{xx},\ u_3 = u_{xxx}$ and so on. The $\varepsilon + \mathbf{u}$ from (2.2) is

$$\varepsilon + \mathbf{u} = \begin{pmatrix} u_1 & & & & & & & \\ 1 & v_1 - u_1 & & & & & & \\ & 1 & z_1 + w_1 - v_1 & & & & & \\ & & 1 & z_1 - w_1 & & & & \\ & & 1 & & -z_1 + w_1 & & & \\ & & & -1 & -1 & -z_1 - w_1 + v_1 & & \\ & & & & & -1 & -v_1 + u_1 & \\ & & & & & & -1 & -u_1 \end{pmatrix}. \tag{4.2}$$

We choose our slice in (2.5) to be

$$\varepsilon + \mathbf{I} = \begin{pmatrix} 0 & & & & I_2 & & I_4 & \\ 1 & 0 & & & & I_3 & & -I_4 \\ & 1 & 0 & & I_1 & & -I_3 & \\ & & 1 & 0 & & -I_1 & & -I_2 \\ & & 1 & & 0 & & & \\ & & & -1 & -1 & 0 & & \\ & & & & & -1 & 0 & \\ & & & & & & -1 & 0 \end{pmatrix}.$$

A unipotent orthogonal matrix $B \in N_{SO(8)}$ is an upper-triangular matrix with 1's on the diagonal and element $b_{i,j}$ at the position (i, j) for $i < j$ satisfying some extra conditions for the orthogonality. In particular we have that $b_{4,5} = 0$ and $b_{3,4} + b_{5,6} = 0$. (Actually one can write out the B using its affine coordinates and solving the orthogonality conditions.) Solving (2.5) by Maple, we get the following expression for the primitive characteristic integrals of the D_4 Toda field theory.

$$\begin{aligned} I_1 &= -(u_2 + v_2 + z_2 + w_2 - {u_1}^2 - {v_1}^2 - {z_1}^2 - {w_1}^2 + u_1 v_1 + z_1 v_1 + w_1 v_1) \\ I_2 &= u_4 + v_4 + 2w_4 + z_2 {v_1}^2 + 3w_2 v_2 - w_2 {v_1}^2 + {w_1}^2 v_2 + v_2 z_2 \\ &\quad - v_2 {z_1}^2 + 2v_2 u_2 - z_2 u_2 - w_2 {u_1}^2 + z_2 {u_1}^2 + v_3 u_1 \\ &\quad - 2u_3 u_1 + 2v_3 w_1 - {w_1}^2 u_2 + w_2 u_2 - 4{w_2}^2 - 2{v_2}^2 \\ &\quad - 2{u_2}^2 + {z_1}^2 u_2 + w_2 v_1 u_1 - z_1 v_1 u_2 + w_1 v_1 u_2 \end{aligned}$$

$$
\begin{aligned}
& -z_2v_1u_1-2v_1w_1v_2+2w_2v_1w_1+2z_1v_1v_2-2v_1z_1z_2 \\
& +u_3v_1-2v_1v_3+v_1z_3-4w_1w_3+v_1w_3-{z_1}^2{u_1}^2+{w_1}^2{u_1}^2 \\
& -{w_1}^2v_1u_1-z_1{v_1}^2u_1-w_1v_1{u_1}^2+{z_1}^2v_1u_1+z_1v_1{u_1}^2+w_1{v_1}^2u_1 \\
I_3 = {} & 2u_4+v_4+w_4+{w_1}^2{u_1}^2-2v_1v_3+v_1z_3-2w_1w_3+v_1w_3 \\
& +u_3v_1+3v_2u_2-w_2{u_1}^2+w_2u_2+2v_3u_1-4u_3u_1+v_3w_1 \\
& -{w_1}^2u_2+v_2z_2-v_2{z_1}^2+z_2{v_1}^2+2w_2v_2-2{v_2}^2-4{u_2}^2 \\
& -2{w_2}^2-{w_1}^2v_1u_1+w_1{v_1}^2u_1+2z_1v_1v_2-2v_1z_1z_2+w_1v_1u_2 \\
& +w_2v_1u_1-w_1v_1{u_1}^2-{v_1}^2u_2+{z_1}^2w_2-{z_1}^2{w_1}^2+v_2{u_1}^2-z_2w_2 \\
& +z_2{w_1}^2+{z_1}^2v_1w_1-{v_1}^2z_1w_1+{w_1}^2v_1z_1-z_2v_1w_1-w_2v_1z_1 \\
& +2v_1u_2u_1-2v_2v_1u_1 \\
I_4 = {} & -u_6-\tfrac{1}{2}v_6-w_6+\text{a lot of other terms which we omit}
\end{aligned}
$$

Actually the usual Pfaffian of $\varepsilon+\mathbf{u}$ in (4.2), which is the product of the first 4 diagonal entries, is contained in the characteristic integral I_2 as the last terms involving only partial derivatives of the first order.

Remark 4.1. In [3], there is a procedure to apply an integration step in using the first fundamental representation (4.1) of D_n to get an analogous formula to Theorem 2.2. That integration step will cause us to lose one characteristic integral, which in the case of D_4 is exactly the above I_2 corresponding to the Pfaffian.

References

[1] Ian M. Anderson, Mark E. Fels and Peter J. Vassiliou, Superposition formulas for exterior differential systems, *Adv. Math.* **221** (2009) 1910–1963.

[2] Olivier Babelon,Denis Bernard and Michel Talon, *Introduction to classical integrable systems*, Cambridge Monographs on Mathematical Physics (Cambridge University Press, Cambridge, 2003).

[3] J. Balog, L. Fehér, L. O'Raifeartaigh, P. Forgács and A. Wipf, Toda theory and $\mathcal{W}$-algebra from a gauged WZNW point of view, *Ann. Physics* **203** (1990) 76–136.

[4] Andreas Čap and Jan Slovák, *Parabolic geometries. I*, Mathematical Surveys and Monographs **154** (American Mathematical Society, Providence, RI, 2009), background and general theory.

[5] D.K. Demskoi, Integrals of open two-dimensional lattices, *Theoret. and Math. Phys.* **163** (2010) 466–471.

[6] V.G. Drinfel'd and V.V. Sokolov, Lie algebras and equations of Korteweg-de Vries type, Current Problems in Mathematics **24** (Itogi Nauki i Tekhniki,Akad. Nauk SSSR Vsesoyuz. Inst. Nauchn. i Tekhn. Inform., Moscow, 1984) 81–180, Russian.

[7] L. Fehér, L. O'Raifeartaigh, P. Ruelle, I. Tsutsui and A. Wipf, Generalized Toda theories and $\mathcal{W}$-algebras associated with integral gradings, *Ann. Physics* **213** (1992) 1–20.

[8] Boris Feigin and Edward Frenkel, Integrals of motion and quantum groups, *Integrable systems and quantum groups (Montecatini Terme, 1993)*, Lecture Notes in Math. **1620** (Springer, Berlin, 1996) 349–418.

[9] William Fulton and Joe Harris, *Representation theory*, Graduate Texts in Mathematics **129** (Springer-Verlag, New York, 1991), a first course; Readings in Mathematics.

[10] A.M. Gur'eva and A.V. Zhiber, Laplace invariants of two-dimensionalized open Toda chains, *Teoret. Mat. Fiz.* **138** (2004) 401–421, Russian, with Russian summary; English translation: *Theoret. and Math. Phys.* **138** (2004) 338–355.

[11] A.M. Gur'eva and A.V. Zhiber, Laplace invariants of Toda lattices with exceptional Cartan matrices, `http://arxiv.org/abs/nlin/0512001` (2005).
[12] Bertram Kostant, The principal three-dimensional subgroup and the Betti numbers of a complex simple Lie group, *Amer. J. Math.* **81** (1959) 973–1032.
[13] Bertram Kostant, Lie group representations on polynomial rings, *Amer. J. Math.* **85** (1963) 327–404.
[14] Erling G.B. Hohler and Kåre Olaussen, Using conservation laws to solve Toda field theories, *Internat. J. Modern Phys. A* **11** (1996) 1831–1853.
[15] A.N. Leznov and M.V. Saveliev, Representation of zero curvature for the system of nonlinear partial differential equations $x_{\alpha,z\bar{z}} = \exp(kx)_\alpha$ and its integrability, *Lett. Math. Phys.* **3** (1979) 489–494.
[16] A.N. Leznov and M.V. Saveliev, *Group-theoretical methods for integration of nonlinear dynamical systems*, Progress in Physics **15** (Birkhäuser Verlag, Basel, 1992), translated and revised from Russian by D.A. Leuites.
[17] G. Mari Beffa, Geometric Hamiltonian structures on flat semisimple homogeneous manifolds, *Asian J. Math.* **12** (2008) 1–33.
[18] A.B. Shabat, Higher symmetries of two-dimensional lattices, *Phys. Lett. A* **200** (1995) 121–133.
[19] A.B. Shabat and R. L. Yamilov, Exponential systems of type II and Cartan matrices, Preprint No. 1 Bashkir Branch USSR Acad. Sci. Ufa (1981).
[20] Keizo Yamaguchi, Differential systems associated with simple graded Lie algebras, *Progress in differential geometry*, Adv. Stud. Pure Math. **22** (Math. Soc. Japan, Tokyo, 1993) 413–494.
[21] A.V. Zhiber and V.V. Sokolov, Exactly integrable hyperbolic equations of Liouville type, *Uspekhi Mat. Nauk* **56** (2001) 63–106, Russian, with Russian summary; English translation: *Russian Math. Surveys* **56** (2001) 61–101.