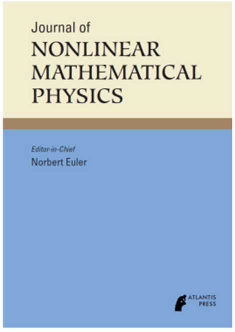

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Noether symmetries and the quantization of a Liénard-type nonlinear oscillator

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The classical quantization of a Liénard-type nonlinear oscillator is achieved by a quantization scheme (M. C. Nucci. *Theor. Math. Phys.*, 168:994–1001, 2011) that preserves the Noether point symmetries of the underlying Lagrangian in order to construct the Schrödinger equation. This method straightforwardly yields the Schrödinger equation in the momentum space as given in (V. Chithiika Ruby, M. Senthilvelan, and M. Lakshmanan. *J. Phys. A: Math. Gen.*, 45:382002, 2012), and sheds light on the apparently remarkable connection with the linear harmonic oscillator.

Keywords: Classical quantization; Lie symmetries; Noether symmetries; Liénard-type nonlinear oscillator

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1. Introduction

The classical method in the passage from classical to quantum mechanics is based on the substitution of the classical coordinates and momenta $(q_i, p_i)_{i=1,\dots,N}$ with the quantum operators:

$$q_i \to q_i, \quad p_i \to -\mathrm{i}\frac{\partial}{\partial q_i}. \tag{1.1}$$

However if the quantization of nonstandard Hamiltonians is pursued then ambiguity may occur due to ordering non-commuting factors. In such cases, the normal ordering method – as described in classical textbooks such as [4, 22] – and the Weyl quantization scheme [41] were devised.

Also since quantum mechanics is essentially a *linear* theory then problems arise when *nonlinear* canonical transformations are involved and there is no guarantee of consistency [1, 5, 25, 38]. For a more recent perspective on the canonical transformations in quantum mechanics see [3] and the references within.

In [34] it was inferred that Lie symmetries should be preserved if a consistent quantization is desired. In [10] [ex. 18, p. 433] an alternative Hamiltonian for the simple harmonic oscillator was presented. It is obtained by applying a nonlinear canonical transformation to the classical Hamiltonian of the harmonic oscillator. That alternative Hamiltonian was used in [33] to demonstrate what nonsense the usual quantization schemes[a] produce. In [28] a quantization scheme that preserves the Noether symmetries was proposed and applied to Goldstein's example in order to derive the correct Schrödinger equation. In [29] the same quantization scheme was applied in order to quantize the second-order Riccati equation while in [30] the quantization of Calogero's goldfish system was achieved.

Let us reformulate the quantization scheme that preserves the Noether symmetries to the case of a linearizable system of N second-order ordinary differential equations, i.e.

$$\ddot{\mathbf{x}}(t) = \mathbf{f}(t, \mathbf{x}, \dot{\mathbf{x}}), \quad \mathbf{x} \in \mathbb{R}^N, \tag{1.2}$$

which possesses the maximal number of admissible Lie point symmetries, namely $N^2 + 4N + 3$. In [11, 12] it was proven that the maximal-dimension Lie symmetry algebra of a system of N equations of second order is isomorphic to $sl(N+2, \mathbb{R})$, and that the corresponding Noether symmetries generate a $(N^2 + 3N + 6)/2$-dimensional Lie algebra g^V whose structure (Levi-Malćev decomposition and realization by means of a matrix algebra) was determined. It was also proven that the corresponding linear system is

$$\mathbf{y}''(s) + 2A_1(s) \cdot \mathbf{y}'(s) + A_0(s) \cdot \mathbf{y}(s) + \mathbf{b}(s) = 0, \tag{1.3}$$

with the condition

$$A_0(s) = A_1'(s) + A_1(s)^2 + a(s)\mathbb{1}, \tag{1.4}$$

where A_0, A_1 are $N \times N$ matrices, and a is a scalar function.

The algorithm that yields the Schrödinger equation can be summarized as follows:

Step 1. Find the linearizing transformation which does not change the time, as prescribed in non-relativistic quantum mechanics.

Step 2. Derive the Lagrangian by applying the linearizing transformation to the standard Lagrangian of the corresponding linear system (1.3), namely that that admits the maximum number of Noether symmetries[b].

Step 3. Apply the linearizing transformation to the Schrödinger equation of the corresponding classical linear problem. This yields the Schrödinger equation corresponding to system (1.2).

This quantization is consistent with the classical properties of the system, namely the Lie symmetries of the obtained Schrödinger equation, i.e.

$$\Omega = \hat{T}(t, \mathbf{x})\partial_t + \sum_{i=1}^{n} \hat{X}_i(t, \mathbf{x})\partial_{x_i} + \hat{\Psi}(t, \mathbf{x}, \psi)\partial_\psi, \tag{1.5}$$

are such that $\hat{T}(t, \mathbf{x})\partial_t + \sum_{i=1}^{n} \hat{X}_i(t, \mathbf{x})\partial_{x_i}$ correspond to the Noether symmetries admitted by the Lagrangian of system (1.2).

[a] Such as normal-ordering [4, 22] and Weyl quantization [41].

[b] In [12] it was shown that any diffeomorphism between two systems of second-order differential equations takes Noether symmetries into Noether symmetries, and therefore the Lagrangian is unique up to a diffeomorphism.

Remark: Since the Schrödinger equation is homogeneous and linear, it admits also the homogeneity symmetry $\psi\partial_\psi$, and the linearity symmetry $F(t,\mathbf{x},\psi)\partial_\psi$, where F is any particular solution of the Schrödinger equation itself.

In this paper we apply this quantization algorithm to a linearizable Liénard equation, i.e.

$$\ddot{x}+kx\dot{x}+\frac{k^2}{9}x^3+\omega^2x=0, \tag{1.6}$$

that has been recently quantized in momentum space [9].

In Section 2 we recall the properties of the linearizable Liénard equation (1.6). In Section 3 we consider the classical analogue of the momentum representation of equation (1.6) as given in [9] and after showing that is linearizable we quantize it by preserving the Noether symmetries. In Section 4 a comparison between the Noether quantization method and that applied in [9] is given, and, in particular, it is shown that the two Schrödinger equations are equal. The last Section contains some final remarks.

2. Classical properties of the Liénard equation (1.6)

The one-dimensional nonlinear oscillator (1.6) is a special case of the general Liénard equation, i.e.

$$\ddot{x}+f(x)\dot{x}+g(x)=0, \tag{2.1}$$

introduced more than 75 years ago [21, 37] for modeling electrical circuits. Since then Liénard equations have been applied to many different areas even in biology [26].

The particular Liénard equation (1.6) has been extensively studied by many authors, the most recent papers being [6–8, 13]. In [23] Lie group analysis was applied to (1.6) with $\omega=0$ and it was shown that it is linearizable, while in [36] the same was proven when $\omega\neq 0$. In fact it was found that (1.6) admits an eight-dimensional Lie point symmetry algebra generated by the following operators:

$$\Gamma_1=x\partial_t-\left(\frac{1}{3}x^3k+\frac{3\omega^2}{k}x\right)\partial_x, \tag{2.2a}$$

$$\Gamma_2=\sin(\omega t)x\partial_t-\left(\frac{k}{3}\sin(\omega t)x^3-\omega\cos(\omega t)x^2\right)\partial_x, \tag{2.2b}$$

$$\Gamma_3=\cos(\omega t)x\partial_t-\left(\frac{k}{3}\cos(\omega t)x^3+\omega\sin(\omega t)x^2\right)\partial_x, \tag{2.2c}$$

$$\begin{aligned}\Gamma_4=&\left(\frac{3\omega}{k}\cos(2\omega t)-\sin(2\omega t)x\right)\partial_t\\&+\left[\left(\frac{k}{3}x^3-\frac{3\omega^2}{k}x\right)\sin(2\omega t)-2\omega x^2\cos(2\omega t)\right]\partial_x,\end{aligned} \tag{2.2d}$$

$$\begin{aligned}\Gamma_5=&-\left(\frac{3\omega}{k}\sin(2\omega t)+\cos(2\omega t)x\right)\partial_t\\&-\left[\left(\frac{k}{3}x^3-\frac{3\omega^2}{k}x\right)\cos(2\omega t)+2\omega x^2\cos(2\omega t)\right]\partial_x,\end{aligned} \tag{2.2e}$$

$$\Gamma_6=\cos(\omega t)\partial_t+\left(\omega\sin(\omega t)x-\frac{3\omega^2}{k}\cos(\omega t)\right)\partial_x, \tag{2.2f}$$

$$\Gamma_7 = -\sin(\omega t)\partial_t + \left(\omega\cos(\omega t)x + \frac{3\omega^2}{k}\sin(\omega t)\right)\partial_x, \tag{2.2g}$$

$$\Gamma_8 = \partial_t. \tag{2.2h}$$

Following Lie [20], the linearizing transformation is given by finding a two-dimensional abelian intransitive subalgebra and putting it into the canonical form $\partial_{\tilde{x}}$, $\tilde{t}\partial_{\tilde{x}}$. Since a two-dimensional abelian intransitive subalgebra is that generated by

$$k\Gamma_2 - 3\omega\Gamma_6, \qquad k\Gamma_3 - 3\omega\Gamma_7, \tag{2.3}$$

then the point transformation that takes (1.6) into the one-dimensional free-particle

$$\frac{\mathrm{d}^2\tilde{x}}{\mathrm{d}\tilde{t}^2} = 0 \tag{2.4}$$

is

$$\tilde{t} = \frac{kx\cos(\omega t) + 3\omega\sin(\omega t)}{kx\sin(\omega t) - 3\omega\cos(\omega t)}, \qquad \tilde{x} = -\frac{1}{9\omega^2}\frac{x}{kx\sin(\omega t) - 3\omega\cos(\omega t)}. \tag{2.5}$$

Indeed the general solution of (1.6) is known to be

$$x = \frac{9\omega^3 A\sin(\omega t + \delta)}{k - 3\omega^2 kA\cos(\omega t + \delta)}, \tag{2.6}$$

with A and δ arbitrary constants.

Thus equation (1.6) represents a non-linear oscillator – at least if $|A| < 1/(3\omega^2)$ – and should be related to the linear harmonic oscillator. Indeed in [7] it was shown that the nonlocal transformation

$$U = xe^{\frac{k}{3}\int x(\tau)\mathrm{d}\tau} \tag{2.7}$$

takes equation (1.6) into the linear harmonic oscillator:

$$\ddot{U} + \omega^2 U = 0. \tag{2.8}$$

Also in [7] the following Lagrangian for equation (1.6) was determined:

$$L = \frac{27\omega^6}{2k^2}\frac{1}{k\dot{x} + \frac{k^2}{3}x^2 + 3\omega^2} + \frac{3\omega^2}{2k}\dot{x} - \frac{9\omega^4}{2k^2}. \tag{2.9}$$

Since the momentum is

$$p \equiv \frac{\partial L}{\partial \dot{x}} = -\frac{27\omega^6}{2k}\frac{1}{\left(k\dot{x} + \frac{k^2}{3}x^2 + 3\omega^2\right)^2} + \frac{3\omega^2}{2k}, \tag{2.10}$$

and consequently the velocity $\dot{x}$ is

$$\dot{x} = -\frac{k}{3}x^2 + 3\omega^2\,\frac{1 - \sqrt{1 - \frac{2k}{3\omega^2}p}}{k\sqrt{1 - \frac{2k}{3\omega^2}p}}, \tag{2.11}$$

then the corresponding Hamiltonian was derived to be

$$H = \frac{9\omega^4}{2k^2}\left[2 - \frac{2k}{3\omega^2}p - 2\left(1 - \frac{2k}{3\omega^2}p\right)^{\frac{1}{2}} + \frac{k^2x^2}{9\omega^2}\left(1 - \frac{2k}{3\omega^2}p\right)\right], \tag{2.12}$$

with the restriction $-\infty < p \leq 3\omega^2/2k$. The substitution of the general solution (2.6) into (2.7) yielded the following canonical transformation between (1.6) and (2.8):

$$x = \frac{U}{1 - \dfrac{k}{3\omega^2}P}, \quad p = P\left(1 - \frac{k}{6\omega^2}P\right). \tag{2.13}$$

Remark: We observe that the last two terms of the Lagrangian (2.9) represent the total derivative of the function:

$$G = \frac{3\omega^2}{2k}x - \frac{9\omega^4}{2k^2}t. \tag{2.14}$$

Although the addition of the total derivative of the particular function G as given in (2.14) may seem useless, it actually allows one to replace the otherwise ambiguous term $\sqrt{-p}$ with $\sqrt{1 - \frac{2k}{3\omega^2}p}$.

Apart an unessential multiplicative constant and the addition of the total derivative of G as given in (2.14), the Lagrangian (2.9) was determined in [35] by means of the Jacobi Last Multiplier [14–17] as a particular case of the Lagrangian for the general Liénard equation (2.1), i.e.

$$L = \left(\dot{x} + \frac{g(x)}{\alpha f(x)}\right)^{2-\frac{1}{\alpha}} + \frac{\mathrm{d}}{\mathrm{d}t}G(t,x), \tag{2.15}$$

when the following relationship holds between $f(x)$ and $g(x)$:

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{g(x)}{f(x)}\right) = \alpha(1-\alpha)f(x), \tag{2.16}$$

where α is a constant $\neq 1$.

In the case of equation (1.6) it was shown in [35] that the relationship (2.16) holds, and consequently the following function was determined:

$$q = \frac{x}{\dot{x} + \dfrac{k}{3}x^2 + \dfrac{3\omega^2}{k}} \tag{2.17}$$

such that $q = xe^{\frac{k}{3}\int x(\tau)\mathrm{d}\tau}$ and satisfies the linear harmonic oscillator equation:

$$\ddot{q} + \omega^2 q = 0, \tag{2.18}$$

which is indeed (2.8) with the identification $q = U$.

3. Quantization of (1.6) in momentum space

In [9] the quantization problem of (1.6) was tackled in the momentum representation since the Hamiltonian (2.12) is quadratic in x. The von Roos' quantization scheme [39,40] was then applied. Instead we begin with the classical Lagrangian equation that comes from the Hamiltonian (2.12) where x is replaced with the momentum p and then apply the Noether symmetry quantization.

The canonical transformation

$$(x,p) \to (X,P) = (p,-x) \tag{3.1}$$

transforms the Hamiltonian (2.12) into the “inverted” Hamiltonian

$$\tilde{H} = \frac{9\omega^4}{2k^2}\left[2 - \frac{2k}{3\omega^2}X - 2\left(1-\frac{2k}{3\omega^2}X\right)^{\frac{1}{2}} + \frac{k^2P^2}{9\omega^2}\left(1-\frac{2k}{3\omega^2}X\right)\right]. \tag{3.2}$$

The corresponding Lagrangian is:

$$\tilde{L} = \frac{\dot{X}^2}{2\omega^2\left(1-\frac{2k}{3\omega^2}X\right)} - \frac{9\omega^4}{2k^2}\left(1-\sqrt{1-\frac{2k}{3\omega^2}X}\right)^2, \tag{3.3}$$

and its Lagrangian equation is:

$$\ddot{X} = \frac{3\omega^4}{k}\left(1-\frac{2k}{3\omega^2}X-\sqrt{1-\frac{2k}{3\omega^2}X}\right) - \frac{k\dot{X}^2}{3\omega^2\left(1-\frac{2k}{3\omega^2}X\right)}. \tag{3.4}$$

Using the REDUCE programs [27] we determine that the Lie symmetry algebra admitted by equation (3.4) is generated by the following operators:

$$\Xi_1 = \partial_t, \tag{3.5a}$$

$$\Xi_2 = \cos(2\omega t)\partial_t + \sin(2\omega t)\frac{3\omega^3}{k}\left[1-\frac{2k}{3\omega^2}X-\sqrt{1-\frac{2k}{3\omega^2}X}\right]\partial_X, \tag{3.5b}$$

$$\Xi_3 = \sin(2\omega t)\partial_t - \cos(2\omega t)\frac{3\omega^3}{k}\left[1-\frac{2k}{3\omega^2}X-\sqrt{1-\frac{2k}{3\omega^2}X}\right]\partial_X, \tag{3.5c}$$

$$\Xi_4 = \sqrt{1-\frac{2k}{3\omega^2}X}\cos(\omega t)\partial_X, \tag{3.5d}$$

$$\Xi_5 = \sqrt{1-\frac{2k}{3\omega^2}X}\sin(\omega t)\partial_X, \tag{3.5e}$$

$$\begin{aligned}\Xi_6 = {} & \cos(\omega t)\left(\sqrt{1-\frac{2k}{3\omega^2}X}-1\right)\partial_t \\ & + \frac{3\omega^3}{k}\sin(\omega t)\left(\sqrt{1-\frac{2k}{3\omega^2}X}-2\right)\left(1-\frac{2k}{3\omega^2}X\right)\partial_X,\end{aligned} \tag{3.5f}$$

$$\Xi_7 = \sin(\omega t)\left(\sqrt{1-\frac{2k}{3\omega^2}X}-1\right)\partial_t \tag{3.5g}$$
$$-\frac{3\omega^3}{k}\cos(\omega t)\left(\sqrt{1-\frac{2k}{3\omega^2}X}-2\right)\left(1-\frac{2k}{3\omega^2}X\right)\partial_X,$$
$$\Xi_8 = \sqrt{1-\frac{2k}{3\omega^2}X}\left(1-\sqrt{1-\frac{2k}{3\omega^2}X}\right)\partial_X. \tag{3.5h}$$

Obviously equation (3.4) is linearizable and the operators Ξ_i give a representation of the Lie algebra $sl(3,\mathbb{R})$ [20]. Therefore the Noether symmetry quantization can be applied step by step.

Step 1. Let us find the linearizing transformation. A two-dimensional abelian intransitive subalgebra is provided by Ξ_4 and Ξ_5 and thus the linearizing transformation that takes (3.4) into the free particle equation, i.e.

$$\frac{\mathrm{d}^2\xi(\tau)}{\mathrm{d}\tau^2} = 0, \tag{3.6}$$

is given by

$$\tau = \tan(\omega t), \quad \xi = \frac{3\omega}{k}\frac{1-\sqrt{1-\frac{2k}{3\omega^2}X}}{\cos(\omega t)}. \tag{3.7}$$

Unfortunately this transformation involves changing the time t.
However we recall that the point transformation between the free particle (3.6) and the linear harmonic oscillator, i.e.

$$\frac{\mathrm{d}^2Z(t)}{\mathrm{d}t^2} + \omega^2 Z(t) = 0, \tag{3.8}$$

is given by:

$$\tau = \tan(\omega t), \quad \xi = \frac{Z}{\cos(\omega t)}. \tag{3.9}$$

Then it is easy to show that the transformation:

$$\eta = \frac{3\omega}{k}\left(1-\sqrt{1-\frac{2k}{3\omega^2}X}\right) \tag{3.10}$$

takes equation (3.4) into the linear harmonic oscillator:

$$\ddot{\eta} + \omega^2\eta = 0. \tag{3.11}$$

Thus the linearization transformation (3.10) yields the general solution of equation (3.4), i.e.

$$X(t) = \frac{A}{6}\cos(\omega t+\delta)\left[6\omega - Ak\cos(\omega t+\delta)\right], \tag{3.12}$$

where A and δ are two arbitrary constants.

Remark: Equation (3.4) and Liénard equation (1.6) are examples of nonlinear oscillators whose amplitudes do not depend on the frequency, unlike other famous nonlinear oscillators, e.g. the Mathews-Lakshmanan oscillator [18, 24].

Step 2. The Lagrangian (3.3) admits five Noether point symmetries, namely Ξ_i with $i = 1, \ldots, 5$ in (3.5).

Step 3. Let us consider the Schrödinger equation for the linear harmonic oscillator:

$$2\mathrm{i}\widetilde{\Phi}_t + \widetilde{\Phi}_{\eta\eta} - \omega^2\eta^2\widetilde{\Phi} = 0, \tag{3.13}$$

with $\widetilde{\Phi} = \widetilde{\Phi}(t,\eta)$. Then applying the transformation (3.10) we obtain the following Schrödinger equation :

$$2\mathrm{i}\widetilde{\Phi}_t + \omega^2\left(1 - \frac{2k}{3\omega^2}X\right)\widetilde{\Phi}_{XX} - \frac{k}{3}\widetilde{\Phi}_X - \frac{9\omega^4}{k^2}\left(1 - \sqrt{1 - \frac{2k}{3\omega^2}X}\right)^2\widetilde{\Phi} = 0, \tag{3.14}$$

with $\widetilde{\Phi} = \widetilde{\Phi}(t,X)$. In order to eliminate the first derivative of Φ with respect to X in (3.14) we apply the following standard transformation[c]:

$$\widetilde{\Phi}(t,X) = \frac{\Phi(t,X)}{\left[3\omega^2\left(1 - \frac{2k}{3\omega^2}X\right)\right]^{\frac{1}{4}}} \tag{3.15}$$

and hence the final form of the Schrödinger equation is:

$$\begin{aligned} &2\mathrm{i}\Phi_t + \omega^2\left(1 - \frac{2k}{3\omega^2}X\right)\Phi_{XX} \\ &\quad + \left[\frac{k^2}{12\omega^2\left(1 - \frac{2k}{3\omega^2}X\right)} - \frac{9\omega^4}{k^2}\left(\sqrt{1 - \frac{2k}{3\omega^2}X} - 1\right)^2\right]\Phi = 0. \end{aligned} \tag{3.16}$$

We now check the classical consistency of the Schrödinger equation (3.16). Using the REDUCE programs [27] we find that its Lie point symmetries are generated by the following operators:

$$\Omega_1 = \Xi_1, \tag{3.17a}$$

$$\Omega_2 = \Xi_2 + \left[\frac{\omega}{2}\frac{\sin(2\omega t)}{\sqrt{1 - \frac{2k}{3\omega^2}X}} - \mathrm{i}9\omega^4\cos(2\omega t)\left(1 + \sqrt{1 - \frac{2k}{3\omega^2}X}\right)^2\right]\Phi\partial_\Phi, \tag{3.17b}$$

$$\Omega_3 = \Xi_3 - \left[\frac{\omega}{2}\frac{\cos(2\omega t)}{\sqrt{1 - \frac{2k}{3\omega^2}X}} - \mathrm{i}9\omega^4\sin(2\omega t)\left(1 + \sqrt{1 - \frac{2k}{3\omega^2}X}\right)^2\right]\Phi\partial_\Phi, \tag{3.17c}$$

[c] Obviously it does not affect the (t,X) part of the admitted finite Lie symmetry algebra of equation (3.14).

$$\Omega_4 = \Xi_4 - \left[\frac{k\cos(\omega t)}{6\omega^2\sqrt{1-\dfrac{2k}{3\omega^2}X}} + 3\mathrm{i}\omega\left(1-\sqrt{1-\frac{2k}{3\omega^2}X}\right)\sin(\omega t)\right]\Phi\partial_\Phi \tag{3.17d}$$

$$\Omega_5 = \Xi_5 - \left[\frac{k\sin(\omega t)}{6\omega^2\sqrt{1-\dfrac{2k}{3\omega^2}X}} - 3\mathrm{i}\omega\left(1-\sqrt{1-\frac{2k}{3\omega^2}X}\right)\cos(\omega t)\right]\Phi\partial_\Phi \tag{3.17e}$$

$$\Omega_6 = \Phi\partial_\Phi, \tag{3.17f}$$

$$\Omega_\chi = \chi(t,X)\partial_\Phi, \tag{3.17g}$$

where $\chi(t,X)$ is any solution of (3.16).

It was shown in [9] that the spectrum of the Liénard equation in momentum space consists of two parts, one positive and one negative.

The positive part is:

$$E_n = \omega\left(n+\frac{1}{2}\right), \quad n \in \mathbb{N},$$

which is the spectrum of the quantum harmonic oscillator. The eigenfunctions of this part of the spectrum satisfy the boundary conditions:

$$\begin{aligned} &\lim_{X\to-\infty}\Phi(t,X) = 0, \quad \text{for every } t \in \mathbb{R}_+,\\ &\Phi\left(t, \frac{3\omega^2}{2k}\right) = 0, \quad \text{for every } t \in \mathbb{R}_+. \end{aligned} \tag{3.18}$$

The negative part is:

$$E_{n^-} = -\omega\left(n+\frac{1}{2}\right), \quad n \in \mathbb{N},$$

which is not the spectrum of the quantum harmonic oscillator, because of the negative sign. The eigenfunctions of this part of the spectrum satisfy the boundary conditions:

$$\begin{aligned} &\lim_{X\to+\infty}\Phi(t,X) = 0, \quad \text{for every } t \in \mathbb{R}_+,\\ &\Phi\left(t, \frac{3\omega^2}{2k}\right) = 0, \quad \text{for every } t \in \mathbb{R}_+. \end{aligned} \tag{3.19}$$

In [2, 19, 31, 32] and more recently in [33] it was shown how to find the eigenfunctions and the eigenvalues of the Schrödinger equation by means of its admitted Lie symmetries.

We apply this method to the Schrödinger equation (3.16) and find the same results as in [9].

Let us rewrite the Lie point symmetries (3.17) of equation (3.16) in complex form, i.e.:

$$\Sigma_1 = \mathrm{i}\partial_t, \tag{3.20a}$$

$$\Sigma_{2\pm} = e^{\pm 2\mathrm{i}\omega t}\partial_t \mp \mathrm{i}e^{\pm 2\mathrm{i}\omega t}\frac{3\omega^3}{k}\left(1-\sqrt{1-\frac{2k}{3\omega^2}X}\right)\sqrt{1-\frac{2k}{3\omega^2}X}\partial_X \tag{3.20b}$$
$$-\mathrm{i}e^{\pm 2\mathrm{i}\omega t}\left[\frac{\omega}{2\sqrt{1-\dfrac{2k}{3\omega^2}X}} \pm 9\omega^4\left(1+\sqrt{1-\frac{2k}{3\omega^2}X}\right)\right]\Phi\partial_\Phi,$$
$$\Sigma_{3\pm} = e^{\pm\mathrm{i}\omega t}\sqrt{1-\frac{2k}{3\omega^2}X}\partial_X \tag{3.20c}$$
$$-\left[\frac{k}{6\omega^2\sqrt{1-\dfrac{2k}{3\omega^2}X}} \pm 3\omega\left(1-\sqrt{1-\frac{2k}{3\omega^2}X}\right)\right]\Phi\partial_\phi,$$
$$\Sigma_4 = \Phi\partial_\Phi, \tag{3.20d}$$
$$\Sigma_\chi = \chi(t,X)\partial_\Phi. \tag{3.20e}$$

The operators $\Sigma_{3\pm}$ act as creation and annihilation operators, namely in the case of the boundary conditions (3.18) Σ_{3+} is the annihilation operator and Σ_{3-} is the creation operator, and viceversa in the case of the boundary conditions (3.19).

Let us now consider the case of the boundary conditions (3.18). The invariant surface of the operator Σ_{3+} is given by

$$F(t,X,\Phi) = f\left(t, \Phi\frac{e^{-\frac{3\omega}{k^2}\left(3\omega^2\sqrt{1-\frac{2k}{3\omega^2}X}+kX\right)}}{\left(1-\frac{2k}{3\omega^2}X\right)^{\frac{1}{4}}}\right) = 0, \tag{3.21}$$

and consequently by means of the Implicit Function Theorem one gets

$$\Phi(t,X) = T(t)\left(1-\frac{2k}{3\omega^2}X\right)^{\frac{1}{4}} e^{\frac{3\omega}{k^2}\left(3\omega^2\sqrt{1-\frac{2k}{3\omega^2}X}+kX\right)}, \tag{3.22}$$

with $T(t)$ arbitrary function of t. Substituting this solution into the Schrödinger equation (3.16) yields $T(t) = e^{-\frac{1}{2}it}$ and thus the ground state solution is:

$$\Phi_0(t,X) = \left(1-\frac{2k}{3\omega^2}X\right)^{\frac{1}{4}} e^{-\frac{1}{2}\mathrm{i}\omega t+\frac{3\omega}{k^2}\left(3\omega^2\sqrt{1-\frac{2k}{3\omega^2}X}+kX\right)}. \tag{3.23}$$

This solution $\Phi_0(t,X)$ satisfies the boundary conditions (3.18) and is indeed the ground state[d] since

$$[\Sigma_{3+}, \Sigma_{\Phi_0}] = 0. \tag{3.24}$$

Thus there are no states under $\Phi_0(t,X)$.

The operator Σ_1 acts like an eigenvalue operator since:

$$\Sigma_1\Phi_0 = \frac{\omega}{2}\Phi_0, \tag{3.25}$$

which yields the ground energy $E_0 = \omega/2$, just like the usual quantum harmonic oscillator.

[d] Apart from an unessential normalization constant.

We use the creation operator Σ_{3-} and Σ_{Φ_0} in order to construct the higher states. Since the commutator:

$$[\Sigma_{3-},\Sigma_{\Phi_0}] = \frac{2 3^{\frac{7}{4}}\omega^{\frac{9}{4}}}{k}\left(\sqrt{1-\frac{2k}{3\omega^2}X}-1\right)\left(1-\frac{2k}{3\omega^2}X\right)^{\frac{1}{4}} e^{-\frac{3}{2}\mathrm{i}t+\frac{3\omega}{k^2}\left(3\omega^2\sqrt{1-\frac{2k}{3\omega^2}X}+kX\right)} \tag{3.26}$$

then:

$$\Phi_1(t,X) = \frac{2 3^{\frac{7}{4}}\omega^{\frac{9}{4}}}{k}\left(\sqrt{1-\frac{2k}{3\omega^2}X}-1\right)\left(1-\frac{2k}{3\omega^2}X\right)^{\frac{1}{4}} e^{-\frac{3}{2}\mathrm{i}t+\frac{3\omega}{k^2}\left(3\omega^2\sqrt{1-\frac{2k}{3\omega^2}X}+kX\right)} \tag{3.27}$$

is another solution of (3.16) that satisfies the boundary conditions (3.18) and has a greater energy eigenvalue $E_1 = \frac{3\omega}{2}$ given by:

$$\Sigma_1\Phi_1 = \frac{3\omega}{2}\Phi_1. \tag{3.28}$$

If we evaluate the commutator between Σ_{3+} and Σ_{Φ_1} then we obtain a multiple of Φ_0, i.e.:

$$[\Sigma_{3+},\Sigma_{\Phi_1}] = -4k\omega\Phi_0\partial_\Phi, \tag{3.29}$$

and thus we have constructed the first excited state.

Iterating the process yields all the eigenfunctions, i.e.

$$\left[\Sigma_{3-},\Sigma_{\Phi_{n-1}}\right] = \Phi_n\partial_\Phi = \Sigma_{\Phi_n}. \tag{3.30}$$

Since we have proven that Σ_{3+} acts as the annihilation operator for the first excited state, then we can easily show by means of Jacobi identity[e] that this holds true for every $n \in \mathbb{N}$, i.e.

$$\begin{aligned}[\Sigma_{3+},\Sigma_{\Phi_n}] &= \left[\Sigma_{3+},\left[\Sigma_{3-},\Sigma_{\Phi_{n-1}}\right]\right]\\ &= \left[\left[\Sigma_{\Phi_{n-1}},\Sigma_{3+}\right],\Sigma_{3-}\right] + \left[[\Sigma_{3+},\Sigma_{3-}],\Sigma_{\Phi_{n-1}}\right]\\ &= -\kappa\left[\Sigma_{\Phi_{n-2}},\Sigma_{3-}\right] = \kappa\Sigma_{\Phi_{n-1}},\end{aligned} \tag{3.31}$$

where κ is a constant.

The generic eigenvalue and eigenfunction can be derived in the following manner. We evaluate the commutator between Σ_{3-} and Σ_χ, where χ is a generic solution of (3.16), i.e.

$$[\Sigma_{3-},\Sigma_\chi] = -\frac{1}{2}\frac{e^{-\mathrm{i}\omega t}}{\sqrt{3}\omega k\sqrt{1-\frac{2kX}{3\omega^2}}}\left[\left(-18\omega^3+18\omega^3\sqrt{1-\frac{2kX}{3\omega^2}}-k^2+12k\omega X\right)\chi\right.$$

[e] And by means of

$$[\Sigma_{3,+},\Sigma_{3,-}] = 6\omega\Sigma_4.$$

$$+\left(4k^2X-6k\omega^2\right)\chi_X\Bigg]\partial_\Phi. \tag{3.32}$$

We define the operator $\hat{O}$:

$$\hat{O}=\frac{e^{-\mathrm{i}\omega t}}{2\sqrt{3}\omega k\sqrt{1-\frac{2kX}{3\omega^2}}}\left[\left(18\omega^3-18\omega^3\sqrt{1-\frac{2kX}{3\omega^2}}+k^2-12k\omega X\right)-\left(4k^2X-6k\omega^2\right)\partial_X\right] \tag{3.33}$$

and then beginning with the ground state Φ_0 generate the nth eigenfunction by using the iteration procedure (3.30), i.e.:

$$\begin{aligned}
\Phi_1 &= e^{-\mathrm{i}\omega t}\hat{O}\Phi_0,\\
\Phi_2 &= e^{-\mathrm{i}\omega t}\hat{O}\Phi_1=e^{-2\mathrm{i}\omega t}\hat{O}^2\Phi_0,\\
&\vdots\\
\Phi_n &= e^{-\mathrm{i}\omega t}\hat{O}\Phi_{n-1}=e^{-\mathrm{i}n\omega t}\hat{O}^n\Phi_0.
\end{aligned} \tag{3.34}$$

Since the operator $\hat{O}$ acts on X only, then

$$\Phi_n=e^{-\mathrm{i}\left(n+\frac{1}{2}\right)\omega t}\hat{O}^n\left(\left(1-\frac{2k}{3\omega^2}X\right)^{\frac{1}{4}}e^{\frac{3\omega}{k^2}\left(3\omega^2\sqrt{1-\frac{2k}{3\omega^2}X}+kX\right)}\right). \tag{3.35}$$

Consequently, applying the eigenvalue operator Σ_1 yields the positive part of the spectrum, i.e.

$$\Sigma_1\Phi_n=\omega\left(n+\frac{1}{2}\right)\Phi_n. \tag{3.36}$$

The negative part of the spectrum can be determined in the same way. We determine the invariant surface of Σ_{3-}, i.e.

$$G(t,X,\Phi)=g\left(t,\Phi\frac{e^{\frac{3\omega}{k^2}\left(3\omega^2\mathrm{i}\sqrt{\frac{2k}{3\omega^2}X-1}+kX\right)}}{\left(\frac{2k}{3\omega^2}X-1\right)^{\frac{1}{4}}}\right), \tag{3.37}$$

and consequently by means of the Implicit Function Theorem one gets:

$$\Phi(t,X)=\tilde{T}(t)\left(\frac{2k}{3\omega^2}X-1\right)^{\frac{1}{4}}e^{-\frac{3\omega}{k^2}\left(3\omega^2\mathrm{i}\sqrt{\frac{2k}{3\omega^2}X-1}+kX\right)}. \tag{3.38}$$

Substituting $\Phi(t,X)$ into (3.16) yields $\tilde{T}(t)=e^{\frac{\omega}{2}\mathrm{i}t}$, i.e. the solution:

$$\Phi_{0^-}(t,X)=\left(\frac{2k}{3\omega^2}X-1\right)^{\frac{1}{4}}e^{\frac{1}{2}\mathrm{i}\omega t-\frac{3\omega}{k^2}\left(3\omega^2\mathrm{i}\sqrt{\frac{2k}{3\omega^2}X-1}+kX\right)}. \tag{3.39}$$

The function $\Phi_{0^-}(t,X)$ (3.39) satisfies the boundary conditions (3.19) and is equivalent to the ground state since

$$\left[\Sigma_{3-},\Sigma_{\Phi_{0^-}}\right]=0. \tag{3.40}$$

Consequently there are no states above Φ_{0^-}.
The operator Σ_1 acts like an eigenvalue operator since:

$$\Sigma_1 \Phi_{0^-} = -\frac{\omega}{2} \Phi_{0^-}, \tag{3.41}$$

which yields the ground energy $E_{0^-} = -\omega/2$, and consequently $\Phi_{0^-}(t,X)$ has a negative eigenvalue. Since $\left[\Sigma_{3+}, \Sigma_{\Phi_{0^-}}\right] = \Sigma_{\Phi_{1^-}}$, we explicitly determine the first negative excited state $\Phi_{1^-}(t,X)$, i.e.:

$$\Phi_{1^-}(t,X) = \frac{2 3^{\frac{7}{4}} \omega^{\frac{9}{4}}}{k} \left(\mathrm{i} \sqrt{\frac{2k}{3\omega^2} X - 1} - 1 \right) \left(\frac{2k}{3\omega^2} X - 1 \right)^{\frac{1}{4}} e^{\frac{3}{2} \mathrm{i} t - \frac{3\omega}{k^2} \left(3\omega^2 \mathrm{i} \sqrt{\frac{2k}{3\omega^2} X - 1} + kX \right)}. \tag{3.42}$$

Applying Σ_1 to $\Phi_{1^-}(t,X)$ we get the corresponding eigenvalue:

$$\Sigma_1 \Phi_{1^-} = -\frac{3\omega}{2} \Phi_{1^-}, \tag{3.43}$$

and by applying the commutator with Σ_{3-} we indeed obtain:

$$[\Sigma_{3-}, \Sigma_{\Phi_{1^-}}] = -6\omega \Sigma_{\Phi_{0^-}}. \tag{3.44}$$

Finally, in analogy with the positive part of the spectrum, we have the following recursive formula yielding all the eigenvalues and eigenfunctions:

$$\left[\Sigma_{3+}, \Sigma_{\Phi_{n-1^-}}\right] = \Phi_{n^-} \partial_\Phi = \Sigma_{\Phi_{n^-}}, \tag{3.45}$$

$$\left[\Sigma_{3-}, \Sigma_{\Phi_{n^-}}\right] = \Phi_{n-1^-} \partial_\Phi = \Sigma_{\Phi_{n-1^-}}, \tag{3.46}$$

$$\Sigma_1 \Phi_{n^-} = -\omega \left(n + \frac{1}{2} \right) \Phi_{n^-}. \tag{3.47}$$

4. Comparison between the two quantization methods for (3.4)

We compare the Noether symmetry quantization method applied to equation (3.4), as shown in the previous Section, with that used in [9]. Since the Hamiltonian (2.12) is a nonstandard one, the classical quantization rule (1.1) cannot be used. Therefore in [9] a simple modification of the quantization scheme proposed by von Roos [39, 40] for position-dependent masses was applied. In [9] the following momentum-dependent mass:

$$m(p) = \frac{1}{\omega^2 \left(1 - \frac{2k}{3\omega^2} p\right)}, \tag{4.1}$$

and the following momentum-dependent potential:

$$U(p) = \frac{9\omega^4}{2k^2} \left(\sqrt{1 - \frac{2k}{3\omega^2} p} - 1 \right)^2 \tag{4.2}$$

were introduced in order to transform the Hamiltonian (2.12) into the form

$$H = \frac{x^2}{2m(p)} + U(p). \tag{4.3}$$

Then the following Schrödinger equation was obtained[f]:

$$
\begin{aligned}
2\mathrm{i}\Psi_t+\omega^2\left(1-\frac{2k}{3\omega^2}p\right)\Psi_{pp}-\frac{2k}{3}\Psi_p \\
+\left[\frac{4k^2\alpha(\alpha+\beta+1)}{9\omega^2\left(1-\dfrac{2k}{3\omega^2}p\right)}-\frac{9\omega^4}{k^2}\left(\sqrt{1-\frac{2k}{3\omega^2}p}-1\right)^2\right]\Psi=0,
\end{aligned}
\tag{4.4}
$$

where the constants α and β are related with the other constant γ by means of the condition

$$
\alpha+\beta+\gamma=-1 \tag{4.5}
$$

as prescribed by the von Roos' method. Moreover the following further restriction was imposed:

$$
4\alpha(\alpha+\beta+1)=-\frac{1}{4} \tag{4.6}
$$

in order to find the solution of equation (4.4) by means of Hermite differential equation.
Since the eigenfunctions are singular on $p=3\omega^2/2k$, consequently in [9] another Schrödinger equation was proposed by considering the following modified Hamiltonian:

$$
\bar{H}=m^d H m^{-d}, \tag{4.7}
$$

and applying to it the von Roos' procedure. Then it was found that d must be equal to $-1/2$, and the following Schrödinger equation was obtained:

$$
\begin{aligned}
2\mathrm{i}\Theta_t+\omega^2\left(1-\frac{2k}{3\omega^2}p\right)\Theta_{pp} \\
+\left[\frac{k^2}{12\omega^2\left(1-\dfrac{2k}{3\omega^2}p\right)}-\frac{9\omega^4}{k^2}\left(\sqrt{1-\frac{2k}{3\omega^2}p}-1\right)^2\right]\Theta=0,
\end{aligned}
\tag{4.8}
$$

which is the same Schrödinger equation (3.16) that we have got in the previous section with the identification $p=X$.
We observe that equation (4.4) and equation (4.8) are related by the trivial point transformation:

$$
\Psi(t,p)=\frac{\Theta(t,p)}{\sqrt{3\omega^2\left(1-\dfrac{2k}{3\omega^2}p\right)}}, \tag{4.9}
$$

that eliminates the first-derivative in (4.4) and does not change the (t,p) part of the admitted finite Lie point symmetry algebra, namely the Noether symmetries of the classical Lagrangian (3.3). We also remark that the condition (4.6) on the constants α and β is equivalent to require that the finite Lie point symmetry algebra of equation (4.4) – and consequently equation (4.8) – be six dimensional[g].

[f] In [9] time-independent Schrödinger equations were derived.
[g] If α and β do not satisfy condition (4.6) then equation (4.4) admits only Γ_1, Γ_6 and Γ_ψ as Lie symmetries.

Indeed we have shown that the Schrödinger equation (4.4) can be obtained by means of the quantization method that preserves the Noether symmetries of the classical problem.

5. Final remarks

A new algorithm for quantization that requires the preservation of Noether symmetries in the passage from classical to quantum mechanics[h] has been recently introduced and applied to both one-dimensional and two-dimensional Lagrangian equations [28–30].

In this paper we have applied this new method to the linearizable Liénard equation (1.6) and compared our results with that determined in [9]. We have found that the Schrödinger equation obtained in [9] can be determined by means of the quantization that preserves the Noether symmetries.

Even in quantum mechanics whenever differential equations are involved, Lie and Noether symmetries have a fundamental role: Noether symmetries yield the correct Schrödinger equation and its Lie symmetries can be algorithmically used to find the eigenvalues and eigenfunctions.

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[h]Namely, the derived Schrödinger equation is such that the independent-variables part of its admitted Lie symmetries corresponds to the Noether symmetries of the classical equations.

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