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On deformation and classification of $\vee$ -systems

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The $\vee$ -systems are special finite sets of covectors which appeared in the theory of the generalized Witten-Dijkgraaf-Verlinde-Verlinde (WDVV) equations. Several families of $\vee$ -systems are known, but their classification is an open problem. We derive the relations describing the infinitesimal deformations of $\vee$ -systems and use them to study the classification problem for $\vee$ -systems in dimension three. We discuss also possible matroidal structures of $\vee$ -systems in relation with projective geometry and give the catalogue of all known irreducible rank three $\vee$ -systems.

Keywords: Root systems; $\vee$ -systems; WDVV equation.

1. Introduction

The $\vee$ -systems are special finite sets of covectors introduced in [19, 20]. The motivation came from the study of certain special solutions of the generalized Witten-Dijkgraaf-Verlinde-Verlinde (WDVV) equations, playing an important role in 2D topological field theory and $N = 2$ SUSY Yang-Mills theory [2, 10].

Let V be a real vector space and $\mathcal{A} \subset V^*$ be a finite set of vectors in the dual space V^* (covectors) spanning V^* . To such a set one can associate the following *canonical form* $G_{\mathcal{A}}$ on V :

$$G_{\mathcal{A}}(x, y) = \sum_{\alpha \in \mathcal{A}} \alpha(x)\alpha(y),$$

where $x, y \in V$, which establishes the isomorphism

$$\varphi_{\mathcal{A}} : V \rightarrow V^*.$$

The inverse $\varphi_{\mathcal{A}}^{-1}(\alpha)$ we denote as $\alpha^{\vee}$. The system $\mathcal{A}$ is called $\vee$ -system if the following relations

$$\sum_{\beta \in \Pi \cap \mathcal{A}} \beta(\alpha^{\vee})\beta^{\vee} = \nu \alpha^{\vee} \quad (1.1)$$

(called $\vee$ -conditions) are satisfied for any $\alpha \in \mathcal{A}$ and any two-dimensional plane $\Pi \subset V^*$ containing α and some ν , which may depend on Π and α . If Π contains more than two covectors, then ν does

not depend on $\alpha \in \Pi$ and the corresponding two forms $G_{\mathcal{A}}$ and

$$G_{\mathcal{A}}^{\Pi}(x, y) := \sum_{\alpha \in \Pi \cap \mathcal{A}} \alpha(x) \alpha(y)$$

are proportional on the plane $\Pi^{\vee} \subset V$ (see [19, 20]). If Π contains only two covectors from $\mathcal{A}$, say α and β , then we must have

$$G_{\mathcal{A}}(\alpha^{\vee}, \beta^{\vee}) = 0.$$

The $\vee$ -conditions are equivalent to the flatness of the corresponding Knizhnik–Zamolodchikov-type $\vee$ -connection

$$\nabla_a = \partial_a + \kappa \sum_{\alpha \in \mathcal{A}} \frac{\langle \alpha, a \rangle}{\langle \alpha, x \rangle} \alpha^{\vee} \otimes \alpha.$$

The examples of $\vee$ -systems include all two-dimensional systems, Coxeter configurations and so-called deformed root systems [11, 17, 19], but the full classification is an open problem. The main results in this direction can be found in [1, 4–6, 9]. In particular, in [5] it was shown that the class of $\vee$ -systems is closed under the operation of restriction, which gives a powerful tool to construct new examples of $\vee$ -systems.

The most comprehensive list of known $\vee$ -systems together with their geometric properties can be found in [4, 5]. The main purpose of this paper is to present some arguments in favour of the completeness of this list in dimension three by studying the infinitesimal deformations of $\vee$ -systems.

We start with a brief review of the general notions from the theory of matroids [13], which provides a natural framework for the problem of classification of $\vee$ -systems. A matroidal approach in this context was also used by Lechtenfeld et al in [9].

Then we study the infinitesimal deformations of the $\vee$ -systems of given matroidal type and derive the corresponding linearised $\vee$ -conditions. This allows us to show that the isolated 3D $\vee$ -systems listed in [4] are indeed isolated.

The main question which still remains open is what are possible matroidal structures of $\vee$ -systems. We discuss this in the context of the projective geometry using the analysis of the known $\vee$ -systems.

In the last section we study the property of the corresponding v -function on the flats of matroid and state the uniqueness conjecture, saying that the matroid and function v on its flats uniquely determine the corresponding $\vee$ -system.

In the Appendix we give the catalogue of all known $\vee$ -systems in dimension three together with the corresponding matroids and v -functions.

2. Vector Configurations and Matroids

The combinatorial structure of the vector configurations can be described using the notion of matroid. The theory of *matroids* was introduced by Whitney in 1935, who was looking for an abstract notion generalising the linear dependence in the vector space.

We review some standard notions from this theory following mainly Oxley [13].

A *matroid* M is a pair $(X, \mathcal{I})$, where X is a finite set and $\mathcal{I}$ is a collection of subsets S of X (called the *independent sets* of M) such that:

- $\mathcal{I}$ is non-empty

- For any $S \in \mathcal{I}$, any $S' \subset S$ one has $S' \in \mathcal{I}$.
- If $A, B \in \mathcal{I}$, $|A| = |B| + 1$ then $\exists x \in A \setminus B$ such that $B \cup \{x\} \in \mathcal{I}$.

The *rank* of the matroid M is defined as $r(M) = \max_{I \in \mathcal{I}} \{|I|\}$. More generally, the rank of the subset $S \subset X$ is defined as $r(S) = \max_{I \in \mathcal{I}} \{|I| : I \subseteq S\}$.

A *direct sum* of matroids $M_1 = (X_1, \mathcal{I}_1)$ and $M_2 = (X_2, \mathcal{I}_2)$ is defined as

$$M_1 \oplus M_2 = (X_1 \cup X_2, \{I_1 \cup I_2 : I_1 \in \mathcal{I}_1, I_2 \in \mathcal{I}_2\}).$$

A matroid is called *connected* if it can not be represented as a direct sum.

The most important class of matroids for us consists of vector matroids. Let A be a real $r \times n$ matrix, $X = \{1, 2, \dots, n\}$ be the set of column labels of A , and $\mathcal{I}$ be the collection of subsets S of X , for which the columns labelled by S are linearly independent over $\mathbb{R}$. Then $(X, \mathcal{I})$ is a matroid, which is called *rank r vector matroid* and denoted by $M[A]$.

The following operations on matrix A do not affect the corresponding vector matroid $M[A]$:

1. Elementary operations with the rows,
2. Multiplication of a column by a non-zero number.

Two matrices A and A' representing the same matroid M are said to be *projectively equivalent representations* of M if A' can be obtained from A by a sequence of these operations. Equivalently, one can say that $A' = CAD$, where C is an invertible $r \times r$ matrix, and D is a diagonal $n \times n$ matrix with non-zero diagonal entries.

Alternatively, one can define the *linear dependence matroid* on the set X as a family $\mathcal{I}_\mathcal{C}$ of minimal dependent subsets C of X (called *circuits*) through the following axioms:

- The empty set is not a circuit.
- No circuit is contained in another circuit.
- If $C_1, C_2 \in \mathcal{I}_\mathcal{C}$ are two circuits sharing an element $e \in X$, then $(C_1 \cup C_2) \setminus e$ is a circuit or contains a circuit.

The rank of a circuit is defined as the dimension of the vector space spanned by its vectors. Circuits spanning the same d -dimensional subspace can be united in so-called d -flats. A set $F \subseteq X$ is a *flat* of the matroid M if

$$r(F \cup \{x\}) = r(F) + 1$$

for all $x \in X \setminus F$, where $r(F)$ is the rank of the flat F . The matroid can be labelled by listing all d -flats.

As an example consider the positive roots of the B_3 -type system. The corresponding matrix (with the first row giving the labelling) is

$$A = \begin{bmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\ 1 & 1 & 0 & 0 & 1 & 1 & 1 & 0 & 0 \\ 1 & -1 & 1 & 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & -1 & 1 & 1 & -1 & 0 & 0 & 1 \end{bmatrix}.$$

Here matroid M is defined on the set $X = \{1, 2, 3, 4, 5, 6, 7, 8, 9\}$, with 2-flats

$$\{(4, 1, 6), (6, 2, 3), (4, 5, 2), (1, 5, 3)\}$$

and

$$\{(3, 4, 8, 9), (1, 2, 7, 8), (5, 6, 7, 9)\}$$

with three and four elements respectively. Together with the 3-flat X this gives the complete list of flats.

Graphically on the projective plane we have

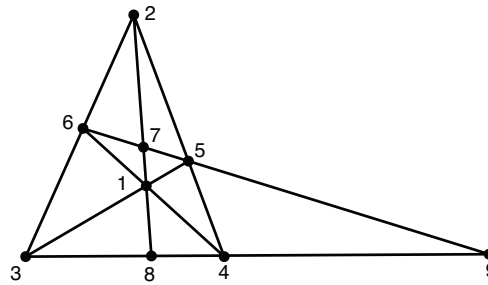


Fig. 1. Graphic representation of B_3 -matroid: lines correspond to rank-2 flats

A matroid is called *simple* if it does not contain one- or two-element circuits. For vector matroids this means that no two vectors are proportional.

Number of matroids up to isomorphism grows very rapidly with $n = |X|$. The following table summarises the results for rank 3-matroids for small n (see [12]).

n	3	4	5	6	7	8	9	10	11	12
all matroids	1	4	13	38	108	325	1275	10037	298491	31899134
simple matroids	1	2	4	9	23	68	383	5249	232928	28872972

Vector matroids build the class of *realisable matroids*. The problem of finding a criterion for realisability is known to be *NP-hard* [14].

Let M be a rank r vector matroid. We say that matroid M is *projectively rigid* if the space of all its rank r vector realisations

$$\mathcal{R}(M) = \{A : M = M[A]\} / \sim$$

modulo projective equivalence is discrete and *strongly projectively rigid* if it consists of only one point (which means that modulo projective equivalence M has a unique vector realisation).

Let G be a finite Coxeter group, which is a finite group generated by the hyperplane reflections in a Euclidean space. We say that matroid M is of *Coxeter type* if it describes the vector configuration of the normals to the corresponding reflection hyperplanes (one for each hyperplane) for such a group. For rank three Coxeter matroids we have the following result.

Theorem 1. *The matroids of Coxeter types A_3 and B_3 are strongly projectively rigid. The matroid of type H_3 is projectively rigid with precisely two projectively non-equivalent vector realisations.*

Proof. Let us prove this first for B_3 case. Since the images a_1, a_2, a_3, a_4 of the elements 1, 2, 3 and 4 in the projective plane form a projective basis it is enough to prove that the remaining a_5, a_6, a_7, a_8, a_9 can be constructed uniquely. From the matroid structure we can see that x_5 must be an intersection point of the lines (2-flats) a_1a_3 and a_2a_4 . We denote this as

$$a_5 = (a_1a_3) \wedge (a_2a_4)$$

using the general lattice theory notation. Similarly we have

$$a_6 = (a_2a_3) \wedge (a_1a_4), \quad a_7 = (a_1a_2) \wedge (a_5a_6),$$

$$a_8 = (a_1a_2) \wedge (a_3a_4), \quad a_9 = (a_3a_4) \wedge (a_5a_6).$$

Similarly one can prove the rigidity in A_3 case (see Fig. 2). In both these cases the space of realisations modulo projective equivalence consists of only one point.

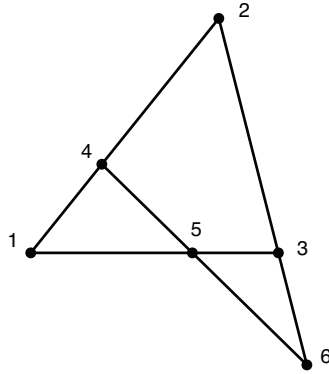


Fig. 2. Graphic representation of A_3 -matroid

The H_3 case is more interesting. Fig. 3 shows the graphic representation of the system H_3 in the real projective plane $\mathbb{RP}^2$.

Recall that on the projective line $\mathbb{RP}^1$ any three points can be mapped into any other three via the action of the group $PGL(2, \mathbb{R})$. For four distinct points p_1, p_2, p_3, p_4 on the projective line $\mathbb{RP}^1$ with homogeneous coordinates $[x_i, y_i]$ there is a projective invariant, namely *cross-ratio* defined as

$$(p_1, p_2; p_3, p_4) = \frac{(x_1y_3 - x_3y_1)(x_2y_4 - x_4y_2)}{(x_1y_4 - x_4y_1)(x_2y_3 - x_3y_2)}.$$

If none of the y_i is zero the cross-ratio can be expressed in terms of the ratios $z_i = \frac{x_i}{y_i}$ as follows:

$$(z_1, z_2; z_3, z_4) = \frac{(z_1 - z_3)(z_2 - z_4)}{(z_1 - z_4)(z_2 - z_3)}.$$

Since any projection from a point in the projective plane preserves the cross-ratio of four points we have the equalities

$$(a_6, a_5; a_9, a_3) = (a_4, a_7; a_{10}, a_3) = (a_5, a_6; a_8, a_3),$$

$$(a_6, a_5; a_9, a_3) = (a_7, a_{11}; a_{10}, a_3) = (a_5, a_8; a_9, a_3).$$

Using elementary manipulations with cross-ratios one can show that that these equalities imply that $x = (a_6, a_5; a_9, a_3)$ satisfies the equation

$$x^2 - x - 1 = 0$$

with two solutions $x_1 = \frac{1+\sqrt{5}}{2}$ and $x_2 = \frac{1-\sqrt{5}}{2}$.

If we fix the positions of the four points a_4, a_5, a_6, a_7 forming a projective basis in $\mathbb{R}P^2$ we can first reconstruct

$$a_1 = (a_5 a_4) \wedge (a_6 a_7), \quad a_2 = (a_5 a_7) \wedge (a_6 a_4), \quad a_3 = (a_5 a_6) \wedge (a_7 a_4).$$

Then using the knowledge of $x = (a_6, a_5; a_9, a_3)$ we can reconstruct a_9 and all the remaining points as

$$a_{14} = (a_2 a_9) \wedge (a_5 a_4), \quad a_{12} = (a_7 a_6) \wedge (a_2 a_9), \quad a_{10} = (a_2 a_9) \wedge (a_3 a_4),$$

$$a_{13} = (a_9 a_4) \wedge (a_6 a_7), \quad a_8 = (a_2 a_{13}) \wedge (a_3 a_6), \quad a_{15} = (a_2 a_{13}) \wedge (a_5 a_4),$$

$$a_{11} = (a_2 a_8) \wedge (a_3 a_4).$$

Thus we have shown that modulo projective group we have only two different vector realisations of matroid H_3 . $\square$

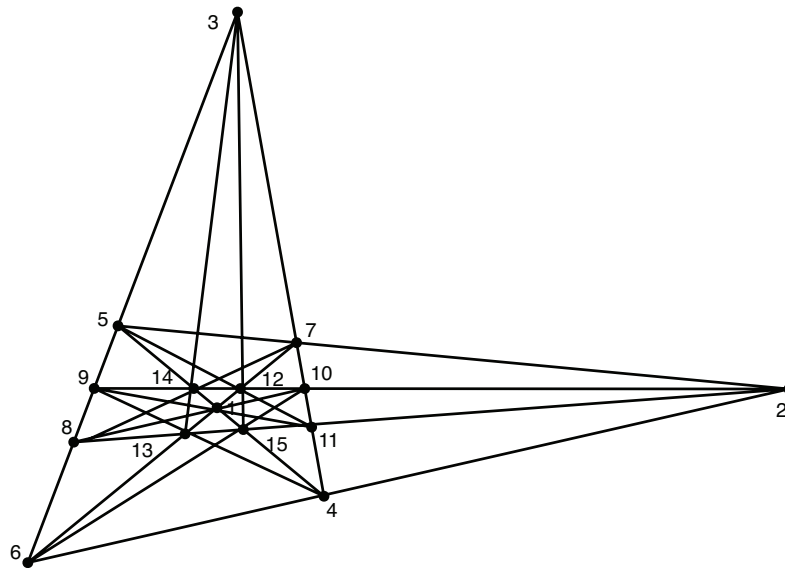


Fig. 3. Graphic representation of H_3 -matroid

Remark. The existence of two projectively non-equivalent realisations is related to the existence of a symmetry of matroid $M(H_3)$, which can not be realised geometrically, see [3]. These two realisations are related by re-ordering of the vectors and thus give rise to the equivalent $\vee$ -systems.

3. Classification Problem for $\vee$ -Systems of Given Matroidal Type

For any $\vee$ -system $\mathcal{A} \subset V^*$ one can consider the corresponding matroid $M(\mathcal{A})$, which encodes a combinatorial structure of $\mathcal{A}$. Conversely, having a matroid M one can look for $\vee$ -system realisations $\mathcal{A}$ of M with given combinatorial structure $M(\mathcal{A}) = M$.

Let $\mathcal{R}^\vee(M)$ be the set of all such realisations modulo group $G = GL(V^*)$ of linear automorphisms of V^* .

If vector matroid $M = M(A)$ is strongly projectively rigid then all its vector realisations modulo G have the form $A' = AD$, or in terms of the columns a_i , $i = 1, \dots, n$ of A ,

$$a'_i = x_i a_i, \quad i = 1, \dots, n$$

with arbitrary non-zero parameters x_i . The $\vee$ -conditions form a system of nonlinear algebraic relations on the parameters $x_i \in \mathbb{R} \setminus 0$ and define $\mathcal{R}^\vee(M)$ as an open set of a real algebraic variety.

For a generic vector matroid this set is actually empty. For example, for n vectors a_i in $\mathbb{R}^3$ in general position the $\vee$ -conditions imply that these vectors must be pairwise orthogonal, which is impossible if $n > 3$.

In the case when the space $\mathcal{R}^\vee(M)$ is known to be non-empty (for example, for all vector matroids M of Coxeter type) we have the question of how to describe this space effectively.

For the case of matroid of Coxeter type A_3 the answer is known [1]. The positive roots of A_3 system are $e_i - e_j$, $1 \leq i < j \leq 4$, where e_i , $i = 1, \dots, 4$ is an orthonormal basis in $\mathbb{R}^4$.

Since matroid A_3 is strongly projectively rigid it is enough to consider the system

$$\mathcal{A} = \{\mu_{ij}(e_i - e_j), 1 \leq i < j \leq 4\}. \quad (3.1)$$

Theorem 2. [1] *The system (3.1) satisfies the $\vee$ -conditions if and only if the parameters satisfy the relations*

$$\mu_{12}\mu_{34} = \mu_{13}\mu_{24} = \mu_{14}\mu_{23}.$$

All the corresponding $\vee$ -systems can be parametrized as

$$A_3(c) = \{\sqrt{c_i c_j}(e_i - e_j), 1 \leq i < j \leq 4\},$$

with arbitrary positive real $c_1, \dots, c_4$.

Without loss of generality, we may choose $c_4 = 1$ and consider the restriction of the system onto the hyperplane $x_4 = 0$. This gives the following parametrisation of the space $\mathcal{R}^\vee(M(A_3))$ by positive real c_1, c_2, c_3 as

$$A_3(c) = \begin{cases} \sqrt{c_i c_j}(e_i - e_j), & 1 \leq i < j \leq 3 \\ \sqrt{c_i}e_i, & i = 1, 2, 3. \end{cases}$$

Consider now the case B_3 , corresponding to the following configuration of vectors in $\mathbb{R}^3$

$$B_3 = \begin{cases} e_i \pm e_j, & 1 \leq i < j \leq 3, \\ e_i & i = 1, \dots, 3. \end{cases}$$

The following 4-parametric family of $\vee$ -systems of B_3 -type was found in [1]:

$$\mathcal{B}(c, \gamma) = \begin{cases} \sqrt{c_i c_j} (e_i \pm e_j), & 1 \leq j < i \leq 3 \\ \sqrt{2c_i(c_i + \gamma)} e_i, & 1 \leq i \leq 3 \end{cases} \quad (3.2)$$

with arbitrary positive c_1, c_2, c_3 and γ such that $c_i + \gamma > 0$ for all $i = 1, 2, 3$.

Theorem 3. *Formula (3.2) gives all rank 3 $\vee$ -systems of matroid type B_3 .*

Proof. The proof is by direct computations, but we present here the details to show the algebraic nature of $\vee$ -conditions in this example.

Since B_3 matroid is strongly projectively rigid, we can assume that the corresponding $\vee$ -system has the form

$$\mathcal{A} = \begin{cases} \alpha_{ij} (e_i + e_j), & 1 \leq i < j \leq 3 \\ \tilde{\alpha}_{ij} (e_i - e_j), & 1 \leq j < i \leq 3 \\ \beta_i e_i, & 1 \leq i \leq 3, \end{cases}$$

where all the parameters can be assumed without loss of generality to be positive.

To write down all $\vee$ -conditions consider all two-dimensional planes containing at least two vectors $v_1, v_2 \in \mathcal{A}$.

There are 3 different types of such planes Π :

1. $\langle e_1, e_2 \pm e_3 \rangle, \langle e_2, e_1 \pm e_3 \rangle, \langle e_3, e_1 \pm e_2 \rangle,$
2. $\langle e_1, e_2, e_1 \pm e_2 \rangle, \langle e_1, e_3, e_1 \pm e_3 \rangle, \langle e_2, e_3, e_2 \pm e_3 \rangle,$
3. $\langle e_1 - e_2, e_2 - e_3, e_1 - e_3 \rangle, \langle e_1 - e_2, e_2 + e_3, e_1 + e_3 \rangle,$
 $\langle e_2 - e_3, e_1 + e_3, e_1 + e_2 \rangle, \langle e_1 - e_3, e_2 + e_3, e_1 + e_2 \rangle.$

The corresponding form G has the matrix

$$G = \begin{pmatrix} \alpha_{13}^2 + \alpha_{12}^2 + \tilde{\alpha}_{13}^2 + \tilde{\alpha}_{12}^2 + \beta_1^2 & \alpha_{12}^2 - \tilde{\alpha}_{12}^2 & \alpha_{13}^2 - \tilde{\alpha}_{13}^2 \\ \alpha_{12}^2 - \tilde{\alpha}_{12}^2 & \alpha_{23}^2 + \alpha_{12}^2 + \tilde{\alpha}_{23}^2 + \tilde{\alpha}_{12}^2 + \beta_2^2 & \alpha_{23}^2 - \tilde{\alpha}_{23}^2 \\ \alpha_{13}^2 - \tilde{\alpha}_{13}^2 & \alpha_{23}^2 - \tilde{\alpha}_{23}^2 & \alpha_{23}^2 + \alpha_{13}^2 + \tilde{\alpha}_{23}^2 + \tilde{\alpha}_{13}^2 + \beta_3^2 \end{pmatrix}$$

In case 1. the $\vee$ -conditions are just the orthogonality conditions $G(\alpha^\vee, \beta^\vee) = 0$ for the corresponding two covectors α and β in the plane Π . We obtain the system

$$\begin{cases} 2(\tilde{\alpha}_{23}^2 \tilde{\alpha}_{13}^2 + \tilde{\alpha}_{23}^2 \tilde{\alpha}_{12}^2 + \tilde{\alpha}_{13}^2 \tilde{\alpha}_{12}^2 - \alpha_{13}^2 \alpha_{12}^2 - \alpha_{13}^2 \tilde{\alpha}_{23}^2 - \alpha_{12}^2 \tilde{\alpha}_{23}^2) - \alpha_{13}^2 \beta_2^2 + \tilde{\alpha}_{13}^2 \beta_2^2 - \alpha_{12}^2 \beta_3^2 + \tilde{\alpha}_{12}^2 \beta_3^2 = 0 \\ 2(\tilde{\alpha}_{23}^2 \tilde{\alpha}_{13}^2 + \tilde{\alpha}_{23}^2 \tilde{\alpha}_{12}^2 + \tilde{\alpha}_{13}^2 \tilde{\alpha}_{12}^2 - \alpha_{23}^2 \alpha_{12}^2 - \alpha_{23}^2 \tilde{\alpha}_{13}^2 - \alpha_{12}^2 \tilde{\alpha}_{13}^2) - \alpha_{23}^2 \beta_1^2 + \tilde{\alpha}_{23}^2 \beta_1^2 - \alpha_{12}^2 \beta_3^2 + \tilde{\alpha}_{12}^2 \beta_3^2 = 0 \\ 2(\tilde{\alpha}_{23}^2 \tilde{\alpha}_{13}^2 - \alpha_{23}^2 \alpha_{13}^2 - \alpha_{23}^2 \tilde{\alpha}_{12}^2 - \alpha_{13}^2 \tilde{\alpha}_{12}^2 + \tilde{\alpha}_{23}^2 \tilde{\alpha}_{12}^2 + \tilde{\alpha}_{13}^2 \tilde{\alpha}_{12}^2) - \alpha_{23}^2 \beta_1^2 + \tilde{\alpha}_{23}^2 \beta_1^2 - \alpha_{13}^2 \beta_2^2 + \tilde{\alpha}_{13}^2 \beta_2^2 = 0 \\ 2(\alpha_{23}^2 \alpha_{12}^2 + \alpha_{23}^2 \tilde{\alpha}_{13}^2 + \alpha_{12}^2 \tilde{\alpha}_{13}^2 - \alpha_{23}^2 \alpha_{13}^2 - \alpha_{23}^2 \tilde{\alpha}_{12}^2 - \alpha_{13}^2 \tilde{\alpha}_{12}^2) - \alpha_{13}^2 \beta_2^2 + \tilde{\alpha}_{13}^2 \beta_2^2 + \alpha_{12}^2 \beta_3^2 - \tilde{\alpha}_{12}^2 \beta_3^2 = 0 \\ 2(\alpha_{13}^2 \alpha_{12}^2 + \alpha_{13}^2 \tilde{\alpha}_{23}^2 + \alpha_{12}^2 \tilde{\alpha}_{23}^2 - \alpha_{23}^2 \tilde{\alpha}_{12}^2 - \alpha_{13}^2 \tilde{\alpha}_{12}^2 - \alpha_{23}^2 \alpha_{13}^2) - \alpha_{23}^2 \beta_1^2 + \tilde{\alpha}_{23}^2 \beta_1^2 + \alpha_{12}^2 \beta_3^2 - \tilde{\alpha}_{12}^2 \beta_3^2 = 0 \\ 2(\alpha_{13}^2 \alpha_{12}^2 + \alpha_{13}^2 \tilde{\alpha}_{23}^2 + \alpha_{12}^2 \tilde{\alpha}_{23}^2 - \alpha_{23}^2 \tilde{\alpha}_{13}^2 - \alpha_{12}^2 \tilde{\alpha}_{13}^2 - \alpha_{23}^2 \alpha_{12}^2) - \alpha_{23}^2 \beta_1^2 + \tilde{\alpha}_{23}^2 \beta_1^2 + \alpha_{13}^2 \beta_2^2 - \tilde{\alpha}_{13}^2 \beta_2^2 = 0, \end{cases}$$

which can be reduced to

$$\begin{cases} (-\alpha_{12}^2 + \tilde{\alpha}_{12}^2)(\alpha_{23}^2 + \alpha_{13}^2 + \tilde{\alpha}_{23}^2 + \tilde{\alpha}_{13}^2 + \beta_3^2 - \frac{(\alpha_{13}^2 - \tilde{\alpha}_{13}^2)^2}{(\alpha_{13}^2 + \alpha_{12}^2 + \tilde{\alpha}_{13}^2 + \tilde{\alpha}_{12}^2 + \beta_1^2)}) = 0 \\ (-\alpha_{13}^2 + \tilde{\alpha}_{13}^2)(\alpha_{23}^2 + \alpha_{12}^2 + \tilde{\alpha}_{23}^2 + \tilde{\alpha}_{12}^2 + \beta_2^2 - \frac{(\alpha_{12}^2 - \tilde{\alpha}_{12}^2)^2}{(\alpha_{13}^2 + \alpha_{12}^2 + \tilde{\alpha}_{13}^2 + \tilde{\alpha}_{12}^2 + \beta_1^2)}) = 0 \\ (-\alpha_{23}^2 + \tilde{\alpha}_{23}^2)(\alpha_{12}^2 + \alpha_{13}^2 + \tilde{\alpha}_{12}^2 + \tilde{\alpha}_{13}^2 + \beta_1^2 - \frac{(\alpha_{13}^2 - \tilde{\alpha}_{13}^2)^2}{(\alpha_{13}^2 + \alpha_{23}^2 + \tilde{\alpha}_{13}^2 + \tilde{\alpha}_{23}^2 + \beta_3^2)}) = 0. \end{cases}$$

Note that the second factors in all equations are ratios of principal minors of matrix G and thus must be positive, since the form G is positive definite. This implies that $\alpha_{ij} = \tilde{\alpha}_{ij}$, which reduces the matrix G to

$$G = \begin{pmatrix} 2(\alpha_{13}^2 + \alpha_{12}^2) + \beta_1^2 & 0 & 0 \\ 0 & 2(\alpha_{23}^2 + \alpha_{12}^2) + \beta_2^2 & 0 \\ 0 & 0 & 2(\alpha_{23}^2 + \alpha_{13}^2) + \beta_3^2 \end{pmatrix}.$$

In cases 2. and 3. we fix for each plane Π a basis $v_1, v_2 \in \mathcal{A} \cap \Pi$. The corresponding dual plane $\Pi^\vee$ is spanned by $v_1^\vee$ and $v_2^\vee$ and the $\vee$ -condition implies the proportionality of the restrictions of the forms G and G_Π onto $\Pi^\vee$. In our case this proportionality turns out to be equivalent to the following system of equations:

$$\begin{cases} \frac{2\alpha_{12}^2}{2(\alpha_{23}^2 + \alpha_{12}^2) + \beta_2^2} - \frac{2\alpha_{13}^2}{2(\alpha_{23}^2 + \alpha_{13}^2) + \beta_3^2} = 0 \\ \frac{2\alpha_{12}^2}{2(\alpha_{13}^2 + \alpha_{12}^2) + \beta_1^2} - \frac{2\alpha_{23}^2}{2(\alpha_{23}^2 + \alpha_{13}^2) + \beta_3^2} = 0 \\ \frac{2\alpha_{13}^2}{2(\alpha_{13}^2 + \alpha_{12}^2) + \beta_1^2} - \frac{2\alpha_{23}^2}{2(\alpha_{23}^2 + \alpha_{12}^2) + \beta_2^2} = 0. \end{cases}$$

Introducing new parameters $c_i, i = 1, 2, 3$ and γ by

$$c_i := \frac{\alpha_{ij}\alpha_{ik}}{\alpha_{jk}}, \quad \gamma := \frac{\beta_3^2 - 2c_3^2}{2c_3}.$$

we can see that these relations imply

$$\alpha_{ij}^2 = c_i c_j, \quad \beta_i^2 = 2c_i(c_i + \gamma),$$

which leads to the parametrisation (3.2). □

For larger matroids the direct analysis of the $\vee$ -conditions is very difficult, so we consider a simpler problem about infinitesimal deformations of $\vee$ -systems.

4. Deformations of $\vee$ -Systems

Let $\mathcal{A} = \{\alpha\} \subset V^*$ be a $\vee$ -system realisation of matroid M . Consider its smooth *scaling deformation* $\mathcal{A}(t)$ of the form

$$\mathcal{A}(t) = \{\alpha_t\}, \quad \alpha_t = \mu_\alpha(t)\alpha, \quad \mu_\alpha(0) = 1. \quad (4.1)$$

For projectively rigid matroids M one can always reduce any deformation to such a form.

Let $\xi_\alpha = \dot{\mu}_\alpha(0)$. We are going to derive the conditions on ξ_α , which can be considered as *linearised $\vee$ -conditions* for such deformations.

Let

$$G_t(x, y) := G_{\mathcal{A}(t)}(x, y) = \sum_{\alpha \in \mathcal{A}} \alpha_t(x) \alpha_t(y)$$

with $G_0 = G = G_{\mathcal{A}}(t)$ and consider its derivative

$$\dot{G}_t(x, y) = \sum_{\alpha \in \mathcal{A}} \dot{\alpha}_t(x) \alpha_t(y) + \sum_{\alpha \in \mathcal{A}} \alpha_t(x) \dot{\alpha}_t(y),$$

which at $t = 0$ gives $\dot{G}_0(x, y) = 2X$, where

$$X = \sum_{\alpha \in \mathcal{A}} \xi_{\alpha} \alpha(x) \alpha(y).$$

Consider now the $\vee$ -conditions.

For any two-dimensional plane containing only two covectors we have

$$G_t(\alpha_t^{\vee}, \beta_t^{\vee}) = 0.$$

Differentiating it in t we have

$$\dot{G}_t(\alpha_t^{\vee}, \beta_t^{\vee}) + G(\dot{\alpha}_t^{\vee}, \beta_t^{\vee}) + G(\alpha_t^{\vee}, \dot{\beta}_t^{\vee}) = 0, \quad (4.2)$$

where here and below by $\dot{\alpha}_t^{\vee}$ we mean $\frac{d}{dt}(\alpha_t^{\vee})$.

To find $G(\dot{\alpha}_t^{\vee}, \beta_t^{\vee})$ note that by definition of $\alpha_t^{\vee}$ $G_t(\alpha_t^{\vee}, v) = \alpha_t(v)$ for any fixed vector $v \in V$. Differentiating this with respect to t we have

$$\dot{G}_t(\alpha_t^{\vee}, v) + G_t(\dot{\alpha}_t^{\vee}, v) = \dot{\alpha}_t(v)$$

which for $t = 0$ gives

$$2X(\alpha^{\vee}, v) + G(\dot{\alpha}_0^{\vee}, v) = \xi_{\alpha} \alpha(v).$$

Thus we have

$$G(\dot{\alpha}_0^{\vee}, v) = \xi_{\alpha} \alpha(v) - 2X(\alpha^{\vee}, v).$$

and thus

$$G(\dot{\alpha}_0^{\vee}, \beta^{\vee}) = \xi_{\alpha} \alpha(\beta^{\vee}) - 2X(\alpha^{\vee}, \beta^{\vee}) = -2X(\alpha^{\vee}, \beta^{\vee})$$

since $\alpha(\beta^{\vee}) = G(\alpha^{\vee}, \beta^{\vee}) = 0$ by the $\vee$ -conditions.

Substituting this into (4.2) we have the first linearised $\vee$ -condition: for α, β being the only two covectors in a plane Π we have

$$X(\alpha^{\vee}, \beta^{\vee}) = 0. \quad (4.3)$$

Let now Π be a two-dimensional plane containing more than two covectors from $\mathcal{A}$ (and hence from $\mathcal{A}_t$). Then from the $\vee$ -conditions there exists $v = v(\Pi) \in \mathbb{R}$ such that for any $\alpha \in \Pi \cap \mathcal{A}$, $v \in V$

we have

$$G^\Pi(\alpha^\vee, \nu) = \nu G(\alpha^\vee, \nu), \quad (4.4)$$

where $G^\Pi(x, y) = G_{\mathcal{A}}^\Pi(x, y) = \sum_{\alpha \in \Pi \cap \mathcal{A}} \alpha(x) \alpha(y)$ (see [5]). Now assuming that $\mathcal{A}$ depends on t as above and differentiating with respect to t at $t = 0$ we have as before for any $\alpha, \beta \in \mathcal{A} \cap \Pi$

$$\dot{G}^\Pi(\alpha^\vee, \beta^\vee) + G^\Pi(\dot{\alpha}^\vee, \beta^\vee) + G^\Pi(\alpha^\vee, \dot{\beta}^\vee) = \dot{\nu} G(\alpha^\vee, \beta^\vee) + \nu \dot{G}(\alpha^\vee, \beta^\vee) + \nu G(\dot{\alpha}^\vee, \beta^\vee) + \nu G(\alpha^\vee, \dot{\beta}^\vee).$$

But from (4.4) we have $G^\Pi(\dot{\alpha}^\vee, \beta^\vee) = \nu G(\dot{\alpha}^\vee, \beta^\vee)$ and $G^\Pi(\alpha^\vee, \dot{\beta}^\vee) = \nu G(\alpha^\vee, \dot{\beta}^\vee)$. Since $\dot{G}^\Pi = 2X^\Pi$, where

$$X^\Pi(x, y) = \sum_{\alpha \in \Pi \cap \mathcal{A}} \xi_\alpha \alpha(x) \alpha(y),$$

we have

$$2X^\Pi(\alpha^\vee, \beta^\vee) = \dot{\nu} G(\alpha^\vee, \beta^\vee) + 2\nu X(\alpha^\vee, \beta^\vee),$$

or, eventually

$$2(X^\Pi - \nu X)((\alpha^\vee, \beta^\vee) = \dot{\nu} G(\alpha^\vee, \beta^\vee). \quad (4.5)$$

Since this is true for all $\alpha, \beta \in \Pi \cap \mathcal{A}$ we have the second linearised $\vee$ -condition: for any plane Π containing more than two covectors from $\mathcal{A}$ we have

$$X^\Pi - \nu X \sim G|_{\Pi^\vee}, \quad (4.6)$$

where the sign $\sim$ means proportionality.

Thus we have proved

Theorem 4. *The deformations of $\vee$ -systems of the form (4.1) are described by the linear $\vee$ -conditions (4.3), (4.6). For projectively rigid matroidal types this describes all infinitesimal deformations of a given $\vee$ -system.*

Case by case check of the $\vee$ -systems from the Appendix leads to the following

Theorem 5. *All rank three vector matroids corresponding to known irreducible 3D $\vee$ -systems are projectively rigid. The H_3 matroid is the only one, which is not strongly projectively rigid.*

Let us show that the largest known case (H_4, A_1) is strongly projectively rigid. We will use the labelling of the points shown at the last figure of the paper. Fix the positions of the four points

6, 25, 27, 30 forming a projective basis in $\mathbb{R}P^2$. After this all the remaining points can be reconstructed uniquely as follows:

$$31 = (25, 30) \wedge (6, 27), \quad 29 = (25, 27) \wedge (6, 30), \quad 12 = (6, 25) \wedge (27, 30),$$

$$9 = (30, 27) \wedge (29, 31), \quad 17 = (25, 21) \wedge (6, 30), \quad 28 = (17, 27) \wedge (12, 25),$$

$$4 = (28, 31) \wedge (12, 30), \quad 24 = (17, 27) \wedge (25, 30), \quad 3 = (25, 30) \wedge (28, 29),$$

$$23 = (3, 28) \wedge (27, 31), \quad 11 = (27, 31) \wedge (28, 30), \quad 19 = (3, 9) \wedge (4, 8),$$

$$1 = (25, 30) \wedge (11, 12), \quad 16 = (7, 11) \wedge (1, 19), \quad 20 = (16, 25) \wedge (4, 8),$$

$$7 = (1, 4) \wedge (25, 27), \quad 21 = (25, 17) \wedge (7, 11), \quad 10 = (7, 31) \wedge (4, 25),$$

$$26 = (7, 31) \wedge (4, 16), \quad 14 = (4, 16) \wedge (11, 31), \quad 18 = (21, 31) \wedge (28, 25),$$

$$22 = (7, 31) \wedge (3, 28), \quad 15 = (7, 31) \wedge (24, 28), \quad 2 = (15, 19) \wedge (10, 11),$$

$$5 = (24, 28) \wedge (21, 31), \quad 13 = (7, 25) \wedge (12, 24), \quad 8 = (1, 12) \wedge (10, 16).$$

A direct computation shows that in case of the classical systems A_3 and B_3 the linear system (4.2), (4.5) has corank four in agreement with the results of the previous section.

The analysis of the linearised $\vee$ -conditions for the families $D_3(t, s)$, $F_3(t)$, $G_3(t)$ and $(AB_4(t), A_1)_{1,2}$ shows that these families of $\vee$ -systems can not be extended.

Consider, for example, the family of $\vee$ -systems $D_3(t, s)$ from [5] with

$$A = \begin{pmatrix} 1 & 1 & 1 & 1 & 0 & 0 & \sqrt{2}\sqrt{s+t-1} \\ 1 & -1 & -1 & 1 & \sqrt{2}\sqrt{\frac{s-t+1}{t}} & 0 & 0 \\ 1 & -1 & 1 & -1 & 0 & \sqrt{2}\sqrt{\frac{-s+t+1}{s}} & 0 \end{pmatrix}$$

with real parameters s, t such that $|s-t| < 1$, $s+t > 1$. Matrices G and X have the form

$$G = \begin{pmatrix} 2(s+t+1) & 0 & 0 \\ 0 & \frac{2(s+t+1)}{t} & 0 \\ 0 & 0 & \frac{2(s+t+1)}{s} \end{pmatrix}$$

$$X = \begin{pmatrix} \xi_1 + \xi_2 + \xi_3 + \xi_4 + 2\xi_7(s+t-1) & \xi_1 - \xi_2 - \xi_3 + \xi_4 & \xi_1 - \xi_2 + \xi_3 - \xi_4 \\ \xi_1 - \xi_2 - \xi_3 + \xi_4 & \xi_1 + \xi_2 + \xi_3 + \xi_4 + \frac{2(s+1)}{t}\xi_5 - 2\xi_5 & \xi_1 + \xi_2 - \xi_3 - \xi_4 \\ \xi_1 - \xi_2 + \xi_3 - \xi_4 & \xi_1 + \xi_2 - \xi_3 - \xi_4 & \xi_1 + \xi_2 + \xi_3 + \xi_4 + \frac{2(-s+t+1)}{s}\xi_6 \end{pmatrix}.$$

For the three covectors $\alpha_5, \alpha_6, \alpha_7$ the first linearised $\vee$ -conditions $X(\alpha_i^\vee, \alpha_j^\vee) = 0$, $i, j = 5, 6, 7$ are equivalent to

$$\xi_1 + \xi_2 - \xi_3 - \xi_4 = 0,$$

$$\xi_1 - \xi_2 - \xi_3 + \xi_4 = 0,$$

$$\xi_1 - \xi_2 + \xi_3 - \xi_4 = 0,$$

which imply that $\xi_1 = \xi_2 = \xi_3 = \xi_4$.

For the planes with more than two covectors we have the linear system

$$(s+t)(\xi_1(s+t+1) + \xi_2(s+t-3) + t(\xi_3(2s+3) + \xi_4(2s-1) - 2(s(\xi_5 + \xi_6) + \xi_5 + \xi_7)) \\ + t^2(-(\xi_3 + \xi_4 - 2\xi_5)) - s(s(\xi_3 + \xi_4 - 2\xi_6) + \xi_3 - 3\xi_4 + 2(\xi_6 + \xi_7)) + 2\xi_7) = 0,$$

$$(s+t)((s-1)(\xi_2 - \xi_7 + s(\xi_6 - \xi_4)) + t(\xi_2 + \xi_3 - \xi_5 - \xi_7 + s(\xi_3 + \xi_4 - \xi_5 - \xi_6)) \\ + t^2(-\xi_3 + \xi_5)) = 0,$$

$$(s+1)t(s(\xi_1 - \xi_2 + \xi_3 + 3\xi_4 - 2(\xi_5 + \xi_6)) + \xi_1 + 3\xi_2 + \xi_3 - \xi_4 - 2(\xi_5 + \xi_7)) + (s+1) \\ t^2(\xi_1 - 3\xi_3 + 2\xi_5) + (s^2 - 1)(-\xi_2(s-1) - \xi_4(s-1) + 2\xi_6s - 2\xi_7) = 0,$$

$$(s+1)t(\xi_2 + s(\xi_3 + \xi_4 - \xi_5 - \xi_6) + \xi_3 - \xi_5 - \xi_7) + (s^2 - 1)(\xi_2 + s(\xi_6 - \xi_4) - \xi_7) \\ + (s+1)t^2(-(\xi_3 - \xi_5)) = 0,$$

$$(t+1)(-t(s(\xi_1 + \xi_4 - 2(\xi_5 + \xi_6)) + \xi_3(3s+2) - 2(\xi_5 + \xi_7)) + s(-s(\xi_1 + 2\xi_6) + \xi_3 \\ + \xi_4(3s-1) + 2(\xi_6 + \xi_7)) - \xi_1s + \xi_2s(t-3) + (t-1)^2) + t^2(\xi_3 - 2\xi_5) + \xi_3 - 2\xi_7 = 0,$$

$$(t+1)(t(\xi_2 + s(\xi_3 + \xi_4 - \xi_5 - \xi_6) + \xi_3 - \xi_5 - \xi_7) + (s-1)(\xi_2 + s(\xi_6 - \xi_4) \\ - \xi_7) + t^2(\xi_5 - \xi_3)) = 0,$$

$$(t+1)(\xi_1(s(t-3) + (t-1)^2) - t(s(\xi_2 + \xi_3 - 2(\xi_5 + \xi_6)) + \xi_4(3s+2) - 2(\xi_5 + \xi_7)) \\ - s(s(\xi_2 - 3\xi_3 + 2\xi_6) + \xi_2 + \xi_3 - \xi_4 - 2(\xi_6 + \xi_7)) + t^2(\xi_4 - 2\xi_5) + \xi_4 - 2\xi_7) = 0,$$

$$(t+1)(t(\xi_1 + s(\xi_3 + \xi_4 - \xi_5 - \xi_6) + \xi_4 - \xi_5 - \xi_7) + (s-1)(\xi_1 + s(\xi_6 - \xi_3) \\ - \xi_7) + t^2(\xi_5 - \xi_4)) = 0,$$

$$(s+1)t(\xi_1(s-3) - s(\xi_2 + 3\xi_3 + \xi_4 - 2(\xi_5 + \xi_6)) - \xi_2 + \xi_3 - \xi_4 + 2(\xi_5 + \xi_7)) \\ + (s^2 - 1)(\xi_1(s-1) + \xi_3(s-1) - 2\xi_6s + 2\xi_7) + (s+1)t^2(-(\xi_2 - 3\xi_4 + 2\xi_5)) = 0,$$

$$(s+1)t(\xi_1 + s(\xi_3 + \xi_4 - \xi_5 - \xi_6) + \xi_4 - \xi_5 - \xi_7) + (s^2 - 1)(\xi_1 + s(\xi_6 - \xi_3) \\ - \xi_7) + (s+1)t^2(-(\xi_4 - \xi_5)) = 0,$$

$$(s+t)(t(\xi_1(2s+3) + \xi_2(2s-1) + \xi_4 - 2(s(\xi_5 + \xi_6) + \xi_5 + \xi_7)) + s(-s(\xi_1 + \xi_2 - 2\xi_6) + \xi_4 - 2(\xi_6 + \xi_7)) + t^2(-(\xi_1 + \xi_2 - 2\xi_5)) - \xi_1 s + 3\xi_2 s + \xi_3(s+t+1) - 3\xi_4 + 2\xi_7) = 0,$$

$$(s+t)(t(s(\xi_1 + \xi_2 - \xi_5 - \xi_6) + \xi_1 + \xi_4 - \xi_5 - \xi_7) + t^2(\xi_5 - \xi_1) + (s-1)(s(\xi_6 - \xi_2) + \xi_4 - \xi_7)) = 0.$$

A check with Mathematica shows that the co-rank of the total system is three for every admissible values of s and t . The free parameters correspond to two deformation parameters s and t and the uniform scaling of the system.

This approach with the use of Mathematica (see the programme in Appendix B to [15]) allows us to prove that the isolated examples of $\vee$ -systems from the list [5] are indeed isolated.

Theorem 6. *There are no non-trivial deformations of the $\vee$ -systems $(E_7, A_1^2 \times A_2)$, $(E_8, A_2 \times A_3)$, $(E_8, A_2^2 \times A_1)$, $(E_8, A_1^3 \times A_2)$, $(E_8, A_1^2 \times A_3)$, $(E_8, A_1 \times A_4)$, (H_4, A_1) and H_3 .*

5. Matroidal Structure of $\vee$ -Systems and Projective Geometry

The main part of the classification problem is to characterise the corresponding class of possible matroids. This question was addressed by Lechtenfeld et al in [9]. They developed a Mathematica program, which generates simple and connected matroids of a given size of the ground set X . If a generated matroid has a vector representation, they have checked first if the orthogonality $\vee$ -conditions are possible to satisfy before verification of the $\vee$ -conditions for the non-trivial planes (all 2-flats). For matroids with $n < 10$ elements the orthogonality conditions are strong enough to identify all matroids corresponding to $\vee$ -systems in dimensions three. All the identified $\vee$ -systems turned out to be part of the list in [4].

For larger matroids this approach seems unworkable because of the unreasonably large computer time required. This means that we need a more conceptual approach, which is still missing.

In this section we collect some partial observations based on the analysis of the known 3D $\vee$ -systems and projective geometry.

We start with the notion of *extension and degeneration* for $\vee$ -systems.

Let $\mathcal{A}_1, \mathcal{A}_2 \subset \mathbf{V}^*$ be two $\vee$ -systems. If $\mathcal{A}_2 \subset \mathcal{A}_1$ we call $\mathcal{A}_1$ an *extension* of $\mathcal{A}_2$.

Let $\vee$ -system $\mathcal{A} = \mathcal{A}_t$ depend on the parameter t . Assume that for some $t = t_0$ one or more of the covectors $\alpha \in \mathcal{A}_{t_0}$ vanishes. In that case the system $\widetilde{\mathcal{A}} = \lim_{t \rightarrow t_0} \mathcal{A}(t)$ is called *degeneration* of $\mathcal{A}(t)$. A reverse process we will call *regeneration*.

In the tables below we give the list of all extensions and degenerations for known three-dimensional $\vee$ -systems from the catalogue in the Appendix.

Table 1. Extensions of known 3D $\vee$ -systems.

$\vee$ -system	Extension	The added covectors
A_3	$F_3(t)$	$\{1, 2, 3, 10, 11, 12, 13\}$
A_3	$(AB_4(t), A_1)_2$	$\{1, 2, 3, 10\}$
(E_6, A_1^3)	$(E_8, A_1^3 \times A_2)$	$\{3, 4, 5, 6, 9, 14, 15, 16, 17\}$
$G_3(\frac{3}{2})$	$(E_8, A_1^3 \times A_2)$	$\{1, 2, 10, 11, 12, 13\}$
H_3	(H_4, A_1)	$\{4, 5, 6, 7, 20, 21, 22, 23, 24, 25, 26, 27, 28, 29, 30\}$

Table 2. Degenerations of known 3D $\vee$ -systems.

$\vee$ -system	Degeneration	The vanishing covectors
$F_3(t)$	$\lim_{t \rightarrow 0} F_3(t) \sim B_3(\sqrt{2})$	$\{11, 12, 13, 14\}$
$F_3(t)$	$\lim_{t \rightarrow \infty} F_3(t) \sim D_3(1, 1)$	$\{8, 9, 10, 11, 12, 13\}$
$B_3(c, c, c; \gamma)$	$\lim_{\gamma \rightarrow c} B_3(c, c, c; \gamma) \sim A_3$	$\{1, 2, 3\}$
$(AB_4(t), A_1)_2$	$\lim_{t \rightarrow \infty} (AB_4(t), A_1)_2 \sim D_3(1, 1)$	$\{5, 7, 9\}$
$(AB_4(t), A_1)_2$	$\lim_{t \rightarrow 0} (AB_4(t), A_1)_2 \sim B_3(\sqrt{2})$	$\{10\}$
$(AB_4(t), A_1)_1$	$\lim_{t \rightarrow \frac{1}{\sqrt{2}}} (AB_4(t), A_1)_1 \sim (E_6, A_1^3)$	$\{3\}$
$\frac{1}{t}(AB_4(t), A_1)_1$	$\lim_{t \rightarrow \infty} (AB_4(t), A_1)_1 \sim B_3(\sqrt{2})$	$\{5, 6\}$
$G_3(t)$	$\lim_{t \rightarrow \frac{1}{2}} G_3(t) \sim (E_6, A_1^3)$	$\{4, 5, 6\}$
$D_3(t, s)$	$\lim_{t \rightarrow (s+1)} D_3(t, s) \sim A_3$	$\{5\}$
$B_3(c_1, c_2, c_3, \gamma)$	$\lim_{c_1 \rightarrow -\gamma} B_3(c_1, c_2, c_3, \gamma) \sim (E_6, A_3)$	$\{1\}$
$B_3(c_1, c_2, c_3, \gamma)$	$\lim_{c_1, c_2 \rightarrow -\gamma} B_3(c_1, c_2, c_3, \gamma) \sim D_3(1, 1)$	$\{1, 2\}$

More relations between $\vee$ -realisable matroids can be seen using projective geometry, in particular projective duality. We will demonstrate this on few examples.

We start with the matroid of the $\vee$ -system of type A_3 . In projective geometry (see e.g. [8]) it is known as the simplest configuration $(6_2 4_3)$ consisting of four lines with three points on each line and two lines passing through every point. Its projective dual is a complete quadrangle $(4_3 6_2)$ consisting of four points, no three of which are collinear and six lines connecting each pair of points (see figure 4). If we extend the dual configuration by adding the remaining three points of intersections of lines (the points marked white in the graphic), we come to the projective configuration of seven points and six lines, corresponding to the matroid of the $\vee$ -system of type D_3 .

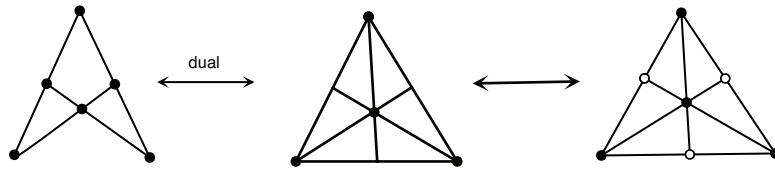


Fig. 4. The projective configuration of A_3 type, its dual and the extended configuration corresponding to $\vee$ -system of type D_3 .

We can proceed the construction by taking the dual of the new obtained configuration and extending it by adding the missing points of intersections of lines. The result is the configuration of nine points and seven lines realisable as B_3 -type $\vee$ -system (see figure 5).

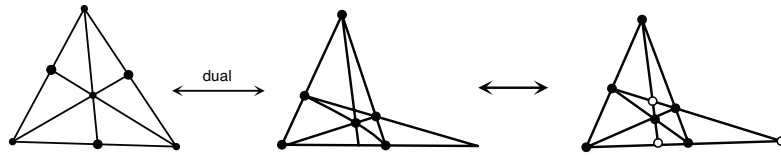


Fig. 5. D_3 configuration, its projective dual and the extended configuration of matroidal type B_3 .

The next step of the construction is demonstrated in figure 6. The dual configuration was obtained from the configuration D_3 by adding all missing lines passing through any pair of points. Applying Desargue's theorem to two marked triangles we see that the white marked points of the extended configuration are collinear. The new configuration of 10 points and 10 lines is self-dual and corresponds to the $\vee$ -system of type $(AB_4(t), A_1)_2$ (see system 9.6 in the Appendix).

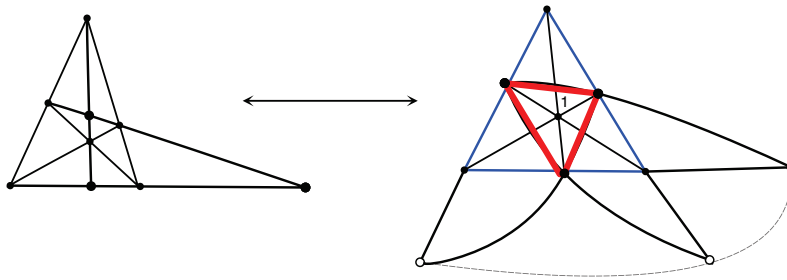


Fig. 6. B_3 -configuration and its schematic extended projective dual, corresponding to $\vee$ -system of type $(AB_4(t), A_1)_2$. Due to Desargue's theorem the three added (white) points are collinear.

One can check that the adding of three intersection points with red lines and three lines connecting them pairwise leads to the configuration of F_3 type (see system 9.10). However, if we add also the three intersection points with dotted line then we come to the configuration which can be shown to be not $\vee$ -realisable.

Although this relation with projective configurations and theorems in projective geometry looks quite promising, we see that the extension procedure is not straightforward and does not guarantee the $\vee$ -realisability of the resulting configuration.

We conclude with the following conjecture about 2-flats with precisely four points. Recall that four points A, B, C, D on a projective line form a *harmonic range* if the cross-ratio $(A, B; C, D) = -1$. The corresponding pencil of four lines on a plane is called *harmonic bundle*. The B_3 configuration provides a geometric way to construct harmonic ranges: on Fig. 1 the points 3,4,9,8 always form a harmonic range. Note that the covectors 8 and 9 are orthogonal and determine the bisectors for the lines corresponding to covectors 3 and 4. Case by case check of the known 3D $\vee$ -systems suggests that the same is true in general.

Conjecture 1. *Let $\mathcal{A}$ be a $\vee$ -system and $\Pi_{\mathcal{A}} \subset V^*$ be two-dimensional plane containing exactly four covectors $\alpha_i \in \mathcal{A}$, $i = 1, \dots, 4$, then the corresponding four lines form a harmonic bundle with two orthogonals.*

6. $\vee$ -Function, Uniqueness and Rigidity Conjectures

Let M be a matroid and $\mathcal{A}$ be its $\vee$ -system realisation. Such a realisation defines the $\vee$ -function on the 2-flats of M , where $\vee$ is the coefficient in the $\vee$ -conditions (1.1) corresponding to the plane Π representing the flat.

Conjecture 2. (Uniqueness Conjecture) *An irreducible $\vee$ -system $\mathcal{A}$ is uniquely determined modulo linear group $GL(V^*)$ by its matroid M and the corresponding $\vee$ -function on its flats.*

A weaker version of the conjecture is

Conjecture 3. (Rigidity Conjecture) *An irreducible $\vee$ -system $\mathcal{A}$ is locally uniquely determined by its matroid M and the corresponding $\vee$ -function on its flats.*

If the function $\vee$ is fixed under deformation then $\dot{\vee} = 0$ and the corresponding $\vee$ -conditions are

$$X(\alpha^\vee, \beta^\vee) = 0 \quad (6.1)$$

for α, β be the only two covectors in the plane, and

$$X^\Pi - \vee X = 0 \mid_{\Pi^\vee} \quad (6.2)$$

for any plane Π containing more than two covectors from $\mathcal{A}$.

Conjecturally this should imply that $X = cG$ corresponding to the global scaling of the system.

Case by case check from the list in the Appendix leads to the following

Theorem 7. *Both conjectures are true for all known $\vee$ -systems in dimension three.*

We have also the following conjecture based on the analysis of the list of extensions of $\vee$ -systems from the previous section.

Conjecture 4. (Extension Conjecture) *For any irreducible $\vee$ -system and its extension the values of the $\vee$ -functions on the corresponding flats are proportional.*

One can check that this is true for all known cases. For example, for the extension $H_3 \subset (H_4, A_1)$ we have the set of values

$$\{3/10, 1/2\} = 3 \times \{1/10, 1/6\}.$$

Now we present some results about $\vee$ -functions for $\vee$ -systems.

First we give the following, more direct geometric way to compute $\vee(\Pi)$. The form $G_{\mathcal{A}}$ on V defines the scalar product on V^* and thus the norm $|\alpha|$, $\alpha \in V^*$.

Theorem 8. *For every plane $\Pi \subset V^*$ containing more than two covectors α from a $\vee$ -system $\mathcal{A}$*

$$\vee(\Pi) = \frac{1}{2} \sum_{\alpha \in \Pi \cap \mathcal{A}} |\alpha|^2. \quad (6.3)$$

Proof. From the $\vee$ -conditions (1.1) we have

$$\sum_{\alpha \in \Pi \cap \mathcal{A}} \alpha^\vee \otimes \alpha \mid_{\Pi^\vee} = v(\Pi) I \mid_{\Pi^\vee}.$$

Taking the trace of both sides gives (6.3). $\square$

Let $\mathcal{A} \subset V^*$ be a $\vee$ -system generating V^* and consider the set $\mathcal{F}_{\mathcal{A}}$ of 2-flats in the corresponding matroid, which the same as the set of 2D planes $\Pi \subset V^*$ containing more than two covectors from $\mathcal{A}$.

We say that the set of weights x_Π , $\Pi \in \mathcal{F}_{\mathcal{A}}$ is *admissible* if for each $\alpha \in \mathcal{A}$

$$\sum_{\Pi \in \mathcal{F}_{\mathcal{A}}: \alpha \in \Pi} x_\Pi = 1. \quad (6.4)$$

Theorem 9. *For every admissible set of weights we have*

$$\sum_{\Pi \in \mathcal{F}_{\mathcal{A}}} x_\Pi v(\Pi) = \frac{n}{2}, \quad (6.5)$$

where n is the dimension of V .

Proof. We have

$$\sum_{\beta \in \mathcal{A}} \beta^\vee \otimes \beta = \sum_{\beta \in \mathcal{A}} \left(\sum_{\Pi \in \mathcal{F}_{\mathcal{A}}: \beta \in \Pi} x_\Pi \right) \beta^\vee \otimes \beta = \sum_{\Pi \in \mathcal{F}_{\mathcal{A}}} x_\Pi \sum_{\beta \in \Pi \cap \mathcal{A}} \beta^\vee \otimes \beta.$$

From the $\vee$ -condition

$$\sum_{\beta \in \Pi \cap \mathcal{A}} \beta^\vee \otimes \beta = v(\Pi) P_\Pi,$$

where P_Π is the orthogonal projector onto $\Pi^\vee$. Taking trace and using the fact that $\sum_{\beta \in \mathcal{A}} \beta^\vee \otimes \beta = Id$ we obtain (6.5). $\square$

We call (6.5) the *universal relation* for values of function v .

For the $\vee$ -system of type A_3 the universal relation completely describes the set of all possible functions v . Indeed, one can easily see from Fig. 2 that $x_\Pi = 1/2$ is the only admissible weight system, which leads to the universal relation

$$\sum_{\Pi \in \mathcal{F}_{\mathcal{A}}} v(\Pi) = 3.$$

This gives us three free parameters, which are exactly three parameters of deformation.

However, in general universal relations are not strong enough to describe possible v -functions. Moreover, a $\vee$ -system $\mathcal{A}$ may not have admissible weights x_Π at all. For instance, this is the case for the $\vee$ -system of $D_3(t)$ -type (this is however the only exception among known 3D $\vee$ -systems).

The list of all known $\vee$ -systems in dimension three with the corresponding v -functions is given in the Appendix.

7. Concluding Remarks

Although the problem of classification of $\vee$ -systems seems to be very hard, in dimension three it does not look hopeless. As we have seen, matroid theory provides a natural framework for the problem of classification of $\vee$ -systems. For a given matroidal structure the $\vee$ -conditions define a set of algebraic relations on the vector realisations. In case when matroid is strongly projectively rigid we have one free parameter for each vector, which makes possible the full classification of $\vee$ -systems with small number of vectors.

The main problem is to describe all possible matroidal types, which we believe form a finite list in any dimension. The results of Lechtenfeld et al [9] show that the direct computer approach is probably unrealistic for $\vee$ -systems with more than 10 covectors, while we have already in dimension three an example with 31 covectors (system (H_4, A_1) , see 7.16 in the Appendix). In dimension three we have an intriguing relation with the theory of configurations on the projective plane and with the theorems in projective geometry, which also suggests that the final list should be finite.

In the theory of matroids and graphs many families have been proved to be closed under taking minors, thus giving a possibility to reduce the problem of classification to the identification of the forbidden minors. We hope that a similar approach could be fruitful for classification of $\vee$ -systems.

Another result from matroid theory, which could be relevant, is Seymour's decomposition theorem [18], which states that all regular matroids can be build up in a simple way as sums of certain type of graphic matroids, their duals, and one special matroid on 10 elements. Our analysis of degenerations and extensions of $\vee$ -systems suggests a possibility of a similar result for the $\vee$ -realisable matroids.

8. Acknowledgements

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9. Appendix. Catalogue of all Known Real 3-Dimensional $\vee$ -Systems

Each 3D $\vee$ -system $\mathcal{A}$ is presented below by the matrix with columns giving the covectors of the system (the first row is simply the labelling of the covectors). We give the graphical representation of the corresponding matroid with the list of orthogonal pairs, 2-flats, the form G and the values of v -function. The ordering of the list is according to the number of covectors in the system. The parameters are assumed to be chosen in such a way that all the covectors are real and non-zero.

Below is a schematic way to present all known $\vee$ -systems in dimension three taken from [5].

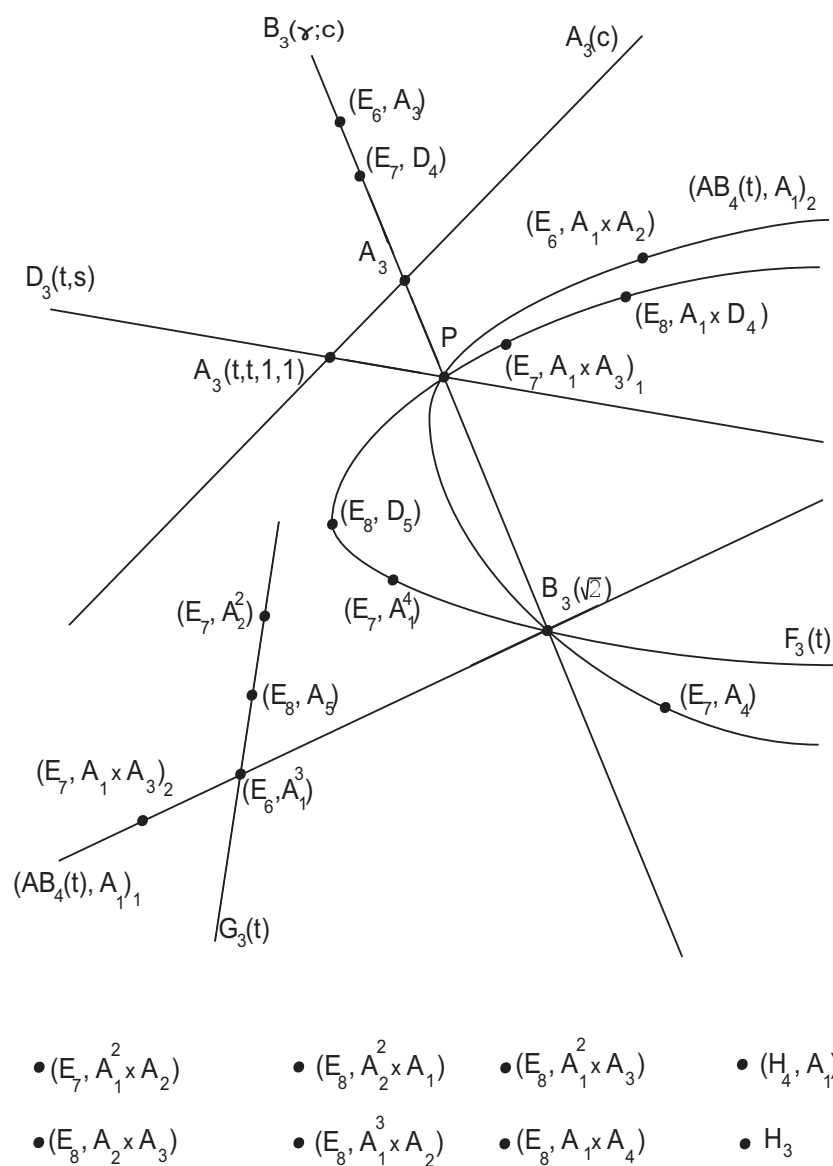


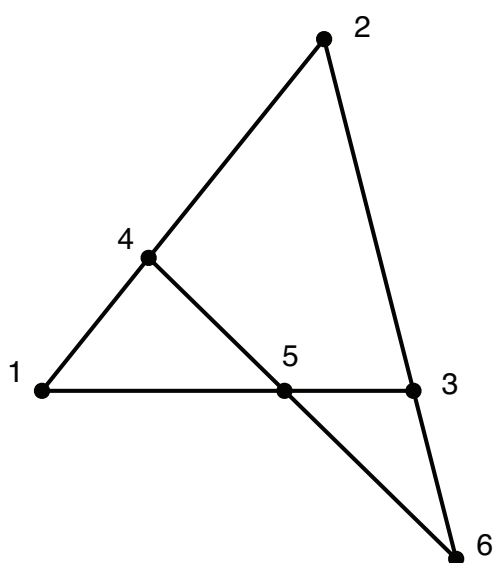
Fig. 7. The map of all known 3-dimensional $\vee$ -systems from [5].

We use here the notations from [4, 5]. In particular, for a Coxeter group G and its parabolic subgroup H (G, H) denotes the corresponding $\vee$ -system given by the restriction procedure [4]. When the type of the subgroup does not fix the subgroup up to a conjugation the index 1 or 2 is used to distinguish them.

The $\vee$ -systems of type AB_4 , G_3 and D_3 are related to the exceptional generalised root systems $AB(1, 3)$, $G(1, 2)$ and $D(2, 1, \lambda)$ appeared in the theory of basic classical Lie superalgebras [16, 17].

9.1. $\vee$ -systems $A_3(c_1, c_2, c_3)$

$$\mathcal{A} = \begin{bmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ \sqrt{c_1} & 0 & 0 & -\sqrt{c_1 c_2} & -\sqrt{c_1 c_3} & 0 \\ 0 & \sqrt{c_2} & 0 & \sqrt{c_1 c_2} & 0 & -\sqrt{c_2 c_3} \\ 0 & 0 & \sqrt{c_3} & 0 & \sqrt{c_1 c_3} & \sqrt{c_2 c_3} \end{bmatrix}$$



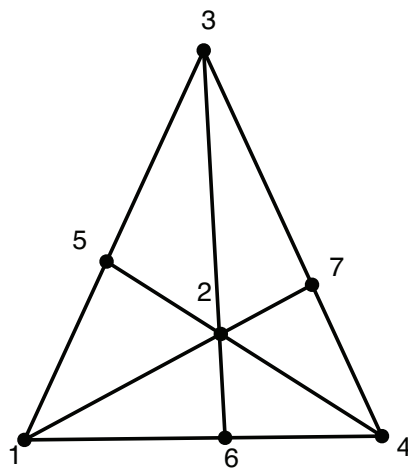
$$G = \begin{bmatrix} c_1(1+c_2+c_3) & -c_1 c_2 & -c_1 c_3 \\ -c_1 c_2 & c_2(1+c_1+c_3) & -c_2 c_3 \\ -c_1 c_3 & -c_2 c_3 & c_3(1+c_1+c_2) \end{bmatrix}$$

$$\mathcal{I}_2 = \{(1, 6), (2, 5), (3, 4)\}$$

$$\mathcal{I}_3 = \begin{cases} (1, 2, 4) & v_3 = \frac{1+c_1+c_2}{1+c} \\ (1, 3, 5) & v_2 = \frac{1+c_1+c_3}{1+c} \\ (2, 3, 6) & v_1 = \frac{1+c_2+c_3}{1+c} \\ (4, 5, 6), & v_4 = \frac{c}{1+c}, c = c_1 + c_2 + c_3. \end{cases}$$

9.2. $\vee$ -system $D_3(t, s)$

$$\mathcal{A} = \begin{bmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 1 & 1 & 1 & 1 & 0 & 0 & \sqrt{2(s+t-1)} \\ 1 & -1 & -1 & 1 & \sqrt{\frac{2(s-t+1)}{t}} & 0 & 0 \\ 1 & -1 & 1 & -1 & 0 & \sqrt{\frac{2(t-s+1)}{s}} & 0 \end{bmatrix}$$



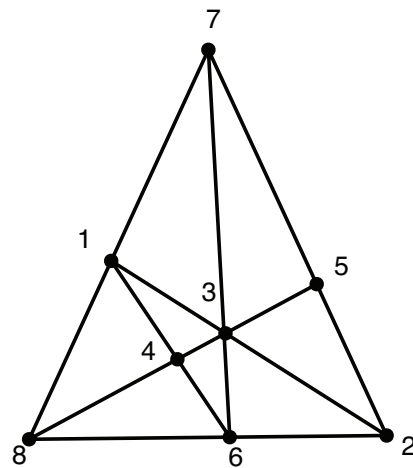
$$G = 2(1+s+t) \begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{t} & 0 \\ 0 & 0 & \frac{1}{s} \end{bmatrix}$$

$$\mathcal{I}_2 = \{(5, 6), (5, 7), (6, 7)\}$$

$$\mathcal{I}_3 = \begin{cases} (1, 2, 7), (3, 4, 7) & v_{31} = \frac{s+t}{1+s+t} \\ (1, 3, 5), (2, 4, 5) & v_{32} = \frac{1+s}{1+s+t} \\ (1, 4, 6), (2, 3, 6) & v_{33} = \frac{1+t}{1+s+t} \end{cases}$$

9.3. $\vee$ -system (E_6, A_3)

$$\mathcal{A} = \begin{bmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 2 & 2 & 2\sqrt{6} & 2 & -2 & 2 & -2 & 0 \\ 2 & -2 & 0 & \frac{1}{2} & \frac{1}{2} & -1 & -1 & \frac{\sqrt{6}}{2} \\ 0 & 0 & 0 & \frac{1}{2} & \frac{1}{2} & 1 & 1 & \frac{\sqrt{6}}{2} \end{bmatrix}$$



$$G = 4 \begin{bmatrix} 12 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

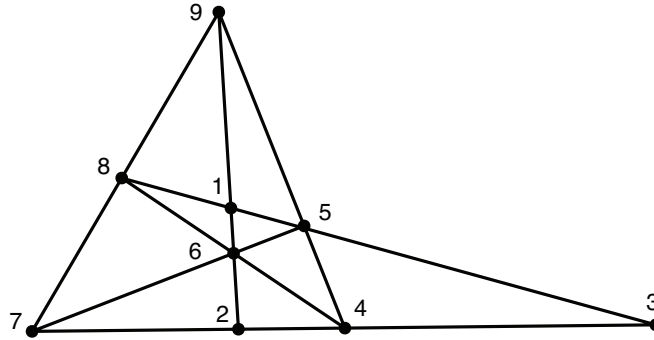
$$\mathcal{I}_2 = \{(1, 5), (2, 4), (4, 7), (5, 6)\}$$

$$\mathcal{I}_3 = \begin{cases} (2, 5, 7), (1, 4, 6) & v_{31} = \frac{1}{2} \\ (6, 3, 7), (8, 1, 7), (1, 2, 3), (8, 6, 2) & v_{32} = \frac{2}{3} \end{cases}$$

$$\mathcal{I}_4 = \{(8, 4, 3, 5) \quad v_4 = \frac{2}{3}$$

9.4. $\vee$ -systems of $B_3(c_1, c_2, c_3, \gamma)$

$$\mathcal{A} = \begin{bmatrix} & 1 & & 2 & & 3 & & & \\ \sqrt{2c_1(c_1+\gamma)} & & 0 & & 0 & & \dots & & \\ & 0 & \sqrt{2c_2(c_2+\gamma)} & & 0 & & & & \\ & 0 & & 0 & \sqrt{2c_3(c_3+\gamma)} & & & & \\ & 4 & 5 & 6 & 7 & 8 & 9 & & \\ \dots & 0 & \sqrt{c_1c_3} & \sqrt{c_1c_2} & 0 & -\sqrt{c_1c_3} & -\sqrt{c_1c_2} & & \\ & \sqrt{c_2c_3} & 0 & \sqrt{c_1c_2} & -\sqrt{c_2c_3} & 0 & \sqrt{c_1c_2} & & \\ & \sqrt{c_2c_3} & \sqrt{c_1c_3} & 0 & \sqrt{c_2c_3} & \sqrt{c_1c_3} & 0 & & \end{bmatrix}$$



$$G = 2(c_1 + c_2 + c_3 + \gamma) \begin{bmatrix} c_1 & 0 & 0 \\ 0 & c_2 & 0 \\ 0 & 0 & c_3 \end{bmatrix}$$

$$\mathcal{J}_2 = \{(1, 4), (1, 7), (2, 5), (2, 8), (3, 6), (3, 9)\}$$

Four 3-point lines:

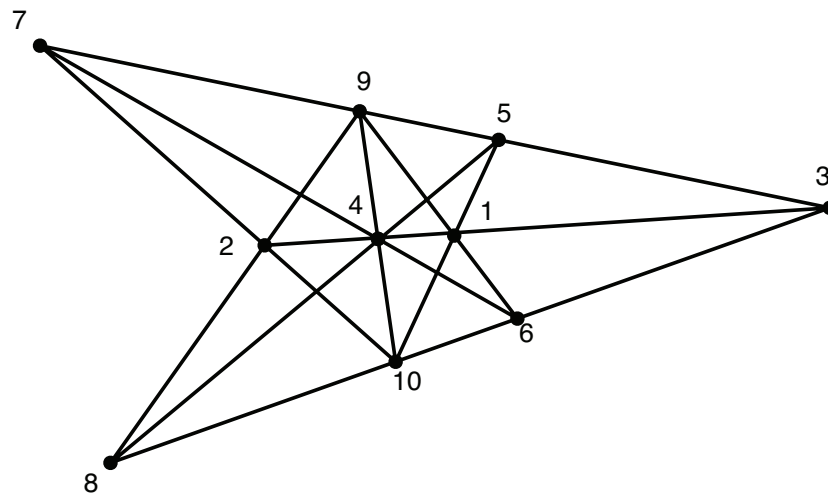
$$\mathcal{J}_3 = \{(4, 5, 9), (4, 6, 8), (5, 6, 7), (7, 8, 9)\}, \quad v_3 = \frac{\sum c_i}{2(\gamma + \sum_{i=1}^3 c_i)}$$

Three 4-point lines:

$$\mathcal{J}_4 = \{(1, 2, 6, 9), (1, 3, 5, 8), (2, 3, 4, 7)\}, \quad v_{4j} = \frac{\gamma - c_j + \sum c_i}{\gamma + \sum_{i=1}^3 c_i}, \quad j=1, 2, 3$$

9.5. $\vee$ -system (E_6, A_1^3)

$$\mathcal{A} = \begin{bmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 \\ \sqrt{2} & \sqrt{2} & 2\sqrt{3} & 0 & \sqrt{2} & -\sqrt{2} & -\sqrt{2} & \sqrt{2} & 0 & 0 \\ \sqrt{2} & -\sqrt{2} & 0 & 2 & \frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} & 1 & -1 \\ 0 & 0 & 0 & 0 & \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} & 1 & 1 \end{bmatrix}$$



$$G = \begin{bmatrix} 24 & 0 & 0 \\ 0 & 12 & 0 \\ 0 & 0 & 4 \end{bmatrix}$$

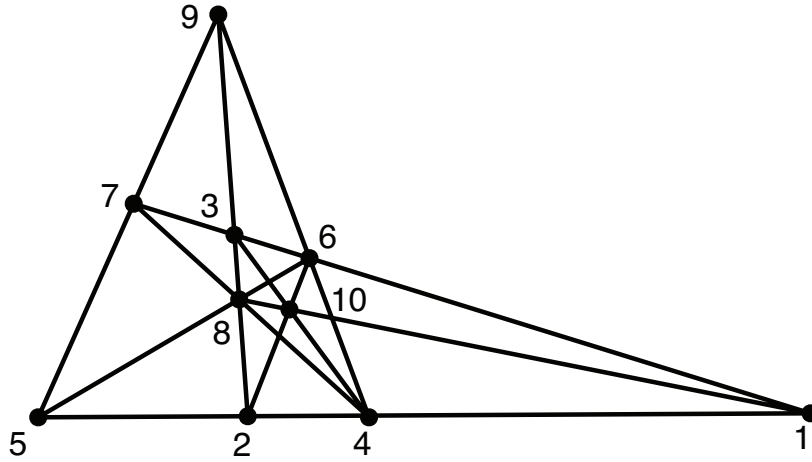
$$\mathcal{I}_2 = \{(1, 7), (1, 8), (2, 5), (2, 6), (5, 6), (7, 8)\}$$

$$\mathcal{I}_3 = \begin{cases} (10, 2, 7), (4, 6, 7), (9, 8, 2), (1, 5, 10), (4, 5, 8), (9, 1, 6), & v_{31} = \frac{5}{12} \\ (4, 9, 10), & v_{32} = \frac{1}{2} \end{cases}$$

$$\mathcal{I}_4 = \{(4, 1, 2, 3), (9, 5, 3, 7), (6, 3, 8, 10)\}, v_4 = \frac{2}{3}$$

9.6. $\vee$ -system $(AB_4(t), A_1)_2$

$$\mathcal{A} = \begin{bmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 \\ \sqrt{2} & 0 & 0 & 1 & \frac{1}{\sqrt{4t^2+1}} & 1 & \frac{1}{\sqrt{4t^2+1}} & 0 & 0 & \frac{t\sqrt{2}}{\sqrt{(t^2+1)}} \\ 0 & \sqrt{2} & 0 & 1 & -\frac{1}{\sqrt{4t^2+1}} & 0 & 0 & 1 & \frac{1}{\sqrt{4t^2+1}} & \frac{t\sqrt{2}}{\sqrt{(t^2+1)}} \\ 0 & 0 & \sqrt{2} & 0 & 0 & 1 & -\frac{1}{\sqrt{4t^2+1}} & 1 & -\frac{1}{\sqrt{4t^2+1}} & \frac{t\sqrt{2}}{\sqrt{(t^2+1)}} \end{bmatrix}$$



$$G = \frac{6(1+2t^2)}{(1+t^2)(1+4t^2)} \begin{bmatrix} 1+2t^2 & t^2 & t^2 \\ t^2 & 1+2t^2 & t^2 \\ t^2 & t^2 & 1+2t^2 \end{bmatrix}$$

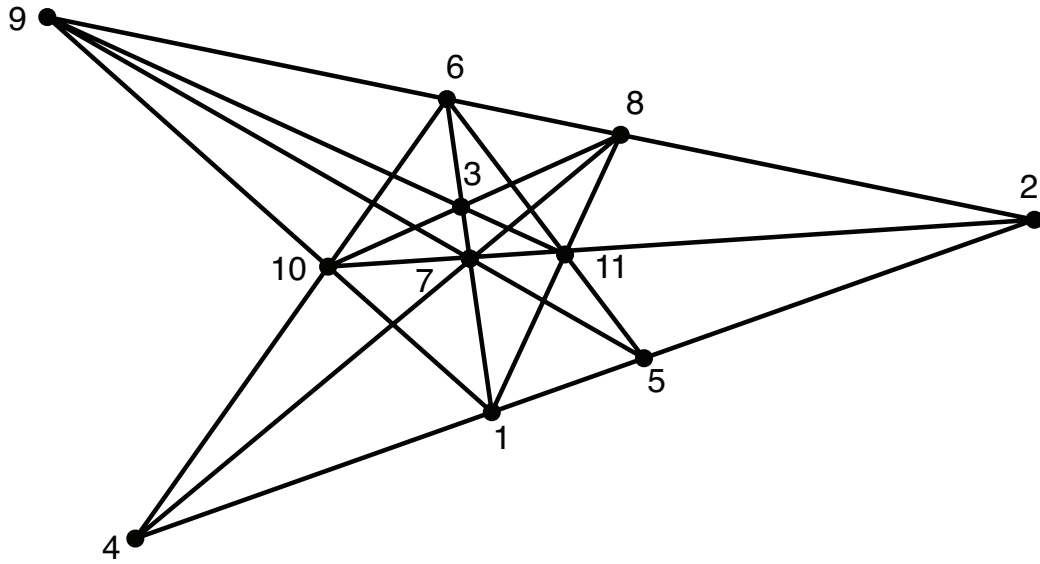
$$\mathcal{J}_2 = \{(1, 9), (2, 7), (3, 5), (5, 10), (7, 10), (9, 10)\}$$

$$\mathcal{J}_3 = \begin{cases} (1, 8, 10), (2, 6, 10), (3, 4, 10) & v_{31} = \frac{1+4t^2}{3(1+2t^2)} \\ (4, 6, 9), (4, 7, 8), (5, 6, 8) & v_{32} = \frac{3+4t^2}{6(1+2t^2)} \\ (5, 7, 9) & v_{33} = \frac{1}{2(1+2t^2)} \end{cases}$$

$$\mathcal{J}_4 = \{(1, 2, 4, 5), (1, 3, 6, 7), (2, 3, 8, 9)\}, v_4 = \frac{2}{3}$$

9.7. $\vee$ -system $(AB_4(t), A_1)_1$

$$\mathcal{A} = \begin{bmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 \\ \sqrt{2(2t^2+1)} & 0 & 0 & \sqrt{2} & \sqrt{2} & t\sqrt{2} & t\sqrt{2} & t & t & t & t \\ 0 & 2\sqrt{2(t^2+1)} & 0 & \sqrt{2}-\sqrt{2} & 0 & 0 & 2t & -2t & 2t & -2t \\ 0 & 0 & t\sqrt{\frac{2(2t^2-1)}{t^2+1}} & 0 & 0 & t\sqrt{2} & -t\sqrt{2} & t & t & -t & -t \end{bmatrix}$$



$$G = 6 \begin{bmatrix} 1+2t^2 & 0 & 0 \\ 0 & 2+4t^2 & 0 \\ 0 & 0 & \frac{t^2+2t^4}{1+t^2} \end{bmatrix}$$

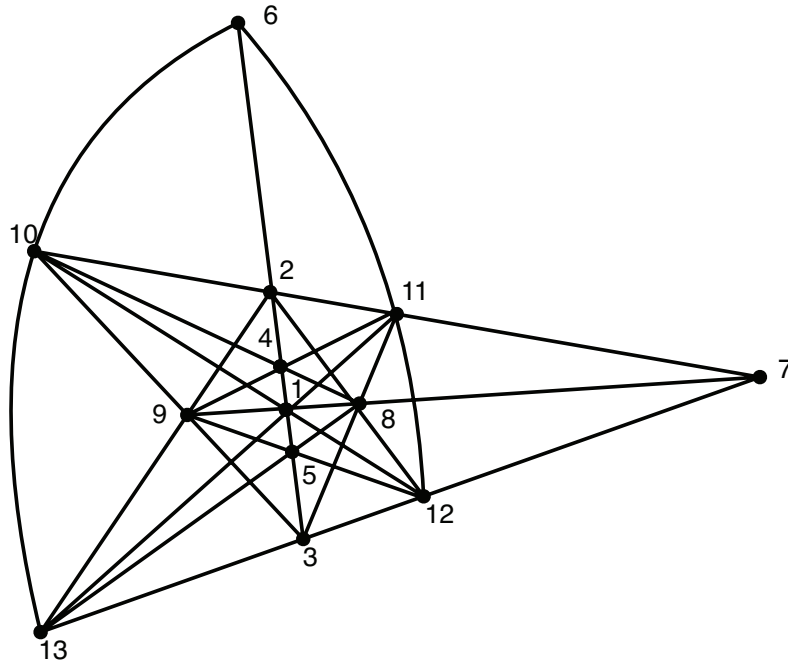
$$\mathcal{J}_2 = \{(2,3), (3,4), (3,5), (4,9), (4,11), (5,8), (5,10)\}$$

$$\mathcal{J}_3 = \begin{cases} (4,6,10), (4,7,8), (5,6,11), (5,7,9) & v_{31} = \frac{3+4t^2}{6(1+2t^2)} \\ (1,8,11), (1,9,10) & v_{32} = \frac{1+3t^2}{3(1+2t^2)} \\ (3,8,10), (3,9,11) & v_{33} = \frac{t^2}{(1+2t^2)} \end{cases}$$

$$\mathcal{J}_4 = \begin{cases} (2,6,8,9), (2,7,10,11) & v_{41} = \frac{2}{3} \\ (1,2,4,5) & v_{42} = \frac{3+2t^2}{3(1+2t^2)} \\ (1,3,6,7) & v_{43} = \frac{1+4t^2}{3(1+2t^2)} \end{cases}$$

9.8. ∇ -system $G_3(t)$

$$\mathcal{A} = \begin{bmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 & 13 \\ \sqrt{2t+1} & 0 & \sqrt{2t+1} & \sqrt{\frac{2t-1}{3}} & 2\sqrt{\frac{2t-1}{3}} & \sqrt{\frac{2t-1}{3}} & 0 & 1 & 1 & 0 & 0 & 1 & 1 \\ 0 & \sqrt{2t+1} & \sqrt{2t+1} & -\sqrt{\frac{2t-1}{3}} & \sqrt{\frac{2t-1}{3}} & 2\sqrt{\frac{2t-1}{3}} & 0 & 0 & 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & \sqrt{\frac{3}{t}} & 1 & -1 & 1 & -1 & 1 & -1 \end{bmatrix}$$



$$G = \begin{bmatrix} 4(1+2t) & 2(1+2t) & 0 \\ 2(1+2t) & 4(1+2t) & 0 \\ 0 & 0 & 3(2+\frac{1}{t}) \end{bmatrix}$$

$$\mathcal{I}_2 = \{(4, 7), (4, 12), (4, 13), (5, 7), (5, 10), (5, 11), (6, 7), (6, 8), (6, 9)\}$$

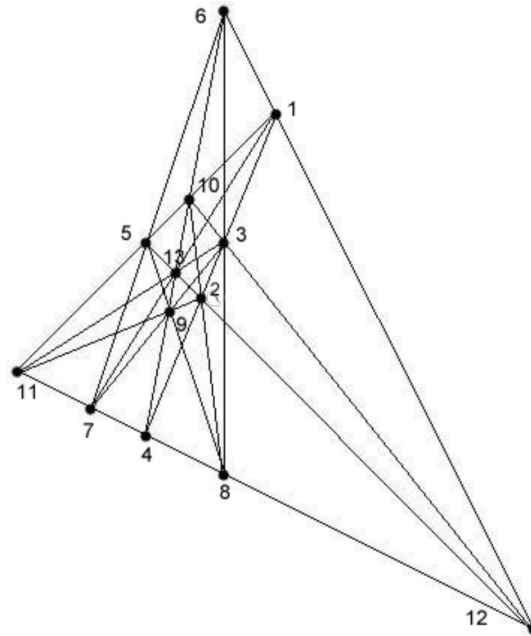
$$\mathcal{I}_3 = \begin{cases} (2, 9, 13), (2, 8, 12), (3, 8, 11), (1, 11, 13), (1, 10, 12), (3, 9, 10) & v_{31} = \frac{3+4t}{6(1+2t)} \\ (4, 8, 10), (6, 11, 12), (6, 10, 13), (5, 9, 12), (5, 8, 13), (4, 9, 11) & v_{32} = \frac{1+4t}{6(1+2t)} \end{cases}$$

$$\mathcal{I}_4 = \{(2, 7, 10, 11), (1, 7, 8, 9), (3, 7, 12, 13)\}, v_4 = \frac{3+2t}{3+6t}$$

$$\mathcal{I}_6 = \{(1, 2, 3, 4, 5, 6)\}, v_6 = \frac{2t}{1+2t}$$

9.9. $\vee$ -system $(E_7, A_1^2 \times A_2)$

$$\mathcal{A} = \begin{bmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 & 13 \\ \sqrt{3} & \sqrt{3} & 2 & 0 & 0 & \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & \sqrt{\frac{3}{2}} & -\sqrt{\frac{3}{2}} & -\sqrt{\frac{3}{2}} & \sqrt{\frac{3}{2}} \\ \sqrt{3} & -\sqrt{3} & 0 & 2\sqrt{6} & 0 & \frac{3}{\sqrt{2}} & -\frac{3}{\sqrt{2}} & \frac{3}{\sqrt{2}} & -\frac{3}{\sqrt{2}} & \sqrt{\frac{3}{2}} & -\sqrt{\frac{3}{2}} & \sqrt{\frac{3}{2}} & -\sqrt{\frac{3}{2}} \\ 0 & 0 & 0 & 0 & 1 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & \sqrt{\frac{3}{2}} & \sqrt{\frac{3}{2}} & \sqrt{\frac{3}{2}} & \sqrt{\frac{3}{2}} \end{bmatrix}$$



$$G = 9 \begin{bmatrix} 2 & 0 & 0 \\ 0 & 6 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\mathcal{I}_2 = \{(1, 8), (1, 9), (2, 6), (2, 7), (3, 5), (4, 5), (6, 11), (7, 10), (8, 13), (9, 12)\}$$

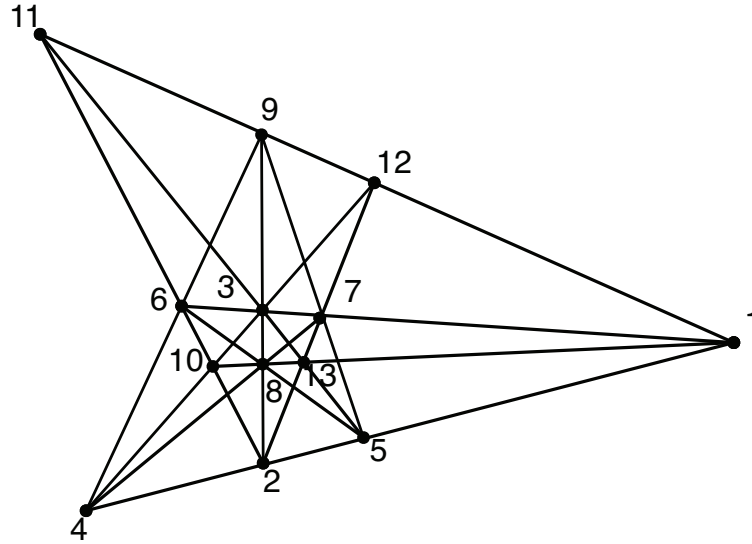
$$\mathcal{I}_3 = \begin{cases} (5, 9, 8), (7, 5, 6) & v_{31} = \frac{2}{9} \\ (10, 3, 12), (11, 13, 3) & v_{32} = \frac{7}{18} \\ (6, 3, 8), (7, 9, 3) & v_{33} = \frac{5}{18} \\ (6, 1, 12), (10, 2, 8), (7, 13, 1), (11, 9, 2) & v_{34} = \frac{1}{3} \end{cases}$$

$$\mathcal{I}_4 = \begin{cases} (5, 13, 2, 12), (11, 5, 10, 1) & v_{41} = \frac{4}{9} \\ (4, 2, 3, 1) & v_{42} = \frac{5}{9} \end{cases}$$

$$\mathcal{I}_5 = \{(11, 7, 4, 8, 12), (6, 10, 13, 9, 4)\}, v_5 = \frac{2}{3}$$

9.10. $\vee$ -system $F_3(t)$

$$\mathcal{A} = \begin{bmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 & 13 \\ \sqrt{4t^2+2} & 0 & 0 & 1 & 1 & 1 & 1 & 0 & 0 & t\sqrt{2} & t\sqrt{2} & t\sqrt{2} & t\sqrt{2} \\ 0 & \sqrt{4t^2+2} & 0 & 1 & -1 & 0 & 0 & 1 & 1 & t\sqrt{2} & -t\sqrt{2} & t\sqrt{2} & -t\sqrt{2} \\ 0 & 0 & \sqrt{4t^2+2} & 0 & 0 & 1 & -1 & 1 & -1 & t\sqrt{2} & t\sqrt{2} & -t\sqrt{2} & -t\sqrt{2} \end{bmatrix}$$



$$G = (6 + 12t^2)I$$

$$\mathcal{I}_2 = \{(4,11), (4,13), (5,10), (5,12), (6,13), (7,10), (7,11), (8,11), (8,12), (9,10), (9,13)\}$$

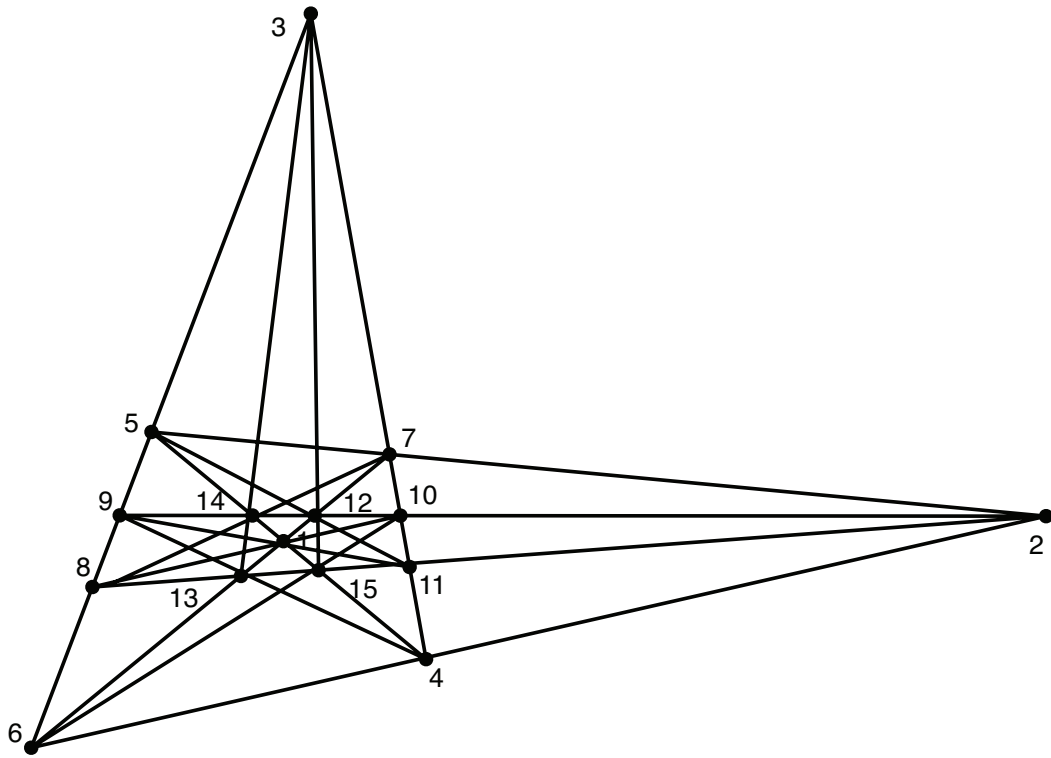
$$\mathcal{I}_3 = \{(4,6,9), (4,7,8), (5,6,8), (5,7,9), \quad v_{31} = \frac{1}{2+4t^2}$$

$$\mathcal{I}_4 = \begin{cases} (1,2,4,5), (1,3,6,7), (2,3,8,9), & v_{32} = \frac{2(1+t^2)}{3+6t^2} \\ (1,8,10,13), (1,9,11,12), (2,6,10,11), (2,7,12,13), & \\ (3,4,10,12), (3,5,11,13), & v_{33} = \frac{1+4t^2}{3+6t^2} \end{cases}$$

9.11. Coxeter $\vee$ -system H_3

$$\mathcal{A} = \begin{bmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 & 13 & 14 & 15 \\ 2\phi & 0 & 0 & 1 & -1 & 1 & 1 & \phi & -\phi & \phi & \phi & \phi^2 & \phi^2 & -\phi^2 & \phi^2 \\ 0 & 2\phi & 0 & \phi & \phi & -\phi & \phi & -\phi^2 & \phi^2 & \phi^2 & \phi^2 & 1 & -1 & 1 & 1 \\ 0 & 0 & 2\phi & \phi^2 & \phi^2 & \phi^2 & -\phi^2 & 1 & 1 & -1 & 1 & -\phi & \phi & \phi & \phi \end{bmatrix}$$

where ϕ is the golden ratio $\phi = \frac{1+\sqrt{5}}{2}$.



$$G = 10(3 + \sqrt{5})I$$

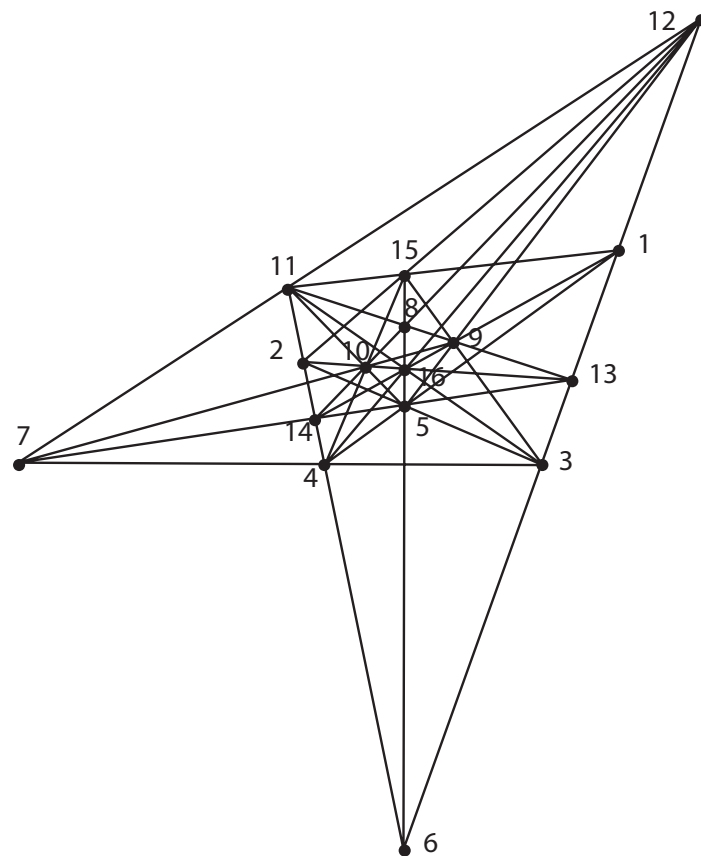
$$\mathcal{J}_2 = \left\{ (1, 2), (1, 3), (2, 3), (4, 8), (4, 12), (5, 10), (5, 13), (6, 11), \right. \\ \left. (6, 14), (7, 9), (7, 15), (8, 12), (9, 15), (10, 13), (11, 14). \right.$$

$$\mathcal{J}_3 = \left\{ (1, 8, 10), (1, 9, 11), (2, 4, 6), (2, 5, 7), (3, 12, 15), \right. \\ \left. (3, 13, 14), (4, 9, 13), (5, 11, 12), (6, 10, 15), (7, 8, 14) \right. \quad v_{31} = \frac{3}{10}$$

$$\mathcal{J}_5 = \left\{ (1, 4, 5, 14, 15), (1, 6, 7, 12, 13), (2, 8, 11, 13, 15), (2, 9, 10, 12, 14), \right. \\ \left. (3, 4, 7, 10, 11), (3, 5, 6, 8, 9) \right. \quad v_5 = \frac{1}{2}$$

9.12. $\vee$ -system $(E_8, A_1 \times A_4)$

$$\mathcal{A} = \begin{bmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 \\ \sqrt{10} & \sqrt{10} & \sqrt{2} & \sqrt{2} & 0 & 0 & 2 & 0 & 1 & -1 & \sqrt{5} & -\sqrt{5} & \dots \\ \sqrt{10} & -\sqrt{10} & 0 & 0 & \sqrt{5} & \sqrt{10} & 0 & 2\sqrt{10} & \frac{2}{5} & \frac{5}{2} & -\frac{3\sqrt{5}}{2} & -\frac{3\sqrt{5}}{2} & \dots \\ 0 & 0 & \sqrt{2} & -\sqrt{2} & \sqrt{5} & -\sqrt{10} & 0 & 0 & \frac{1}{2} & \frac{1}{2} & \frac{\sqrt{5}}{2} & \frac{\sqrt{5}}{2} & \dots \\ 13 & 14 & 15 & 16 \\ \dots & \sqrt{10} & -\sqrt{10} & 0 & 0 \\ \frac{\sqrt{10}}{2} & \frac{\sqrt{10}}{2} & -\frac{5}{\sqrt{2}} & \frac{3\sqrt{10}}{2} \\ \frac{\sqrt{10}}{2} & \frac{\sqrt{10}}{2} & \frac{1}{\sqrt{2}} & \frac{\sqrt{10}}{2} \end{bmatrix}$$



$$G = 30 \begin{bmatrix} 2 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\mathcal{I}_2 = \{(1, 10), (2, 9), (3, 8), (3, 10), (3, 14), (4, 8), (4, 9), (4, 13), (6, 7), (6, 9), (6, 10), (7, 15), (7, 16), (13, 15), (14, 15)\}$$

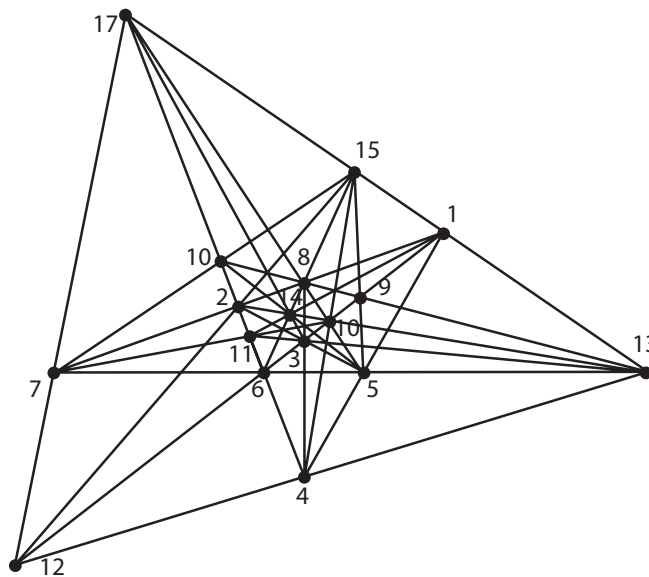
$$\mathcal{I}_3 = \begin{cases} (5, 10, 11), (5, 9, 12), (7, 11, 12) & v_{31} = \frac{7}{30} \\ (7, 9, 10) & v_{32} = \frac{1}{10} \\ (3, 16, 11), (4, 5, 1), (4, 16, 12), (3, 5, 2), (11, 15, 1), (2, 15, 12) & v_{33} = \frac{4}{15} \\ (4, 10, 15), (3, 9, 15), (3, 4, 7) & v_{34} = \frac{2}{15} \end{cases}$$

$$\mathcal{I}_4 = \{(7, 2, 8, 1), (12, 8, 10, 14), (14, 16, 9, 1), (13, 16, 10, 2), (13, 5, 14, 7), (13, 9, 8, 11)\}, \quad v_4 = \frac{2}{5}$$

$$\mathcal{I}_5 = \{(15, 8, 16, 5, 6), (12, 1, 13, 3, 6), (11, 2, 14, 4, 6)\}, \quad v_5 = \frac{3}{5}$$

9.13. $\vee$ -system $(E_8, A_2 \times A_3)$

$$\mathcal{A} = \begin{bmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 & 13 \\ 2\sqrt{3} & 2\sqrt{3} & 0 & 0 & \sqrt{3} & \sqrt{\frac{15}{2}} & 2\sqrt{3} & 0 & \frac{3}{2} & \frac{3}{2} & -3 & -3 & \frac{3\sqrt{6}}{2} \dots \\ 2\sqrt{3} & -2\sqrt{3} & 2 & 2 & 0 & 0 & 0 & 2\sqrt{6} & 2 & -2 & 2 & -2 & 0 \\ 0 & 0 & 2 & -2 & \sqrt{3} & -\sqrt{\frac{15}{2}} & 0 & 0 & \frac{1}{2} & \frac{1}{2} & 1 & 1 & \frac{\sqrt{6}}{2} \\ 14 & 15 & 16 & 17 \\ \dots & -\frac{\sqrt{3}}{2} & -\frac{\sqrt{3}}{2} & \sqrt{3} & \sqrt{3} \\ 2\sqrt{3} & -2\sqrt{3} & 2\sqrt{3} & -2\sqrt{3} \\ \frac{\sqrt{3}}{2} & -\frac{\sqrt{3}}{2} & \sqrt{3} & \sqrt{3} \end{bmatrix}$$



$$G = 30 \begin{bmatrix} 3 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\mathcal{I}_2 = \{(1, 10), (2, 9), (3, 7), (3, 10), (3, 15), (4, 7), (4, 9), (4, 14), (5, 11), (5, 12), (9, 11), (9, 17), (10, 12), (10, 16), (11, 15), (12, 14)\}$$

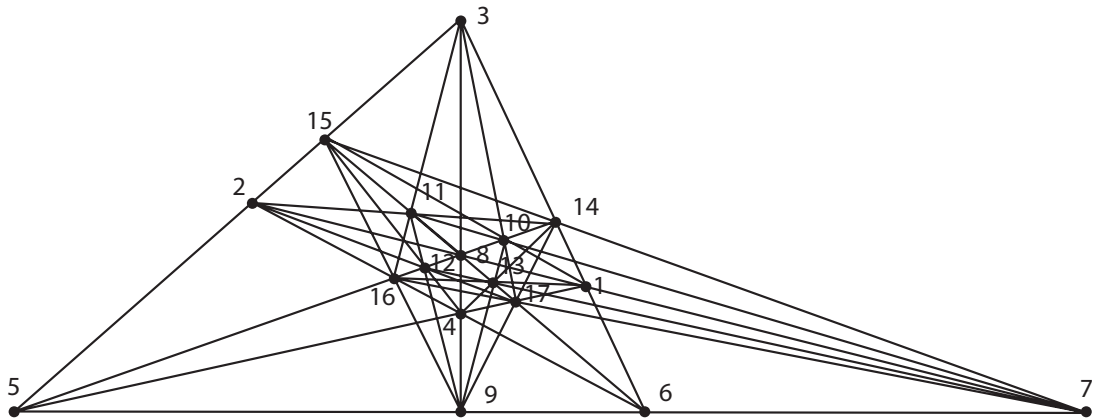
$$\mathcal{I}_3 = \begin{cases} \{(5, 9, 15), (5, 10, 14), (7, 9, 14), (7, 10, 15)\} & v_{31} = \frac{1}{6} \\ \{(3, 14, 17), (4, 12, 13), (4, 15, 16), (7, 11, 16), (7, 12, 17), (8, 11, 12), \\ (1, 4, 5), (1, 11, 14), (2, 3, 5), (2, 12, 15), (3, 4, 8), (3, 11, 13)\} & v_{32} = \frac{4}{15} \end{cases}$$

$$\mathcal{I}_4 = \begin{cases} (8, 9, 10, 13) & v_{41} = \frac{4}{15} \\ (1, 2, 7, 8), (1, 13, 15, 17), (2, 13, 14, 16), (5, 6, 7, 13), (5, 8, 16, 17), (6, 8, 14, 15) & v_{42} = \frac{2}{5} \end{cases}$$

$$\mathcal{I}_6 = \{(1, 3, 6, 9, 12, 16), (2, 4, 6, 10, 11, 17)\}, v_6 = \frac{3}{5}$$

9.14. $\vee$ -system $(E_8, A_1^2 \times A_3)$

$$\mathcal{A} = \begin{bmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 & 13 \\ 2 & 2 & 0 & 0 & 2 & 2 & 2 & 0 & 0 & \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} & \dots \\ 2 & -2 & 2 & 2 & 0 & 0 & 0 & 2\sqrt{10} & 0 & 2\sqrt{2} & -2\sqrt{2} & -2\sqrt{2} & 2\sqrt{2} \\ 0 & 0 & 2 & -2 & 2 & -2 & 0 & 0 & 2 & \frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} \\ 14 & 15 & 16 & 17 \\ \dots & \sqrt{2} & \sqrt{2} & \sqrt{2} & \sqrt{2} \\ 2\sqrt{2} & -2\sqrt{2} & -2\sqrt{2} & 2\sqrt{2} \\ \sqrt{2} & -\sqrt{2} & \sqrt{2} & -\sqrt{2} \end{bmatrix}$$



$$G = 30 \begin{bmatrix} 1 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\mathcal{I}_2 = \{(1, 9), (1, 11), (1, 12), (2, 9), (2, 10), (2, 13), (3, 7), (3, 12), (3, 13), (4, 7), (4, 10), (4, 11), (5, 11), (5, 13), (6, 10), (6, 12)\}$$

$$\mathcal{I}_3 = \begin{cases} (7, 10, 11), (7, 12, 13), (9, 10, 13), (9, 11, 12) & v_{31} = \frac{1}{6} \\ (9, 15, 16), (9, 14, 17), (7, 16, 17), (7, 14, 15) & v_{32} = \frac{4}{15} \\ (1, 10, 15), (1, 13, 16), (2, 11, 14), (2, 12, 17), (3, 10, 17), \\ (3, 11, 16), (4, 12, 15), (4, 13, 14) & v_{33} = \frac{7}{30} \end{cases}$$

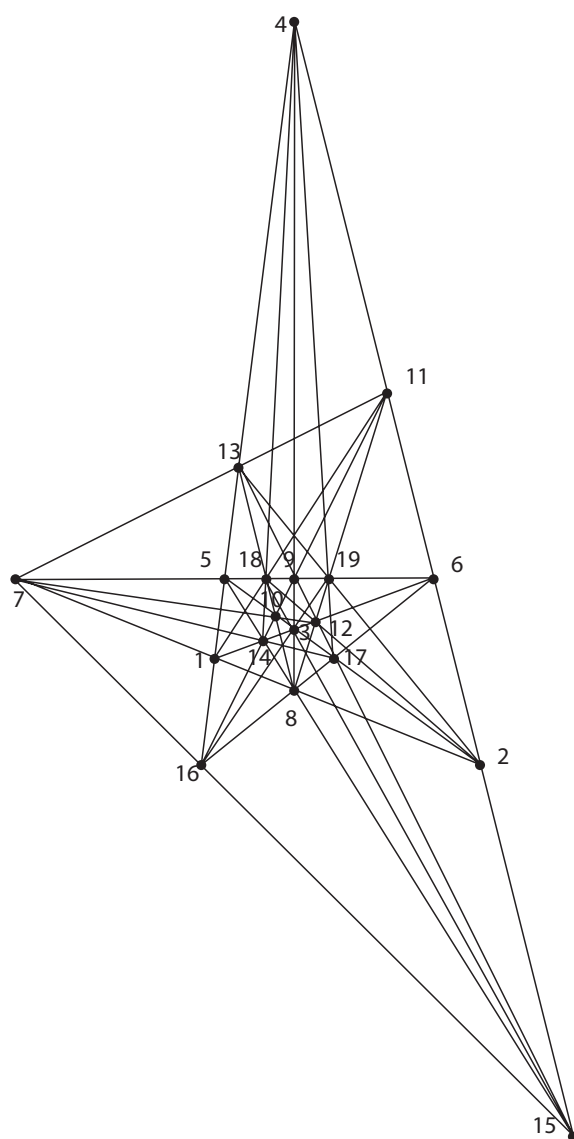
$$\mathcal{I}_4 = \{(1, 2, 7, 8), (1, 3, 6, 14), (1, 4, 5, 17), (2, 3, 5, 15), (2, 4, 6, 16), (3, 4, 8, 9), (5, 6, 7, 9)\},$$

$$v_4 = \frac{2}{5}$$

$$\mathcal{I}_6 = \{(6, 8, 11, 13, 15, 17), (5, 8, 10, 12, 14, 16)\}, v_6 = \frac{3}{5}$$

9.15. $\vee$ -system $(E_8, A_1^3 \times A_2)$

$$\mathcal{A} = \begin{bmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 & 13 & 14 & 15 \\ \sqrt{3} & 3 & 0 & 1 & \sqrt{3} & 0 & \sqrt{6} & 0 & 0 & \sqrt{6} & \sqrt{3} & 3 & 1 & \sqrt{3} & 0 & \dots \\ \sqrt{3} & 0 & 3 & -1 & 0 & \sqrt{3} & 0 & \sqrt{6} & 0 & \sqrt{6} & \sqrt{3} & 3 & 2 & 0 & \sqrt{3} \\ 0 & 3 & 3 & 0 & -\sqrt{3} & -\sqrt{3} & 0 & 0 & 6\sqrt{2} & 3\sqrt{6} & 4\sqrt{3} & 6 & 3 & 3\sqrt{3} & 3\sqrt{3} \\ 16 & 17 & 18 & 19 \\ \dots & 2 & \sqrt{6} & 0 & \sqrt{6} \\ 1 & 0 & \sqrt{6} & \sqrt{6} \\ 3 & 2\sqrt{6} & 2\sqrt{6} & \sqrt{6} \end{bmatrix}$$



$$G = 30 \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}$$

$$\begin{aligned} \mathcal{J}_2 = \{ & (1, 9), (1, 15), (1, 17), (2, 9), (2, 14), (2, 16), (3, 7), (3, 11), \\ & (3, 13), (4, 7), (4, 10), (4, 12), (5, 11), (5, 12), (6, 10), (6, 13), (10, 15), (11, 17), (12, 16), \\ & (13, 14), (14, 19), (15, 19), (16, 18), (17, 18) \} \end{aligned}$$

$$\mathcal{I}_3 = \begin{cases} (3, 15, 18), (3, 16, 19), (4, 14, 18), (4, 17, 19), (7, 14, 17), (7, 15, 16) & v_{31} = \frac{7}{30} \\ (7, 13, 11), (2, 12, 18), (2, 13, 19), (1, 10, 19), (1, 11, 18), (7, 10, 12) & v_{32} = \frac{1}{6} \end{cases}$$

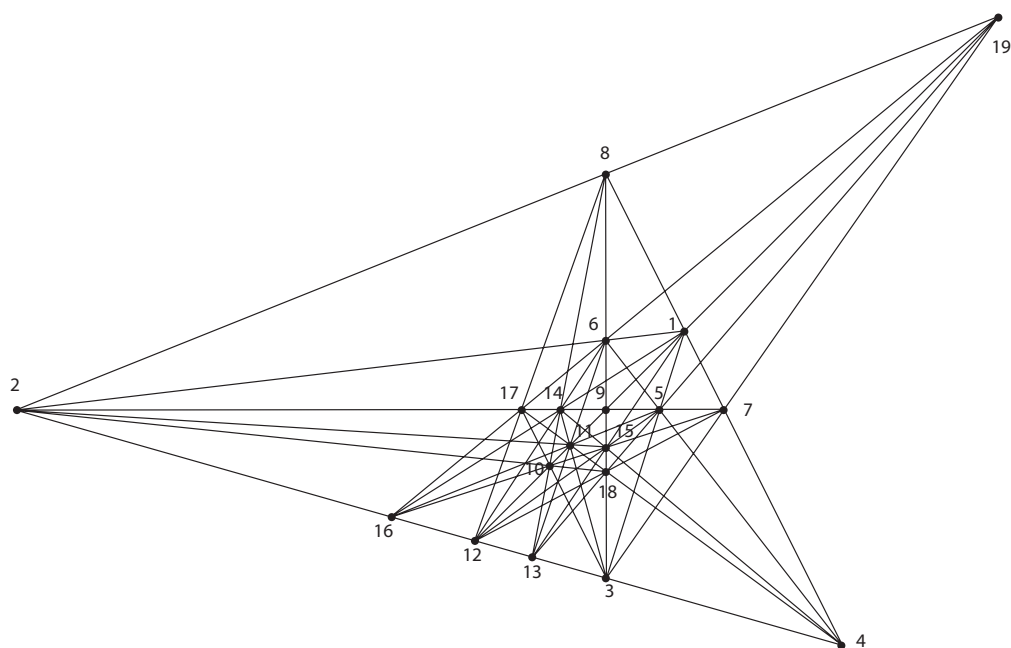
$$\mathcal{I}_4 = \begin{cases} (1, 2, 7, 8), (8, 10, 13, 18), (8, 11, 12, 19) & v_{31} = \frac{4}{15} \\ (3, 4, 8, 9), (5, 8, 14, 15), (6, 8, 16, 17) & v_{32} = \frac{2}{5} \end{cases}$$

$$\mathcal{I}_5 = \begin{cases} (9, 12, 13, 15, 17), (9, 10, 11, 14, 16), (2, 4, 6, 11, 15), \\ (2, 3, 5, 10, 17), (1, 4, 5, 13, 16), (1, 3, 6, 12, 14) \end{cases} \quad v_5 = \frac{2}{5}$$

$$\mathcal{I}_6 = \{(5, 6, 7, 9, 18, 19)\}, v_6 = \frac{3}{5}$$

9.16. $\vee$ -system $(E_8, A_2^2 \times A_1)$

$$\mathcal{A} = \begin{bmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 & 13 & 14 & 15 \\ \sqrt{3} & 3 & 0 & 1 & \sqrt{3} & 0 & \sqrt{6} & 0 & 0 & \sqrt{6} & \sqrt{3} & 3 & 1 & \sqrt{3} & 0 & \dots \\ \sqrt{3} & 0 & 3 & -1 & 0 & \sqrt{3} & 0 & \sqrt{6} & 0 & \sqrt{6} & \sqrt{3} & 3 & 2 & 0 & \sqrt{3} & \\ 0 & 3 & 3 & 0 & -\sqrt{3} & -\sqrt{3} & 0 & 0 & 6\sqrt{2} & 3\sqrt{6} & 4\sqrt{3} & 6 & 3 & 3\sqrt{3} & 3\sqrt{3} & \\ 16 & 17 & 18 & 19 \\ \dots & 2 & \sqrt{6} & 0 & \sqrt{6} \\ 1 & 0 & \sqrt{6} & \sqrt{6} \\ 3 & 2\sqrt{6} & 2\sqrt{6} & \sqrt{6} \end{bmatrix}$$



$$G = 30 \begin{bmatrix} 2 & 1 & 3 \\ 1 & 2 & 3 \\ 3 & 3 & 12 \end{bmatrix}$$

$$\mathcal{I}_2 = \{(1, 17), (1, 18), (4, 9), (4, 10), (4, 19), (5, 8), (5, 10), (6, 7), (6, 10), (7, 11), (7, 13), \\ (8, 11), (8, 16), (9, 13), (9, 16), (13, 17), (14, 18), (14, 19), (15, 17), (15, 19), (16, 18)\}$$

$$\mathcal{I}_3 = \begin{cases} (1, 2, 6), (1, 3, 5), (2, 11, 15), (3, 11, 14), (5, 12, 15), (6, 12, 14) & v_{31} = \frac{7}{30} \\ (7, 12, 18), (8, 12, 17), (3, 10, 17), (3, 7, 19), (2, 10, 18), (2, 8, 19) & v_{32} = \frac{4}{15} \\ (1, 13, 15), (1, 14, 16), (4, 5, 6), (4, 14, 15), (5, 11, 16), (6, 11, 13) & v_{33} = \frac{1}{6} \end{cases}$$

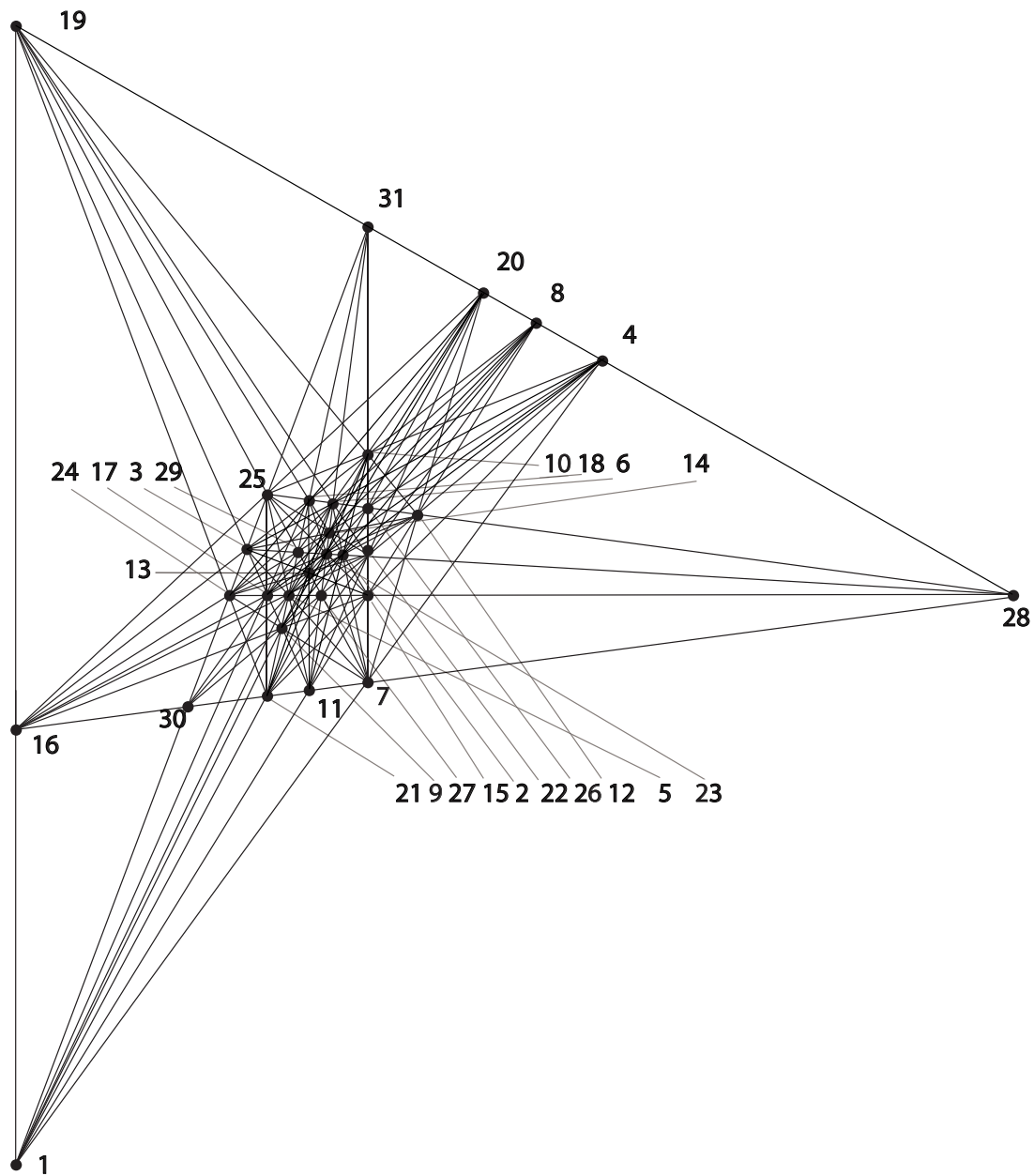
$$\mathcal{I}_4 = \begin{cases} (1, 4, 7, 8), (4, 11, 17, 18), (5, 13, 18, 19), (6, 16, 17, 19), \\ (7, 10, 15, 16), (8, 10, 13, 14) \end{cases} \quad v_4 = \frac{4}{15}$$

$$\mathcal{I}_6 = \begin{cases} (2, 3, 4, 12, 13, 16); & v_{61} = \frac{2}{5} \\ (2, 5, 7, 9, 14, 17), (1, 9, 10, 11, 12, 19), (3, 6, 8, 9, 15, 18) & v_{62} = \frac{3}{5} \end{cases}$$

9.17. $\vee$ -system (H_4, A_1)

$$\mathcal{A} = \begin{bmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 & 13 & 14 & 15 & 16 \\ 1 & 0 & 0 & \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} & a & a & a & a & b & b & b & b & \frac{1}{2} & \dots \\ 0 & 1 & 0 & \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} & \frac{1}{2} & \frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} & a & a & -a & -a & b \\ 0 & 0 & 1 & \frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} & b & -b & b & -b & \frac{1}{2} & -\frac{1}{2} & \frac{1}{2} & -\frac{1}{2} & a \\ 17 & 18 & 19 & 20 & 21 & 22 & 23 & 24 & 25 & 26 & 27 \\ \dots & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & a\sqrt{2} & a\sqrt{2} & 0 & 0 & b\sqrt{2} & b\sqrt{2} & \sqrt{b\sqrt{5}} & \sqrt{b\sqrt{5}} & \dots \\ & b & -b & -b & b\sqrt{2} & -b\sqrt{2} & a\sqrt{2} & a\sqrt{2} & 0 & 0 & 2a\sqrt{b\sqrt{5}} & -2a\sqrt{b\sqrt{5}} \\ & -a & a & -a & 0 & 0 & b\sqrt{2} & -b\sqrt{2} & a\sqrt{2} & -a\sqrt{2} & 0 & 0 \\ & 28 & 29 & 30 & 31 \\ \dots & 0 & 0 & 2a\sqrt{b\sqrt{5}} & -2a\sqrt{b\sqrt{5}} \\ & \sqrt{b\sqrt{5}} & \sqrt{b\sqrt{5}} & 0 & 0 \\ & 2a\sqrt{b\sqrt{5}} & -2a\sqrt{b\sqrt{5}} & \sqrt{b\sqrt{5}} & \sqrt{b\sqrt{5}} \end{bmatrix}$$

with $a = \frac{1+\sqrt{5}}{4}$ and $b = \frac{-1+\sqrt{5}}{4}$.



$$G = I$$

$$\begin{aligned} \mathcal{I}_2 = & \{(1, 22), (1, 23), (1, 28), (1, 29), (2, 24), (2, 25), (2, 30), (2, 31), (3, 20), (3, 21), (3, 26), (3, 27), \\ & (4, 11), (4, 13), (4, 18), (5, 10), (5, 12), (5, 19), (6, 9), (6, 15), (6, 16), (7, 8), (7, 14), (7, 17), (8, 25), \\ & (8, 29), (8, 27), (9, 25), (9, 26), (9, 28), (10, 24), (10, 27), (10, 28), (11, 24), (11, 26), (11, 29), \end{aligned}$$

$$\begin{aligned}
 & (12, 29), (12, 31), (13, 20), (13, 28), (13, 31), (14, 21), (14, 28), (12, 21), (14, 30), (15, 20), \\
 & (15, 29), (15, 30), (16, 23), (16, 27), (16, 31), (17, 22), (17, 26), \\
 & (17, 31), (18, 22), (18, 27), (18, 30), (19, 23), (19, 26), (19, 30) \} \\
 \mathcal{I}_3 = & \begin{cases} \{(1, 4, 7), (1, 5, 6), (2, 4, 5), (2, 6, 7), (3, 4, 6), (3, 5, 7), (4, 10, 25), \\ (4, 15, 21), (4, 17, 23), (5, 11, 25), (17, 21, 25), (8, 21, 22), (18, 20, 24), \\ (16, 20, 25), (19, 21, 24), (5, 14, 20), (5, 16, 22), (6, 8, 24), (6, 13, 21), \\ (6, 19, 22), (7, 9, 24), (7, 12, 20), (7, 18, 23), (9, 20, 23), (10, 21, 23), \\ (11, 20, 22), (12, 23, 24), (13, 22, 24), (14, 22, 25), (15, 23, 25)\} & v_{31} = \frac{2}{15} \\ \{(1, 16, 19), (1, 17, 18), (2, 8, 9), (2, 10, 11), (3, 12, 14), (3, 13, 15), \\ (8, 14, 17), (11, 13, 18), (9, 15, 16), (10, 12, 19)\} & v_{32} = \frac{1}{10} \end{cases} \\
 \mathcal{I}_5 = & \{(1, 8, 11, 12, 15), (1, 9, 10, 13, 14), (2, 12, 13, 16, 17), (2, 14, 15, 18, 19), \\ & (3, 8, 10, 16, 18), (3, 9, 11, 17, 19)\} \quad v_5 = \frac{1}{6} \\
 \mathcal{I}_6 = & \{(1, 2, 20, 21, 26, 27), (1, 3, 24, 25, 30, 31), (2, 3, 22, 23, 28, 29), (4, 8, 19, 20, 28, 31), \\ & (5, 8, 13, 23, 26, 30), (5, 9, 18, 21, 29, 31), (5, 15, 17, 24, 27, 28), (6, 10, 17, 20, 29, 30), \\ & (6, 11, 14, 23, 27, 31), (6, 12, 18, 25, 26, 28), (4, 9, 12, 22, 27, 30), (4, 14, 16, 24, 26, 29), \\ & (7, 10, 15, 22, 26, 31), (7, 11, 16, 21, 28, 30), (7, 13, 19, 25, 27, 29)\} \quad v_6 = \frac{1}{3}
 \end{aligned}$$