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## Bi-Hamiltonian structure of multi-component Novikov equation

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In this paper, we present the multi-component Novikov equation and derive its bi-Hamiltonian structure.

*Keywords:* bi-Hamiltonian structure; multi-component Novikov equation.

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### 1. Introduction

The Camassa-Holm (CH) equation

$$m_t + um_x + 2u_xm = 0, \quad m = u - u_{xx}, \quad (1.1)$$

was derived by Camassa and Holm from an approximation to the incompressible Euler equations [3], and found to be completely integrable with a Lax pair and associated bi-Hamiltonian structure [4]. The CH equation has been studied in a large number of papers [1, 2, 5, 6, 9, 10, 14–18]. Interestingly, the CH equation is linked with the first negative flow of the KdV hierarchy by reciprocal transformation [10]. But unlike KdV equation, the CH equation admits peaked soliton solutions [1–4]. Besides the CH equation, many other systems with peaked soliton solutions have been constructed.

In 1999, Degasperis-Procesi presented a new equation with peaked solutions

$$m_t + um_x + 3u_xm = 0, \quad m = u - u_{xx}, \quad (1.2)$$

which is known as DP equation [7]. The DP equation is integrable with a bi-Hamiltonian structure and a Lax pair associated with a third-order spectral problem [8]. Both CH equation and DP equation have nonlinear terms that are quadratic.

Recently, Vladimir Novikov found a new equation with cubic nonlinearity

$$m_t + u^2m_x + 3u u_xm = 0, \quad m = u - u_{xx}, \quad (1.3)$$

from his symmetry classification study of nonlocal partial differential equations [20]. In [12], Hone and Wang gave a matrix Lax pair, infinitely many conserved quantities as well as a bi-Hamiltonian

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structure of the Eq. (1.3) which is also named Novikov equation. Very recently, Geng and Xue constructed a two-component generalization for the Novikov equation (1.3)

$$\begin{aligned} m_t + 3u_x vm + uv m_x &= 0, \\ n_t + 3v_x un + uv n_x &= 0, \\ m &= u - u_{xx}, \quad n = v - v_{xx}, \end{aligned} \quad (1.4)$$

which was associated with a  $3 \times 3$  matrix spectral problem, they also gave the  $N$  peakons, infinite sequence of conserved quantities and a Hamiltonian structure [11]. In 2013, Li and Liu showed the system (1.4) was indeed a bi-Hamiltonian structure and got the Hamiltonian operators found by Hone and Wang for the Novikov equation (1.3) using the proper Dirac reduction [19].

The purpose of this paper is to construct the bi-Hamiltonian system for the multi-component Novikov equation

$$\begin{aligned} q_{it} &= \sum_{j=1}^n [-2q_i u_{jx} v_j - q_i u_j v_{jx} - q_{ix} u_j v_j - u_{ix} q_j v_j + u_i q_j v_{jx}], \\ r_{it} &= \sum_{j=1}^n [-2r_i u_j v_{jx} - r_i u_{jx} v_j - r_{ix} u_j v_j - v_{ix} r_j u_j + v_i r_j u_{jx}], \\ q_i &= u_i - u_{ixx}, \quad r_i = v_i - v_{ixx}, \quad i = 1, 2, \dots, n, \end{aligned} \quad (1.5)$$

where  $q_{it} = \frac{\partial q_i(x,t)}{\partial t}$ ,  $r_{it} = \frac{\partial r_i(x,t)}{\partial t}$ ,  $u_{jx} = \frac{\partial u_j(x,t)}{\partial x}$ ,  $v_{jx} = \frac{\partial v_j(x,t)}{\partial x}$ ,  $i, j = 1, 2, \dots, n$ , and so on. When  $n = 1, q = m, r = n$  the Eq. (1.5) reduces to the two-component system (1.4). Moreover the system (1.5) can reduce to the Eqs. (1.2) and (1.3) as  $n = 1, q = m, v = 1$  and  $n = 1, q = m, v = u$  respectively. It's worthwhile to note that there is also research on other multi-component CH-type equations [13, 22, 23].

## 2. Bi-Hamiltonian structure of multi-component Novikov equation

Possession of the bi-Hamiltonian structure is an important property of soliton equations and all soliton equations are turn out to be bi-Hamiltonian systems. In this section, we derive the bi-Hamiltonian structure of multi-component Novikov equation (1.5).

The multi-component Novikov equation (1.5) has the equivalent form

$$\begin{aligned} Q_t &= -2\langle U_x, V \rangle Q - \langle U, V_x \rangle Q - \langle U, V \rangle Q_x - \langle Q, V \rangle U_x + \langle Q, V_x \rangle U, \\ R_t &= -2\langle U, V_x \rangle R - \langle U_x, V \rangle R - \langle U, V \rangle R_x - \langle R, U \rangle V_x + \langle R, U_x \rangle V, \end{aligned} \quad (2.1)$$

where  $\langle \cdot, \cdot \rangle$  denotes the inner product and  $Q, R, U, V$  are the  $n$ -component vector potentials defined as

$$\begin{aligned} Q &= (q_1, q_2, \dots, q_n)^T, \quad R = (r_1, r_2, \dots, r_n)^T, \quad U = (u_1, u_2, \dots, u_n)^T, \quad V = (v_1, v_2, \dots, v_n)^T, \\ Q &= U - U_{xx}, \quad R = V - V_{xx}, \end{aligned}$$

and  $T$  is the transpose of a vector, and  $U_x^T = \frac{\partial U^T}{\partial x}$  as well.

The system (2.1) arises as a zero-curve equation

$$M_t - N_x + [M, N] = 0, \quad (2.2)$$

this being the compatibility condition of the  $(n+2) \times (n+2)$  matrix spectral problem

$$\varphi_x = M\varphi, \quad \varphi_t = N\varphi, \quad (2.3)$$

with

$$M = \begin{pmatrix} 0 & \lambda Q^T & 1 \\ \mathbf{0}^T & \mathbf{0}_{n \times n} & \lambda R \\ 1 & \mathbf{0} & 0 \end{pmatrix}, \quad N = \begin{pmatrix} -U_x^T V & \frac{U_x^T}{\lambda} - \lambda U^T V Q^T & U_x^T V_x \\ \frac{V}{\lambda} & -\frac{I_n}{\lambda^2} + V U_x^T - V_x U^T & -\frac{V_x}{\lambda} - \lambda U^T V R \\ -U^T V & \frac{U^T}{\lambda} & U^T V_x \end{pmatrix}, \quad (2.4)$$

where  $\mathbf{0}$  and  $\mathbf{0}_{n \times n}$  are respectively  $n$  dimension row vector and  $n \times n$  zero matrix,  $\lambda$  is a constant spectral parameter and  $I_n$  denotes the  $n \times n$  identity matrix. It is worth noting that the matrix spectral problem (2.3) is the vector prolongation of the spectral problem in [11], so the system (2.1) is also a negative flow in the hierarchy.

To compute the bi-Hamiltonian structure of the system (2.1), we consider an integrable hierarchy which consists of (2.1), i.e., the  $N$  in time part of (2.3) is

$$N = \begin{pmatrix} N_{1,1} & A & N_{1,n+2} \\ B & S & C \\ N_{n+2,1} & D & N_{n+2,n+2} \end{pmatrix}, \quad (2.5)$$

where  $B$ ,  $C$  and  $A$ ,  $D$  are respectively  $n$  dimension column and row vectors depending on vector potentials  $Q$ ,  $R$  and spectral parameter  $\lambda$ .  $S$  and the remaining entries are respectively  $n \times n$  matrix and functions depending on vector potentials  $Q$ ,  $R$  and spectral parameter  $\lambda$ .

Substituting  $M$  and  $N$  respectively in (2.4) and (2.5) into (2.2), we get

$$\begin{aligned} C &= -B_x + \lambda R N_{n+2,1}, \quad A = D_x + \lambda N_{n+2,1} Q^T, \quad N_{11} = (N_{n+2,1})_x + N_{n+2,n+2}, \\ S &= \lambda \partial^{-1} (R D - B Q^T), \quad N_{n+2,1} = \lambda (\partial^3 - 4\partial)^{-1} (3Q^T B_x + Q_x^T B + 3D_x R + D R_x), \\ (N_{1,n+2})_x + (N_{n+2,1})_x + \lambda (Q^T B_x + D_x R) &= 0, \quad N_{n+2,n+2} = -\frac{1}{2} (N_{n+2,1})_x + \frac{1}{2} \lambda \partial^{-1} (Q^T B - D R), \end{aligned}$$

and

$$\begin{pmatrix} Q \\ R \end{pmatrix}_t = (\lambda^{-1} \mathcal{K} + \lambda \mathcal{J}) \begin{pmatrix} B \\ D^T \end{pmatrix}, \quad (2.6)$$

where

$$\mathcal{K} = \begin{pmatrix} 0 & (\partial^2 - 1)I_n \\ (1 - \partial^2)I_n & 0 \end{pmatrix}, \quad (2.7)$$

$$\mathcal{J} = \mathcal{J}_1 + \mathcal{J}_2, \quad (2.8)$$

with

$$\begin{aligned} \mathcal{J}_1 &= \begin{pmatrix} \frac{3}{2} Q \partial + Q_x \\ \frac{3}{2} R \partial + R_x \end{pmatrix} (\partial^3 - 4\partial)^{-1} (3Q^T \partial + Q_x^T 3R^T \partial + R_x^T), \\ \mathcal{J}_2 &= \begin{pmatrix} \frac{1}{2} Q \partial^{-1} Q^T + (Q \partial^{-1} Q^T)^T & -\frac{1}{2} Q \partial^{-1} R^T - Q^T \partial^{-1} R I_n \\ -\frac{1}{2} R \partial^{-1} Q^T - R^T \partial^{-1} Q I_n & \frac{1}{2} R \partial^{-1} R^T + (R \partial^{-1} R^T)^T \end{pmatrix}. \end{aligned}$$

Obviously, the operators  $\mathcal{K}$  and  $\mathcal{J}$  are skew-symmetric, furthermore the operator  $\mathcal{K}$  is a Hamiltonian operator. In the following, we show how the Jacobi identity for the operator  $\mathcal{J}$  and compatibility for the operators  $\mathcal{K}$  and  $\mathcal{J}$  may be checked by the multivector approach to Hamiltonian systems in infinite dimensions, as described in the work of Olver [21].

Our main results are summarized as

**Theorem 2.1.** *The multi-component Novikov system (2.1) may be reformulated as a bi-Hamiltonian system*

$$\begin{pmatrix} Q \\ R \end{pmatrix}_t = \mathcal{K} \begin{pmatrix} \frac{\delta H_0}{\delta m} \\ \frac{\delta H_0}{\delta n} \end{pmatrix} = \mathcal{J} \begin{pmatrix} \frac{\delta H_1}{\delta m} \\ \frac{\delta H_1}{\delta n} \end{pmatrix} \quad (2.9)$$

where the operators  $\mathcal{K}$  and  $\mathcal{J}$  are given by (2.7) and (2.8) respectively, and

$$\begin{aligned} H_0 &= \frac{1}{2} \int \langle Q, V \rangle \langle U_x, V \rangle - \langle R, U \rangle \langle V_x, U \rangle + (\langle R, U_x \rangle - \langle Q, V_x \rangle) \langle U, V \rangle dx, \\ H_1 &= \frac{1}{2} \int \langle Q, V \rangle + \langle R, U \rangle dx. \end{aligned}$$

Before the proof of the Theorem 2.1, please allow us give a brief explanation of the Olver's technique [21]. Let  $\theta$  denote the basic uni-vector corresponding to potential,  $\mathcal{D}$  is any skew-symmetry operator depending on a spatial variable  $x$  and the potential. In the proof procedure, we have mainly used the following three properties:

- the basic property of wedge product

$$\int \xi \wedge \eta dx = (-1)^{mn} \int \eta \wedge \xi dx, \quad (2.10)$$

for any  $m$ -form  $\xi$  and  $n$ -form  $\eta$ .

- the skew-symmetry of the operator  $\mathcal{D}$

$$\int \xi \wedge \mathcal{D}\eta dx = - \int (\mathcal{D}\xi) \wedge \eta dx. \quad (2.11)$$

- the prolongation

$$-\text{PrV}_{\mathcal{D}\theta}(\theta \wedge \mathcal{D}\theta) = \theta \wedge \text{PrV}_{\mathcal{D}\theta}(\mathcal{D}) \wedge \theta, \quad (2.12)$$

the minus sign coming from the fact that we have interchanged a wedge product of  $\theta$ 's using the formula (2.10).

**Proof.** Assume that  $\theta_1 = (\theta_{11}, \theta_{12}, \dots, \theta_{1n})^T$ ,  $\theta_2 = (\theta_{21}, \theta_{22}, \dots, \theta_{2n})^T$  are the basic uni-vectors corresponding to  $Q$  and  $R$  respectively. We know that the operator  $\mathcal{J}$  is the Hamiltonian if and only if

$$\text{PrV}_{\mathcal{J}\theta}(\Theta_{\mathcal{J}}) = \text{PrV}_{\mathcal{J}\theta}(\Theta_{\mathcal{J}_1}) + \text{PrV}_{\mathcal{J}\theta}(\Theta_{\mathcal{J}_2}) = 0, \quad (2.13)$$

where  $\theta = \begin{pmatrix} \theta_1 \\ \theta_2 \end{pmatrix}$  and

$$\Theta_{\mathcal{J}} = \frac{1}{2} \int (\theta \wedge \mathcal{J}\theta) dx = \Theta_{\mathcal{J}_1} + \Theta_{\mathcal{J}_2} = \frac{1}{2} \int (\theta \wedge \mathcal{J}_1\theta) dx + \frac{1}{2} \int (\theta \wedge \mathcal{J}_2\theta) dx,$$

is the associated bi-vector of  $\mathcal{J}$ .

To check whether  $\mathcal{H}$  and  $\mathcal{J}$  form a bi-Hamiltonian pair, we only need to prove

$$\text{PrV}_{\mathcal{H}\theta}(\Theta_{\mathcal{J}}) = \text{PrV}_{\mathcal{H}\theta}(\Theta_{\mathcal{J}_1}) + \text{PrV}_{\mathcal{H}\theta}(\Theta_{\mathcal{J}_2}) = 0. \quad (2.14)$$

The proof of the Theorem 2.1 is rather technical and lengthy, so are given in Appendix A.  $\square$

According to the bi-Hamiltonian theory, the Hamiltonian pair  $\mathcal{H}, \mathcal{J}$  gives rise to the hereditary recursion operator  $\mathcal{R} = \mathcal{J}\mathcal{H}^{-1}$ . The recursion operator acting on a seed symmetry of the soliton equation can generate an infinite sequence of symmetries. Assume the seed symmetry of (2.1) is  $(\mathbf{0}, \mathbf{0})^T$ , then we get the sequence of symmetries

$$\sigma_n = \mathcal{R}^n(\mathbf{0}, \mathbf{0})^T, \quad n = 0, 1, 2, \dots \quad (2.15)$$

As  $n = 1$ , the above expression (2.15) is the local symmetry

$$\sigma_1 = \begin{pmatrix} -2\langle U_x, V \rangle Q - \langle U, V_x \rangle Q - \langle U, V \rangle Q_x - \langle Q, V \rangle U_x + \langle Q, V_x \rangle U \\ -2\langle U, V_x \rangle R - \langle U_x, V \rangle R - \langle U, V \rangle R_x - \langle R, U \rangle V_x + \langle R, U_x \rangle V \end{pmatrix}$$

which is just the right side of the equality (2.1). This is natural.

But when  $n = 2$ , the recursion formula (2.15) leads to the nonlocal symmetry

$$\sigma_2 = \begin{pmatrix} \sigma_{21} \\ \sigma_{22} \end{pmatrix}, \quad (2.16)$$

where

$$\begin{aligned} \sigma_{21} &= \left( \frac{3}{2} Q \partial + Q_x \right) (\partial^3 - 4\partial)^{-1} (3\langle Q, \Phi_{1x} \rangle + \langle Q_x, \Phi_1 \rangle - 3\langle R, \Phi_{2x} \rangle - \langle R_x, \Phi_2 \rangle) \\ &\quad + \frac{1}{2} Q \partial^{-1} (\langle Q, \Phi_1 \rangle + \langle R, \Phi_2 \rangle) + (Q \partial^{-1} Q^T)^T \Phi_1 + Q^T \partial^{-1} R I_n \Phi_2, \\ \sigma_{22} &= \left( \frac{3}{2} R \partial + R_x \right) (\partial^3 - 4\partial)^{-1} (3\langle Q, \Phi_{1x} \rangle + \langle Q_x, \Phi_1 \rangle - 3\langle R, \Phi_{2x} \rangle - \langle R_x, \Phi_2 \rangle) \\ &\quad - \frac{1}{2} R \partial^{-1} (\langle Q, \Phi_1 \rangle + \langle R, \Phi_2 \rangle) - (R \partial^{-1} R^T)^T \Phi_2 - R^T \partial^{-1} Q I_n \Phi_1, \end{aligned}$$

and  $\Phi_1, \Phi_2$  are nonlocal variables defined by

$$\Phi_1 = (1 - \partial^2)^{-1} R_t, \quad \Phi_2 = (1 - \partial^2)^{-1} Q_t.$$

For example, let us consider the reduction: the  $Q$  and  $R$  are both one-dimensional scalar functions, and  $R = Q$  as well. A local symmetry under the reduction must be local. We will demonstrate the symmetry  $\sigma_2$  under the reduction is nonlocal. Set the symmetry  $\sigma_2$  in (2.16) under the constraint is  $\widehat{\sigma}_2$ , then

$$\widehat{\sigma}_2 = \begin{pmatrix} 3Q\partial^{-1}(Q\Psi) \\ -3Q\partial^{-1}(Q\Psi) \end{pmatrix},$$

where  $\Psi = (1 - \partial^2)^{-1} Q_t$  with  $Q_t = -4u^2 u_x + u^2 u_{xxx} + 3uu_x u_{xx}$ . If  $\widehat{\sigma}_2$  is local,  $\Psi$  must be a local variable, i.e., there is a function  $f(x, u, u_x)$  that satisfies

$$(1 - \partial^2)^{-1} (-4u^2 u_x + u^2 u_{xxx} + 3uu_x u_{xx}) = f(x, u, u_x). \quad (2.17)$$

But after calculation, we find there is no function  $f(x, u, u_x)$  that satisfies the equality (2.17), so  $\widehat{\sigma}_2$  is nonlocal and then  $\sigma_2$  is a nonlocal symmetry.

Therefore, from the recursion formula (2.15), we can obtain an infinite sequence of higher order nonlocal symmetries of the system (2.1).

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## Appendix A.

First, we prove that the operator  $\mathcal{J}$  is Hamiltonian, namely to verify (2.13). To simplify the presentation and calculations, we introduce  $\tilde{Q}$  and  $\tilde{R}$  as

$$\tilde{Q} = (\partial^3 - 4\partial)^{-1}(3Q^T \theta_{1x} + Q_x^T \theta_1), \quad \tilde{R} = (\partial^3 - 4\partial)^{-1}(3R^T \theta_{2x} + R_x^T \theta_2). \quad (\text{A.1})$$

From (2.8), we have

$$\mathcal{J}_1 \theta = \begin{pmatrix} \frac{3}{2}Q(\tilde{Q} + \tilde{R})_x + Q_x(\tilde{Q} + \tilde{R}) \\ \frac{3}{2}R(\tilde{Q} + \tilde{R})_x + R_x(\tilde{Q} + \tilde{R}) \end{pmatrix} = \begin{pmatrix} \frac{3}{2}q_1(\tilde{Q} + \tilde{R})_x + q_{1x}(\tilde{Q} + \tilde{R}) \\ \vdots \\ \frac{3}{2}q_n(\tilde{Q} + \tilde{R})_x + q_{nx}(\tilde{Q} + \tilde{R}) \\ \frac{3}{2}r_1(\tilde{Q} + \tilde{R})_x + r_{1x}(\tilde{Q} + \tilde{R}) \\ \vdots \\ \frac{3}{2}r_n(\tilde{Q} + \tilde{R})_x + r_{nx}(\tilde{Q} + \tilde{R}) \end{pmatrix}, \quad (\text{A.2})$$

and

$$\begin{aligned} \mathcal{J}_2 \theta &= \begin{pmatrix} \frac{1}{2}Q\partial^{-1}(Q^T \theta_1 - R^T \theta_2) + (Q\partial^{-1}Q^T)^T \theta_1 - Q^T \partial^{-1}R I_n \theta_2 \\ -\frac{1}{2}R\partial^{-1}(Q^T \theta_1 - R^T \theta_2) - R^T \partial^{-1}Q I_n \theta_1 + (R\partial^{-1}R^T)^T \theta_2 \end{pmatrix} \\ &= \begin{pmatrix} \frac{1}{2}q_1\partial^{-1}(Q^T \theta_1 - R^T \theta_2) + \sum_{i=1}^n q_i\partial^{-1}(q_1\theta_{1i} - r_i\theta_{21}) \\ \vdots \\ \frac{1}{2}q_n\partial^{-1}(Q^T \theta_1 - R^T \theta_2) + \sum_{i=1}^n q_i\partial^{-1}(q_n\theta_{1i} - r_i\theta_{2n}) \\ -\frac{1}{2}r_1\partial^{-1}(Q^T \theta_1 - R^T \theta_2) - \sum_{i=1}^n r_i\partial^{-1}(q_i\theta_{11} - r_1\theta_{2i}) \\ \vdots \\ -\frac{1}{2}r_n\partial^{-1}(Q^T \theta_1 - R^T \theta_2) - \sum_{i=1}^n r_i\partial^{-1}(q_i\theta_{1n} - r_n\theta_{2i}) \end{pmatrix}. \end{aligned} \quad (\text{A.3})$$

Then the associated bi-vectors for  $\mathcal{J}_1$  and  $\mathcal{J}_2$  are respectively

$$\begin{aligned} \Theta_{\mathcal{J}_1} &= \frac{1}{2} \int (\theta \wedge \mathcal{J}_1 \theta) dx \\ &= \frac{1}{2} \int \begin{pmatrix} \theta_1 \\ \theta_2 \end{pmatrix} \wedge \begin{pmatrix} \frac{3}{2}Q(\tilde{Q} + \tilde{R})_x + Q_x(\tilde{Q} + \tilde{R}) \\ \frac{3}{2}R(\tilde{Q} + \tilde{R})_x + R_x(\tilde{Q} + \tilde{R}) \end{pmatrix} dx \\ &= \frac{1}{2} \sum_{j=1}^n \int \left[ \frac{3}{2}(\theta_{1j} \wedge q_j + \theta_{2j} \wedge r_j)(\tilde{Q} + \tilde{R})_x + (\theta_{1j} \wedge q_{jx} + \theta_{2j} \wedge r_{jx})(\tilde{Q} + \tilde{R}) \right] dx \\ &= -\frac{1}{4} \sum_{j=1}^n \int (q_{jx}\theta_{1j} + 3q_j\theta_{1jx} + r_{jx}\theta_{2j} + 3r_j\theta_{2jx}) \wedge (\tilde{Q} + \tilde{R}) dx, \end{aligned} \quad (\text{A.4})$$



and

$$\begin{aligned}
 \Theta_{\mathcal{J}_2} &= \frac{1}{2} \int (\theta \wedge \mathcal{J}_2 \theta) dx \\
 &= \frac{1}{2} \int \begin{pmatrix} \theta_1 \\ \theta_2 \end{pmatrix} \wedge \begin{pmatrix} \frac{1}{2} Q \partial^{-1} (Q^T \theta_1 - R^T \theta_2) + (Q \partial^{-1} Q^T)^T \theta_1 - Q^T \partial^{-1} R I_n \theta_2 \\ -\frac{1}{2} R \partial^{-1} (Q^T \theta_1 - R^T \theta_2) - R^T \partial^{-1} Q I_n \theta_1 + (R \partial^{-1} R^T)^T \theta_2 \end{pmatrix} dx \\
 &= \frac{1}{2} \int \left[ \frac{1}{2} (\theta_1 \wedge Q - \theta_2 \wedge R) \partial^{-1} (Q^T \theta_1 - R^T \theta_2) + \theta_1 \wedge (Q \partial^{-1} Q^T)^T \theta_1 - \theta_1 \wedge Q^T \partial^{-1} R I_n \theta_2 \right. \\
 &\quad \left. + \theta_2 \wedge (R \partial^{-1} R^T)^T \theta_2 - \theta_2 \wedge R^T \partial^{-1} Q I_n \theta_1 \right] dx \\
 &= \frac{1}{2} \sum_{i,j=1}^n \int \left[ \frac{1}{2} (\theta_{1j} \wedge q_j - \theta_{2j} \wedge r_j) \partial^{-1} (q_i \theta_{1i} - r_i \theta_{2i}) + \theta_{1j} \wedge q_i \partial^{-1} q_j \theta_{1i} + \theta_{2j} \wedge r_i \partial^{-1} r_j \theta_{2i} \right. \\
 &\quad \left. - \theta_{1j} \wedge q_i \partial^{-1} r_i \theta_{2j} - \theta_{2j} \wedge r_i \partial^{-1} q_i \theta_{1j} \right] dx \\
 &= \frac{1}{2} \sum_{i,j=1}^n \int \left[ \frac{1}{2} (\theta_{1j} \wedge q_j - \theta_{2j} \wedge r_j) \partial^{-1} (q_i \theta_{1i} - r_i \theta_{2i}) + \theta_{1j} \wedge q_i \partial^{-1} (q_j \theta_{1i} - r_j \theta_{2j}) \right. \\
 &\quad \left. + \theta_{2j} \wedge r_i \partial^{-1} (r_j \theta_{2i} - q_j \theta_{1j}) \right] dx. \tag{A.5}
 \end{aligned}$$

In the equality (A.4), we have applied integration by parts which is a special case of (2.11) to the terms which contain explicitly  $(\tilde{Q} + \tilde{R})_x$ .

We calculate

$$\begin{aligned}
 \text{PrV}_{\mathcal{J}\theta}(\Theta_{\mathcal{J}_1}) &= \sum_{j=1}^n \int \left[ -\frac{3}{2} \theta_{1jx} \wedge \left( \frac{3}{2} q_j (\tilde{Q} + \tilde{R})_x + q_{jx} (\tilde{Q} + \tilde{R}) + \frac{1}{2} q_j \partial^{-1} (Q^T \theta_1 - R^T \theta_2) \right) \right. \\
 &\quad \left. + \sum_{i=1}^n q_i \partial^{-1} (q_j \theta_{1i} - r_i \theta_{2j}) \right) - \frac{1}{2} \theta_{1j} \wedge \left( \frac{3}{2} q_j (\tilde{Q} + \tilde{R})_x + q_{jx} (\tilde{Q} + \tilde{R}) \right) \\
 &\quad \left. + \frac{1}{2} q_j \partial^{-1} (Q^T \theta_1 - R^T \theta_2) + \sum_{i=1}^n q_i \partial^{-1} (q_j \theta_{1i} - r_i \theta_{2j}) \right)_x \\
 &\quad - \frac{3}{2} \theta_{2jx} \wedge \left( \frac{3}{2} r_j (\tilde{Q} + \tilde{R})_x + r_{jx} (\tilde{Q} + \tilde{R}) - \frac{1}{2} r_j \partial^{-1} (Q^T \theta_1 - R^T \theta_2) \right) \\
 &\quad - \sum_{i=1}^n r_i \partial^{-1} (q_i \theta_{1j} - r_j \theta_{2i}) \right) - \frac{1}{2} \theta_{2j} \wedge \left( \frac{3}{2} r_j (\tilde{Q} + \tilde{R})_x + r_{jx} (\tilde{Q} + \tilde{R}) \right) \\
 &\quad \left. - \frac{1}{2} r_j \partial^{-1} (Q^T \theta_1 - R^T \theta_2) - \sum_{i=1}^n r_i \partial^{-1} (q_i \theta_{1j} - r_j \theta_{2i}) \right)_x \wedge (\tilde{Q} + \tilde{R}) dx \\
 &= \sum_{j=1}^n \int \left[ \left( -\frac{9}{4} q_j \theta_{1jx} - \frac{5}{4} q_{jx} \theta_{1j} - \frac{9}{4} r_j \theta_{2jx} - \frac{5}{4} r_{jx} \theta_{2j} \right) \wedge (\tilde{Q} + \tilde{R})_x \right. \\
 &\quad \left. + \left( -\frac{3}{4} q_j \theta_{1j} - \frac{3}{4} r_j \theta_{2j} \right) \wedge (\tilde{Q} + \tilde{R})_{xx} - \frac{1}{4} (q_j \theta_{1j} - r_j \theta_{2j}) \wedge (Q^T \theta_1 - R^T \theta_2) \right. \\
 &\quad \left. + \left( -\frac{3}{4} q_j \theta_{1jx} - \frac{1}{4} q_{jx} \theta_{1j} + \frac{3}{4} r_j \theta_{2jx} + \frac{1}{4} r_{jx} \theta_{2j} \right) \wedge \partial^{-1} (Q^T \theta_1 - R^T \theta_2) \right. \\
 &\quad \left. + \sum_{i=1}^n \left( \left( -\frac{3}{2} q_i \theta_{1jx} - \frac{1}{2} q_{ix} \theta_{1j} \right) \wedge \partial^{-1} (q_j \theta_{1i} - r_i \theta_{2j}) - \frac{1}{2} q_i \theta_{1j} \wedge q_j \theta_{1i} \right. \right. \\
 &\quad \left. \left. + \left( \frac{3}{2} r_i \theta_{2jx} + \frac{1}{2} r_{ix} \theta_{2j} \right) \wedge \partial^{-1} (q_i \theta_{1j} - r_j \theta_{2i}) - \frac{1}{2} r_i \theta_{2j} \wedge r_j \theta_{2i} \right) \right] \wedge (\tilde{Q} + \tilde{R}) dx
 \end{aligned}$$

$$\begin{aligned}
 &= \sum_{j=1}^n \int \left[ \left( -\frac{9}{4} q_j \theta_{1jx} - \frac{5}{4} q_{jx} \theta_{1j} - \frac{9}{4} r_j \theta_{2jx} - \frac{5}{4} r_{jx} \theta_{2j} \right) \wedge (\tilde{Q} + \tilde{R})_x \right. \\
 &\quad + \left( \frac{3}{4} q_{jx} \theta_{1j} + \frac{3}{4} q_j \theta_{1jx} + \frac{3}{4} r_{jx} \theta_{2j} + \frac{3}{4} r_j \theta_{2jx} \right) \wedge (\tilde{Q} + \tilde{R})_x \\
 &\quad + \left( -\frac{3}{4} q_j \theta_{1jx} - \frac{1}{4} q_{jx} \theta_{1j} + \frac{3}{4} r_j \theta_{2jx} + \frac{1}{4} r_{jx} \theta_{2j} \right) \wedge \partial^{-1} (Q^T \theta_1 - R^T \theta_2) \\
 &\quad + \sum_{i=1}^n \left( \left( -\frac{3}{2} q_i \theta_{1jx} - \frac{1}{2} q_{ix} \theta_{1j} \right) \wedge \partial^{-1} (q_j \theta_{1i} - r_i \theta_{2j}) \right. \\
 &\quad \left. \left. + \left( \frac{3}{2} r_i \theta_{2jx} + \frac{1}{2} r_{ix} \theta_{2j} \right) \wedge \partial^{-1} (q_i \theta_{1j} - r_j \theta_{2i}) \right) \right] \wedge (\tilde{Q} + \tilde{R}) dx.
 \end{aligned}$$

In the above, we have used the formula (2.12) and dropped some terms through (2.10), as well as the expression  $\sum_{j=1}^n (q_j \theta_{1j} - r_j \theta_{2j}) = Q^T \theta_1 - R^T \theta_2$ . Moreover, we have integrated the terms which contain explicitly  $(\tilde{Q} + \tilde{R})_{xx}$  by parts.

Owing to

$$\begin{aligned}
 &\sum_{j=1}^n \int \left[ \left( -\frac{9}{4} q_j \theta_{1jx} - \frac{5}{4} q_{jx} \theta_{1j} - \frac{9}{4} r_j \theta_{2jx} - \frac{5}{4} r_{jx} \theta_{2j} \right) \wedge (\tilde{Q} + \tilde{R})_x \right. \\
 &\quad \left. + \left( \frac{3}{4} q_{jx} \theta_{1j} + \frac{3}{4} q_j \theta_{1jx} + \frac{3}{4} r_{jx} \theta_{2j} + \frac{3}{4} r_j \theta_{2jx} \right) \wedge (\tilde{Q} + \tilde{R})_x \right] \wedge (\tilde{Q} + \tilde{R}) dx \\
 &= \sum_{j=1}^n \int \left[ -\frac{1}{2} (3q_j \theta_{1jx} + q_{jx} \theta_{1j} + 3r_j \theta_{2jx} + r_{jx} \theta_{2j}) \wedge (\tilde{Q} + \tilde{R})_x \right] \wedge (\tilde{Q} + \tilde{R}) dx \\
 &= \int -\frac{1}{2} (\partial^3 - 4\partial) (\tilde{Q} + \tilde{R}) \wedge (\tilde{Q} + \tilde{R})_x \wedge (\tilde{Q} + \tilde{R}) dx \\
 &= 0,
 \end{aligned} \tag{A.6}$$

we obtain

$$\begin{aligned}
 \text{PrV}_{\mathcal{J}\theta}(\Theta_{\mathcal{J}_1}) &= \sum_{j=1}^n \int \left[ \left( -\frac{3}{4} q_j \theta_{1jx} - \frac{1}{4} q_{jx} \theta_{1j} + \frac{3}{4} r_j \theta_{2jx} + \frac{1}{4} r_{jx} \theta_{2j} \right) \wedge \partial^{-1} (Q^T \theta_1 - R^T \theta_2) \right. \\
 &\quad + \sum_{i=1}^n \left( \left( -\frac{3}{2} q_i \theta_{1jx} - \frac{1}{2} q_{ix} \theta_{1j} \right) \wedge \partial^{-1} (q_j \theta_{1i} - r_i \theta_{2j}) \right. \\
 &\quad \left. \left. + \left( \frac{3}{2} r_i \theta_{2jx} + \frac{1}{2} r_{ix} \theta_{2j} \right) \wedge \partial^{-1} (q_i \theta_{1j} - r_j \theta_{2i}) \right) \right] \wedge (\tilde{Q} + \tilde{R}) dx.
 \end{aligned} \tag{A.7}$$

On the other hand, we have

$$\begin{aligned}
 \text{PrV}_{\mathcal{J}\theta}(\Theta_{\mathcal{J}_2}) &= \sum_{j=1}^n \int \left\{ \frac{1}{2} [\theta_{1j} \wedge \left( \frac{3}{2} q_j (\tilde{Q} + \tilde{R})_x + q_{jx} (\tilde{Q} + \tilde{R}) + \frac{1}{2} q_j \partial^{-1} (Q^T \theta_1 - R^T \theta_2) \right) \right. \right. \\
 &\quad + \sum_{k=1}^n q_k \partial^{-1} (q_j \theta_{1k} - r_k \theta_{2j}) - \theta_{2j} \wedge \left( \frac{3}{2} r_j (\tilde{Q} + \tilde{R})_x + r_{jx} (\tilde{Q} + \tilde{R}) \right. \\
 &\quad \left. \left. - \frac{1}{2} r_j \partial^{-1} (Q^T \theta_1 - R^T \theta_2) + \sum_{k=1}^n r_k \partial^{-1} (r_j \theta_{2k} - q_k \theta_{1j}) \right) \right] \wedge \partial^{-1} (Q^T \theta_1 - R^T \theta_2) \\
 &\quad + \theta_{1j} \wedge \sum_{i=1}^n \left[ \frac{3}{2} q_i (\tilde{Q} + \tilde{R})_x + q_{ix} (\tilde{Q} + \tilde{R}) + \frac{1}{2} q_i \partial^{-1} (Q^T \theta_1 - R^T \theta_2) \right.
 \end{aligned}$$

$$\begin{aligned}
 & + \sum_{k=1}^n q_k \partial^{-1} (q_i \theta_{1k} - r_k \theta_{2i}) \wedge \partial^{-1} (q_j \theta_{1i} - r_i \theta_{2j}) \\
 & + \theta_{2j} \wedge \sum_{i=1}^n \left[ \frac{3}{2} r_i (\tilde{Q} + \tilde{R})_x + r_{ix} (\tilde{Q} + \tilde{R}) - \frac{1}{2} r_i \partial^{-1} (Q^T \theta_1 - R^T \theta_2) \right. \\
 & \left. - \sum_{k=1}^n r_k \partial^{-1} (q_k \theta_{1i} - r_i \theta_{2k}) \right] \wedge \partial^{-1} (r_j \theta_{2i} - q_i \theta_{1j}) \} dx. \quad (\text{A.8})
 \end{aligned}$$

In order to understand, we divide the equality (A.8) into two parts **I** and **II**. The part **I** is the terms which contain explicitly only  $\tilde{Q} + \tilde{R}$  or  $(\tilde{Q} + \tilde{R})_x$ , i.e.,

$$\begin{aligned}
 \text{I} &= \sum_{j=1}^n \int \left[ \left( \frac{3}{4} q_j \theta_{1j} - \frac{3}{4} r_j \theta_{2j} \right) \wedge (\tilde{Q} + \tilde{R})_x + \left( \frac{1}{2} q_{jx} \theta_{1j} - \frac{1}{2} r_{jx} \theta_{2j} \right) \wedge (\tilde{Q} + \tilde{R}) \right] \wedge \partial^{-1} (Q^T \theta_1 - R^T \theta_2) \\
 & + \sum_{i=1}^n \left( \left( \frac{3}{2} q_i \theta_{1j} \wedge (\tilde{Q} + \tilde{R})_x + q_{ix} \theta_{1j} \wedge (\tilde{Q} + \tilde{R}) \right) \wedge \partial^{-1} (q_j \theta_{1i} - r_i \theta_{2j}) \right. \\
 & \left. + \left( \frac{3}{2} r_i \theta_{2j} \wedge (\tilde{Q} + \tilde{R})_x + r_{ix} \theta_{2j} \wedge (\tilde{Q} + \tilde{R}) \right) \wedge \partial^{-1} (r_j \theta_{2i} - q_i \theta_{1j}) \right) dx \\
 &= \sum_{j=1}^n \int \left[ \left( -\frac{3}{4} q_j \theta_{1jx} - \frac{1}{4} q_{jx} \theta_{1j} + \frac{3}{4} r_j \theta_{2jx} + \frac{1}{4} r_{jx} \theta_{2j} \right) \wedge (\tilde{Q} + \tilde{R}) \wedge \partial^{-1} (Q^T \theta_1 - R^T \theta_2) \right. \\
 & + \sum_{i=1}^n \left( \left( -\frac{3}{2} q_i \theta_{1jx} - \frac{1}{2} q_{ix} \theta_{1j} \right) \wedge (\tilde{Q} + \tilde{R}) \wedge \partial^{-1} (q_j \theta_{1i} - r_i \theta_{2j}) \right. \\
 & \left. + \left( \frac{3}{2} r_i \theta_{2jx} + \frac{1}{2} r_{ix} \theta_{2j} \right) \wedge (\tilde{Q} + \tilde{R}) \wedge \partial^{-1} (q_i \theta_{1j} - r_j \theta_{2i}) \right) dx. \quad (\text{A.9})
 \end{aligned}$$

The rest of (A.8) is as follows

$$\begin{aligned}
 \text{II} &= \sum_{j=1}^n \int \left\{ \frac{1}{2} \sum_{k=1}^n [\theta_{1j} \wedge q_k \partial^{-1} (q_j \theta_{1k} - r_k \theta_{2j}) + \theta_{2j} \wedge r_k \partial^{-1} (q_k \theta_{1j} - r_j \theta_{2k})] \wedge \partial^{-1} (Q^T \theta_1 - R^T \theta_2) \right. \\
 & + \theta_{1j} \wedge \sum_{i=1}^n \left[ \frac{1}{2} q_i \partial^{-1} (Q^T \theta_1 - R^T \theta_2) + \sum_{k=1}^n q_k \partial^{-1} (q_i \theta_{1k} - r_k \theta_{2i}) \right] \wedge \partial^{-1} (q_j \theta_{1i} - r_i \theta_{2j}) \\
 & \left. - \theta_{2j} \wedge \sum_{i=1}^n \left[ -\frac{1}{2} r_i \partial^{-1} (Q^T \theta_1 - R^T \theta_2) - \sum_{k=1}^n r_k \partial^{-1} (q_k \theta_{1i} - r_i \theta_{2k}) \right] \wedge \partial^{-1} (q_i \theta_{1j} - r_j \theta_{2i}) \right\} dx \\
 &= \sum_{j=1}^n \int \left[ \theta_{1j} \wedge \sum_{i,k=1}^n q_k \partial^{-1} (q_i \theta_{1k} - r_k \theta_{2i}) \wedge \partial^{-1} (q_j \theta_{1i} - r_i \theta_{2j}) \right. \\
 & \left. + \theta_{2j} \wedge \sum_{i,k=1}^n r_k \partial^{-1} (q_k \theta_{1i} - r_i \theta_{2k}) \wedge \partial^{-1} (q_i \theta_{1j} - r_j \theta_{2i}) \right] dx \\
 &= \sum_{i,j,k=1}^n \int [q_k \theta_{1j} \wedge \partial^{-1} (q_i \theta_{1k} - r_k \theta_{2i}) \wedge \partial^{-1} (q_j \theta_{1i} - r_i \theta_{2j}) \\
 & - r_j \theta_{2k} \wedge \partial^{-1} (q_i \theta_{1k} - r_k \theta_{2i}) \wedge \partial^{-1} (q_j \theta_{1i} - r_i \theta_{2j}) \\
 & + r_k \theta_{2j} \wedge \partial^{-1} (q_k \theta_{1i} - r_i \theta_{2k}) \wedge \partial^{-1} (q_i \theta_{1j} - r_j \theta_{2i}) \\
 & + r_j \theta_{2k} \wedge \partial^{-1} (q_i \theta_{1k} - r_k \theta_{2i}) \wedge \partial^{-1} (q_j \theta_{1i} - r_i \theta_{2j})] dx
 \end{aligned}$$

$$\begin{aligned}
 &= \sum_{i,j,k=1}^n \int (q_k \theta_{1j} - r_j \theta_{2k}) \wedge \partial^{-1} (q_i \theta_{1k} - r_k \theta_{2i}) \wedge \partial^{-1} (q_j \theta_{1i} - r_i \theta_{2j}) dx \\
 &= 0.
 \end{aligned} \tag{A.10}$$

From (A.9) and (A.10), we have

$$\text{PrV}_{\mathcal{J}\theta}(\Theta_{\mathcal{J}_2}) = \mathbf{I}. \tag{A.11}$$

Combining (A.7) and (A.11) gives

$$\text{PrV}_{\mathcal{J}\theta}(\Theta_{\mathcal{J}}) = 0, \tag{A.12}$$

so the operator  $\mathcal{J}$  is Hamiltonian.

Secondly, we will show the compatibility of the operators  $\mathcal{K}$  and  $\mathcal{J}$ , i.e., the equality (2.14).

Notice that

$$\mathcal{K}\theta = \begin{pmatrix} \theta_{2xx} - \theta_2 \\ \theta_1 - \theta_{1xx} \end{pmatrix}, \tag{A.13}$$

so from the equalities (A.4) and (A.5), we obtain

$$\begin{aligned}
 \text{PrV}_{\mathcal{K}\theta}(\Theta_{\mathcal{J}_1}) &= \sum_{j=1}^n \int \left[ -\frac{3}{2} \theta_{1jx} \wedge (\theta_{2jxx} - \theta_{2j}) - \frac{1}{2} \theta_{1j} \wedge (\theta_{2jxx} - \theta_{2j})_x \right. \\
 &\quad \left. - \frac{3}{2} \theta_{2jx} \wedge (\theta_{1j} - \theta_{1jxx}) - \frac{1}{2} \theta_{2j} \wedge (\theta_{1j} - \theta_{1jxx})_x \right] \wedge (\tilde{Q} + \tilde{R}) dx \\
 &= \sum_{j=1}^n \int \left[ -\frac{1}{2} (\partial^3 - 4\partial)(\theta_{1j} \wedge \theta_{2j}) \wedge (\tilde{Q} + \tilde{R}) \right] dx \\
 &= \frac{1}{2} \sum_{j=1}^n \int (\theta_{1j} \wedge \theta_{2j}) \wedge (3Q^T \theta_{1x} + Q_x^T \theta_1 + 3R^T \theta_{2x} + R_x^T \theta_2) dx \\
 &= \frac{1}{2} \sum_{i,j=1}^n \int (\theta_{1j} \wedge \theta_{2j}) \wedge (3q_i \theta_{1ix} + q_{ix} \theta_{1i} + 3r_i \theta_{2ix} + r_{ix} \theta_{2i}) dx,
 \end{aligned} \tag{A.14}$$

and

$$\begin{aligned}
 \text{PrV}_{\mathcal{K}\theta}(\Theta_{\mathcal{J}_2}) &= \sum_{i,j=1}^n \int \left[ \frac{1}{2} (\theta_{1j} \wedge (\theta_{2jxx} - \theta_{2j}) - \theta_{2j} \wedge (\theta_{1j} - \theta_{1jxx})) \wedge \partial^{-1} (q_i \theta_{1i} - r_i \theta_{2i}) \right. \\
 &\quad \left. + \theta_{1j} \wedge (\theta_{2ixx} - \theta_{2i}) \wedge \partial^{-1} (q_j \theta_{1i} - r_i \theta_{2j}) \right. \\
 &\quad \left. + \theta_{2j} \wedge (\theta_{1i} - \theta_{1ixx}) \wedge \partial^{-1} (r_j \theta_{2i} - q_i \theta_{1j}) \right] dx \\
 &= \sum_{i,j=1}^n \int \left[ \frac{1}{2} (\theta_{1j} \wedge \theta_{2jxx} - \theta_{1jxx} \wedge \theta_{2j}) \wedge \partial^{-1} (q_i \theta_{1i} - r_i \theta_{2i}) \right. \\
 &\quad \left. + (\theta_{1j} \wedge \theta_{2ixx} - \theta_{1jxx} \wedge \theta_{2i}) \wedge \partial^{-1} (q_j \theta_{1i} - r_i \theta_{2j}) \right] dx \\
 &= \sum_{i,j=1}^n \int \left[ -\frac{1}{2} (\theta_{1j} \wedge \theta_{2jx} - \theta_{1jx} \wedge \theta_{2j}) \wedge (q_i \theta_{1i} - r_i \theta_{2i}) \right. \\
 &\quad \left. - (\theta_{1j} \wedge \theta_{2ix} - \theta_{1jx} \wedge \theta_{2i}) \wedge (q_j \theta_{1i} - r_i \theta_{2j}) \right] dx
 \end{aligned}$$

$$\begin{aligned}
 &= \sum_{i,j=1}^n \int \left[ \frac{1}{2} (\theta_{1j} \wedge \theta_{2jx} + \theta_{1jx} \wedge \theta_{2j}) \wedge (q_i \theta_{1i} + r_i \theta_{2i}) \right. \\
 &\quad \left. - \theta_{1j} \wedge \theta_{2j} \wedge (q_i \theta_{1ix} + r_i \theta_{2ix}) \right] dx \\
 &= -\frac{1}{2} \sum_{i,j=1}^n \int [(\theta_{1j} \wedge \theta_{2j}) \wedge (3q_i \theta_{1ix} + q_{ix} \theta_{1i} + 3r_i \theta_{2ix} + r_{ix} \theta_{2i})] dx. \tag{A.15}
 \end{aligned}$$

The Eqs. (A.14) and (A.15) lead to

$$\text{PrV}_{\mathcal{H}\theta}(\Theta_{\mathcal{J}}) = \text{PrV}_{\mathcal{H}\theta}(\Theta_{\mathcal{J}_1}) + \text{PrV}_{\mathcal{H}\theta}(\Theta_{\mathcal{J}_2}) = 0, \tag{A.16}$$

so the operators  $\mathcal{H}$  and  $\mathcal{J}$  are compatible Hamiltonian operators.

Thus, we complete the proof of the Theorem 2.1.