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## Integrability on generalized $q$ -Toda equation and hierarchy

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In this paper, we construct a new integrable equation which is a generalization of  $q$ -Toda equation. Meanwhile its soliton solutions are constructed to show its integrable property. Further the Lax pairs of the generalized  $q$ -Toda equation and a whole integrable generalized  $q$ -Toda hierarchy are also constructed. To show the integrability, the Bi-Hamiltonian structure and tau symmetry of the generalized  $q$ -Toda hierarchy are given and this leads to the tau function.

**Keywords:** generalized  $q$ -Toda lattice, soliton solutions, Lax equation, generalized  $q$ -Toda hierarchy, tau function.

2000 Mathematics Subject Classification: 37K05, 37K10, 37K20

### 1. Introduction

The Toda lattice equation is a completely integrable system which has many important applications in mathematics and physics including the theory of Lie algebra representation, orthogonal polynomials and random matrix model [3, 19, 20, 23, 24]. Toda system has many kinds of reduction or extension, for example extended Toda hierarchy (ETH) [2], bigraded Toda hierarchy (BTH) [1]- [8] and so on. These generalized Toda hierarchies have important application in Gromov-Witten theory on  $\mathbb{CP}^1$  and orbifold.

The  $q$ -calculus (also called quantum calculus) traces back to the early 20th century and attracted important works in the area of  $q$ -calculus [6, 7] and  $q$ -hypergeometric series. The  $q$ -deformation of classical nonlinear integrable system started in 1990's by means of  $q$ -derivative  $\partial_q$  instead of usual derivative with respect spatial variable in the classical system. Several  $q$ -deformed integrable systems have been presented, for example the  $q$ -deformed Kadomtsev-Petviashvili ( $q$ -KP) hierarchy is a subject of intensive study in the literatures [16]- [14]. The  $q$ -Toda equation was also studied in [17, 21] but not for a whole hierarchy. This paper will be devoted to the further studies on a generalized  $q$ -Toda equation (GQTE) and generalized  $q$ -Toda hierarchy (GQTH).

To show the complete integrability of nonlinear evolution, it is necessary to test whether the equation has Hirota bilinear equation, three-soliton solution, Lax pair, Bi-Hamiltonian structure

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and even tau symmetry. This paper will show the integrability on the Generalized  $q$ -Toda hierarchy from the above several directions.

## 2. $q$ -difference operator and its generalization

As we all know, in common sense an integrable equation can always be rewritten in form of a Hirota bilinear equation using Hirota direct method. Therefore firstly we introduce some basic notation including Hirota derivatives as a preparation for introducing the Hirota bilinear equation of the generalized  $q$ -Toda equation.

Let  $F$  be a space of differentiable functions  $f, g : \mathbb{R}^n \rightarrow \mathbb{R}$ . The Hirota  $D$ -operator  $D : F \times F \rightarrow F$  is defined as

$$[D_x^{m_1} D_t^{m_2} \dots] f \cdot g = [(\partial_x - \partial_{x'})^{m_1} (\partial_t - \partial_{t'})^{m_2} \dots] f(x, t, \dots) g(x', t', \dots) |_{x'=x, t'=t, \dots} \quad (2.1)$$

Then one can find the following standard statement holds. Let  $P(D)$  be an arbitrary polynomial in  $D$  acting on two differentiable functions  $f(x, t, \dots)$  and  $g(x, t, \dots)$ , then the following equations hold

$$P(D)f \cdot g = P(-D)g \cdot f, \quad (2.2)$$

$$P(D)f \cdot 1 = P(\partial)f; \quad P(D)1 \cdot f = P(-\partial)f, \quad (2.3)$$

where  $\partial$  is the usual differential operator with respect to spatial variable  $x$ . The virtue of exponential identity can appropriately be as following form in terms of the Hirota  $D$ -operator

$$e^{\varepsilon D_x} f(x) g(x) = f(x + \varepsilon) g(x - \varepsilon). \quad (2.4)$$

If  $\varepsilon$  is parameter and  $f, g$  belong to continuously differentiable functions, like in [17], then define

$$\sigma_\varepsilon(x) = e^{\varepsilon x(x)\partial_x}. \quad (2.5)$$

Then

$$e^{\varepsilon x(x)\partial_x} u(x) = u(e^{\varepsilon x(x)\partial_x} x) = u(\sigma_\varepsilon(x)), \quad \varepsilon > 0. \quad (2.6)$$

If  $\sigma_\varepsilon(u(x)) = e^{\varepsilon \partial_x} u(x) = u(x + \varepsilon)$ , the system introduced later will lead to original Toda lattice. If  $\sigma_\varepsilon(x) = e^{\varepsilon x \partial_x} x = e^\varepsilon x$ , it implies  $e^{\varepsilon x \partial_x} u(x) = u(e^\varepsilon x)$ . Then the system will lead to  $q$ -Toda lattice in [17]. Considering that the vector field of the form  $x(x)\partial_x = x^n \partial_x$  on  $\mathbb{R}$ , it will be the general generalized  $q$ -Toda lattice. In this paper, we only give the case  $n = 2$ , and we just name the leading system later the generalized  $q$ -Toda equation.

**Proposition 2.1.** *The  $q$ -exponential identity acts on arbitrary continuous differentiable functions  $f(x), g(x)$  as the rule*

$$e^{\varepsilon x^2 D_x} f(x) g(x) = \Lambda_\varepsilon f(x) \Lambda_\varepsilon^{-1} g(x), x \in \mathbb{R} \quad (2.7)$$

where the forward and backward shift operators are separately represented by  $\Lambda_\varepsilon$  and  $\Lambda_\varepsilon^{-1}$ , respectively acting as

$$\Lambda_\varepsilon f(x) = f\left(\frac{x}{1 - x\varepsilon}\right), \quad \Lambda_\varepsilon^{-1} g(x) = g\left(\frac{x}{1 + x\varepsilon}\right). \quad (2.8)$$

**Proof.** Making use of the change of variable  $x^2 D_x = D_{x'}$ ,  $x' = -\frac{1}{x}$  is the idea to prove the identity, i.e.

$$e^{\varepsilon x^2 D_x} f(x) g(x) = e^{\varepsilon D_{x'}} f\left(-\frac{1}{x'}\right) g\left(-\frac{1}{x'}\right). \quad (2.9)$$

Using eq.(2.4) for the right hand side of eq.(2.9), we end up the proof with

$$e^{\varepsilon x^2 D_x} f(x) g(x) = f\left(-\frac{1}{x'+\varepsilon}\right) g\left(-\frac{1}{x'-\varepsilon}\right) = f\left(\frac{1}{\frac{1}{x}-\varepsilon}\right) g\left(\frac{1}{\frac{1}{x}+\varepsilon}\right) = \Lambda_\varepsilon f(x) \Lambda_\varepsilon^{-1} g(x).$$

□

To give the definition of the generalized  $q$ -Toda equation, we need the following central generalized difference operator.

**Definition 2.1.** The central  $q$ -difference operator  $\Delta_x^2$  acts on an arbitrary function  $f(x)$ ,  $x \in \mathbb{R}$ , as

$$\Delta_x^2 f(x) = f\left(\frac{x}{1-x\varepsilon}\right) + f\left(\frac{x}{1+x\varepsilon}\right) - 2f(x), \quad (2.10)$$

which is easily rewritten as  $\Delta_x^2 f(x) = (\Lambda_\varepsilon + \Lambda_\varepsilon^{-1} - 2)f(x)$ .

In the next section, we will try to use the above defined generalize  $q$ -shift operator to define the generalized  $q$ -Toda equation.

### 3. The generalized $q$ -Toda equation

The well-known Toda equation represents the motion of the one-dimensional particles by

$$\frac{d^2 y_n}{dt^2} = e^{y_{n-1}-y_n} - e^{y_n-y_{n+1}}. \quad (3.1)$$

By introducing the force

$$U_n = e^{y_{n-1}-y_n} - 1, \quad (3.2)$$

the Toda equation eq.(3.1) turns out to be

$$\frac{d^2}{dt^2} \log(1 + U_n) = U_{n+1} + U_{n-1} - 2U_n. \quad (3.3)$$

Similarly as Toda equation, we define the generalized  $q$ -Toda equation(GQTE) as follows

$$\varepsilon^2 \frac{d^2 \phi(x)}{dt^2} = e^{\phi(\frac{x}{1+x\varepsilon}) - \phi(x)} - e^{\phi(x) - \phi(\frac{x}{1-x\varepsilon})}. \quad (3.4)$$

By introducing the force

$$V = e^{\phi(\frac{x}{1+x\varepsilon}) - \phi(x)} - 1, \quad (3.5)$$

then the GQTE becomes

$$\varepsilon^2 \frac{d^2}{dt^2} \log(1 + V(x, t)) = \Delta_x^2 V(x, t) = V\left(\frac{x}{1-x\varepsilon}, t\right) + V\left(\frac{x}{1+x\varepsilon}, t\right) - 2V(x, t). \quad (3.6)$$

It is necessary to introduce the dependent variable transformation as

$$V(x, t) = \frac{d^2}{dt^2} \log f(x, t). \quad (3.7)$$

Then the bilinear form for  $f(x, t)$  is evolved as

$$V(x, t) = \frac{f_{tt}f - f_t^2}{f^2} = \frac{f(\frac{x}{1-x\varepsilon}, t)f(\frac{x}{1+x\varepsilon}, t)}{f^2} - 1. \quad (3.8)$$

Then the generalized  $q$ -Toda equation can be rewritten as a Hirota bilinear form in terms of Hirota  $D$ -operator as

$$P(D)f(x, t) \cdot f(x, t) = [D_t^2 - (e^{\varepsilon x^2 D_x} + e^{-\varepsilon x^2 D_x} - 2)]f(x, t) \cdot f(x, t) = 0, \quad (3.9)$$

by multiplying eq.(3.8) by  $2f^2(x, t)$  and using the  $q$ -exponential identity eq.(2.7). Supposing function  $f$  has finite perturbation expansion around a formal perturbation parameter  $\varepsilon$  as

$$f(x, t) = 1 + \varepsilon f^{(1)}(x, t) + \varepsilon^2 f^{(2)}(x, t) + \dots \quad (3.10)$$

Substituting eq.(3.10) into generalized  $q$ -Toda equation

$$P(D)f(x, t) \cdot f(x, t) = 0, \quad (3.11)$$

we have

$$\begin{aligned} & P(D)f(x, t) \cdot f(x, t) \\ &= P(D)[1 \cdot 1 + \varepsilon 1 \cdot f^{(1)} + f^{(1)} \cdot 1 + \varepsilon^2(1 \cdot f^{(2)} + f^{(2)} \cdot 1 + f^{(1)} \cdot f^{(1)}) \\ &+ \varepsilon^3(1 \cdot f^{(3)} + f^{(3)} \cdot 1 + f^{(1)} \cdot f^{(2)} + f^{(2)} \cdot f^{(1)}) \\ &+ \varepsilon^4(1 \cdot f^{(4)} + f^{(4)} \cdot 1 + f^{(1)} \cdot f^{(3)} + f^{(3)} \cdot f^{(1)} + f^{(2)} \cdot f^{(2)}) + \dots]. \end{aligned} \quad (3.12)$$

The coefficient of the first term  $\varepsilon^0$  is trivial. For the coefficient of  $\varepsilon^1$ , we get

$$P(D)1 \cdot f^{(1)} + f^{(1)} \cdot 1 = 2P(\partial)f^{(1)} = 2[\partial_t^2 - (e^{\varepsilon x^2 \partial_x} + e^{-\varepsilon x^2 \partial_x} - 2)]f^{(1)} = 0. \quad (3.13)$$

Then the equation  $f^{(1)}$  has exponential type solution as

$$f^{(1)}(x, t) = e^{-\frac{\alpha}{x} + \beta t + \eta}, \quad (3.14)$$

where  $\alpha, \beta, \eta$  are arbitrary constants with the dispersion relation as

$$\beta^2 = e^{\alpha \varepsilon} + e^{-\alpha \varepsilon} - 2. \quad (3.15)$$

Comparing the coefficients of  $\varepsilon^2$  in eq.(3.12) will yield

$$P(D)1 \cdot f^{(2)} + f^{(2)} \cdot 1 + f^{(1)} \cdot f^{(1)} = 2P(\partial)f^{(2)} + P(D)f^{(1)} \cdot f^{(1)} = 0, \quad (3.16)$$

which implies exactly

$$\begin{aligned} & [D_t^2 - (e^{\varepsilon x^2 D_x} + e^{-\varepsilon x^2 D_x} - 2)]f^{(1)}(x, t) \cdot f^{(1)}(x, t) \\ &= -2[\partial_t^2 - (e^{\varepsilon x^2 \partial_x} + e^{-\varepsilon x^2 \partial_x} - 2)]f^{(2)}(x, t). \end{aligned} \quad (3.17)$$

Since  $f^{(1)}$  given in eq.(3.14) satisfies the form of eq.(3.17) by considering eq.(3.15), it is logical to take all order terms as zero, i.e.  $f^{(j)} = 0, j \geq 2$ . Therefore without loss of generality, we let  $\varepsilon = 1$ .

Then one- $q$ -soliton is constructed by the virtue of eq.(3.14) and eq.(3.15) as

$$V(x, t) = \frac{\beta^2 e^{-\frac{\alpha}{x} + \beta t + \eta}}{(1 + e^{-\frac{\alpha}{x} + \beta t + \eta})^2}. \quad (3.18)$$

The solution of one- $q$ -soliton  $V$  can be seen from Figure 1.

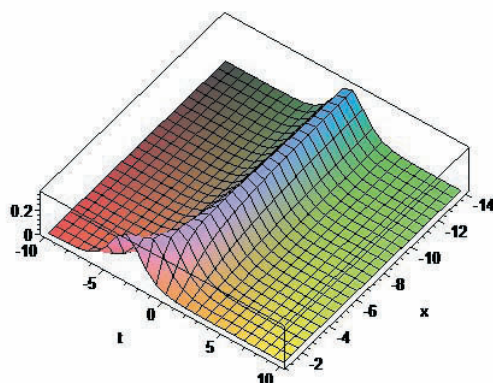


Fig. 1. One- $q$ -soliton solution  $V$  of generalized  $q$ -Toda equation with  $e^\varepsilon = 1.25$ ,  $\alpha_1 = -5$ ,  $\beta_1 = -1.1745$ .

We pick the starting solution of eq.(3.13) as the assumption of two-soliton solutions.

$$f^{(1)} = 2 \cosh\left(-\frac{\alpha_1}{x} + \beta_1 t + \eta_1\right), \quad (3.19)$$

where  $\alpha_i, \eta_i, i = 1, 2$  are arbitrary constants with the related dispersion relation

$$\beta_i^2 = e^{\alpha_i \varepsilon} + e^{-\alpha_i \varepsilon} - 2, i = 1, 2. \quad (3.20)$$

Apparently the use of vector notation

$$p_1 \pm p_2 = (\beta_1 \pm \beta_2, \alpha_1 \pm \alpha_2, \eta_1 \pm \eta_2), \quad (3.21)$$

can lead to dispersion relation eq.(3.20) as  $P(p_i) = 0, i = 1, 2, \dots$ . Then we get

$$-P(\partial)f^{(2)} = [(\beta_1 - \beta_2)^2 - (e^{(\alpha_1 - \alpha_2)\varepsilon} + e^{(\alpha_2 - \alpha_1)\varepsilon} - 2)]e^{-\frac{\alpha_1 + \alpha_2}{x} + (\beta_1 + \beta_2)t + \eta_1 + \eta_2}. \quad (3.22)$$

Therefore, the form of  $f^{(2)}$  can be

$$f^{(2)} = A(1, 2)e^{-\frac{\alpha_1 + \alpha_2}{x} + (\beta_1 + \beta_2)t + \eta_1 + \eta_2}. \quad (3.23)$$

Substituting such  $f^{(2)}$  into eq.(3.22) will help us determine the position of two- $q$ -soliton as

$$A(1, 2) = -\frac{(\beta_1 - \beta_2)^2 - (e^{(\alpha_1 - \alpha_2)\varepsilon} + e^{(\alpha_2 - \alpha_1)\varepsilon} - 2)}{(\beta_1 + \beta_2)^2 - (e^{(\alpha_1 + \alpha_2)\varepsilon} + e^{-(\alpha_1 + \alpha_2)\varepsilon} - 2)} = -\frac{P(p_1 - p_2)}{P(p_1 + p_2)}. \quad (3.24)$$

Supposing  $f^{(3)} = 0$ , by the use of the dispersion relation eq.(3.20) the coefficient of  $\varepsilon^3$  vanishes trivially and so do the rest of  $\varepsilon^j, j > 3$ . That means we have a good truncation up to  $\varepsilon^3$  which leads

to the two-q-soliton solution as

$$f(x, t) = 1 + e^{-\frac{\alpha_1}{x} + \beta_1 t + \eta_1} + e^{-\frac{\alpha_2}{x} + \beta_2 t + \eta_2} + A(1, 2)e^{-\frac{\alpha_1 + \alpha_2}{x} + (\beta_1 + \beta_2)t + \eta_1 + \eta_2}. \quad (3.25)$$

Therefore, we illustrate the collision of two-q-solitons as Figure 2.

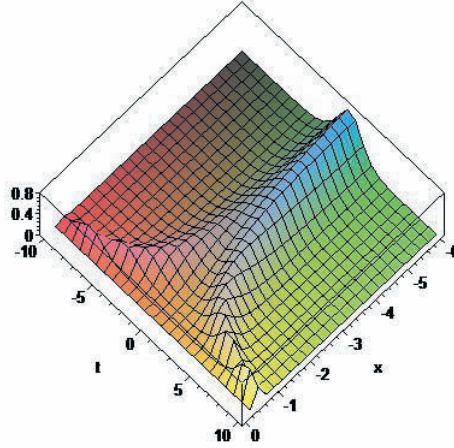


Fig. 2. Two-q-soliton solution  $V$  of generalized  $q$ -Toda equation with  $e^\varepsilon = 1.25$ ,  $\alpha_1 = -5$ ,  $\alpha_2 = 6$ .

To further derive three-soliton solution, we choose the starting solution of eq.(3.13) as the assumption as

$$f^{(1)} = \sum_{i=1}^3 e^{-\frac{\alpha_i}{x} + \beta_i t + \eta_i}, \quad (3.26)$$

where  $\alpha_i, \eta_i$  are arbitrary constants for  $i = 1, 2, 3$ . Similarly to the precious arguments, the coefficient of  $\varepsilon^0$  vanishes trivially. From the coefficient of  $\varepsilon^1$ , we have the corresponding dispersion relation

$$\beta_i^2 = e^{\alpha_i \varepsilon} + e^{-\alpha_i \varepsilon} - 2, i = 1, 2, 3. \quad (3.27)$$

From the coefficient of  $\varepsilon^2$ , we can obtain

$$-P(\partial)f^{(2)} = \sum_{i < j}^{(3)} [(\beta_i - \beta_j)^2 - (e^{(\alpha_i - \alpha_j)\varepsilon} + e^{(\alpha_i - \alpha_j)\varepsilon} - 2)] e^{-\frac{\alpha_i + \alpha_j}{x} + (\beta_i + \beta_j)t + \eta_i + \eta_j}. \quad (3.28)$$

The equation eq.(3.28) implies the explicit form of  $f^{(2)}$

$$f^{(2)} = \sum_{i < j}^{(3)} A(i, j) e^{-\frac{\alpha_i + \alpha_j}{x} + (\beta_i + \beta_j)t + \eta_i + \eta_j}, \quad (3.29)$$

with

$$A(i, j) = -\frac{P(p_i - p_j)}{P(p_i + p_j)} = -\frac{(\beta_i - \beta_j)^2 - (e^{(\alpha_i - \alpha_j)\varepsilon} + e^{(\alpha_i - \alpha_j)\varepsilon} - 2)}{(\beta_i + \beta_j)^2 - (e^{(\alpha_i + \alpha_j)\varepsilon} + e^{-(\alpha_i + \alpha_j)\varepsilon} - 2)}. \quad (3.30)$$

For the coefficient of  $\varepsilon^3$ , we have

$$P(D)1 \cdot f^{(3)} + f^{(3)} \cdot 1 + f^{(1)} \cdot f^{(2)} + f^{(2)} \cdot f^{(1)} = 0.$$

We can also rewrite them as

$$\begin{aligned} -P(\partial)f^{(3)} &= (A(1,2)P(p_3 - p_1 - p_2) + A(1,3)P(p_2 - p_1 - p_3) + A(2,3)P(p_1 - p_2 - p_3)) \\ &\quad \times e^{-\frac{\alpha_1 + \alpha_2 + \alpha_3}{x} + (\beta_1 + \beta_2 + \beta_3)t + \eta_1 + \eta_2 + \eta_3}. \end{aligned} \quad (3.31)$$

Suppose that  $f^{(3)}$  is of the form

$$f^{(3)} = A(1,2,3)e^{-\frac{\alpha_1 + \alpha_2 + \alpha_3}{x} + (\beta_1 + \beta_2 + \beta_3)t + \eta_1 + \eta_2 + \eta_3}, \quad (3.32)$$

then one can find

$$A(1,2,3) = -\frac{A(1,2)P(p_3 - p_1 - p_2) + A(1,3)P(p_2 - p_1 - p_3) + A(2,3)P(p_1 - p_2 - p_3)}{P(p_1 + p_2 + p_3)}. \quad (3.33)$$

Following the steps, one can find we can suppose the vanishing of  $f^{(4)}$  and it is a reasonable truncation to terms of  $\varepsilon^4$ , i.e. the form of the equation (3.12) becomes

$$2P(D)f^{(1)} \cdot f^{(3)} + P(D)f^{(2)} \cdot f^{(2)} = 0, \quad (3.34)$$

which means the following condition holds

$$A(1,2,3) = A(1,2)A(1,3)A(2,3). \quad (3.35)$$

Then we can express the solution of three- $q$ -soliton(see Figure 3) as

$$\begin{aligned} f(x,t) &= 1 + \sum_{i=1}^3 e^{-\frac{\alpha_i}{x} + \beta_i t + \eta_i} + \sum_{i < j}^3 A(i,j) e^{-\frac{\alpha_i + \alpha_j}{x} + (\beta_i + \beta_j)t + \eta_i + \eta_j} \\ &\quad + A(1,2)A(1,3)A(2,3) e^{-\frac{\alpha_1 + \alpha_2 + \alpha_3}{x} + (\beta_1 + \beta_2 + \beta_3)t + \eta_1 + \eta_2 + \eta_3}. \end{aligned} \quad (3.36)$$

The above three-soliton solutions show the great integrable possibility in a certain sense. To deeply prove the integrability, we will give the Lax pair of the generalized  $q$ -Toda equation and further generalize it to a whole integrable hierarchy in the next section.

#### 4. The generalized $q$ -Toda hierarchy

Now we will consider that the algebra of the shift operator  $\Lambda_\varepsilon := e^{\varepsilon x^2 \partial_x}$ . A Left multiplication by  $X$  is as  $X\Lambda_\varepsilon^j$ ,  $(X\Lambda_\varepsilon^j)(g)(x) := X(x) \circ g(\frac{x}{1-j\varepsilon x})$  with defining the product  $(X(x)\Lambda_\varepsilon^i) \circ (Y(x)\Lambda_\varepsilon^j) := X(x)Y(\frac{x}{1-i\varepsilon x})\Lambda_\varepsilon^{i+j}$ .

Now we introduce the following free operators  $W_0, \bar{W}_0$

$$W_0 := e^{\sum_{j=0}^{\infty} t_j \frac{\Lambda_\varepsilon^j}{\varepsilon j!}}, \quad (4.1)$$

$$\bar{W}_0 := e^{-\sum_{j=0}^{\infty} t_j \frac{\Lambda_\varepsilon^{-j}}{\varepsilon j!}}, \quad (4.2)$$

where  $t_j \in \mathbb{R}$  will play the role of continuous times.



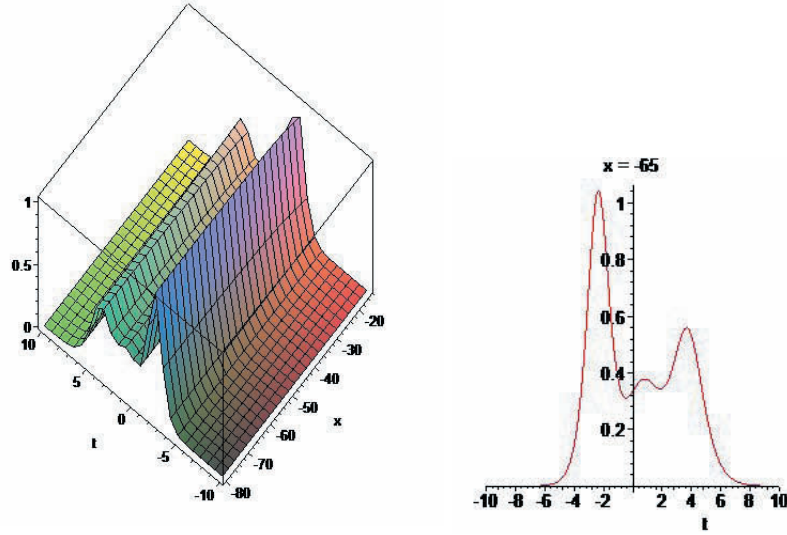


Fig. 3. Three-q-soliton solution  $V$  of generalized  $q$ -Toda equation with  $e^\varepsilon = 1.25$ ,  $\alpha_1 = -5$ ,  $\alpha_2 = 6$ ,  $\alpha_3 = -7.9141$ ,  $\beta_1 = -1.1745$ ,  $\beta_2 = -1.4411$ ,  $\beta_3 = 2.0045$

We define the dressing operators  $W, \bar{W}$  as follows

$$W := S \circ W_0, \quad \bar{W} := \bar{S} \circ \bar{W}_0, \quad (4.3)$$

where  $S, \bar{S}$  have expansions as

$$\begin{aligned} S &= 1 + \omega_1(x)\Lambda_\varepsilon^{-1} + \omega_2(x)\Lambda_\varepsilon^{-2} + \cdots, \\ \bar{S} &= \bar{\omega}_0(x) + \bar{\omega}_1(x)\Lambda_\varepsilon + \bar{\omega}_2(x)\Lambda_\varepsilon^2 + \cdots. \end{aligned} \quad (4.4)$$

The inverse operators  $S^{-1}, \bar{S}^{-1}$  of operators  $S, \bar{S}$  have expansions of the form

$$\begin{aligned} S^{-1} &= 1 + \omega'_1(x)\Lambda_\varepsilon^{-1} + \omega'_2(x)\Lambda_\varepsilon^{-2} + \cdots, \\ \bar{S}^{-1} &= \bar{\omega}'_0(x) + \bar{\omega}'_1(x)\Lambda_\varepsilon + \bar{\omega}'_2(x)\Lambda_\varepsilon^2 + \cdots. \end{aligned} \quad (4.5)$$

The Lax operator  $\mathcal{L}$  of the generalized  $q$ -deformed Toda hierarchy is defined by

$$\mathcal{L} := W \circ \Lambda_\varepsilon \circ W^{-1} = \bar{W} \circ \Lambda_\varepsilon^{-1} \circ \bar{W}^{-1}, \quad (4.6)$$

and have the following expansions

$$\mathcal{L} = \Lambda_\varepsilon + U(x) + V(x)\Lambda_\varepsilon^{-1}. \quad (4.7)$$

In fact the Lax operators  $\mathcal{L}$  are also be equivalently defined by

$$\mathcal{L} := S \circ \Lambda_\varepsilon \circ S^{-1} = \bar{S} \circ \Lambda_\varepsilon^{-1} \circ \bar{S}^{-1}. \quad (4.8)$$

#### 4.1. Lax equations of the GQTH

In this section we will give the Lax equations of the GQTH. Let us firstly introduce some convenient notation such as the operators  $B_j$  defined as  $B_j := \frac{\mathcal{L}^j}{j!}$ . Now we give the definition of the generalized  $q$ -Toda hierarchy(GQTH).

**Definition 4.1.** The generalized  $q$ -Toda hierarchy is a hierarchy in which the dressing operators  $S, \bar{S}$  satisfy following Sato equations

$$\varepsilon \partial_{t_j} S = -(B_j)_- S, \quad \varepsilon \partial_{t_j} \bar{S} = (B_j)_+ \bar{S}. \quad (4.9)$$

Then one can easily get the following proposition about  $W, \bar{W}$ .

**Proposition 4.1.** The dressing operators  $W, \bar{W}$  are subject to following Sato equations

$$\varepsilon \partial_{t_j} W = (B_j)_+ W, \quad \varepsilon \partial_{t_j} \bar{W} = -(B_j)_- \bar{W}. \quad (4.10)$$

From the previous proposition one can derive the following Lax equations for the Lax operators.

**Proposition 4.2.** The Lax equations of the GQTH are as follows

$$\varepsilon \partial_{t_j} \mathcal{L} = [(B_j)_+, \mathcal{L}]. \quad (4.11)$$

To see this kind of hierarchy more clearly, the generalized  $q$ -Toda equations as the  $t_1$  flow equations will be given in the next subsection.

#### 4.2. The generalized $q$ -Toda equations

As a consequence Sato equations, after taking into account that  $S$  and  $\bar{S}$ , the  $t_1$  flow of  $\mathcal{L}$  in the form of  $\mathcal{L} = \Lambda_\varepsilon + U + V \Lambda_\varepsilon^{-1}$  is as

$$\varepsilon \partial_{t_1} \mathcal{L} = [\Lambda_\varepsilon + U, V \Lambda_\varepsilon^{-1}], \quad (4.12)$$

which lead to generalized  $q$ -Toda equation

$$\varepsilon \partial_{t_1} U = V \left( \frac{x}{1 - \varepsilon x} \right) - V(x), \quad (4.13)$$

$$\varepsilon \partial_{t_1} V = U(x) V(x) - V(x) U \left( \frac{x}{1 + \varepsilon x} \right). \quad (4.14)$$

From Sato equation we deduce the following set of nonlinear partial differential-difference equations

$$\begin{cases} \omega_1(x) - \omega_1 \left( \frac{x}{1 + \varepsilon x} \right) = \varepsilon \partial_{t_1} (e^{\phi(x)}) \cdot e^{-\phi(x)}, \\ \varepsilon \partial_{t_1} \omega_1(x) = -e^{\phi(x)} e^{-\phi \left( \frac{x}{1 - \varepsilon x} \right)}. \end{cases} \quad (4.15)$$

Observe that if we cross the two first equations, then we get the generalized  $q$ -Toda equation (3.5). To give a linear description of the GQTH, we introduce wave functions  $\psi, \bar{\psi}$  defined by

$$\psi = W \cdot \chi, \quad \bar{\psi} = \bar{W} \cdot \bar{\chi}, \quad (4.16)$$

where

$$\chi(z) := z^{-\frac{1}{\varepsilon}}, \quad \bar{\chi}(z) := z^{\frac{1}{\varepsilon}}, \quad (4.17)$$

and the “ $\cdot$ ” means the action of an operator on a function. Note that  $\Lambda_\varepsilon \cdot \chi = z\chi$  and the following asymptotic expansions can be defined

$$\begin{aligned} \psi &= (1 + \omega_1(x)z^{-1} + \cdots) \psi_0(z), & \psi_0 &:= z^{-\frac{1}{\varepsilon}} e^{\sum_{j=1}^{\infty} t_j \frac{z^j}{\varepsilon j!}}, \\ \bar{\psi} &= (\bar{\omega}_0(x) + \bar{\omega}_1(x)z + \cdots) \bar{\psi}_0(z), & \bar{\psi}_0 &:= z^{\frac{1}{\varepsilon}} e^{-\sum_{j=0}^{\infty} t_j \frac{z^{-j}}{\varepsilon j!}}. \end{aligned} \quad (4.18)$$

We can further get linear equations of the GQTH in the following proposition.

**Proposition 4.3.** *The wave functions  $\psi, \bar{\psi}$  are subject to following Sato equations*

$$\mathcal{L} \cdot \psi = z\psi, \quad \mathcal{L} \cdot \bar{\psi} = z\bar{\psi}, \quad (4.19)$$

$$\varepsilon \partial_{t_j} \psi = (B_j)_+ \cdot \psi, \quad \varepsilon \partial_{t_j} \bar{\psi} = -(B_j)_- \cdot \bar{\psi}. \quad (4.20)$$

## 5. Bi-Hamiltonian structure and tau symmetry

To describe the integrability of the GQTH, we will construct the Bi-Hamiltonian structure and tau symmetry of the GQTH in this section. In this section, we will consider the GQTH on Lax operator

$$\mathcal{L} = \Lambda_\varepsilon + u + \varepsilon^v \Lambda_\varepsilon^{-1}. \quad (5.1)$$

Then for  $\bar{f} = \int f dx, \bar{g} = \int g dx$ , we can define the Hamiltonian bracket as

$$\{\bar{f}, \bar{g}\} = \int \sum_{w, w'} \frac{\delta f}{\delta w} \{w, w'\} \frac{\delta g}{\delta w'} dx, \quad w, w' = u \text{ or } v. \quad (5.2)$$

The bi-Hamiltonian structure for the GQTH can be given by the following two compatible Poisson brackets similar as [2, 8]

$$\begin{aligned} \{v(x), v(y)\}_1 &= \{u(x), u(y)\}_1 = 0, \\ \{u(x), v(y)\}_1 &= \frac{1}{\varepsilon} \left[ e^{\varepsilon x^2 \partial_x} - 1 \right] \delta(x - y), \end{aligned} \quad (5.3)$$

$$\begin{aligned} \{u(x), u(y)\}_2 &= \frac{1}{\varepsilon} \left[ e^{\varepsilon x^2 \partial_x} e^{v(x)} - e^{v(x)} e^{-\varepsilon x^2 \partial_x} \right] \delta(x - y), \\ \{u(x), v(y)\}_2 &= \frac{1}{\varepsilon} u(x) \left[ e^{\varepsilon x^2 \partial_x} - 1 \right] \delta(x - y), \\ \{v(x), v(y)\}_2 &= \frac{1}{\varepsilon} \left[ e^{\varepsilon x^2 \partial_x} - e^{-\varepsilon x^2 \partial_x} \right] \delta(x - y). \end{aligned} \quad (5.4)$$

For any difference operator  $A = \sum_k A_k \Lambda_\varepsilon^k$ , define residue  $\text{Res} A = A_0$ . In the following theorem, we will prove the above Poisson structure can be as the the Bi-Hamiltonian structures of the GQTH.

**Theorem 5.1.** *The flows of the GQTH are Hamiltonian systems of the form*

$$\frac{\partial u}{\partial t_j} = \{u, H_j\}_1, \quad j \geq 0. \quad (5.5)$$

*They satisfy the following bi-Hamiltonian recursion relation*

$$\{\cdot, H_{n-1}\}_2 = n \{\cdot, H_n\}_1.$$

*Here the Hamiltonians have the form*

$$H_j = \int h_j(u, v; u_x, v_x; \dots; \varepsilon) dx, \quad j \geq 1, \quad (5.6)$$

*with*

$$h_j = \frac{1}{j!} \text{Res} \mathcal{L}^j. \quad (5.7)$$

**Proof.** The proof is similar as the proof in [2, 8]. Here we will prove that the flows  $\frac{\partial}{\partial t_n}$  are also Hamiltonian systems with respect to the first Poisson bracket.

Suppose

$$B_n = \sum_k a_{n;k} \Lambda_\varepsilon^k, \quad (5.8)$$

and from

$$\frac{\partial \mathcal{L}}{\partial t_n} = [(B_n)_+, \mathcal{L}] = [-(B_n)_-, \mathcal{L}], \quad (5.9)$$

we can derive equation

$$\varepsilon \frac{\partial u}{\partial t_n} = a_{n;1} \left( \frac{x}{1 - \varepsilon x} \right) - a_{n;1}(x), \quad (5.10)$$

$$\varepsilon \frac{\partial v}{\partial t_n} = a_{n;0} \left( \frac{x}{1 + \varepsilon x} \right) e^{v(x)} - a_{n;0}(x) e^{v(\frac{x}{1 - \varepsilon x})}. \quad (5.11)$$

By the following calculation

$$\begin{aligned} dh_n &= \frac{1}{n!} d \operatorname{Res} [\mathcal{L}^n] = \frac{1}{n!} \operatorname{Res} [\mathcal{L}^n d\mathcal{L}] \\ &= \operatorname{Res} \left[ a_{n;0}(x) du + a_{n;1} \left( \frac{x}{1 + \varepsilon x} \right) e^{v(x)} dv \right], \end{aligned} \quad (5.12)$$

it yields the following identities

$$\frac{\delta H_n}{\delta u} = a_{n;0}(x), \quad \frac{\delta H_n}{\delta v} = a_{n;1} \left( \frac{x}{1 + \varepsilon x} \right) e^{v(x)}. \quad (5.13)$$

This agree with Lax equation

$$\frac{\partial u}{\partial t_n} = \{u, H_n\}_1 = \frac{1}{\varepsilon} \left[ e^{\varepsilon x^2 \partial_x} - 1 \right] \frac{\delta H_n}{\delta v} = \frac{1}{\varepsilon} (a_{n;1} \left( \frac{x}{1 - \varepsilon x} \right) - a_{n;1}(x)), \quad (5.14)$$

$$\frac{\partial v}{\partial t_n} = \{v, H_n\}_1 = \frac{1}{\varepsilon} \left[ 1 - e^{\varepsilon x^2 \partial_x} \right] \frac{\delta H_n}{\delta u} = \frac{1}{\varepsilon} \left[ a_{n;0} \left( \frac{x}{1 + \varepsilon x} \right) e^{v(x)} - a_{n;0}(x) e^{v(\frac{x}{1 - \varepsilon x})} \right]. \quad (5.15)$$

From the above identities we see that the flows  $\frac{\partial}{\partial t_n}$  are Hamiltonian systems with the first Hamiltonian structure. The recursion relation follows from the following trivial identities

$$n \frac{1}{n!} \mathcal{L}^n = \mathcal{L} \frac{1}{(n-1)!} \mathcal{L}^{n-1} = \frac{1}{(n-1)!} \mathcal{L}^{n-1} \mathcal{L}.$$

Then we get,

$$\begin{aligned} na_{n;1}(x) &= a_{n-1;0} \left( \frac{x}{1 - \varepsilon x} \right) + u a_{n-1;1}(x) + e^v a_{n-1;2} \left( \frac{x}{1 + \varepsilon x} \right) \\ &= a_{n-1;0}(x) + u \left( \frac{x}{1 - \varepsilon x} \right) a_{n-1;1}(x) + e^{v(\frac{x}{1 - 2\varepsilon x})} a_{n-1;2}(x). \end{aligned}$$

This further leads to

$$\begin{aligned} \{u, H_{n-1}\}_2 &= \left\{ \left[ \Lambda_\varepsilon e^{v(x)} - e^{v(x)} \Lambda_\varepsilon^{-1} \right] a_{n-1;0}(x) + u(x) [\Lambda_\varepsilon - 1] a_{n-1;1} \left( \frac{x}{1+\varepsilon x} \right) e^{v(x)} \right\} \\ &= n \left[ a_{n;1}(x) e^{v\left(\frac{x}{1-\varepsilon x}\right)} - a_{n;1} \left( \frac{x}{1+\varepsilon x} \right) e^{v(x)} \right]. \end{aligned}$$

This is exactly the recursion relation on flows for  $u$ . The similar recursion flow on  $v$  can be similarly derived. Theorem is proved till now.  $\square$

Similarly as [2], the tau symmetry of the GQTH can be proved in the following theorem.

**Theorem 5.2.** *The GQTH has the following tau-symmetry property:*

$$\frac{\partial h_m}{\partial t_n} = \frac{\partial h_n}{\partial t_m}, \quad m, n \geq 1. \quad (5.16)$$

**Proof.** Let us prove the theorem in a direct way

$$\begin{aligned} \frac{\partial h_m}{\partial t_n} &= \frac{1}{m!n!} \text{Res}[-(\mathcal{L}^n)_-, \mathcal{L}^m] \\ &= \frac{1}{m!n!} \text{Res}[(\mathcal{L}^m)_+, (\mathcal{L}^n)_-] \\ &= \frac{1}{m!n!} \text{Res}[(\mathcal{L}^m)_+, \mathcal{L}^n] = \frac{\partial h_n}{\partial t_m}. \end{aligned} \quad (5.17)$$

Theorem is proved.  $\square$

This property justifies the definition of the tau function for the GQTH as in the following proposition.

**Proposition 5.1.** *The tau function of the GQTH can also be defined by the following expressions in terms of the densities of the Hamiltonians:*

$$h_n = \varepsilon(\Lambda_\varepsilon - 1) \frac{\partial \log \tau}{\partial t_n}, \quad n \geq 0. \quad (5.18)$$

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