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Periodic orbits associated to Hamiltonian functions of degree four

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We consider the Hamiltonian polynomial function $H$ of degree fourth given by either

$$H(x, y, p_x, p_y) = \frac{1}{2}(p_x^2 + p_y^2) + \frac{1}{4}(x^2 + y^2) + V_3(x, y) + V_4(x, y),$$

or

$$H(x, y, p_x, p_y) = \frac{1}{2}(-p_x^2 + p_y^2) + \frac{1}{4}(-x^2 + y^2) + V_3(x, y) + V_4(x, y),$$

where $V_3(x, y)$ and $V_4(x, y)$ are homogeneous polynomials of degree three and four, respectively. Our main objective is to prove the existence and stability of periodic solutions associated to $H$ using the classical averaging method.

Keywords: Hamiltonian systems; Periodic orbits; Stability; Averaging theory.

2000 Mathematics Subject Classification: Primary 34C29, 37J25, 34C25; Secondary 85A05.

1. Introduction

It is known that the periodic orbits are the most simple non-trivial solutions of an ordinary differential system, and that their study is of particular interest because the motion in a neighborhood can be determined by their type of stability, [22]. In this paper we study the existence and stability of periodic solutions in Hamiltonian systems defined for two families of polynomial Hamiltonians of degree four on the plane. More specifically, we consider the following polynomial Hamiltonian:

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functions:
\[ H^+ = \frac{1}{2} (p_x^2 + p_y^2) + \frac{1}{2} (x^2 + y^2) + V_3(x,y) + V_4(x,y), \]
and
\[ H^- = \frac{1}{2} (-p_x^2 + p_y^2) + \frac{1}{2} (-x^2 + y^2) + V_3(x,y) + V_4(x,y), \]
where the polynomials \( V_3 \) and \( V_4 \) are homogeneous polynomials of the third and fourth degree, respectively and which are given by
\[ V_3(x,y) = \frac{A_3}{3} x^3 + B_3 x^2 y, \]
and
\[ V_4(x,y) = \frac{A_4}{4} x^4 + \frac{m}{2} x^2 y^2 + \frac{\Lambda}{4} y^4, \]
where \( A, B, \Lambda, m, \lambda \) are real parameters.

In this work we will make use of the averaging method to find families of periodic orbits associated to the Hamiltonian given by (1.1) and (1.2), respectively, for appropriate conditions on the parameters \( A, B, m, \lambda \) and \( \Lambda \). The essential tools that we use are the definitions and notations on the averaging method which are given in [6] (see also [27]). Others studies related to the Hénon-Heiles system can be found in: [1], [2], [4], [5], [8], [9], [10], [12], [13], [14], [15], [18], [19], [23], [26] and references therein. In [7], the authors study the dynamics associated to (1.1) in a much more general sense but the case of the cosmological model (case \( H^- \)) is not considered.

Before to enunciate the main results in this paper, we introduce the appropriate notations in relation to the parameters of the Hamiltonian (1.1) and (1.2), respectively. For the constants \( A, B, m, \lambda, \Lambda \), we define
\[
M_1 = 10B(A + B) + 9(\lambda - m)
\]
\[
M_2 = 10(A - 3B)(A + B) - 9(\lambda + \Lambda - 2m)
\]
\[
M_3 = 10(A - 2B)(A + B) + 9(m - \Lambda)
\]
\[
M_4 = 2(A - 6B)B + 3m
\]
\[
M_5 = 14B(A - B) + 3(3\lambda - m)
\]
\[
M_6 = 2(5A - 9B)(A - B) - 9(\lambda + \Lambda) + 6m
\]
\[
M_7 = 2(5A - 2B)(A - B) - 3(3\Lambda - m).
\]
Once fixed \( h > 0 \) and the parameters \( A, B, m, \lambda, \Lambda \), we consider the following sets:
\[ \Lambda_h^1 = \left\{ (A, B, m, \lambda, \Lambda) \in \mathbb{R}^5 : \frac{M_1}{M_2} < 0, \frac{M_3}{M_2} > 0, M_3 M_4 \neq 0 \right\} \]
and
\[ \Lambda_h^2 = \left\{ (A, B, m, \lambda, \Lambda) \in \mathbb{R}^5 : \frac{M_5}{M_6} < 0, \frac{M_7}{M_6} > 0, M_4 M_7 \neq 0 \right\}, \]
then we get the following result:
Theorem 1.1. Let $h > 0$ and $H$ be as in (1.1). Then:

1. there is at least one family of periodic orbits if $(A, B, m, \lambda, \Lambda) \in \Lambda^1_h \setminus \Lambda^2_h$.
2. there are at least two families of periodic orbits if $(A, B, m, \lambda, \Lambda) \in \Lambda^2_h \setminus \Lambda^3_h$.
3. there are at least three families of periodic orbits if $(A, B, m, \lambda, \Lambda) \in \Lambda^3_h \cap \Lambda^2_h$.

The next theorem gives information on the stability and unstability of the periodic solutions obtained in Theorem 1.1.

Theorem 1.2. Under the same hypotheses of Theorem 1.1 we have that:

1. The family in (a) is linearly stable if $M_3M_4 > 0$ and unstable if $M_3M_4 < 0$.
2. The two families in (b) are linearly stable if $M_4M_7 < 0$ and unstable if $M_4M_7 > 0$.
3. The three families in (c) are linearly stable if $M_3M_4 > 0$ and $M_5M_7 < 0$, and unstable if either
   - $(A1) M_3M_4 > 0$ and $M_6M_7 > 0$, or
   - $(A2) M_3M_4 < 0$ and $M_6M_7 > 0$, or
   - $(A3) M_3M_4 < 0$ and $M_6M_7 < 0$.

In a similar way, for $A, B, m, \lambda, \Lambda$, we define the following expressions:

- $N_1 = 10B(A - B) + 9(\lambda + m)$
- $N_2 = 10(A - B)(A + 3B) + 9(\lambda + \Lambda + 2m)$
- $N_3 = 10(A - B)(A + 2B) + 9(\Lambda + m)$
- $N_4 = 2B(A + 6B) - 3m$
- $N_5 = 14B(A + B) + 3(3\lambda + m)$
- $N_6 = 2(A + B)(5A + 9B) + 9(\lambda + \Lambda) + 6m$
- $N_7 = 2(A + B)(5A + 2B) + 3(3\lambda + m)$.

Fixed $h \neq 0$ small and the parameters $A, B, m, \lambda, \Lambda$, we consider the following sets:

- $\Omega^1_h = \left\{ (A, B, m, \lambda, \Lambda) \in \mathbb{R}^5 : -2h\frac{N_1}{N_2} > 0, 2h\frac{N_3}{N_2} > 0, N_3N_4 \neq 0 \right\}$
- $\Omega^2_h = \left\{ (A, B, m, \lambda, \Lambda) \in \mathbb{R}^5 : -2h\frac{N_5}{N_6} > 0, 2h\frac{N_7}{N_6} > 0, N_4N_7 \neq 0 \right\}$

then we get the following result:

Theorem 1.3. For any $h \neq 0$, the Hamiltonian system associated to (1.2) has at least:

1. one family of period orbits if $(A, B, m, \lambda, \Lambda) \in \Omega^1_h \setminus \Omega^2_h$.
2. two families of period orbits if $(A, B, m, \lambda, \Lambda) \in \Omega^2_h \setminus \Omega^1_h$.
3. three families of period orbits if $(A, B, m, \lambda, \Lambda) \in \Omega^1_h \cap \Omega^2_h$.

The next theorem gives us information on the stability or unstability of the periodic orbits given in Theorem 1.3.

Theorem 1.4. Under the same hypotheses of Theorem 1.3 we have that:
The family in (a) is linearly stable if $h \frac{N_3 N_4}{N_4} > 0$ and unstable if $h \frac{N_3 N_4}{N_4} < 0$.

(ii) The two families in (b) are linearly stable if $h \frac{N_4 N_7}{N_7} < 0$ and unstable if $h \frac{N_4 N_7}{N_7} > 0$.

(iii) The three families in (c) are linearly stable if $h \frac{N_3 N_4}{N_4} > 0$ and $h \frac{N_4 N_7}{N_7} < 0$, and unstable if either

(A1) $h \frac{N_3 N_4}{N_4} > 0$ and $h \frac{N_4 N_7}{N_7} > 0$, or

(A2) $h \frac{N_3 N_4}{N_4} < 0$ and $h \frac{N_4 N_7}{N_7} > 0$, or

(A3) $h \frac{N_3 N_4}{N_4} < 0$ and $h \frac{N_4 N_7}{N_7} < 0$.

Theorems 1.1 and 1.2 are improvement of results given in [18] and Theorems 1.3 and 1.4 are improvement of results given in [1]. The strategy to add more terms to the classical models of cosmological scalar fields in [18] is because such perturbation increases the regions of existence of periodic orbits.

Now, we point out that since our potential $V = V_3 + V_4$ is the addition of the homogeneous polynomials of degree three and four defined in (1.3) and (1.4) they have the property of symmetry, in fact, the reflection $y \rightarrow -y$. Thus, we can apply symmetry arguments, but will not be studied in this work. On the other hand, since the Hamiltonian function $H^+$ (and $H^-$) has the origin $(0, 0, 0, 0)$ as an equilibrium point whose eigenvalues of the linearization are $\pm i$ with multiplicity two and the linear part is diagonalizable and the Hessian evaluated at $(0, 0, 0, 0)$ is positive definite. Then by Weinstein’s Theorem (see [22]) we can prove the existence of at least two (families) periodic solutions whose periods are close to $2\pi$ and the energy level $h = H^+$ is sufficiently close to 0. But, we cannot apply this theorem for $H^-$.

It is important to observe that the used averaging method is closely reminiscent of the normal form approach: therefore, methods very similar to those exploited by us have been used in many occasions before, especially to find periodic orbits, see for example [3], [11], [20], [21] and [28]. On the other hand, the natural systems with indefinite kinetic energy (the “cosmological Hamiltonian” in the language of our work) have recently been investigated by [24] and [25].

The organization of the paper is as follows: In Sections 2 and 3, we will describe the motion equations to apply the Averaging theory. The main results will be proved in Section 4 and 5. Finally, in Section 6 we will give examples where the conditions of the main results are not empty.

2. Statement of the problem and equations of motion for $H^+$

The equations of motion associated to the system (1.1) and are

\[
\begin{align*}
\dot{x} &= p_x, \\
\dot{y} &= p_y, \\
\dot{p}_x &= -x - \frac{\partial V_3}{\partial x} - \frac{\partial V_4}{\partial x}, \\
\dot{p}_y &= -y - \frac{\partial V_3}{\partial y} - \frac{\partial V_4}{\partial y}.
\end{align*}
\]
Let \( x = \sqrt{\varepsilon} X, \ y = \sqrt{\varepsilon} Y, \ p_x = \sqrt{\varepsilon} p_X \) and \( p_y = \sqrt{\varepsilon} p_Y \) be the change of variable, which is \( \varepsilon^{-2} \)-symplectic, therefore the system (2.1) becomes

\[
\begin{aligned}
\dot{X} &= p_X, \\
\dot{Y} &= p_Y, \\
\dot{p}_X &= -X - \varepsilon \frac{\partial V_3}{\partial X} - \varepsilon^2 \frac{\partial V_4}{\partial X}, \\
\dot{p}_Y &= -Y - \varepsilon \frac{\partial V_3}{\partial Y} - \varepsilon^2 \frac{\partial V_4}{\partial Y}.
\end{aligned}
\]  

(2.2)

The Hamiltonian function associated to (2.1) and (2.2) is

\[
K^+ = \frac{1}{2}(p_x^2 + p_y^2) + \frac{1}{2}(X^2 + Y^2) + \varepsilon V_3 + \varepsilon^2 V_4,
\]  

(2.3)

where \( K^+ = \varepsilon^{-2} H^+ = h. \)

By the standard theory of Hamiltonian dynamical systems, for all \( \varepsilon \) different from zero, the original system (2.1) and the new system (2.2) have essentially the same phase portrait, and additionally the system (2.2), for \( \varepsilon \) sufficiently small, is close to an integrable system.

Now, we introduce the change of variables \((r, \rho, \theta, \alpha)\) by the relations

\[
\begin{aligned}
X &= r \cos \theta, & Y &= \rho \cos(\theta + \alpha), & p_X &= r \sin \theta, & p_Y &= \rho \sin(\theta + \alpha).
\end{aligned}
\]

Recall that this is a well defined change of variables when \( r > 0 \) and \( \rho > 0 \). Clearly this change of variables is not canonical, so we lose the Hamiltonian structure of the system of differential equations. Next, differentiating directly and using the expressions given in (2.2) we obtain

\[
\begin{aligned}
\dot{r} &= -\varepsilon \sin \theta \frac{\partial V_3}{\partial X} - \varepsilon^2 \sin \theta \frac{\partial V_4}{\partial X}, \\
\dot{\theta} &= -1 - \varepsilon \frac{\cos \theta}{r} \frac{\partial V_3}{\partial X} - \varepsilon^2 \frac{\cos \theta}{r} \frac{\partial V_4}{\partial X}, \\
\dot{\rho} &= -\varepsilon \sin(\theta + \alpha) \frac{\partial V_3}{\partial Y} - \varepsilon^2 \sin(\theta + \alpha) \frac{\partial V_4}{\partial Y}, \\
\dot{\alpha} &= \varepsilon \left[ \frac{\cos \theta}{r} \frac{\partial V_3}{\partial X} - \frac{\cos(\theta + \alpha)}{\rho} \frac{\partial V_3}{\partial Y} \right] + \varepsilon^2 \left[ \frac{\cos \theta}{r} \frac{\partial V_4}{\partial X} - \frac{\cos(\theta + \alpha)}{\rho} \frac{\partial V_4}{\partial Y} \right],
\end{aligned}
\]  

(2.4)

where the partial derivatives of \( V_3 \) and \( V_4 \) are evaluated at the point \((r \cos \theta, \rho \cos(\theta + \alpha))\). Moreover note that

\[
\begin{aligned}
\frac{\partial V_k}{\partial r} &= \cos \theta \frac{\partial V_k}{\partial X}, \\
\frac{\partial V_k}{\partial \theta} &= -r \sin \theta \frac{\partial V_k}{\partial X} - \rho \sin(\theta + \alpha) \frac{\partial V_k}{\partial Y}, \\
\frac{\partial V_k}{\partial \rho} &= \cos(\theta + \alpha) \frac{\partial V_k}{\partial Y}, \\
\frac{\partial V_k}{\partial \alpha} &= -\rho \sin(\theta + \alpha) \frac{\partial V_k}{\partial Y}.
\end{aligned}
\]
for \( k = 3, 4 \). Therefore (2.4) can be written as

\[
\dot{r} = \varepsilon \frac{1}{r} \left( \frac{\partial V_3}{\partial \theta} - \frac{\partial V_3}{\partial \alpha} \right) + \varepsilon^2 \frac{1}{r} \left( \frac{\partial V_4}{\partial \theta} - \frac{\partial V_4}{\partial \alpha} \right),
\]
\[
\dot{\theta} = -1 - \varepsilon \frac{1}{r} \frac{\partial V_3}{\partial r} - \varepsilon^2 \frac{1}{r} \frac{\partial V_4}{\partial r},
\]
\[
\dot{\rho} = \varepsilon \frac{1}{\rho} \frac{\partial V_3}{\partial \alpha} + \varepsilon^2 \frac{1}{\rho} \frac{\partial V_4}{\partial \alpha}.
\]
\[
\alpha = \varepsilon \left( \frac{1}{r} \frac{\partial V_3}{\partial r} - \frac{1}{\rho} \frac{\partial V_3}{\partial \rho} \right) + \varepsilon^2 \left( \frac{1}{r} \frac{\partial V_4}{\partial r} - \frac{1}{\rho} \frac{\partial V_4}{\partial \rho} \right).
\]  

(2.5)

We observe that for a fixed value \( h \) of \( K^+ \), in polar coordinates, it assumes the form

\[
h = \frac{1}{2} (r^2 + \rho^2) + \varepsilon V_3 (r \cos \theta, \rho \cos (\theta + \alpha)) + \varepsilon^2 V_4 (r \cos \theta, \rho \cos (\theta + \alpha)).
\]  

(2.6)

In order to put our system as a periodic ordinary differential equation, we introduce the \( \theta \) variable as independent (new time), and we use the notation prime to denote the derivative with respect to \( \theta \). It is observed that the angular variable \( \alpha \) cannot be used as the independent variable, since the new differential system do not have the appropriate form in order to apply the averaging method. Now, note that

\[
\frac{dr}{d\theta} = -\frac{\varepsilon \left( \frac{\partial V_3}{\partial \theta} - \frac{\partial V_3}{\partial \alpha} \right) + \varepsilon^2 \left( \frac{\partial V_4}{\partial \theta} - \frac{\partial V_4}{\partial \alpha} \right)}{r + \varepsilon \frac{\partial V_3}{\partial r} + \varepsilon^2 \frac{\partial V_4}{\partial r}}.
\]

Remember that the function

\[
f(\varepsilon) = \frac{1}{1 + a_1 \varepsilon + a_2 \varepsilon^2} = 1 - a_1 \varepsilon + (a_1^2 - a_2) \varepsilon^2 + O(\varepsilon^3),
\]

by developing a Taylor series in powers of \( \varepsilon \) around \( \varepsilon = 0 \). Replacing \( a_1 = \frac{1}{r} \frac{\partial V_3}{\partial r} \) and \( a_2 = \frac{1}{r} \frac{\partial V_4}{\partial r} \) we get

\[
\frac{1}{1 + \frac{1}{r} \frac{\partial V_3}{\partial r} \varepsilon + \frac{1}{r} \frac{\partial V_4}{\partial r} \varepsilon^2} = 1 - \frac{1}{r} \frac{\partial V_3}{\partial r} \varepsilon + \left[ \left( \frac{1}{r} \frac{\partial V_3}{\partial r} \right)^2 - \frac{1}{r^2} \frac{\partial V_4}{\partial r} \right] \varepsilon^2 + O(\varepsilon^3).
\]

Therefore,

\[
r' = -\frac{1}{r} \left( \frac{\partial V_3}{\partial \theta} - \frac{\partial V_3}{\partial \alpha} \right) \varepsilon + \frac{1}{r} \left[ \frac{1}{r} \frac{\partial V_3}{\partial \theta} \left( \frac{\partial V_3}{\partial \theta} - \frac{\partial V_3}{\partial \alpha} \right) - \left( \frac{\partial V_4}{\partial \theta} - \frac{\partial V_4}{\partial \alpha} \right) \right] \varepsilon^2 + O(\varepsilon^3).
\]

Moreover,

\[
\frac{d\rho}{d\theta} = -\frac{\varepsilon \frac{1}{\rho} \frac{\partial V_3}{\partial \alpha} + \varepsilon^2 \frac{1}{\rho} \frac{\partial V_4}{\partial \alpha}}{1 + \frac{1}{r} \frac{\partial V_3}{\partial r} \varepsilon + \frac{1}{r} \frac{\partial V_4}{\partial r} \varepsilon^2}.
\]
Thus, from (2.8), (2.9) and (2.10) the development of the function \( \varepsilon \) Derivating (2.8) with respect to \( \varepsilon \) we have
\[
\frac{d\alpha}{d\theta} = -\frac{\varepsilon}{1 + \varepsilon} \left( \frac{1}{r} \frac{\partial V_3}{\partial \varphi} - \frac{1}{\rho} \frac{\partial V_3}{\partial \varphi} \right) + \varepsilon^2 \left( \frac{1}{r} \frac{\partial V_4}{\partial \varphi} - \frac{1}{\rho} \frac{\partial V_4}{\partial \varphi} \right),
\]
then
\[
\alpha' = -\left( \frac{1}{r} \frac{\partial V_3}{\partial \varphi} - \frac{1}{\rho} \frac{\partial V_3}{\partial \varphi} \right) \varepsilon + \left[ \frac{1}{r} \frac{\partial V_4}{\partial \varphi} \left( \frac{1}{r} \frac{\partial V_3}{\partial \varphi} - \frac{1}{\rho} \frac{\partial V_3}{\partial \varphi} \right) - \left( \frac{1}{r} \frac{\partial V_3}{\partial \varphi} - \frac{1}{\rho} \frac{\partial V_3}{\partial \varphi} \right) \right] \varepsilon^2 + O(\varepsilon^3).
\]
If we write the previous system as a Taylor series in powers of \( \varepsilon \), we have that
\[
r' = -\frac{1}{r} \left( \frac{\partial V_3}{\partial \varphi} - \frac{\partial V_3}{\partial \varphi} \right) \varepsilon + \frac{1}{r} \left[ \frac{1}{r} \frac{\partial V_4}{\partial \varphi} \left( \frac{1}{r} \frac{\partial V_3}{\partial \varphi} - \frac{1}{\rho} \frac{\partial V_3}{\partial \varphi} \right) - \left( \frac{1}{r} \frac{\partial V_3}{\partial \varphi} - \frac{1}{\rho} \frac{\partial V_3}{\partial \varphi} \right) \right] \varepsilon^2 + O(\varepsilon^3),
\]
\[
\rho' = -\frac{1}{r} \frac{\partial V_3}{\partial \alpha} \varepsilon + \frac{1}{r} \left[ \frac{1}{r} \frac{\partial V_4}{\partial \alpha} \left( \frac{1}{r} \frac{\partial V_3}{\partial \alpha} - \frac{1}{\rho} \frac{\partial V_3}{\partial \alpha} \right) - \left( \frac{1}{r} \frac{\partial V_3}{\partial \alpha} - \frac{1}{\rho} \frac{\partial V_3}{\partial \alpha} \right) \right] \varepsilon^2 + O(\varepsilon^3),
\]
\[
\alpha' = -\left( \frac{1}{r} \frac{\partial V_3}{\partial \alpha} - \frac{1}{\rho} \frac{\partial V_3}{\partial \alpha} \right) \varepsilon + \left[ \frac{1}{r} \frac{\partial V_4}{\partial \alpha} \left( \frac{1}{r} \frac{\partial V_3}{\partial \alpha} - \frac{1}{\rho} \frac{\partial V_3}{\partial \alpha} \right) - \left( \frac{1}{r} \frac{\partial V_3}{\partial \alpha} - \frac{1}{\rho} \frac{\partial V_3}{\partial \alpha} \right) \right] \varepsilon^2 + O(\varepsilon^3).
\]
In order to apply the averaging theory we will fix the value of the first integral of \( K = h > 0 \). We write the Taylor expansion of the function \( \rho \) of second order in \( \varepsilon \) and rewrite the system (2.7) in the variables \( r \) and \( \alpha \). The function \( \rho \) in its Taylor form is
\[
\rho(\varepsilon) = \rho(0) + \rho'(0)\varepsilon + O(\varepsilon^2). \tag{2.8}
\]
Regarding (2.8), we obtain for \( \varepsilon = 0 \),
\[
\rho(0) = \sqrt{2h-r^2}. \tag{2.9}
\]
Derivating (2.8) with respect to \( \varepsilon \), we have
\[
0 = \rho \rho' + V_3(r \cos \theta, \rho \cos(\theta + \alpha)) + \varepsilon \frac{\partial V_3}{\partial \varphi} \rho' \cos(\theta + \alpha) +
2 \varepsilon V_4(r \cos \theta, \rho \cos(\theta + \alpha)) + \varepsilon^2 \frac{\partial V_4}{\partial \varphi} \rho' \cos(\theta + \alpha)
\]
and evaluating at \( \varepsilon = 0 \), we obtain
\[
\rho'(0) = -\frac{1}{\sqrt{2h-r^2}} V_3 \left( r \cos \theta, \sqrt{2h-r^2} \cos(\theta + \alpha) \right). \tag{2.10}
\]
Thus, from (2.8), (2.9) and (2.10) the development of the function \( \rho \) in Taylor series has the form
\[
\rho(\varepsilon) = \sqrt{2h-r^2} - \frac{1}{\sqrt{2h-r^2}} V_3(r \cos \theta, \sqrt{2h-r^2} \cos(\theta + \alpha)) \varepsilon + O(\varepsilon^2).
\]
Since $V_3$ depends on the variables $(r, \rho, \theta, \alpha)$, the derivatives of $V_3$ respect to their variables in the Taylor series are:

- $\frac{\partial V_3}{\partial \theta} = \frac{\partial V_3}{\partial \theta} + \rho'(0) \frac{\partial^2 V_3}{\partial \rho \partial \theta} \epsilon + O(\epsilon^2),$

- $\frac{\partial V_3}{\partial \alpha} = \frac{\partial V_3}{\partial \alpha} + \rho'(0) \frac{\partial^2 V_3}{\partial \rho \partial \alpha} \epsilon + O(\epsilon^2),$

- $\frac{\partial V_3}{\partial r} = \frac{\partial V_3}{\partial r} + \rho'(0) \frac{\partial^2 V_3}{\partial \rho \partial r} \epsilon + O(\epsilon^2),$

- $\frac{\partial V_3}{\partial \rho} = \frac{\partial V_3}{\partial \rho} + \rho'(0) \frac{\partial^2 V_3}{\partial \rho^2} \epsilon + O(\epsilon^2).$

Moreover,

- $\frac{1}{\rho} = \frac{1}{\rho(0)} - \rho'(0) \frac{\partial V_3}{\partial \rho} \epsilon + O(\epsilon^2),$

- $\frac{1}{\rho} \frac{\partial V_3}{\partial \rho} = \frac{1}{\rho(0)} \frac{\partial V_3}{\partial \rho} + \left[ \frac{\rho'(0)}{\rho(0)} \frac{\partial^2 V_3}{\partial \rho \partial \rho} - \frac{\rho'(0) \partial V_3}{\partial \rho} \right] \epsilon + O(\epsilon^2).$

Recall that the function $V_3$ and its partial derivatives are evaluated at the point $(r \cos \theta, \rho(0) \cos(\theta + \alpha))$, with $\rho(0)$ as in (2.9). Therefore,

$$\frac{1}{r} \left( \frac{\partial V_3}{\partial \theta} - \frac{\partial V_3}{\partial \alpha} \right) = \frac{1}{r} \left( \frac{\partial V_3}{\partial \theta} - \frac{\partial V_3}{\partial \alpha} \right) + \frac{\rho'(0)}{r} \left( \frac{\partial^2 V_3}{\partial \rho \partial \theta} - \frac{\partial^2 V_3}{\partial \rho \partial \alpha} \right) \epsilon \quad (\text{2.11})$$

Then substituting the latter expression into equation (2.7), we obtain the two differential equations

$$r' = -\frac{1}{r} \left( \frac{\partial V_3}{\partial \theta} - \frac{\partial V_3}{\partial \alpha} \right) \epsilon + \frac{1}{r} \left[ \frac{1}{r} \frac{\partial V_3}{\partial r} \left( \frac{\partial V_3}{\partial r} - \frac{\partial V_3}{\partial \theta} \right) - \rho'(0) \left( \frac{\partial^2 V_3}{\partial \rho \partial \theta} - \frac{\partial^2 V_3}{\partial \rho \partial \alpha} \right) \right] \epsilon^2 + O(\epsilon^3) \quad (\text{2.12})$$

$$\alpha' = -\frac{1}{r} \left( \frac{\partial V_3}{\partial \alpha} - \frac{r}{\rho(0)} \frac{\partial V_3}{\partial r} \right) \epsilon + \left[ -\frac{\rho'(0)}{r} \frac{\partial^2 V_3}{\partial \rho \partial \theta} + \left( \frac{\rho'(0)}{\rho(0)} \right) \frac{\partial^2 V_3}{\partial \rho^2} - \frac{\rho'(0) \partial V_3}{\partial \rho} \right] \epsilon^2 + O(\epsilon^3),$$

where the partial derivatives of $V_3$ and $V_4$ are evaluated at the point $\rho = \rho(0)$ (as in (2.9) and $(r \cos \theta, \rho(0) \cos(\theta + \alpha))$).

The system (2.12) has the general form

$$r' = F_{11} \epsilon + F_{21} \epsilon^2 + O(\epsilon^3),$$

$$\alpha' = F_{12} \epsilon + F_{22} \epsilon^2 + O(\epsilon^3), \quad (\text{2.13})$$
where

\[ F_{11} = -\frac{1}{r} \left( \frac{\partial V_3}{\partial \theta} - \frac{\partial V_3}{\partial \alpha} \right), \]

\[ F_{12} = -\frac{1}{r} \left( \frac{\partial V_3}{\partial r} - \frac{r}{\rho(0)} \frac{\partial V_3}{\partial \rho} \right), \]

\[ F_{21} = \frac{1}{r} \left[ \frac{1}{r} \frac{\partial V_3}{\partial \theta} \left( \frac{\partial V_3}{\partial \theta} - \frac{\partial V_3}{\partial \alpha} \right) - \rho'(0) \left( \frac{\partial^2 V_3}{\partial \rho \partial \theta} - \frac{\partial^2 V_3}{\partial \rho \partial \alpha} \right) \right], \]

\[ F_{22} = \left[ -\frac{\rho'(0)}{r} \frac{\partial^2 V_3}{\partial \rho \partial r} + \left( \frac{\rho'(0)}{\rho(0)} \frac{\partial^2 V_3}{\partial \rho^2} - \frac{\rho'(0)}{\rho^2(0)} \frac{\partial V_3}{\partial \rho} \right) \right] + \frac{1}{r} \frac{\partial V_3}{\partial r} \left( \frac{1}{r} \frac{\partial V_3}{\partial r} - \frac{1}{\rho} \frac{\partial V_3}{\partial \rho} \right) - \left( \frac{1}{r} \frac{\partial V_3}{\partial r} - \frac{1}{\rho} \frac{\partial V_3}{\partial \rho} \right). \]

Also, we denote

\[ f_1(r, \alpha) := (f_{11}, f_{12}) = \frac{1}{2\pi} \int_0^{2\pi} (F_{11}, F_{12}) d\theta \]

and

\[ f_2(r, \alpha) := (f_{21}, f_{22}) = \frac{1}{2\pi} \int_0^{2\pi} \left[ D_{r\alpha} F_1(\theta, r, \alpha) \cdot \int_0^\theta F_1(t, r, \alpha) dt + F_2(\theta, r, \alpha) \right] d\theta, \]

where \( F_1 = (F_{11}, F_{12}) \), \( F_2 = (F_{21}, F_{22}) \) and \( D_{r\alpha} F_1(\theta, r, \alpha) \) is the Jacobian matrix given by

\[ D_{r\alpha} F_1(\theta, r, \alpha) = \begin{pmatrix} \frac{\partial F_{11}}{\partial r} & \frac{\partial F_{11}}{\partial \alpha} \\ \frac{\partial F_{12}}{\partial r} & \frac{\partial F_{12}}{\partial \alpha} \end{pmatrix}. \]

The following lemma is related to the expressions (2.13), (2.15), (2.16) and will be very useful for future computations.

**Lemma 2.1.** \( f_1(r, \alpha) = (0, 0) \)

**Proof.** Note that

\[ F_{11} = \left[ B \left( 2h - r^2 \right) \cos^2(\alpha + \theta) + Ar^2 \cos^2 \theta \right] \sin \theta, \]

\[ F_{12} = -\frac{1}{r} \left[ B \left( 2h - 3r^2 \right) \cos^2(\alpha + \theta) + Ar^2 \cos^2 \theta \right] \cos \theta. \]

As each of the integrals in (2.15) involves the presence of combinations of the functions \( \cos^3(\theta) \), \( \cos^2(\theta) \sin(\theta) \), \( \cos(\theta) \sin^2(\theta) \), \( \sin^3(\theta) \), we arrive to the equation \( f_1(r, \alpha) \equiv (0, 0) \).

\[ \square \]
3. Statement of the problem and equations of motion for $H^-$

The equations of motion associated to the system (1.2) are

$$
\begin{align*}
\dot{x} &= -p_x, \\
\dot{y} &= p_y, \\
\dot{p}_x &= x - \frac{\partial V_3}{\partial x} - \frac{\partial V_4}{\partial x}, \\
\dot{p}_y &= -y - \frac{\partial V_3}{\partial y} - \frac{\partial V_4}{\partial y}.
\end{align*}
$$

(3.1)

Let $x = \sqrt{\varepsilon}X$, $y = \sqrt{\varepsilon}Y$, $p_x = \sqrt{\varepsilon}p_X$, and $p_y = \sqrt{\varepsilon}p_Y$ be the change of variable, which is $\varepsilon^{-2}$-symplectic, therefore the system (3.1) becomes

$$
\begin{align*}
\dot{X} &= -p_X, \\
\dot{Y} &= p_Y, \\
\dot{p}_X &= X - \varepsilon \frac{\partial V_3}{\partial X} - \varepsilon^2 \frac{\partial V_4}{\partial X}, \\
\dot{p}_Y &= -Y - \varepsilon \frac{\partial V_3}{\partial Y} - \varepsilon^2 \frac{\partial V_4}{\partial Y}.
\end{align*}
$$

(3.2)

The Hamiltonian function associated to (3.1) and (3.2) is

$$
K^- = \frac{1}{2}(-p_X^2 + p_Y^2) + \frac{1}{2}(-X^2 + Y^2) + \varepsilon V_3 + \varepsilon^2 V_4,
$$

(3.3)

where $K^- = \varepsilon^{-2}H^- = h$.

By the standard theory of Hamiltonian dynamical systems, for all $\varepsilon$ different from zero, the original system (3.1) and the new system (3.2) have essentially the same phase portrait, and additionally the system (3.2), for $\varepsilon$ sufficiently small, is close to an integrable system.

Now, analogously to the previous case we introduce the convenient change of coordinates $(r, \rho, \theta, \alpha)$ by the relations

$$
X = r \cos \theta, \quad Y = \rho \cos(-\theta + \alpha), \quad p_X = r \sin \theta, \quad p_Y = \rho \sin(-\theta + \alpha).
$$

Recall that this is a well defined change of variables when $r > 0$ and $\rho > 0$. We observe that for a fixed value $h$ of $K$, in polar coordinates,

$$
h = \frac{1}{2}(-r^2 + \rho^2) + \varepsilon V_3(r \cos \theta, \rho \cos(-\theta + \alpha)) + \varepsilon^2 V_4(r \cos \theta, \rho \cos(-\theta + \alpha)).
$$

(3.4)
Differentiating directly, we arrive to the fact that the equations of motion assume the form

\[ \begin{align*}
\dot{r} &= -\varepsilon \sin \theta \frac{\partial V_3}{\partial X} - \varepsilon^2 \sin \theta \frac{\partial V_4}{\partial X}, \\
\dot{\theta} &= 1 - \varepsilon \cos \theta \frac{\partial V_3}{r \partial X} - \varepsilon^2 \cos \theta \frac{\partial V_4}{r \partial X}, \\
\dot{\rho} &= \varepsilon \sin(-\theta + \alpha) \frac{\partial V_3}{\partial Y} + \varepsilon^2 \sin(-\theta + \alpha) \frac{\partial V_4}{\partial Y}, \\
\dot{\alpha} &= \varepsilon \left( -\frac{\cos \theta}{r} \frac{\partial V_3}{\partial X} - \frac{\cos(-\theta + \alpha)}{\rho} \frac{\partial V_3}{\partial Y} \right) + \varepsilon^3 \left( -\frac{\cos \theta}{r} \frac{\partial V_4}{\partial X} - \frac{\cos(-\theta + \alpha)}{\rho} \frac{\partial V_4}{\partial Y} \right).
\end{align*} \tag{3.5} \]

Moreover, note that

\[ \begin{align*}
\frac{\partial V_k}{\partial r} &= \cos \theta \frac{\partial V_k}{\partial X}, \\
\frac{\partial V_k}{\partial \theta} &= -r \sin \theta \frac{\partial V_k}{\partial X} + \rho \sin(-\theta + \alpha) \frac{\partial V_k}{\partial Y}, \\
\frac{\partial V_k}{\partial \rho} &= \cos(-\theta + \alpha) \frac{\partial V_k}{\partial Y}, \\
\frac{\partial V_k}{\partial \alpha} &= -\rho \sin(-\theta + \alpha) \frac{\partial V_k}{\partial Y},
\end{align*} \]

for \( k = 3, 4 \). Therefore (2.4) can be written as

\[ \begin{align*}
\dot{r} &= \varepsilon \frac{1}{r} \left( \frac{\partial V_3}{\partial \theta} + \frac{\partial V_3}{\partial \alpha} \right) + \varepsilon^2 \frac{1}{r} \left( \frac{\partial V_4}{\partial \theta} + \frac{\partial V_4}{\partial \alpha} \right), \\
\dot{\theta} &= 1 - \varepsilon \frac{1}{r} \frac{\partial V_3}{\partial r} - \varepsilon^2 \frac{1}{r} \frac{\partial V_4}{\partial r}, \\
\dot{\rho} &= -\varepsilon \frac{1}{\rho} \frac{\partial V_3}{\partial \alpha} - \varepsilon^2 \frac{1}{\rho} \frac{\partial V_4}{\partial \alpha}, \\
\dot{\alpha} &= \varepsilon \left( -\frac{1}{r} \frac{\partial V_3}{\partial \theta} - \frac{1}{\rho} \frac{\partial V_3}{\partial \theta} \right) + \varepsilon^2 \left( -\frac{1}{r} \frac{\partial V_4}{\partial \theta} - \frac{1}{\rho} \frac{\partial V_4}{\partial \theta} \right),
\end{align*} \tag{3.6} \]

where the partial derivatives of \( V_3 \) and \( V_4 \) are evaluated at the point \((r \cos \theta, \rho \cos(-\theta + \alpha))\). In order to put our system as a periodic system of ordinary differential equations, we introduce the \( \theta \) variable as independent (new time), and we use the notation prime to denote the derivative with respect to \( \theta \). It is observed that the angular variable \( \alpha \) cannot be used as the independent variable, since the new differential system does not have the appropriate form in order to apply the averaging method. Now, note that

\[ \frac{dr}{d\theta} = \frac{\varepsilon \left( \frac{\partial V_3}{\partial \theta} + \frac{\partial V_3}{\partial \alpha} \right) + \varepsilon^2 \left( \frac{\partial V_4}{\partial \theta} + \frac{\partial V_4}{\partial \alpha} \right)}{r - \varepsilon \frac{\partial V_3}{\partial r} - \varepsilon^2 \frac{\partial V_4}{\partial r}}. \]

Remember that the function

\[ f(\varepsilon) = \frac{1}{1 - a_1 \varepsilon - a_2 \varepsilon^2} = 1 + a_1 \varepsilon + (a_1^2 + a_2) \varepsilon^2 + O(\varepsilon^3), \]

Periodic orbits associated to Hamiltonian functions of degree four.
by developing a Taylor series in powers of $\epsilon$ around $\epsilon = 0$. Replacing $a_1 = \frac{1}{r} \frac{\partial V_3}{\partial r}$ and $a_2 = \frac{1}{r} \frac{\partial V_4}{\partial r}$ we get

\[
\frac{1}{1 - \frac{1}{r} \frac{\partial V_3}{\partial r} \epsilon - \frac{1}{r} \frac{\partial V_4}{\partial r} \epsilon^2} = 1 + \frac{1}{r} \frac{\partial V_3}{\partial r} \epsilon + \left[ \left( \frac{1}{r} \frac{\partial V_3}{\partial r} \right)^2 + \frac{1}{r} \frac{\partial V_4}{\partial r} \right] \epsilon^2 + O(\epsilon^3).
\]

Therefore,

\[
r' = \frac{1}{r} \left( \frac{\partial V_3}{\partial \theta} + \frac{\partial V_3}{\partial \alpha} \right) \epsilon + \frac{1}{r} \left( \frac{1}{r} \frac{\partial V_3}{\partial r} \left( \frac{\partial V_3}{\partial \theta} + \frac{\partial V_3}{\partial \alpha} \right) \right) + \left( \frac{1}{r} \frac{\partial V_4}{\partial \theta} + \frac{\partial V_4}{\partial \alpha} \right) \right] \epsilon^2 + O(\epsilon^3).
\]

Moreover,

\[
\frac{d\rho}{d\theta} = -\frac{\epsilon}{\rho} \frac{1}{r} \frac{\partial V_3}{\partial \alpha} \epsilon + \epsilon^2 \frac{1}{r} \frac{\partial V_4}{\partial \alpha},
\]

thus

\[
\rho' = -\frac{1}{\rho} \frac{\partial V_3}{\partial \alpha} \epsilon - \left[ \frac{1}{r \rho} \frac{\partial V_3}{\partial \alpha} + \frac{1}{\rho} \frac{\partial V_4}{\partial \alpha} \right] \epsilon^2 + O(\epsilon^3).
\]

Also

\[
\frac{d\alpha}{d\theta} = \epsilon \left( -\frac{1}{r} \frac{\partial V_3}{\partial r} - \frac{1}{r} \frac{\partial V_3}{\partial \rho} \right) + \epsilon^2 \left( -\frac{1}{r} \frac{\partial V_4}{\partial r} - \frac{1}{\rho} \frac{\partial V_4}{\partial \rho} \right),
\]

then

\[
\alpha' = \left( -\frac{1}{r} \frac{\partial V_3}{\partial r} - \frac{1}{r} \frac{\partial V_3}{\partial \rho} \right) \epsilon + \left[ \frac{1}{r} \frac{\partial V_3}{\partial \rho} \right] + \left( -\frac{1}{r} \frac{\partial V_4}{\partial r} - \frac{1}{\rho} \frac{\partial V_4}{\partial \rho} \right) \right] \epsilon^2 + O(\epsilon^3).
\]

If we write the previous system as a Taylor series in powers of $\epsilon$, we have that

\[
(3.7)
\]

\[
r' = \frac{1}{r} \left( \frac{\partial V_3}{\partial \theta} + \frac{\partial V_3}{\partial \alpha} \right) \epsilon + \left[ \frac{1}{r} \frac{\partial V_3}{\partial r} \left( \frac{\partial V_3}{\partial \theta} + \frac{\partial V_3}{\partial \alpha} \right) \right] + \left( \frac{1}{r} \frac{\partial V_4}{\partial \theta} + \frac{\partial V_4}{\partial \alpha} \right) \right] \epsilon^2 + O(\epsilon^3),
\]

\[
\rho' = -\frac{1}{\rho} \frac{\partial V_3}{\partial \alpha} \epsilon - \left[ \frac{1}{r \rho} \frac{\partial V_3}{\partial \alpha} + \frac{1}{\rho} \frac{\partial V_4}{\partial \alpha} \right] \epsilon^2 + O(\epsilon^3),
\]

\[
\alpha' = \left( -\frac{1}{r} \frac{\partial V_3}{\partial r} - \frac{1}{r} \frac{\partial V_3}{\partial \rho} \right) \epsilon + \left[ \frac{1}{r} \frac{\partial V_3}{\partial \rho} \right] + \left( -\frac{1}{r} \frac{\partial V_4}{\partial r} - \frac{1}{\rho} \frac{\partial V_4}{\partial \rho} \right) \right] \epsilon^2 + O(\epsilon^3).
\]

In order to apply the averaging theory we will fix the value of the first integral of $K = h > 0$. We write the Taylor expansion of the function $\rho$ of second order in $\epsilon$ and rewrite the system (3.7) in the
variables \( r \) and \( \alpha \). The function \( \rho \) in its Taylor form is

\[
\rho(\varepsilon) = \rho(0) + \rho'(0)\varepsilon + O(\varepsilon^2). \tag{3.8}
\]

Regarding (3.4), we obtain for \( \varepsilon = 0 \),

\[
\rho(0) = \sqrt{2h + r^2}. \tag{3.9}
\]

Derivating (3.4) with respect to \( \varepsilon \), we have

\[
0 = \rho \rho' + V_3(r \cos \theta, \rho \cos(-\theta + \alpha)) + \varepsilon \frac{\partial V_3}{\partial Y} \rho' \cos(-\theta + \alpha) + 2\varepsilon V_4(r \cos \theta, \rho \cos(-\theta + \alpha)) + \varepsilon^2 \frac{\partial V_4}{\partial Y} \rho' \cos(-\theta + \alpha)
\]

and evaluating at \( \varepsilon = 0 \), we obtain

\[
\rho'(0) = -\frac{1}{\sqrt{2h + r^2}} V_3(r \cos \theta, \sqrt{2h + r^2} \cos(-\theta + \alpha)). \tag{3.10}
\]

Thus, from (3.8), (3.9) and (3.10) the development of the function \( \rho \) in a Taylor series has the form

\[
\rho(\varepsilon) = \sqrt{2h + r^2} - \frac{1}{\sqrt{2h + r^2}} V_3(r \cos \theta, \sqrt{2h + r^2} \cos(-\theta + \alpha)) \varepsilon + O(\varepsilon^2).
\]

Since \( V_3 \) depends on the variables \( (r, \rho, \theta, \alpha) \), the derivatives of \( V_3 \) respect to their variables in the Taylor series are:

- \( \frac{\partial V_3}{\partial \theta} = \frac{\partial V_3}{\partial \theta} + \rho'(0) \frac{\partial^2 V_3}{\partial \rho \partial \theta} \varepsilon + O(\varepsilon^2) \),
- \( \frac{\partial V_3}{\partial \alpha} = \frac{\partial V_3}{\partial \alpha} + \rho'(0) \frac{\partial^2 V_3}{\partial \rho \partial \alpha} \varepsilon + O(\varepsilon^2) \),
- \( \frac{\partial V_3}{\partial r} = \frac{\partial V_3}{\partial r} + \rho'(0) \frac{\partial^2 V_3}{\partial \rho \partial r} \varepsilon + O(\varepsilon^2) \),
- \( \frac{\partial V_3}{\partial \rho} = \frac{\partial V_3}{\partial \rho} + \rho'(0) \frac{\partial^2 V_3}{\partial \rho^2} \varepsilon + O(\varepsilon^2) \).

Moreover,

- \( \frac{1}{\rho} = \frac{1}{\rho(0)} - \frac{\rho'(0)}{\rho(0)^2} \varepsilon + O(\varepsilon^2) \),
- \( \frac{1}{\rho} \frac{\partial V_3}{\partial \rho} = \frac{1}{\rho(0)} \frac{\partial V_3}{\partial \rho} + \left[ \frac{\rho'(0)}{\rho(0)} \frac{\partial^2 V_3}{\partial \rho^2} - \frac{\rho'(0)}{\rho(0)^2} \frac{\partial V_3}{\partial \rho} \right] \varepsilon + O(\varepsilon^2) \).
Recall that all the previous functions are evaluated at the point \((r \cos \theta, \rho(0) \cos(-\theta + \alpha))\). Therefore,

\[
\frac{1}{r} \left( \frac{\partial V_3}{\partial \theta} + \frac{\partial V_3}{\partial \alpha} \right) = \frac{1}{r} \left( \frac{\partial V_3}{\partial \theta} + \frac{\partial V_3}{\partial \alpha} \right) + \rho'(0) \left( \frac{\partial^2 V_3}{\partial \rho \partial \theta} + \frac{\partial^2 V_3}{\partial \rho \partial \alpha} \right) \varepsilon,
\]

\[
\left( -\frac{1}{r} \frac{\partial V_3}{\partial r} - \frac{1}{\rho} \frac{\partial V_3}{\partial \rho} \right) = \left( -\frac{1}{r} \frac{\partial V_3}{\partial r} - \frac{1}{\rho(0)} \frac{\partial V_3}{\partial \rho} \right) - \left[ \rho'(0) \frac{\partial^2 V_3}{\partial \rho^2} + \left( \frac{\rho'(0)}{\rho(0)} \frac{\partial^2 V_3}{\partial \rho \partial \rho} \right) \right] \varepsilon. \tag{3.11}
\]

Then substituting the latter expression into equation (3.7), we obtain the two differential equations

\[
r' = \frac{1}{r} \left( \frac{\partial V_3}{\partial \theta} + \frac{\partial V_3}{\partial \alpha} \right) \varepsilon + \frac{1}{r} r \left( \frac{\partial V_3}{\partial \theta} + \frac{\partial V_3}{\partial \alpha} \right) + \rho'(0) \left( \frac{\partial^2 V_3}{\partial \rho \partial \theta} + \frac{\partial^2 V_3}{\partial \rho \partial \alpha} \right) + \\
\left( \frac{\partial V_4}{\partial \theta} + \frac{\partial V_4}{\partial \alpha} \right) \varepsilon^2 + O(\varepsilon^3),
\]

\[
\alpha' = \frac{1}{r} \left( -\frac{\partial V_3}{\partial r} - \frac{r}{\rho(0)} \frac{\partial V_3}{\partial \rho} \right) \varepsilon + \left[ -\frac{\rho'(0)}{r} \frac{\partial^2 V_3}{\partial \rho \partial r} - \left( \frac{\rho'(0)}{\rho(0)} \frac{\partial^2 V_3}{\partial \rho \partial r} \right) \right] \varepsilon^2 + O(\varepsilon^3),
\]

where the last system is evaluated at the point \(\rho = \rho(0)\) as in (3.9) and \((r \cos \theta, \rho(0) \cos(-\theta + \alpha))\).

The system (3.12) has the general form

\[
r' = F_{11} \varepsilon + F_{21} \varepsilon^2 + O(\varepsilon^3),
\]

\[
\alpha' = F_{12} \varepsilon + F_{22} \varepsilon^2 + O(\varepsilon^3),
\]

where

\[
F_{11} = \frac{1}{r} \left( \frac{\partial V_3}{\partial \theta} + \frac{\partial V_3}{\partial \alpha} \right),
\]

\[
F_{12} = -\frac{1}{r} \left( \frac{\partial V_3}{\partial r} + \frac{r}{\rho(0)} \frac{\partial V_3}{\partial \rho} \right),
\]

\[
F_{21} = \frac{1}{r} \left( \frac{1}{r} \frac{\partial V_3}{\partial r} \left( \frac{\partial V_3}{\partial \theta} + \frac{\partial V_3}{\partial \alpha} \right) + \rho'(0) \left( \frac{\partial^2 V_3}{\partial \rho \partial \theta} + \frac{\partial^2 V_3}{\partial \rho \partial \alpha} \right) + \\
\left( \frac{\partial V_4}{\partial \theta} + \frac{\partial V_4}{\partial \alpha} \right) \right],
\]

\[
F_{22} = \left[ -\frac{\rho'(0)}{r} \frac{\partial^2 V_3}{\partial \rho \partial r} - \left( \frac{\rho'(0)}{\rho(0)} \frac{\partial^2 V_3}{\partial \rho \partial r} \right) \right] \varepsilon^2 + O(\varepsilon^3).
\]

Also, we denote

\[
f_1(r, \alpha) := (f_{11}, f_{12}) = \frac{1}{2\pi} \int_0^{2\pi} (F_{11}, F_{12}) d\theta \tag{3.15}
\]
and

\[ f_2(r, \alpha) := (f_{21}, f_{22}) \]

\[ = \frac{1}{2\pi} \int_0^{2\pi} [D_{r\alpha} F_1(\theta, r, \alpha) \cdot \int_0^\theta F_1(t, r, \alpha) dt + F_2(\theta, r, \alpha)] d\theta, \]

(3.16)

where \( F_1 = (F_{11}, F_{12}) \) and \( F_2 = (F_{21}, F_{22}) \) and \( D_{r\alpha} F_1(\theta, r, \alpha) \) is the jacobian matrix given by

\[
D_{r\alpha} F_1(\theta, r, \alpha) = \begin{pmatrix}
\frac{\partial F_{11}}{\partial r} & \frac{\partial F_{11}}{\partial \alpha} \\
\frac{\partial F_{12}}{\partial r} & \frac{\partial F_{12}}{\partial \alpha}
\end{pmatrix}.
\]

(3.17)

As the expression given in \( F_{11} \) and \( F_{22} \) involves the presence of combinations of the functions \( \cos^3 \theta \), \( \cos^2 \theta \sin \theta \), \( \cos \theta \sin^2 \theta \), \( \sin^3 \theta \), we arrive to the equation \( f_1(r, \alpha) = (0, 0) \). This result is the same as in Lemma 2.1.

4. Proof of Theorem 1.1 and Theorem 1.2, the case \( H^+ \)

**Proof.** We use the same notation as in [6] to apply the averaging method. Let \( H \) be as in (1.1) such that \( h \) be any small positive number. Moreover, the expressions of \( f_{21} \) and \( f_{22} \) given by (2.16) are of the form:

\[
f_{21} = \frac{1}{24} r (r^2 - 2h) \left[ 2AB + 3 \left( m - 4B^2 \right) \right] \sin(2\alpha)
\]

\[
f_{22} = \frac{1}{24} \left[ 10A^2 r^2 + 24AB(h - r^2) - 2B^2(2h + 3r^2) - 3 \left( h(4m - 6\lambda) + r^2(-4m + 3(\lambda + \Lambda)) \right) - 2(2A - 6B)B + 3m(h - r^2) \cos(2\alpha) \right].
\]

Let \( M_j \), for \( j = 1, \ldots, 7 \) be the expressions given in Section 1. In order to apply the averaging method, first we solve the system \( f_{21} = 0 \) and \( f_{22} = 0 \) for the variables \( r \) and \( \alpha \). Therefore, we obtain the solutions \((r_j, \alpha_j)\), for \( j = 1, 2, 3 \), given by

\[
(r_1, \alpha_1) = \left( \sqrt{-2h}, 0 \right),
\]

\[
(r_2, \alpha_2) = \left( \sqrt{-2h}, -\frac{\pi}{2} \right),
\]

\[
(r_2, \alpha_2) = \left( \sqrt{-2h}, \frac{\pi}{2} \right).
\]

Remember that \( \rho_j = \sqrt{2h - r_j^2} \), for \( j = 1, 2, 3 \). Therefore, this expressions are

\[
\rho_1 = \sqrt{2h},
\]

\[
\rho_2 = \sqrt{2h},
\]

\[
\rho_3 = \rho_2.
\]
The next step is to find the region of the parameters where the determinant $J_j$ of the Jacobian matrix, $D_{r,\alpha}(f_1, f_2)$ evaluated in the zeros $(r_j, \alpha_j)$ is non null. We find that

$$J_1 = \frac{hr_1^2}{72} M_3 M_4,$$

$$J_2 = -\frac{hr_2^2}{72} M_5 M_7,$$

$$J_3 = J_2.$$

Now, we finish the proof of Theorem 1.1. Indeed, if $(A, B, m, \lambda, \Lambda) \in \Lambda_k^1 \setminus \Lambda_k^2$, then $r_1 > 0$, $\rho_1 > 0$ and $J_1 \neq 0$ and therefore we obtain at least one-family of period orbits. If $(A, B, m, \lambda, \Lambda) \in \Lambda_k^2 \setminus \Lambda_k^1$, then $r_2 > 0$, $\rho_2 > 0$ and $J_2 \neq 0$, so we get at least two-families of periodic orbits. Finally, if $(A, B, m, \lambda, \Lambda) \in \Lambda_k^1 \cap \Lambda_k^2$, then $r_1 > 0$, $\rho_1 > 0$, $\rho_2 > 0$, $J_1 \neq 0$ and $J_2 \neq 0$, so we get at least three-families of periodic orbits. This completes the proof of Theorem 1.1.

For the proof of Theorem 1.2. Observe that if we denote by $\lambda_1^1$ and $\lambda_2^1$ the eigenvalue of the Jacobian matrix $D_{r,\alpha}(f_1, f_2)$ evaluate in $(r_1, \alpha_1)$, we have that

$$\lambda_1^1 = -\frac{1}{6\sqrt{2}} r_1 \sqrt{h \sqrt{-M_3 M_4}}, \quad \lambda_2^1 = -\lambda_1^1.$$

Moreover, as $J_1 = \lambda_1^1 \lambda_2^1$, we obtain that $J_1 = -(\lambda_1^1)^2$. Therefore, for $(A, B, m, \lambda, \Lambda) \in \Lambda_k^1 \setminus \Lambda_k^2$ such that $M_3 M_4 > 0$ we obtain at least one-family linearly stable and linearly unstable if $M_3 M_4 < 0$. This proves the first part of Theorem 1.2. For the second part of Theorem 1.2, we denote by $\lambda_1^2$ and $\lambda_2^2$ the eigenvalues of the Jacobian matrix $D_{r,\alpha}(f_1, f_2)$ evaluated in $(r_2, \alpha_2)$, we have that

$$\lambda_1^2 = -\frac{1}{6\sqrt{2}} r_2 \sqrt{h \sqrt{-M_4 M_7}}, \quad \lambda_2^2 = -\lambda_1^2.$$

Moreover, as $J_2 = \lambda_1^2 \lambda_2^2$, we obtain that $J_2 = -(\lambda_1^2)^2$. Therefore, for $(A, B, m, \lambda, \Lambda) \in \Lambda_k^2 \setminus \Lambda_k^1$ such that $M_4 M_7 < 0$ we obtain at least two-families linearly stable and linearly unstable if $M_4 M_7 > 0$. Finally, for the proof of the third part of Theorem 1.2, for $(A, B, m, \lambda, \Lambda) \in \Lambda_k^1 \cap \Lambda_k^2$ such that $M_3 M_4 > 0$ and $M_4 M_7 < 0$, we obtain three-families linearly stable and three-families linearly unstable in the other conditions. Therefore, the proof is completed.

5. **Proof of Theorem 1.3 and Theorem 1.4, the case $H^-$**

**Proof.** We use the same notation as in [6] to apply the averaging method. Let $H$ be as in (1.2) such that $h$ be any small number. Moreover, the functions $f_{21}$ and $f_{22}$ given by (3.16) are:

$$f_{21} = \frac{1}{24} r \left(2h + r^2\right) \left(2AB + 12B^2 - 3m\right) \sin(2\alpha),$$

$$f_{22} = \frac{1}{24} \left[-10A^2 r^2 - 24AB(h+r^2) + B^2(-4h + 6r^2) - 3(h(4m + 6\lambda) + r^2(4m + 3(\lambda + \Lambda))) + 2\left(2B(A + 6B) - 3m \right)(h + r^2) \cos(2\alpha)\right].$$

Let $N_j$ be for $j = 1, \ldots, 7$ the expressions given in Section 1. To apply the averaging method, first we solve the system $f_{21} = 0$ and $f_{22} = 0$ for the variables $r$ and $\alpha$. Therefore, we obtain the solutions
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\((r_j, \alpha_j)\), for \(j = 1, 2, 3\), given by

\[
(r_1, \alpha_1) = \left(\sqrt{-2h \frac{N_1}{N_2}}, 0\right),
\]

\[
(r_2, \alpha_2) = \left(\sqrt{-2h \frac{N_5}{N_6} - \frac{\pi}{2}}, 0\right),
\]

\[
(r_3, \alpha_3) = \left(r_2, \frac{\pi}{2}\right).
\]

Remember that in the case \(H^-\), \(\rho_j = \sqrt{2h + r_j^2}\), for \(j = 1, 2, 3\). Therefore, these expressions are

\[
\rho_1 = \sqrt{2h \frac{N_1}{N_2}},
\]

\[
\rho_2 = \sqrt{2h \frac{N_5}{N_6}},
\]

\[
\rho_3 = \rho_2.
\]

The next step is to find the region of the parameters where the determinant \(J_j\) of the Jacobian matrix, \(D_{r,\alpha}(f_1, f_2)\) evaluated in the zeros \((r_j, \alpha_j)\) is non null. We find that

\[
J_1 = \frac{hr_1^2}{72} N_3 N_4,
\]

\[
J_2 = -\frac{hr_2^2}{72} N_4 N_7,
\]

\[
J_3 = J_2.
\]

Now, we finish the proof of Theorem 1.3. Indeed, if \((A, B, m, \lambda, \Lambda) \in \Omega^4_h \setminus \Omega^2_h\), then \(r_1 > 0, \rho_1 > 0\) and \(J_1 \neq 0\) and therefore we obtain at least one-family of periodic orbits. If \((A, B, m, \lambda, \Lambda) \in \Omega^2_h \setminus \Omega^4_h\), then \(r_2 > 0, \rho_2 > 0\) and \(J_2 \neq 0\), so we get at least two-families of periodic orbits. Finally, if \((A, B, m, \lambda, \Lambda) \in \Omega^1_h \cap \Omega^2_h\), then \(r_1 > 0, r_2 > 0, \rho_1 > 0, \rho_2 > 0, J_1 \neq 0\) and \(J_2 \neq 0\), so we get at least three-families of periodic orbits. This completes the proof of Theorem 1.3.

For the proof of Theorem 1.4, observe that if we denote by \(\lambda_1^1\) and \(\lambda_2^1\) the eigenvalues of the Jacobian matrix \(D_{r,\alpha}(f_1, f_2)\) evaluate in \((r_1, \alpha_1)\), we have that

\[
\lambda_1^1 = -\frac{1}{6\sqrt{2}} r_1 \sqrt{h} \sqrt{-N_3 N_4}, \quad \lambda_2^1 = -\lambda_1^1.
\]

Moreover, as \(J_1 = \lambda_1^1 \lambda_2^1\), we obtain that \(J_1 = - (\lambda_1^1)^2\). Therefore, for \((A, B, m, \lambda, \Lambda) \in \Omega^4_h \setminus \Omega^2_h\) such that \(N_3 N_4 > 0\) we obtain at least one-family linearly stable and linearly unstable if \(N_3 N_4 < 0\). This proves the first part of Theorem 1.4. For the second part of Theorem 1.4, we denote by \(\lambda_1^2\) and \(\lambda_2^2\) the eigenvalues of the Jacobian matrix \(D_{r,\alpha}(f_1, f_2)\) evaluated in \((r_2, \alpha_2)\), we have that

\[
\lambda_1^2 = -\frac{1}{6\sqrt{2}} r_2 \sqrt{h} \sqrt{N_4 N_7}, \quad \lambda_2^2 = -\lambda_1^2.
\]

Moreover, as \(J_2 = \lambda_1^2 \lambda_2^2\), we obtain that \(J_2 = - (\lambda_1^2)^2\). Therefore, for \((A, B, m, \lambda, \Lambda) \in \Omega^2_h \setminus \Omega^4_h\) such that \(N_4 N_7 < 0\) we obtain at least two-families linearly stable and linearly unstable if \(N_4 N_7 > 0\). Finally, for the proof of the third part of Theorem 1.4, for \((A, B, m, \lambda, \Lambda) \in \Omega^1_h \cap \Omega^2_h\) such that \(N_3 N_4 > 0\)
0 and $N_2N_3 < 0$ we obtain three-families linearly stable and three-families linearly unstable in the other conditions. Therefore, the proof is completed. \hfill $\square$


6.1. Examples of families of periodic orbits for $H^+$

Taking $\lambda = \Lambda = m = 0$, that is, $V_4 \equiv 0$, we recuperate the results obtained by Jiménez-Lara and Llibre in [17]. Moreover, we emphasize that adding to the classical Henón-Heiles system a homogeneous polynomial of degree four, we obtain periodic orbits outside the region of the existence of periodic solutions found in [17]. The importance of adding to classical Henón-Heiles system a homogeneous polynomial of degree fourth, lies in to find another family of periodic orbits in a region where the conditions given by Llibre and Jiménez-Lara in [17] are not valid.

Indeed, we point out the following examples.

For a one periodic orbit: Taking $A = 1, B = 4, m = 1.5$ and $\Lambda = -\lambda = 3$, we obtain $r_1 = 0.780988 \sqrt{h}, \rho_1 = 1.17901 \sqrt{h}$ and $J_1 = 552.774h^2$ for every $h > 0$ small. Therefore, the Hamiltonian

$$H = \frac{1}{2} (p_x^2 + p_y^2) + \frac{1}{2} (x^2 + y^2) + \frac{1}{3} x^3 + 4xy^2 + \frac{3}{4} x^4 + \frac{3}{4} y^2 - \frac{3}{4} y^4$$

has at least one family of periodic orbits for every $h > 0$ small. However, note that

$$(2B - 5A)(2B - A) = 21 > 0,$$

therefore, $A$ and $B$ do not satisfy the condition given in [17].

For two periodic orbits: Taking $A = B = 2, m = -3$ and $\Lambda = \lambda = -3$, we obtain $r_2 = r_3 = \sqrt{h}, \rho_2 = \rho_3 = \sqrt{h}$ and $J_2 = J_3 = \frac{49}{4} h^2$ for every $h > 0$ small. Therefore, the Hamiltonian

$$H = \frac{1}{2} (p_x^2 + p_y^2) + \frac{1}{2} (x^2 + y^2) + \frac{2}{3} x^3 + 2xy^2 - \frac{3}{4} x^4 - \frac{3}{2} y^2 - \frac{3}{4} y^4$$

has at least one family of periodic orbits for every $h > 0$ small. However, note that

$$A + B = 4 \neq 0,$$

therefore, $A$ and $B$ do not satisfy the condition given in [17].

For three periodic orbits: Taking $A = 5, B = 4, m = -3$ and $\Lambda = -\lambda = 3$, we obtain, $r_1 = 2\sqrt{\frac{5}{19}} \sqrt{h}, r_2 = r_3 = \sqrt{\frac{19}{8}} \sqrt{h}, J_1 = \frac{14490}{19} h^2, J_2 = J_3 = \frac{3059}{360} h^2$ for every $h > 0$ small. Therefore, the Hamiltonian

$$H = \frac{1}{2} (p_x^2 + p_y^2) + \frac{1}{2} (x^2 + y^2) + \frac{5}{3} x^3 + 4xy^2 + \frac{3}{4} x^4 - \frac{3}{2} y^2 - \frac{3}{4} y^4$$

has at least one family of periodic orbits for every $h > 0$ small. However, note that

$$B(2B - 5A) = -68 < 0,$$

therefore, $A$ and $B$ do not satisfy the condition given in [17].
6.2. Examples of family of periodic orbits for $H^-$

Now, we are going to exhibit that the necessary conditions given in Theorem 1.1 are not empty, in fact, we will consider different particular elections of the parameters $A, B, m, \Lambda, \lambda$ such that the constraints are satisfied.

Firstly, if we take $A = B = 4$, $m = 1.5$ and $\Lambda = -\lambda = 3$, then there exist $r_1 = \sqrt{h}$, $\rho_1 = 1.73205 \sqrt{h}$ and $J_1 = 123.469 h^2$ for every $h > 0$ small, thus the Hamiltonian

$$H = \frac{1}{2} (-p_x^2 + p_y^2) + \frac{1}{2} (-x^2 + y^2) + \frac{4}{3} x^3 + 4xy^2 + \frac{3}{4} x^4 + \frac{3}{4} y^2 - \frac{3}{4} y^4$$

has at least two families of periodic orbits for every $h > 0$ small.

Secondly, considering $A = 30, B = -10$, $m = -4$ and $\Lambda = \lambda = -3$, we obtain that there is $r_2 = r_3 = \frac{1}{3} \sqrt{\frac{2839}{129}} \sqrt{h}$, $\rho_2 = \rho_3 = \frac{1}{3} \sqrt{\frac{3161}{129}} \sqrt{h}$ and $J_2 = J_3 = -\frac{249085343}{2122} h^2$ for every $h > 0$ small. Therefore, the Hamiltonian

$$H = \frac{1}{2} (-p_x^2 + p_y^2) + \frac{1}{2} (-x^2 + y^2) + \frac{30}{3} x^3 - 10xy^2 - \frac{3}{4} x^4 - 2x^2 y^2 - \frac{3}{4} y^4$$

has at least two families of periodic orbits for every $h > 0$ small.

Thirdly, taking $A = 40, B = -10$, $m = -4$ and $\Lambda = \lambda = -3$, we obtain $r_1 = \sqrt{\frac{2063}{2347}} \sqrt{h}$, $r_2 = r_3 = 3 \sqrt{\frac{157}{1087}} \sqrt{h}$, $\rho_1 = \sqrt{\frac{9937}{2347}} \sqrt{h}$, $\rho_2 = \rho_3 = \sqrt{\frac{3587}{1087}} \sqrt{h}$, $J_1 = \frac{5182036193h^2}{43866}$ and $J_2 = J_3 = -\frac{174016131h^2}{2174}$ for every $h > 0$ small. Therefore, the Hamiltonian

$$H = \frac{1}{2} (-p_x^2 + p_y^2) + \frac{1}{2} (-x^2 + y^2) + \frac{40}{3} x^3 - 10xy^2 - \frac{3}{4} x^4 - 2x^2 y^2 - \frac{3}{4} y^4$$

has at least three families of periodic orbits for every $h > 0$ small.

Another important remark is the following which permit us to find regions for the existence of periodic orbits in a simple sense.

**Remark 6.1.** Let consider $m = \lambda = \Lambda$ and $A, B$ such that $m \neq -\frac{2}{3} B(A - 6B)$, $A + B \neq 0$, $A - 2B \neq 0$ and $0 < \frac{B}{A - 3B} < 1$. Then there exists at least a family of periodic orbits, for all $h > 0$. Indeed, if $m = \lambda = \Lambda$ then

$$r_1 = \sqrt{2h} \sqrt{\frac{-B}{A - 3B}} > 0$$

and

$$d_1 = \frac{5h \lambda^2}{36} (A - 2B)(A + B) \left( 2B(A - 6B) + 3m \right) \neq 0.$$ 

Therefore, there exists at least one family of periodic orbits.

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