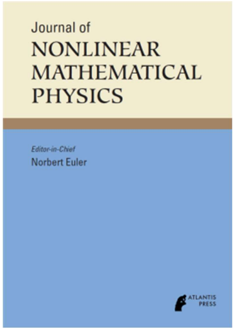

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On nonlinear coherent states properties for electron-phonon dynamics

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This work addresses a construction of a dual pair of nonlinear coherent states (NCS) in the context of changes of bases in the underlying Hilbert space for a model pertaining to an electron-phonon model in the condensed matter physics, obeying a f-deformed Heisenberg algebra. The existence and properties of reproducing kernel in the NCS Hilbert space are studied and discussed; the probability density and its dynamics in the basis of constructed coherent states are provided. A Glauber-Sudarshan P-representation of the density matrix and relevant issues related to the reproducing kernel properties are presented. Moreover, a NCS quantization of classical phase space observables is performed and illustrated in a concrete example of q-deformed coherent states. Finally, an exposition of quantum optical properties is given.

Keywords: Deformed Heisenberg algebra; Electron-phonon dynamics; Coherent states; Coherent state quantization.

1. Introduction

Coherent states (CS), known as the closest states to classical behavior of a system, play an important role in theoretical and experimental physics. First introduced as venerable objects by Schrödinger [29] in 1926 for the harmonic oscillator potential, they are since at the core of research directions in quantum optics, which is an ideal testing ground for ideas of quantum theory. Much work has been done through their theoretical generalizations including their experimental generations and applications. Then, the generalization based on group symmetry approach has led to define CS for arbitrary Lie algebras such as $su(1,1)$, $su(2)$ which have found numerous applications in quantum optics [26], [34].

Further generalizations also extend to so-called nonlinear CS (NCS) or f-CS [20] induced by nonlinear algebras referring to f-deformed oscillator algebras initially introduced by Jannussis *et al.* [16] and Man'ko *et al.* [20], and spanned by the ordinary Fock-Heisenberg generators $\{a, a^\dagger, N\}$ coupled to a free continuous regular function f of the number operator N, such that

$$\begin{aligned} A^- = af(N), \qquad A^\dagger = f(N)a^\dagger \\ N = A^\dagger A^- = Nf^2(N), \qquad [A^-, A^\dagger] = \{N+1\} - \{N\}. \end{aligned} \tag{1.1}$$

The f-deformed quantum algebras offer the advantage to be well represented in the ordinary Fock-Hilbert space. Although they are not merely mathematical objects, NCS are useful to analyse many quantum mechanical systems. For instance, in [21], it has been shown that they are important in the description of the motion of a trapped ion and have some "non-classical properties" such as squeezing, amplitude-squared squeezing, antibuching, sub-Poissonian behavior.

NCS construction uses an operator similar to the displacement operator, generally denoted by $D(z), z \in \mathbb{C}$ with [28]

$$D(z) = e^{za^\dagger - \bar{z}a}, \quad [a, a^\dagger] = \mathbb{I} \tag{1.2}$$

such that the CS parametrized by z are given by the action of $D(z)$ on the ground state $|0\rangle$:

$$|z\rangle = D(z)|0\rangle. \tag{1.3}$$

This is in connection with the method proposed in [3] to define NCS by changing bases in the underlying Hilbert space, involving an interesting duality between pairs of constructed generalized CS.

Another class of CS describing quantum optical models encompasses vector coherent states (VCS) ([8], [11], [2]), used for example in the study of spectra of two-level atomic systems placed in electromagnetic fields like the Jaynes-Cummings model [17]. For more details, see [7] (and references therein) presenting a formulation of VCS for nonlinear spin-orbit Hamiltonian model in terms of the matrix eigenvalue problem for generalized annihilation operators. A formalism of VCS construction for a system of M Fermi-type modes associated with N bosonic modes [22]- [33] was also given in [4]. The defined VCS satisfy, in this context, required mathematical properties of continuity, resolution of identity, temporal stability and action identity.

This work, generalizing previous investigations (see [4] and references therein), develops a construction of a dual pair of NCS, based on a f-deformed algebra, to describe such phenomena like the electron-phonon dynamics in condensed matter physics.

The paper is organized as follows. In Section 2, the physical model and the formalism of nonlinear coherent states (NCS) construction are described. Section 3 deals with the probability density and its dynamics in the basis of constructed coherent states. The existence and properties of reproducing kernel in the NCS Hilbert space are studied and discussed in Section 4. In Section 5, the Glauber-Sudarshan P-representation of the density matrix is elaborated in both the NCS and nonlinear VCS (NVCS); the associated reproducing kernel properties are studied. Section 6 presents the NCS quantization of the complex plane. In Section 7, relevant quantum optical properties are analyzed. The last section is devoted to concluding remarks.

2. CS for electron-phonon dynamics

In this section, we construct the CS for a physical model describing the electron-phonon dynamics given by the following Hamiltonian (with $\hbar = 1$) [22]- [4]:

$$H = \sum_{i=1}^{N_B} \omega_i a_i^\dagger a_i + \sum_{j=1}^{M} \varepsilon_j c_j^\dagger c_j + \sum_{i=1}^{N_B} \sum_{l=1}^{M} g_l c_l^\dagger c_l (a_i^\dagger + a_i) \tag{2.1}$$

where the following commutation rules hold:

$$[a_i, a_k^\dagger] = \delta_{ik}\mathbb{I} \qquad \{c_j^\dagger, c_l\} = \delta_{jl}\mathbb{I}$$

$$[a_i, a_k] = 0 \qquad [a_i^\dagger, a_k^\dagger] = 0 \qquad \{c_j, c_l\} = 0 \qquad \{c_j^\dagger, c_l^\dagger\} = 0 \tag{2.2}$$

with $1 \leq i,k \leq N_B$, $1 \leq j,l \leq M$. The set $\{a_i, a_i^\dagger, \mathbb{I}\}$ for each $i(1 \leq i \leq N_B)$ spans the ordinary Fock-Heisenberg oscillator algebra. A similar model, describing an interaction between a single mode, $(a, a^\dagger)$, of the radiation field with two Fermi type modes, was also investigated by Simon and Geller who outlined some physical aspects of the ensemble-averaged excited-state population dynamics [30] and showed its relevance in the study of electron-phonon dynamics in an ensemble of nearly isolated nanoparticles, in the context of quantum effects in condensed matter systems. The vibrational spectrum of a nanoparticle is here provided by the localized electronic impurity states in doped nanocrystal [22, 33]. The impurity states are used to probe the energy relaxation by phonon emission.

The deformed version of the Hamiltonian (2.1) is obtained by performing the correspondences

$$a_l \to a_l f_l(N_l) \qquad a_l^\dagger \to f_l(N_l) a_l^\dagger \qquad 1 \leq l \leq N_B \tag{2.3}$$

with the following nonlinear commutation rules for the bosonic operators:

$$\begin{aligned} &[N_l, a_l f_l(N_l)] = -a_l f_l(N_l), \quad [N_l, f_l(N_l) a_l^\dagger] = f_l(N_l) a_l^\dagger, \\ &[a_l f_l(N_l), f_l(N_l) a_l^\dagger] = (N_l + 1) f_l^2(N_l + 1) - N_l f_l^2(N_l), \end{aligned} \tag{2.4}$$

$f_l(N_l)$ being a reasonably well behaved real function of the number operator N_l. If $f_l(N_l) = 1$, then the nonlinear algebra in (2.4) reduces to the ordinary non-deformed oscillator algebra.

As a matter of obtaining the eigenvalues and eigenvectors for our system, the Hamiltonian (2.1) can be conveniently diagonalized by introducing the following self adjoint operator:

$$B_{[\mathbf{k}]_l} = \omega_l A_{[\mathbf{k}]_l}^\dagger A_{[\mathbf{k}]_l} + \frac{\varepsilon_{[\mathbf{k}]}}{N_B} - \frac{g_{[\mathbf{k}]}^2}{\omega_l} \quad \text{with} \quad A_{[\mathbf{k}]_l} = a_l f_l(N_l) + \frac{g_{[\mathbf{k}]}}{\omega_l} \tag{2.5}$$

where $g_{[\mathbf{k}]}, \varepsilon_{[\mathbf{k}]}$ are defined as [4]

$$g_{[\mathbf{k}]} := \sum_{k_j \in [\mathbf{k}]} k_j g_j \qquad \varepsilon_{[\mathbf{k}]} := \sum_{k_j \in [\mathbf{k}]} k_j \varepsilon_j \qquad 1 \leq j \leq M \qquad [\mathbf{k}] \in \Gamma, \tag{2.6}$$

$$\Gamma = \{(0,0,\cdots,0,0), (1,0,\cdots,0,0), \ldots, (1,1\cdots,1,1)\}. \tag{2.7}$$

Then, using the relations (2.5)-(2.7) leads to the reduced Hamiltonian in the form:

$$B_{[\mathbf{k}]_{[\mathbf{n}]}} = \sum_{l=1}^{N_B} \omega_l A_{[\mathbf{k}]_l}^\dagger A_{[\mathbf{k}]_l} + \varepsilon_{[\mathbf{k}]} - g_{[\mathbf{k}]}^2 \sum_{l=1}^{N_B} \frac{1}{\omega_l} \quad \text{with} \quad [\mathbf{n}] := n_1, n_2, \ldots, n_{N_B}. \tag{2.8}$$

In the next subsection, we define the operators useful for our construction by transforming CS into their deformed correspondents with an operator T, and conversely into their duals with the inverse operator T^{-1}, such that $TT^{-1} = \mathbb{I}$, the identity operator of the considered Hilbert space.

2.1. *Rescaled basis states, eigenvalues and eigenvectors*

For all l, $1 \leq l \leq N_B$, $\mathfrak{H}_l$ denotes the separable Hilbert space spanned by the eigenvectors $\phi_{n_l}, n_l = 0,1,2,\ldots,\infty$ of the number operator N_l.

On $\mathfrak{H}_l$, let T be an operator densely defined and closed in the domain denoted by $\mathscr{D}(T)$. Suppose that T^{-1} exists and is densely defined with domain $\mathscr{D}(T^{-1})$. Moreover, the vectors $\phi_{n_l} \in \mathscr{D}(T) \cap \mathscr{D}(T^{-1})$ for all n_l and there exist non-empty open sets $\mathscr{D}_T$ and $\mathscr{D}_{T^{-1}}$ in $\mathbb{C}$ such that $\eta_{z_l} \in \mathscr{D}(T), \forall z_l \in \mathscr{D}_T$ and $\eta_{z_l} \in \mathscr{D}(T^{-1}), \forall z_l \in \mathscr{D}_{T^{-1}}$.

The operator T being densely defined and closed, then the operator $\mathscr{S}$ such that $\mathscr{S} = T^*T$, with domain $\mathscr{D}(\mathscr{S})$, where T^* is the operator adjoint of T, is self adjoint.

Let $\mathfrak{H}_l^{\mathscr{S}}$ be the completion of $\mathscr{D}(T)$ in the scalar product

$$\langle \phi_{n_l} | \psi_{n_l} \rangle_{\mathfrak{H}_l^{\mathscr{S}}} = \langle \phi_{n_l} | T^*T \psi_{n_l} \rangle_{\mathfrak{H}_l} = \langle \phi_{n_l} | \mathscr{S} \psi_{n_l} \rangle_{\mathfrak{H}_l}, \quad |\psi_{n_l}\rangle \in \mathscr{D}(\mathscr{S}), \tag{2.9}$$

where $\mathfrak{H}_l^{\mathscr{S}}$ is spanned by the vectors

$$|\phi_{n_l}^{\mathscr{S}}\rangle = T^{-1}|\phi_{n_l}\rangle, \quad |\phi_{n_l}\rangle \in \mathscr{D}(T^{-1}), \tag{2.10}$$

and $\mathfrak{H}_l^{\mathscr{S}^{-1}}$ the completion of $\mathscr{D}(T^{*-1})$ in the scalar product

$$\langle \phi_{n_l} | \psi_{n_l} \rangle_{\mathfrak{H}_l^{\mathscr{S}^{-1}}} = \langle \phi_{n_l} | T^{-1}T^{*-1} \psi_{n_l} \rangle_{\mathfrak{H}_l} = \langle \phi_{n_l} | \mathscr{S}^{-1} \psi_{n_l} \rangle_{\mathfrak{H}_l}, \quad |\psi_{n_l}\rangle \in \mathscr{D}(\mathscr{S}^{-1}), \tag{2.11}$$

where $\mathscr{D}(\mathscr{S}^{-1})$ is the domain of $\mathscr{S}^{-1}$, with $\mathfrak{H}_l^{\mathscr{S}^{-1}}$ spanned by the vectors

$$|\phi_{n_l}^{\mathscr{S}^{-1}}\rangle = T|\phi_{n_l}\rangle, \quad |\phi_{n_l}\rangle \in \mathscr{D}(T). \tag{2.12}$$

Moreover, if the spectrum of $\mathscr{S}$ is bounded away from zero, then $\mathscr{S}^{-1}$ is bounded and there follow the inclusions

$$\mathfrak{H}_l^{\mathscr{S}} \subset \mathfrak{H}_l \subset \mathfrak{H}_l^{\mathscr{S}^{-1}}. \tag{2.13}$$

The so-defined Hilbert spaces $\mathfrak{H}_l^{\mathscr{S}}, \mathfrak{H}_l$ and $\mathfrak{H}_l^{\mathscr{S}^{-1}}$ are called a *Gelfand triple*.

On $\mathfrak{H}_l^{\mathscr{S}}$ the transformed counterparts of the operators $a_l, a_l^\dagger, N_l$ defined on $\mathfrak{H}_l$ are [3]

$$a_l^{\mathscr{S}} = T^{-1}a_lT, \qquad a_l^{\mathscr{S}\dagger} = T^{-1}a_l^\dagger T, \qquad N_l^{\mathscr{S}} = T^{-1}N_lT, \tag{2.14}$$

and their actions on $\mathfrak{H}_l^{\mathscr{S}}$ defined by

$$a_l^{\mathscr{S}}|\phi_{n_l}^{\mathscr{S}}\rangle := \sqrt{n_l}|\phi_{n_l-1}^{\mathscr{S}}\rangle, \quad a_l^{\mathscr{S}\dagger}|\phi_{n_l}^{\mathscr{S}}\rangle := \sqrt{n_l+1}|\phi_{n_l+1}^{\mathscr{S}}\rangle, \quad N_l^{\mathscr{S}}|\phi_{n_l}^{\mathscr{S}}\rangle := n_l|\phi_{n_l}^{\mathscr{S}}\rangle. \tag{2.15}$$

These operators satisfy the same commutation relations as $a_l, a_l^\dagger$ and N_l:

$$[a_l^{\mathscr{S}}, a_l^{\mathscr{S}\dagger}] = \mathbb{I}, \quad [a_l^{\mathscr{S}}, N_l^{\mathscr{S}}] = a_l^{\mathscr{S}}, \quad [a_l^{\mathscr{S}\dagger}, N_l^{\mathscr{S}}] = -a_l^{\mathscr{S}\dagger}. \tag{2.16}$$

The transformed operators $a_l^{\mathscr{S}^{-1}}, a_l^{\mathscr{S}^{-1}\dagger}, N_l^{\mathscr{S}^{-1}}$ of the operators $a_l, a_l^\dagger, N_l$ defined on $\mathfrak{H}_l$, are characterized as follows:

$$a_l^{\mathscr{S}^{-1}} = Ta_lT^{-1}, \qquad a_l^{\mathscr{S}^{-1}\dagger} = Ta_l^\dagger T^{-1}, \qquad N_l^{\mathscr{S}^{-1}} = TN_lT^{-1} \tag{2.17}$$

acting on $\mathfrak{H}_l^{\mathscr{S}^{-1}}$ in a similar manner as in (2.15) and satisfying the commutation relations

$$[a_l^{\mathscr{S}^{-1}}, a_l^{\mathscr{S}^{-1}\dagger}] = \mathbb{I}, \quad [a_l^{\mathscr{S}^{-1}}, N_l^{\mathscr{S}^{-1}}] = a_l^{\mathscr{S}^{-1}}, \quad [a_l^{\mathscr{S}^{-1}\dagger}, N_l^{\mathscr{S}^{-1}}] = -a_l^{\mathscr{S}^{-1}\dagger}. \tag{2.18}$$

The operators $T, T^{-1}, \mathscr{S}$ can be written as follows:

$$T := T(N_l) = \sum_{n_l=0}^{\infty} t(n_l)|\phi_{n_l}\rangle\langle\phi_{n_l}|, \quad T^{-1} := T(N_l)^{-1} = \sum_{n_l=0}^{\infty} \frac{1}{t(n_l)}|\phi_{n_l}\rangle\langle\phi_{n_l}|,$$
$$\mathscr{S} := \mathscr{S}(N_l) = \sum_{n_l=0}^{\infty} t(n_l)^2|\phi_{n_l}\rangle\langle\phi_{n_l}|. \tag{2.19}$$

For all l, the action of $f_l(N_l)$ on the basis vectors $|\phi_{n_l}\rangle$ of the separable Hilbert space $\mathfrak{H}_l$ can be also defined as:

$$f_l(N_l)|\phi_{n_l}\rangle := \frac{t(n_l)}{t(n_l-1)}|\phi_{n_l}\rangle = f_l(n_l)|\phi_{n_l}\rangle, \tag{2.20}$$

where

$$t(n_l) = f_l(n_l)f_l(n_l-1)\cdots f(1) := f_l(n_l)! \tag{2.21}$$

are real numbers, possessing the properties:

- $t(0) = 1$ and $t(n_l) = t(n_l')$ if and only if $n_l = n_l'$;
- $0 < t(n_l) < \infty$;
- the Cauchy criterion implies that the finiteness condition for the limit

$$\lim_{n_l\to\infty}\left[\frac{t(n_l)}{t(n_l+1)}\right]^2 \frac{1}{n_l+1} = \rho < \infty, \tag{2.22}$$

  holds. This condition implies that the series $S(r^2) := \sum_{n_l=0}^{\infty} \frac{r^{2n_l}}{[t(n_l)]^2 n_l!}$ converges for all $r < L = 1/\sqrt{\rho}$.

For all l, we define the function of the operator $N_l, f_l(N_l)^{-1}$ acting on the basis vectors $|\phi_{n_l}\rangle$ of $\mathfrak{H}_l$ in the following manner:

$$f_l(N_l)^{-1}|\phi_{n_l}\rangle := \frac{t(n_l-1)}{t(n_l)}|\phi_{n_l}\rangle = \frac{1}{f_l(n_l)}|\phi_{n_l}\rangle. \tag{2.23}$$

The real numbers $t(n_l)$ satisfying the above two first properties are such that the finiteness condition for the limit

$$\lim_{n_l\to\infty}\left[\frac{t(n_l+1)}{t(n_l)}\right]^2 \frac{1}{n_l+1} = \tilde{\rho} < \infty, \tag{2.24}$$

implies that the series $S'(r^2) := \sum_{n_l=0}^{\infty} \frac{r^{2n_l}(t(n_l))^2}{n_l!}$ converges for all $r < \tilde{L} = 1/\sqrt{\tilde{\rho}}$. Since $|\phi_{n_l}^{\mathscr{S}^{-1}}\rangle = T|\phi_{n_l}\rangle$, then

$$|\phi_{n_l}^{\mathscr{S}^{-1}}\rangle = t(n_l)|\phi_{n_l}\rangle = f_l(n_l)!|\phi_{n_l}\rangle. \tag{2.25}$$

As matter of clarity, recall $\mathfrak{H}_l^{\mathscr{S}} \subset \mathfrak{H}_l \subset \mathfrak{H}_l^{\mathscr{S}^{-1}}$ and the operators $a_l^{\mathscr{S}}$ and $a_l^{\mathscr{S}^{-1}}$ act in $\mathfrak{H}_l^{\mathscr{S}}$ and $\mathfrak{H}_l^{\mathscr{S}^{-1}}$, respectively. To avoid any confusion in what follows, we denote $A_l = a_l^{\mathscr{S}}$ when $a_l^{\mathscr{S}}$ acts on $\mathfrak{H}_l$, i.e. prolongating the action of $a_l^{\mathscr{S}}$ into $\mathfrak{H}_l$, and $A_l' = a_l^{\mathscr{S}^{-1}}|_{\mathfrak{H}_l}$ restricting the action of $a_l^{\mathscr{S}^{-1}}$ to $\mathfrak{H}_l$.

Since T, T^{-1} and $\mathscr{S}$ are positive operators, the adjoints of A_l and A'_l on $\mathfrak{H}_l$ take the form:

$$A_l^\dagger = Ta_l^\dagger T^{-1}, \qquad A_l'^\dagger = T^{-1}a_l^\dagger T. \tag{2.26}$$

From the relations

$$f_l(N_l)|\phi_{n_l}\rangle = f_l(n_l)|\phi_{n_l}\rangle, |\phi_{n_l}^{\mathscr{S}}\rangle = \frac{1}{f_l(n_l)!}|\phi_{n_l}\rangle \tag{2.27}$$

and

$$f_l(N_l)^{-1}|\phi_{n_l}\rangle = \frac{1}{f_l(n_l)}|\phi_{n_l}\rangle, |\phi_{n_l}^{\mathscr{S}^{-1}}\rangle = f_l(n_l)!|\phi_{n_l}\rangle, \tag{2.28}$$

we infer their actions as

$$A_l|\phi_{n_l}\rangle = \sqrt{n_l}f_l(n_l)|\phi_{n_l-1}\rangle, \qquad A_l^\dagger|\phi_{n_l}\rangle = \sqrt{n_l+1}f_l(n_l+1)|\phi_{n_l+1}\rangle. \tag{2.29}$$

The vectors $|\phi_{n_l}^{\mathscr{S}}\rangle$ and $|\phi_{n_l}^{\mathscr{S}^{-1}}\rangle$, defined in (2.27) and (2.28), are called *rescaled basis states* [3].

Then we get

$$A_l = a_l f_l(N_l),\, A_l^\dagger = f_l(N_l)a_l^\dagger,\, A'_l = a_l f_l(N_l)^{-1},\, A_l'^\dagger = f_l(N_l)^{-1}a_l^\dagger \tag{2.30}$$

with the following relations

$$[A_l, A_l'^\dagger] = \mathbb{I} \qquad [A'_l, A_l^\dagger] = \mathbb{I} \qquad 1 \le l \le N_B. \tag{2.31}$$

From the relations (2.29), the operators $a_l^{\mathscr{S}}, a_l^{\mathscr{S}^{-1}\dagger}$ and N_l can be expressed as

$$\begin{aligned} A_l &= \sum_{n_l=0}^{\infty} \sqrt{n_l}f_l(n_l)|\phi_{n_l-1}\rangle\langle\phi_{n_l}|, \qquad A_l^\dagger = \sum_{n_l=0}^{\infty} \sqrt{n_l}f(n_l)|\phi_{n_l}\rangle\langle\phi_{n_l-1}| \\ N_l &= \sum_{n_l=0}^{\infty} n_l|\phi_{n_l}\rangle\langle\phi_{n_l}|. \end{aligned} \tag{2.32}$$

For all l, we define, on the Hilbert space $\mathfrak{H}_l$, the momentum operator P_l as

$$P_l := \frac{a_l f_l(N_l) - f_l(N_l)^{-1}a_l^\dagger}{i\sqrt{2}} \qquad \text{with} \qquad [a_l f_l(N_l), \sqrt{2}P_l] = i\mathbb{I}. \tag{2.33}$$

The eigenvectors of $B_{[\mathbf{k}]_l}$ are given, with $|\Phi_{0_l}^{[\mathbf{k}]}\rangle = e^{i\sqrt{2}\frac{g_{[\mathbf{k}]}}{\omega_l}P_l}|0\rangle$, by

$$|\Phi_{n_l}^{[\mathbf{k}]}\rangle = e^{i\sqrt{2}\frac{g_{[\mathbf{k}]}}{\omega_l}P_l}|\phi_{n_l}\rangle = \frac{(A_{[\mathbf{k}]_l}^\dagger)^{n_l}}{\sqrt{n_l!}f_l(n_l)!}|\Phi_{0_l}^{[\mathbf{k}]}\rangle, \quad A_{[\mathbf{k}]_l} = e^{i\sqrt{2}\frac{g_{[\mathbf{k}]}}{\omega_l}P_l}a_l f_l(N_l)e^{-i\sqrt{2}\frac{g_{[\mathbf{k}]}}{\omega_l}P_l} \tag{2.34}$$

while for the operators $B_{[\mathbf{k}]}$ the eigenvalues $E_{\mathbf{n}}^{[\mathbf{k}]}$ and eigenfunctions $|\Phi_{\mathbf{n}}^{[\mathbf{k}]}\rangle$ are computed as follows:

$$E_{\mathbf{n}}^{[\mathbf{k}]} = \sum_{l=1}^{N_B}\omega_l n_l f_l^2(n_l) + \varepsilon_{[\mathbf{k}]} - g_{[\mathbf{k}]}^2\sum_{l=1}^{N_B}\frac{1}{\omega_l}, \qquad |\Phi_{\mathbf{n}}^{[\mathbf{k}]}\rangle = \bigotimes_{l=1}^{N_B}|\Phi_{n_l}^{[\mathbf{k}]}\rangle. \tag{2.35}$$

Let us denote the *duals* of the operators $B_{[\mathbf{k}]_l}, A_{[\mathbf{k}]_l}, A^{\dagger}_{[\mathbf{k}]_l}$ by $B'_{[\mathbf{k}]_l}, A'_{[\mathbf{k}]_l}, A'^{\dagger}_{[\mathbf{k}]_l}$, respectively, such that

$$B'_{[\mathbf{k}]_l} := \omega_l A'^{\dagger}_{[\mathbf{k}]_l} A'_{[\mathbf{k}]_l} + \frac{\varepsilon_{[\mathbf{k}]}}{N_B} - \frac{g^2_{[\mathbf{k}]}}{\omega_l}, \qquad A'_{[\mathbf{k}]_l} := a_l f_l(N_l)^{-1} + \frac{g_{[\mathbf{k}]}}{\omega_l}, \tag{2.36}$$

and define, on the separable Hilbert space $\mathfrak{H}_l$, the *dual* of the momentum operator P_l by

$$P'_l := \frac{a_l f_l(N_l)^{-1} - f_l(N_l) a^{\dagger}_l}{i\sqrt{2}} \quad \text{with} \quad [a_l f_l(N_l)^{-1}, \sqrt{2}P'_l] = i\mathbb{I}. \tag{2.37}$$

Then, the eigenvectors of $B'_{[\mathbf{k}]_l}$ are given, with $|\Phi'^{[\mathbf{k}]}_{0_l}\rangle = e^{i\sqrt{2}\frac{g_{[\mathbf{k}]}}{\omega_l}P'_l}|0\rangle$, by

$$|\Phi'^{[\mathbf{k}]}_{n_l}\rangle = e^{i\sqrt{2}\frac{g_{[\mathbf{k}]}}{\omega_l}P'_l}|\phi_{n_l}\rangle = \frac{(A'^{\dagger}_{[\mathbf{k}]_l})^{n_l}}{\sqrt{n_l!}} f_l(n_l)!|\Phi'^{[\mathbf{k}]}_{0_l}\rangle, \quad A'_{[\mathbf{k}]_l} = e^{i\sqrt{2}\frac{g_{[\mathbf{k}]}}{\omega_l}P'_l} a_l f_l(N_l)^{-1} e^{-i\sqrt{2}\frac{g_{[\mathbf{k}]}}{\omega_l}P'_l} \tag{2.38}$$

corresponding to the eigenvalues

$$E'^{[\mathbf{k}]}_{\mathbf{n}} = \sum_{l=1}^{N_B} \omega_l n_l f_l^{-2}(n_l) + \varepsilon_{[\mathbf{k}]} - g^2_{[\mathbf{k}]} \sum_{l=1}^{N_B} \frac{1}{\omega_l}. \tag{2.39}$$

It then comes that the eigenvectors of H associated to the eigenvalues $E^{[\mathbf{k}]}_{\mathbf{n}}$ are given by

$$|\varphi^{[\mathbf{k}]}_{\mathbf{n}}\rangle = |\Phi^{[\mathbf{k}]}_{\mathbf{n}}\rangle \otimes |\Psi_{[\mathbf{k}]}\rangle, \qquad |\Psi_{[\mathbf{k}]}\rangle = (c^{\dagger}_1)^{k_1}(c^{\dagger}_2)^{k_2}\cdots(c^{\dagger}_j)^{k_j}\cdots(c^{\dagger}_M)^{k_M}|\Psi_{(0,0,\ldots,0)}\rangle. \tag{2.40}$$

Since the operators $e^{i\sqrt{2}\frac{g_{[\mathbf{k}]}}{\omega_l}P_l}$, $e^{i\sqrt{2}\frac{g_{[\mathbf{k}]}}{\omega_l}P'_l}$ are unitary and act on $\mathfrak{H}_l$, the sets $\{|\Phi^{[\mathbf{k}]}_{n_l}\rangle\}_{n_l=0}^{\infty}$ and $\{|\Phi'^{[\mathbf{k}]}_{n_l}\rangle\}_{n_l=0}^{\infty}$ are both basis vectors of $\mathfrak{H}_l$.

2.2. *Construction of the nonlinear coherent states (NCS)*

From the relations (2.34) and (2.38), we arrive at the following definitions:

$$\{n_l\}! := n_l![f_l(n_l)!]^2, \qquad \{n_l\}_d! := \frac{n_l!}{[f_l(n_l)!]^2}. \tag{2.41}$$

For all l, we consider now, on $\mathfrak{H}_l$, the displacement operators, related to the dual pair $\mathfrak{H}^F_l, \mathfrak{H}^{F^{-1}}_l$, as:

$$D_l(z_l) := e^{\imath\sqrt{2}\frac{g_{[\mathbf{k}]}}{\omega_l}P_l} e^{z_l A'^{\dagger}_l - \bar{z}_l A_l}, z_l \in \mathscr{D}_l = \{z_l \in \mathbb{C}/|z_l| < L = 1/\sqrt{\rho}\}, \tag{2.42}$$

$$D'_l(z_l) := e^{\imath\sqrt{2}\frac{g_{[\mathbf{k}]}}{\omega_l}P'_l} e^{z_l A^{\dagger}_l - \bar{z}_l A'_l}, z_l \in \tilde{\mathscr{D}}_l = \{z_l \in \mathbb{C}/|z_l| < \tilde{L} = 1/\sqrt{\tilde{\rho}}\}. \tag{2.43}$$

Since the operators $e^{\imath\sqrt{2}\frac{g_{[\mathbf{k}]}}{\omega_l}P_l}$ and $e^{\imath\sqrt{2}\frac{g_{[\mathbf{k}]}}{\omega_l}P'_l}$ are unitary on $\mathfrak{H}_l$, the above operators are also unitary on $\mathfrak{H}_l$, and, in view of the relations

$$D_l(z_l)D_l(z'_l) = e^{\imath 2\sqrt{2}\frac{g_{[\mathbf{k}]}}{\omega_l}P_l} e^{\imath Im(\bar{z}_l z'_l)} D_l(z_l + z'_l),$$

$$D'_l(z_l)D'_l(z'_l) = e^{\imath 2\sqrt{2}\frac{g_{[\mathbf{k}]}}{\omega_l}P'_l} e^{\imath Im(\bar{z}_l z'_l)} D'_l(z_l + z'_l) \tag{2.44}$$

which hold if $z_l, z'_l, z_l + z'_l \in \mathscr{D}_l$ (resp. $\in \tilde{\mathscr{D}}_l$), they realize a unitary projective representation of the Weyl-Heisenberg group on $\mathfrak{H}_l$.

By using the Baker-Campbell-Hausdorff identity, we obtain

$$D_l(z_l) = e^{\imath\sqrt{2}\frac{g_{[\mathbf{k}]}}{\omega_l}P_l} e^{-\frac{1}{2}|z_l|^2} e^{z_l A_l'^{\dagger}} e^{-\bar{z}_l A_l}, \tag{2.45}$$

$$D'_l(z_l) = e^{\imath\sqrt{2}\frac{g_{[\mathbf{k}]}}{\omega_l}P'_l} e^{-\frac{1}{2}|z_l|^2} e^{z_l A_l^{\dagger}} e^{-\bar{z}_l A'_l}. \tag{2.46}$$

From (2.45), for all $z_l \in \mathscr{D}_l$, the NCS

$$D_l(z_l)|0\rangle_l = e^{-\frac{1}{2}|z_l|^2} e^{\imath\sqrt{2}\frac{g_{[\mathbf{k}]}}{\omega_l}P_l} \sum_{n_l=0}^{\infty} \frac{z_l^{n_l}}{\sqrt{\{n_l\}!}} |\phi_{n_l}\rangle \tag{2.47}$$

become in the domain $\mathscr{D}_l$ given in (2.42):

$$D_l(z_l)|0\rangle_l := \left(\mathscr{N}(|z_l|^2)\right)^{-1/2} \sum_{n_l=0}^{\infty} \frac{z_l^{n_l}}{\sqrt{\{n_l\}!}} |\Phi_{n_l}^{[\mathbf{k}]}\rangle \tag{2.48}$$

with the normalization factor, by using (2.41), given by

$$\mathscr{N}(|z_l|^2) = \sum_{n_l=0}^{\infty} \frac{(\bar{z}_l z_l)^{n_l}}{\{n_l\}!}. \tag{2.49}$$

These vectors correspond to the well-known *NCS* of quantum optics [20], related to the basis vectors $\{|\Phi_{n_l}^{[\mathbf{k}]}\rangle\}_{n_l=0}^{\infty}$ of the separable Hilbert space $\mathfrak{H}_l$.

Similarly, from (2.46), we determine, for all $z_l \in \tilde{\mathscr{D}}_l$, the NCS

$$D'_l(z_l)|0\rangle_l = \left(\mathscr{N}'(|z_l|^2)\right)^{-1/2} \sum_{n_l=0}^{\infty} \frac{z_l^{n_l}}{\sqrt{\{n_l\}_d!}} |\Phi_{n_l}'^{[\mathbf{k}]}\rangle \tag{2.50}$$

with

$$\mathscr{N}'(|z_l|^2) = \sum_{n_l=0}^{\infty} \frac{(\bar{z}_l z_l)^{n_l}}{\{n_l\}_d!}. \tag{2.51}$$

Setting now

$$|0\rangle = |0\rangle_1 \otimes |0\rangle_2 \otimes \cdots \otimes |0\rangle_l \cdots \otimes |0\rangle_{N_B} \tag{2.52}$$

and considering the displacement operators $D_{\mathbf{n}}(\mathbf{z})$ and $D'_{\mathbf{n}}(\mathbf{z})$:

$$D_{\mathbf{n}}(\mathbf{z}) := \bigotimes_{l=1}^{N_B} D_l(z_l), \qquad D'_{\mathbf{n}}(\mathbf{z}) := \bigotimes_{l=1}^{N_B} D'_l(z_l), \tag{2.53}$$

the action of $D_{\mathbf{n}}(\mathbf{z})$ on $|0\rangle$ can be realized as:

$$D_{\mathbf{n}}(\mathbf{z})|0\rangle = \bigotimes_{l=1}^{N_B} D_l(z_l)(|0\rangle_1 \otimes |0\rangle_2 \otimes \cdots \otimes |0\rangle_l \otimes \cdots \otimes |0\rangle_{N_B})$$

$$
\begin{aligned}
&= D_1(z_1)|0\rangle_1 \otimes D_2(z_2)|0\rangle_2 \otimes \cdots \otimes D_l(z_l)|0\rangle_l \otimes \cdots \otimes D_{N_B}(z_{N_B})|0\rangle_{N_B} \\
&= \left(\mathscr{N}(|z_1|^2)\right)^{-1/2} \sum_{n_1=0}^{\infty} \frac{z_1^{n_1}}{\sqrt{\{n_1\}!}} \left(\mathscr{N}(|z_2|^2)\right)^{-1/2} \sum_{n_2=0}^{\infty} \frac{z_2^{n_2}}{\sqrt{\{n_2\}!}} \\
&\quad \cdots \left(\mathscr{N}(|z_l|^2)\right)^{-1/2} \sum_{n_l=0}^{\infty} \frac{z_l^{n_l}}{\sqrt{\{n_l\}!}} \cdots \left(\mathscr{N}(|z_{N_B}|^2)\right)^{-1/2} \sum_{n_{N_B}=0}^{\infty} \frac{z_{N_B}^{n_{N_B}}}{\sqrt{\{n_{N_B}\}!}} \\
&\quad \times (|\Phi_{n_1}^{[\mathbf{k}]}\rangle \otimes |\Phi_{n_2}^{[\mathbf{k}]}\rangle \otimes \cdots \otimes |\Phi_{n_l}^{[\mathbf{k}]}\rangle \otimes \cdots \otimes |\Phi_{n_{N_B}}^{[\mathbf{k}]}\rangle).
\end{aligned}
\tag{2.54}
$$

Putting

$$
\mathscr{N}(|\mathbf{z}|^2) := \prod_{l=1}^{N_B} \mathscr{N}_l(|z_l|^2), \qquad \mathbf{z}^{\mathbf{n}} := \prod_{l=1}^{N_B} z_l^{n_l}, \tag{2.55}
$$

and

$$
\mathbf{n}! := n_1! n_2! \cdots n_{N_B}!, \qquad f(\mathbf{n})! := f(n_1)! f(n_2)! \cdots f(n_{N_B})!, \tag{2.56}
$$

we derive the NCS as follows:

$$
|\eta_{\mathbf{z}}\rangle := D_{\mathbf{n}}(\mathbf{z})|0\rangle = \left(\mathscr{N}(|\mathbf{z}|^2)\right)^{-1/2} \sum_{\mathbf{n}=0}^{\infty} \frac{\mathbf{z}^{\mathbf{n}}}{\sqrt{\{\mathbf{n}\}!}} |\Phi_{\mathbf{n}}^{[\mathbf{k}]}\rangle, \qquad \mathbf{z} \in \mathscr{D} \tag{2.57}
$$

and

$$
|\eta'_{\mathbf{z}}\rangle := D'_{\mathbf{n}}(\mathbf{z})|0\rangle = \left(\mathscr{N}'(|\mathbf{z}|^2)\right)^{-1/2} \sum_{\mathbf{n}=0}^{\infty} \frac{\mathbf{z}^{\mathbf{n}}}{\sqrt{\{\mathbf{n}\}_d!}} |\Phi_{\mathbf{n}}'^{[\mathbf{k}]}\rangle, \qquad \mathbf{z} \in \tilde{\mathscr{D}}. \tag{2.58}
$$

Remark 2.1. In the previous definitions, for each $l, l = 1, 2, \ldots, N_B$, the variables z_l are assumed to be mutually independent.

On $\mathfrak{H}$ spanned by the bosonic eigenvectors $|\Phi_{\mathbf{n}}^{[\mathbf{k}]}\rangle$ and $|\Phi_{\mathbf{n}}'^{[\mathbf{k}]}\rangle$, we have the following resolutions of the identity:

$$
\int_{\mathscr{D}} |\eta_{\mathbf{z}}\rangle\langle\eta_{\mathbf{z}}| \, d\mu(\mathbf{z}, \bar{\mathbf{z}}) \mathscr{N}(|\mathbf{z}|^2) = I_{\mathfrak{H}}, \tag{2.59}
$$

$$
\int_{\tilde{\mathscr{D}}} |\eta'_{\mathbf{z}}\rangle\langle\eta'_{\mathbf{z}}| \, d\mu(\mathbf{z}, \bar{\mathbf{z}}) \mathscr{N}'(|\mathbf{z}|^2) = I_{\mathfrak{H}}. \tag{2.60}
$$

The measure $d\mu(\mathbf{z}, \bar{\mathbf{z}})$ is such that $d\mu(\mathbf{z}, \bar{\mathbf{z}}) = d\lambda(r) d\theta$, $\mathbf{z} = re^{\imath\theta}$, where $d\lambda(r)$ is determined through the moment problem:

$$
2\pi \int_0^L r^{2\mathbf{n}} d\lambda(r) = \{\mathbf{n}\}!, \qquad \mathbf{n} = 0, 1, 2, \cdots \tag{2.61}
$$

in the case of the vectors $D_{\mathbf{n}}(\mathbf{z})|0\rangle$; for the vectors $D'_{\mathbf{n}}(\mathbf{z})|0\rangle$, with $d\mu(\mathbf{z}, \bar{\mathbf{z}}) = d\rho(r) d\theta$,

$$
2\pi \int_0^{\tilde{L}} r^{2\mathbf{n}} d\rho(r) = \{\mathbf{n}\}_d!, \qquad \mathbf{n} = 0, 1, 2, \cdots. \tag{2.62}
$$

3. Probability density computation

This section discusses the probability density and its time evolution evaluated in the constructed NCS basis.

We start computing the quantity $\langle \eta_{\mathbf{z}} | \eta_{\mathbf{z_0}} \rangle$. Making use of (2.57), (2.49) and (2.55)-(2.56), we obtain

$$\begin{aligned}\langle \eta_{\mathbf{z}} | \eta_{\mathbf{z_0}} \rangle &= \left[\mathscr{N}(|\mathbf{z}|^2) \mathscr{N}(|\mathbf{z_0}|^2) \right]^{-1/2} \sum_{\mathbf{n}=0}^{\infty} \frac{(\bar{\mathbf{z}}\mathbf{z_0})^{\mathbf{n}}}{\{\mathbf{n}\}!} \\ &= \frac{\mathscr{N}(\bar{\mathbf{z}}\mathbf{z_0})}{\sqrt{\mathscr{N}(|\mathbf{z}|^2)\mathscr{N}(|\mathbf{z_0}|^2)}} \end{aligned} \tag{3.1}$$

giving

$$|\langle \eta_{\mathbf{z}} | \eta_{\mathbf{z_0}} \rangle|^2 = \frac{|\mathscr{N}(\bar{\mathbf{z}}\mathbf{z_0})|^2}{\mathscr{N}(|\mathbf{z}|^2)\mathscr{N}(|\mathbf{z_0}|^2)}, \qquad |\mathscr{N}(\bar{\mathbf{z}}\mathbf{z_0})|^2 := \mathscr{N}(\bar{\mathbf{z}}\mathbf{z_0})\mathscr{N}(\mathbf{z}\bar{\mathbf{z}}_0). \tag{3.2}$$

Then, the probability density is defined as the map

$$\mathscr{D} \to \mathbb{R}_+, \qquad \mathbf{z} \mapsto \rho_{\mathbf{z_0}}(\mathbf{z}) := |\langle \eta_{\mathbf{z}} | \eta_{\mathbf{z_0}} \rangle|^2 = \frac{|\mathscr{N}(\bar{\mathbf{z}}\mathbf{z_0})|^2}{\mathscr{N}(|\mathbf{z}|^2)\mathscr{N}(|\mathbf{z_0}|^2)}. \tag{3.3}$$

Consider the following spectral decomposition in the eigenstates $|\Phi_{\mathbf{n}}^{[\mathbf{k}]}\rangle$ given by

$$\tilde{H} = \sum_{\mathbf{n}=0}^{\infty} \Omega_{N_B} \mathscr{E}_{\mathbf{n}} |\Phi_{\mathbf{n}}^{[\mathbf{k}]}\rangle\langle\Phi_{\mathbf{n}}^{[\mathbf{k}]}|, \quad \mathscr{E}_{\mathbf{n}} = \sum_{l=1}^{N_B} n_l f_l(n_l) \tag{3.4}$$

with

$$\Omega_{N_B} := \left(\sum_{l=1}^{N_B} \omega_l n_l f_l(n_l) \right) \Bigg/ \left(\sum_{l=1}^{N_B} n_l f_l(n_l) \right). \tag{3.5}$$

Let

$$\mathbf{J}^{\mathbf{n}/2} := \prod_{l=1}^{N_B} J_l^{n_l/2}, \quad \varepsilon_{\mathbf{n}} := (n_1 f_1(n_1), n_2 f_2(n_2), \cdots, n_{N_B} f_{N_B}(n_{N_B})), \quad \gamma := \begin{pmatrix} \gamma_1 \\ \gamma_2 \\ \vdots \\ \gamma_{N_B} \end{pmatrix}, \tag{3.6}$$

where $\mathbf{z_0}^{\mathbf{n}} := \mathbf{J_0}^{\mathbf{n}/2} e^{-\imath \varepsilon_{\mathbf{n}} \gamma_0}$, such that we get

$$e^{-\imath\tilde{H}t}|\eta_{\mathbf{z_0}}\rangle = \left(\mathscr{N}(|\mathbf{z_0}|^2)\right)^{-1/2}\sum_{\mathbf{n}=0}^{\infty}\frac{\mathbf{J_0}^{\mathbf{n}/2}e^{-\imath\varepsilon_{\mathbf{n}}\gamma_0}}{\sqrt{\{\mathbf{n}\}!}}e^{-\imath\left(\Omega_{N_B}\mathscr{E}_{\mathbf{n}}\right)t}|\Phi_{\mathbf{n}}^{[\mathbf{k}]}\rangle. \tag{3.7}$$

Since

$$\varepsilon_{\mathbf{n}}\beta = (n_1 f_1(n_1), n_2 f_2(n_2), \cdots, n_{N_B} f_{N_B}(n_{N_B}))\begin{pmatrix}1\\ \vdots\\ 1\\ \vdots\\ 1\end{pmatrix} = \sum_{l=1}^{N_B} n_l f_l(n_l) = \mathscr{E}_{\mathbf{n}}, \tag{3.8}$$

we have

$$e^{-\imath\tilde{H}t}|\eta_{\mathbf{z_0}}\rangle = |\mathbf{J}_0, \gamma_0 + \Omega_{N_B}t\beta\rangle = |\eta_{\mathbf{z_0(t)}}\rangle, \qquad (\mathbf{z_0}(t))^{\mathbf{n}} := \mathbf{J_0}^{\mathbf{n}/2}e^{-\imath\varepsilon_{\mathbf{n}}(\gamma_0+\Omega_{N_B}t\beta)}. \tag{3.9}$$

From (3.7) and (3.9), we obtain

$$\begin{aligned}\langle\eta_{\mathbf{z}}|e^{-iHt}|\eta_{\mathbf{z_0}}\rangle &= \left[\mathscr{N}(|\mathbf{z}|^2)\mathscr{N}(|\mathbf{z_0}|^2)\right]^{-1/2}\sum_{\mathbf{m},\mathbf{n}=0}^{\infty}\frac{\mathbf{J}^{\mathbf{m}/2}e^{\imath\varepsilon_{[\mathbf{m}]}\gamma}}{\sqrt{\{\mathbf{m}\}!}}\frac{\mathbf{J_0}^{\mathbf{n}/2}e^{-\imath\varepsilon_{[\mathbf{n}]}(\gamma_0+\Omega_N t\beta)}}{\sqrt{\{\mathbf{n}\}!}}\\ &\quad\langle\Phi_{[\mathbf{m}]}^{[\mathbf{k}]}|\Phi_{[\mathbf{n}]}^{[\mathbf{k}]}\rangle\\ &= \left[\mathscr{N}(|\mathbf{z}|^2)\mathscr{N}(|\mathbf{z_0}|^2)\right]^{-1/2}\sum_{\mathbf{n}=0}^{\infty}\frac{\mathbf{J}^{\mathbf{n}/2}\mathbf{J_0}^{\mathbf{n}/2}e^{-\imath\varepsilon_{[\mathbf{n}]}(\gamma_0+\Omega_N t\beta-\gamma)}}{\{\mathbf{n}\}!}\end{aligned} \tag{3.10}$$

and thereby

$$|\langle\eta_{\mathbf{z}}|e^{-i\tilde{H}t}|\eta_{\mathbf{z_0}}\rangle|^2 = \frac{|\mathscr{N}(\bar{\mathbf{z}}\mathbf{z_0}(t))|^2}{\mathscr{N}(|\mathbf{z}|^2)\mathscr{N}(|\mathbf{z_0}|^2)}. \tag{3.11}$$

Thus, the time evolution behavior of $\rho_{z_0}(z)$ is provided by the mapping

$$\mathbf{z}\mapsto\rho_{\mathbf{z_0}}(\mathbf{z},t) := |\langle\eta_{\mathbf{z}}|e^{-i\tilde{H}t}|\eta_{\mathbf{z_0}}\rangle|^2 = \frac{|\mathscr{N}(\bar{\mathbf{z}}\mathbf{z_0}(t))|^2}{\mathscr{N}(|\mathbf{z}|^2)\mathscr{N}(|\mathbf{z_0}|^2)} \tag{3.12}$$

while the dynamics of the NCS $|\eta_{\mathbf{z}}\rangle$ is governed by the relation:

$$|\eta_{\mathbf{z}};t\rangle = e^{-i\tilde{H}t}|\eta_{\mathbf{z}}\rangle = |\eta_{\mathbf{z}(t)}\rangle, \qquad (\mathbf{z}(t))^{\mathbf{n}} := \mathbf{J}^{\mathbf{n}/2}e^{-\imath\varepsilon_{\mathbf{n}}(\gamma+\Omega_{N_B}t\beta)}. \tag{3.13}$$

Finally, as

$$\langle\eta'_{\mathbf{z}}|\eta'_{\mathbf{z_0}}\rangle = \left[\mathscr{N}(|\mathbf{z}|^2)\mathscr{N}(|\mathbf{z_0}|^2)\right]^{-1/2}\sum_{\mathbf{n}=0}^{\infty}\frac{(\bar{\mathbf{z}}\mathbf{z_0})^{\mathbf{n}}}{\{\mathbf{n}\}_d!}, \tag{3.14}$$

the NCS $|\eta'_{\mathbf{z}}\rangle$ obey the same relations as (3.12) and (3.13).

4. Reproducing kernel

In this section, we discuss another important property of the NCS, i.e. the reproducing kernel on the Hilbert space $\mathfrak{H}$, based on the completeness relation (2.59).

From the expression (3.1), we introduce the quantity

$$\mathscr{K}(\mathbf{z},\mathbf{z}') := \langle \eta_{\mathbf{z}'}|\eta_{\mathbf{z}}\rangle = \frac{\mathscr{N}(\bar{\mathbf{z}}'\mathbf{z})}{\sqrt{\mathscr{N}(|\mathbf{z}'|^2)\mathscr{N}(|\mathbf{z}|^2)}} \tag{4.1}$$

standing for a reproducing kernel [1]. Indeed,

Proposition 4.1.

The following properties are satisfied by the function $\mathscr{K}$ on the Hilbert space $\mathfrak{H}$:

(i) Hermiticity

$$\mathscr{K}(\mathbf{z},\mathbf{z}') = \overline{\mathscr{K}(\mathbf{z}',\mathbf{z})}; \tag{4.2}$$

(ii) Positivity

$$\mathscr{K}(\mathbf{z},\mathbf{z}) > 0; \tag{4.3}$$

(iii) Idempotence

$$\int_{\mathscr{D}} d\mu(\mathbf{z}'',\bar{\mathbf{z}}'')\mathscr{K}(\mathbf{z},\mathbf{z}'')\mathscr{K}(\mathbf{z}'',\mathbf{z}') = \mathscr{K}(\mathbf{z},\mathbf{z}'). \tag{4.4}$$

The condition (4.4), also called *the square integrability property* of the reproducing kernel [1], is a consequence of (2.59).

Proof. See Appendix. □

For any $|\Psi\rangle \in \mathfrak{H}$, we have

$$|\Psi\rangle = \int_{\mathscr{D}} d\mu(\mathbf{z},\bar{\mathbf{z}})\Psi(\mathbf{z})|\eta_{\mathbf{z}}\rangle \tag{4.5}$$

where $\Psi(\mathbf{z}) := \langle \eta_{\mathbf{z}}|\Psi\rangle$. The following reproducing property

$$\Psi(\mathbf{z}) = \int_{\mathscr{D}} d\mu(\mathbf{z}',\bar{\mathbf{z}}')\mathscr{K}(\mathbf{z},\mathbf{z}')\Psi(\mathbf{z}') \tag{4.6}$$

is immediate from the definition of the reproducing kernel (4.1).

5. Diagonal representation of the density matrix

In quantum mechanics, the probability distribution on the states of a physical system can be characterized by a statistical operator called density matrix, generally denoted by ρ. The latter reveals to be an important tool used for examining the physical and chemical properties of a system (see for e.g. [5] and references listed therein). In the Fock representation, the density matrix is given by its matrix elements, $\rho_{m,n} = \langle m|\rho|n\rangle$, as [13]: $\rho = \sum_{m,n}\rho_{m,n}|m\rangle\langle n|$. In this section, we investigate the statistical properties of the NCS making use of the density matrix. The results issued from the previous section are exploited as key ingredients to construct a Glauber-Sudarshan P-representation of the density matrix following [9, 10, 13] and [24], where a q-analogue has been performed. We first provide this P-representation in the electron-phonon system bosonic states and NCS basis, and then extend it to matrix domain with respect to the physical model VCS basis.

Assuming that our quantum model obeys the conditions of the quantum canonical distribution, the density matrix, for a fixed $[\mathbf{k}]$, is given, in terms of the bosonic states, as

$$\rho_{[\mathbf{k}]} = \sum_{\mathbf{n},\mathbf{m}=0}^{\infty} \rho_{[\mathbf{k}]}(\mathbf{n},\mathbf{m}) |\Phi_{\mathbf{n}}^{[\mathbf{k}]}\rangle\langle\Phi_{\mathbf{m}}^{[\mathbf{k}]}| \tag{5.1}$$

where the $\rho_{[\mathbf{k}]}(\mathbf{n},\mathbf{m})$ are the matrix elements.

In a matrix form, we get, by using (5.1), the following expression for $\hat{\rho}$:

$$\hat{\rho} = \sum_{[\mathbf{k}]\in\Gamma} \sum_{\mathbf{n},\mathbf{m}=0}^{\infty} \rho_{[\mathbf{k}]}(\mathbf{n},\mathbf{m}) |\Psi_{[\mathbf{k}]} \otimes \Phi_{[\mathbf{n}]}^{[\mathbf{k}]}\rangle\langle\Psi_{[\mathbf{k}]} \otimes \Phi_{[\mathbf{m}]}^{[\mathbf{k}]}| \tag{5.2}$$

which takes on the Hilbert space $\mathbb{C}^{2^M} \otimes \mathfrak{H}$ the form

$$\hat{\rho} = diag(\rho_{(0,0,\ldots,0,0)}, \rho_{(1,0,\ldots 0,0)}, \cdots, \rho_{[\mathbf{k}]}, \cdots, \rho_{(1,1,\ldots,1,1)}) \tag{5.3}$$

where $[\mathbf{n}] := n_1, n_2, \ldots, n_{N_B}$ and $\mathfrak{H} = \bigoplus_{[\mathbf{k}]\in\Gamma} \mathfrak{H}_{[\mathbf{k}]}$, $\mathfrak{H}_{[\mathbf{k}]}$ is the subspace of $\mathfrak{H}$ spanned by the vectors $|\Phi_{[\mathbf{n}]}^{[\mathbf{k}]}\rangle$ with $\sum_{[\mathbf{k}]\in\Gamma} |\Psi_{[\mathbf{k}]}\rangle\langle\Psi_{[\mathbf{k}]}| = \mathbb{I}_{2^M}$ with Γ defined in (2.7).

Keeping in mind the radial parametrization $\mathbf{z} = re^{\imath\theta}$ we get the following quantity $\mathcal{N}(|\mathbf{z}|^2)|re^{\imath\theta}\rangle\langle re^{\imath\theta}|$:

$$\mathcal{N}(|\mathbf{z}|^2)|re^{\imath\theta}\rangle\langle re^{\imath\theta}| = \sum_{\mathbf{n},\mathbf{m}=0}^{\infty} \frac{r^{\mathbf{m}+\mathbf{n}} e^{\imath(\mathbf{n}-\mathbf{m})\theta}}{\sqrt{\{\mathbf{n}\}!\{\mathbf{m}\}!}} |\Phi_{\mathbf{n}}^{[\mathbf{k}]}\rangle\langle\Phi_{\mathbf{m}}^{[\mathbf{k}]}| \tag{5.4}$$

which leads to the integral

$$\int_0^{2\pi} \frac{d\theta}{2\pi} e^{\imath\mathbf{l}\theta} \mathcal{N}(|\mathbf{z}|^2)|re^{\imath\theta}\rangle\langle re^{\imath\theta}| = \sum_{\mathbf{n},\mathbf{m}=0}^{\infty} \frac{r^{\mathbf{m}+\mathbf{n}}}{\sqrt{\{\mathbf{n}\}!\{\mathbf{m}\}!}} \delta(\mathbf{n}-\mathbf{m}+\mathbf{l}) |\Phi_{\mathbf{n}}^{[\mathbf{k}]}\rangle\langle\Phi_{\mathbf{m}}^{[\mathbf{k}]}|. \tag{5.5}$$

We obtain the expression

$$\begin{aligned}\left(\frac{d}{dr}\right)^{\mathbf{q}} \int_0^{2\pi} \frac{d\theta}{2\pi} e^{\imath\mathbf{l}\theta} \mathcal{N}(|\mathbf{z}|^2)|re^{\imath\theta}\rangle\langle re^{\imath\theta}| = \sum_{\mathbf{n},\mathbf{m}=0}^{\infty} & \frac{r^{\mathbf{m}+\mathbf{n}-\mathbf{q}}}{\sqrt{\{\mathbf{n}\}!\{\mathbf{m}\}!}} \frac{(\mathbf{m}+\mathbf{n})!}{(\mathbf{m}+\mathbf{n}-\mathbf{q})!} \\ & \times\delta(\mathbf{n}-\mathbf{m}+\mathbf{l}) |\Phi_{\mathbf{n}}^{[\mathbf{k}]}\rangle\langle\Phi_{\mathbf{m}}^{[\mathbf{k}]}|\end{aligned} \tag{5.6}$$

such that the following quantity

$$\left\{\left(\frac{d}{dr}\right)^{\mathbf{q}} \int_0^{2\pi} \frac{d\theta}{2\pi} e^{\imath\mathbf{l}\theta} \mathcal{N}(|\mathbf{z}|^2)|re^{\imath\theta}\rangle\langle re^{\imath\theta}|\right\}_{r=0}, \tag{5.7}$$

where, at $r = 0$ in the right-hand side of (5.6), the term with $\mathbf{m}+\mathbf{n}-\mathbf{q} = 0$ alone survives, provides

$$|\Phi^{[\mathbf{k}]}_{\mathbf{n}}\rangle\langle\Phi^{[\mathbf{k}]}_{\mathbf{m}}| = \frac{\sqrt{\{\mathbf{n}\}!\{\mathbf{m}\}!}}{(\mathbf{m}+\mathbf{n})!}\left(\frac{d}{dr}\right)^{\mathbf{m}+\mathbf{n}}\int_0^{2\pi}\frac{d\theta}{2\pi}e^{\imath(\mathbf{m}-\mathbf{n})\theta}\mathscr{N}(|\mathbf{z}|^2)|re^{\imath\theta}\rangle\langle re^{\imath\theta}|\Big|_{r=0}. \tag{5.8}$$

Then, a f-deformed Glauber-Sudarshan P-representation of the density matrix in the bosonic states basis is given by

$$\rho_{[\mathbf{k}]} = \sum_{\mathbf{n},\mathbf{m}=0}^{\infty}\rho_{[\mathbf{k}]}(\mathbf{n},\mathbf{m})\frac{\sqrt{\{\mathbf{n}\}!\{\mathbf{m}\}!}}{(\mathbf{m}+\mathbf{n})!}\left(\frac{d}{dr}\right)^{\mathbf{m}+\mathbf{n}}\int_0^{2\pi}\frac{d\theta}{2\pi}e^{\imath(\mathbf{m}-\mathbf{n})\theta}\mathscr{N}(|\mathbf{z}|^2)|re^{\imath\theta}\rangle\langle re^{\imath\theta}|\Big|_{r=0}. \tag{5.9}$$

In the case of the dual NCS $|\eta'_{\mathbf{z}}\rangle$, we get

$$|\Phi^{[\mathbf{k}]}_{\mathbf{n}}\rangle\langle\Phi^{[\mathbf{k}]}_{\mathbf{m}}| = \frac{\sqrt{\{\mathbf{n}\}_d!\{\mathbf{m}\}_d!}}{(\mathbf{m}+\mathbf{n})!}\left(\frac{d}{dr}\right)^{\mathbf{m}+\mathbf{n}}\int_0^{2\pi}\frac{d\theta}{2\pi}e^{\imath(\mathbf{m}-\mathbf{n})\theta}\mathscr{N}(|\mathbf{z}|^2)|re^{\imath\theta}\rangle\langle re^{\imath\theta}|\Big|_{r=0}, \tag{5.10}$$

and

$$\tilde{\rho}_{[\mathbf{k}]} = \sum_{\mathbf{n},\mathbf{m}=0}^{\infty}\tilde{\rho}_{[\mathbf{k}]}(\mathbf{n},\mathbf{m})\frac{\sqrt{\{\mathbf{n}\}_d!\{\mathbf{m}\}_d!}}{(\mathbf{m}+\mathbf{n})!}\left(\frac{d}{dr}\right)^{\mathbf{m}+\mathbf{n}}\int_0^{2\pi}\frac{d\theta}{2\pi}e^{\imath(\mathbf{m}-\mathbf{n})\theta}\mathscr{N}(|\mathbf{z}|^2)|re^{\imath\theta}\rangle\langle re^{\imath\theta}|\Big|_{r=0}. \tag{5.11}$$

Using the definition (5.1), the matrix elements of $\rho_{[\mathbf{k}]}$ computed with two NCS is found to be

$$\langle\eta_{\mathbf{z}'}|\rho_{[\mathbf{k}]}|\eta_{\mathbf{z}}\rangle = \left[\mathscr{N}(|\mathbf{z}'|^2)\mathscr{N}(|\mathbf{z}|^2)\right]^{-1/2}\sum_{\mathbf{m},\mathbf{n}=0}^{\infty}\rho_{[\mathbf{k}]}(\mathbf{n},\mathbf{m})\frac{\bar{\mathbf{z}}'^{\mathbf{n}}}{\sqrt{\{\mathbf{n}\}!}}\frac{\mathbf{z}^{\mathbf{m}}}{\sqrt{\{\mathbf{m}\}!}} \tag{5.12}$$

which corresponds at $\mathbf{z}' = \mathbf{z}$, to the diagonal element

$$\langle\eta_{\mathbf{z}}|\rho_{[\mathbf{k}]}|\eta_{\mathbf{z}}\rangle = \left(\mathscr{N}(|\mathbf{z}|^2)\right)^{-1}\sum_{\mathbf{m},\mathbf{n}=0}^{\infty}\rho_{[\mathbf{k}]}(\mathbf{n},\mathbf{m})\frac{\bar{\mathbf{z}}^{\mathbf{n}}}{\sqrt{\{\mathbf{n}\}!}}\frac{\mathbf{z}^{\mathbf{m}}}{\sqrt{\{\mathbf{m}\}!}}. \tag{5.13}$$

The quantity $\langle\eta_{\mathbf{z}}|\rho_{[\mathbf{k}]}|\eta_{\mathbf{z}}\rangle$ is analog to the "semi-classical "phase space distribution function $\mu(x,p) = \langle z|\rho|z\rangle$ associated to the density matrix ρ (here $\rho_{[\mathbf{k}]}$) of the system which is normalized as $\int(dxdp/2\pi\hbar)\mu(x,p) = 1$, and often referred to as the Husimi distribution [15]. Remark that an immediate issue of the representations (5.12) and (5.13) with respect to the results of Section 4, will be evidenced in the use of (5.12) when discussing the reproducing kernel insights of the density matrix.

In terms of the NCS, the density matrix is given by

$$\rho_{[\mathbf{k}]} = \int_{\mathscr{D}}d\mu(\mathbf{z},\bar{\mathbf{z}})\mathscr{N}(|\mathbf{z}|^2)P(|\mathbf{z}|^2)\,|\eta_{\mathbf{z}}\rangle\langle\eta_{\mathbf{z}}|, \tag{5.14}$$

where $P(|\mathbf{z}|^2)$ satisfies the normalization condition

$$\int_{\mathscr{D}} d\mu(\mathbf{z},\bar{\mathbf{z}})P(|\mathbf{z}|^2) = 1. \tag{5.15}$$

Note that (5.14) is exploited to derive the quantum-statistical or thermal averages for an operator as shown in [5] (see also references therein).

Then,

$$\langle\Phi_{\mathbf{n}}^{[\mathbf{k}]}|\rho_{[\mathbf{k}]}|\Phi_{\mathbf{n}}^{[\mathbf{k}]}\rangle = \int_{\mathscr{D}} d\mu(\mathbf{z},\bar{\mathbf{z}})\mathscr{N}(|\mathbf{z}|^2)P(|\mathbf{z}|^2)\,|\langle\Phi_{\mathbf{n}}^{[\mathbf{k}]}|\eta_{\mathbf{z}}\rangle|^2 \tag{5.16}$$

where the quantity $|\langle\Phi_{\mathbf{n}}^{[\mathbf{k}]}|\eta_{\mathbf{z}}\rangle|^2$ calculated by use of (2.57) as follows:

$$\begin{aligned}
|\langle\Phi_{\mathbf{n}}^{[\mathbf{k}]}|\eta_{\mathbf{z}}\rangle|^2 &= \langle\Phi_{\mathbf{n}}^{[\mathbf{k}]}|\eta_{\mathbf{z}}\rangle\langle\eta_{\mathbf{z}}|\Phi_{\mathbf{n}}^{[\mathbf{k}]}\rangle \\
&= \left\{\langle\Phi_{\mathbf{n}}^{[\mathbf{k}]}|\left(\mathscr{N}(|\mathbf{z}|^2)\right)^{-1/2}\sum_{\mathbf{j}=0}^{\infty}\frac{\mathbf{z}^{\mathbf{j}}}{\sqrt{\{\mathbf{j}\}!}}|\Phi_{\mathbf{j}}^{[\mathbf{k}]}\rangle\right\} \\
&\quad\times\left\{\left(\mathscr{N}(|\mathbf{z}|^2)\right)^{-1/2}\sum_{\mathbf{p}=0}^{\infty}\langle\Phi_{\mathbf{p}}^{[\mathbf{k}]}|\frac{\bar{\mathbf{z}}^{\mathbf{p}}}{\sqrt{\{\mathbf{p}\}!}}|\Phi_{\mathbf{n}}^{[\mathbf{k}]}\rangle\right\} \\
&= \left\{\left(\mathscr{N}(|\mathbf{z}|^2)\right)^{-1/2}\frac{\mathbf{z}^{\mathbf{n}}}{\sqrt{\{\mathbf{n}\}!}}\right\}\times\left\{\left(\mathscr{N}(|\mathbf{z}|^2)\right)^{-1/2}\frac{\bar{\mathbf{z}}^{\mathbf{n}}}{\sqrt{\{\mathbf{n}\}!}}\right\} \\
&= \left(\mathscr{N}(|\mathbf{z}|^2)\right)^{-1}\frac{|\mathbf{z}|^{2\mathbf{n}}}{\{\mathbf{n}\}!}
\end{aligned} \tag{5.17}$$

corresponds to a f-deformed photon number equal to

$$\mathscr{P}(\mathbf{n},\mathbf{z}) = e^{-|\mathbf{z}|^2}\frac{|\mathbf{z}|^{2\mathbf{n}}}{\mathbf{n}!}, \tag{5.18}$$

when $f(\mathbf{n}) \to 1$, which is a Poisson distribution. Therefore, we say that the NCS obey a f-Poisson distribution.

In the VCS formalism, we write that

$$\hat{\rho} = \sum_{[\mathbf{k}]\in\Gamma}\int_{\mathscr{D}^{2M}} d\mu(\mathfrak{z},\bar{\mathfrak{z}})\mathscr{N}(|\mathbf{z}_{[\mathbf{k}]}|^2)P(|\mathbf{z}_{[\mathbf{k}]}|^2)|\eta_{\mathfrak{z}};[\mathbf{k}]\rangle\langle\eta_{\mathfrak{z}};[\mathbf{k}]| \tag{5.19}$$

where

$$|\eta_{\mathfrak{z}};[\mathbf{k}]\rangle = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ |\eta_{\mathbf{z}_{[\mathbf{k}]}}\rangle \\ 0 \\ \vdots \\ 0 \end{pmatrix} \tag{5.20}$$

coincides, in the special case of $f(N_i) = 1\,(1 \le i \le N_B)$, with the VCS developed in [4], and

$$|\eta_{\mathbf{z}_{[\mathbf{k}]}}\rangle =: \left(\mathscr{N}(|\mathbf{z}_{[\mathbf{k}]}|^2)\right)^{-1/2}\sum_{\mathbf{n}=0}^{\infty}\frac{\mathbf{z}_{[\mathbf{k}]}^{\mathbf{n}}}{\sqrt{\{\mathbf{n}\}!}}|\Psi_{[\mathbf{k}]}\otimes\Phi_{[\mathbf{n}]}^{[\mathbf{k}]}\rangle \tag{5.21}$$

such that, with respect to the measure $d\mu(\mathfrak{z},\bar{\mathfrak{z}}) = \prod_{[\mathbf{k}]\in\Gamma} d\mu(\mathbf{z}_{[\mathbf{k}]},\bar{\mathbf{z}}_{[\mathbf{k}]})$, the resolution of the identity provided as in [4],

$$\begin{aligned}\sum_{[\mathbf{k}]\in\Gamma}\int_{\mathscr{D}^{2^M}} d\mu(\mathfrak{z},\bar{\mathfrak{z}})\mathscr{N}(|\mathbf{z}_{[\mathbf{k}]}|^2)|\eta_{\mathfrak{z}};[\mathbf{k}]\rangle\langle\eta_{\mathfrak{z}};[\mathbf{k}]| &= \sum_{[\mathbf{k}]\in\Gamma}\sum_{[\mathbf{n}]=0}^{\infty}|\Psi_{[\mathbf{k}]}\otimes\Phi_{[\mathbf{n}]}^{[\mathbf{k}]}\rangle\langle\Psi_{[\mathbf{k}]}\otimes\Phi_{[\mathbf{n}]}^{[\mathbf{k}]}| \\ &= \mathbb{I}_{2^M}\otimes I_{\mathfrak{H}},\end{aligned} \tag{5.22}$$

is satisfied.

Keeping in mind the following expression

$$|\eta_{\mathbf{z}}\rangle\langle\eta_{\mathbf{z}}| = \left(\mathscr{N}(|\mathbf{z}|^2)\right)^{-1}\sum_{\mathbf{j},\mathbf{p}=0}^{\infty}\frac{\mathbf{z}^{\mathbf{j}}}{\sqrt{\{\mathbf{j}\}!}}\frac{\bar{\mathbf{z}}^{\mathbf{p}}}{\sqrt{\{\mathbf{p}\}!}}|\Phi_{\mathbf{j}}^{[\mathbf{k}]}\rangle\langle\Phi_{\mathbf{p}}^{[\mathbf{k}]}| \tag{5.23}$$

and using the relation (5.14), the density matrix can be explicitly computed as

$$\rho_{[\mathbf{k}]} = \sum_{\mathbf{j},\mathbf{p}=0}^{\infty}\int_{\mathscr{D}} d\mu(\mathbf{z},\bar{\mathbf{z}})P(|\mathbf{z}|^2)\left\{\frac{\mathbf{z}^{\mathbf{j}}}{\sqrt{\{\mathbf{j}\}!}}\frac{\bar{\mathbf{z}}^{\mathbf{p}}}{\sqrt{\{\mathbf{p}\}!}}|\Phi_{\mathbf{j}}^{[\mathbf{k}]}\rangle\langle\Phi_{\mathbf{p}}^{[\mathbf{k}]}|\right\}. \tag{5.24}$$

From (5.1) and (5.24) together, it follows that

$$\langle\Phi_{\mathbf{n}}^{[\mathbf{k}]}|\rho_{[\mathbf{k}]}|\Phi_{\mathbf{n}}^{[\mathbf{k}]}\rangle := \rho_{[\mathbf{k}]}(\mathbf{n},\mathbf{n}) = \int_{\mathscr{D}} d\mu(\mathbf{z},\bar{\mathbf{z}})P(|\mathbf{z}|^2)\,\frac{|\mathbf{z}|^{2\mathbf{n}}}{\{\mathbf{n}\}!}, \tag{5.25}$$

and the matrix elements of the density operator evaluated in the bosonic states $|\Phi_{\mathbf{n}}^{[\mathbf{k}]}\rangle$ and $|\Phi_{\mathbf{m}}^{[\mathbf{k}]}\rangle$ are provided by the expression

$$\langle\Phi_{\mathbf{n}}^{[\mathbf{k}]}|\rho_{[\mathbf{k}]}|\Phi_{\mathbf{m}}^{[\mathbf{k}]}\rangle := \rho_{[\mathbf{k}]}(\mathbf{n},\mathbf{m}) = \int_{\mathscr{D}} d\mu(\mathbf{z},\bar{\mathbf{z}})P(|\mathbf{z}|^2)\left\{\frac{\mathbf{z}^{\mathbf{n}}}{\sqrt{\{\mathbf{n}\}!}}\frac{\bar{\mathbf{z}}^{\mathbf{m}}}{\sqrt{\{\mathbf{m}\}!}}\right\} \tag{5.26}$$

which reduces to (5.25) at $\mathbf{m} = \mathbf{n}$.

Taking

$$\rho_{[\mathbf{k}]} = \int_{\mathscr{D}} d\mu(\mathbf{z},\bar{\mathbf{z}})P(|\mathbf{z}|^2)\,|\eta_{\mathbf{z}}\rangle\langle\eta_{\mathbf{z}}| \tag{5.27}$$

instead of the relation (5.14), giving explicitly

$$\rho_{[\mathbf{k}]} = \int_{\mathscr{D}} d\mu(\mathbf{z},\bar{\mathbf{z}})P(|\mathbf{z}|^2)\,|\eta_{\mathbf{z}}\rangle\langle\eta_{\mathbf{z}}|$$

$$= \int_0^L \int_0^{2\pi} d\lambda(r) d\theta P(|\mathbf{z}|^2) \left\{ \left(\mathscr{N}(|\mathbf{z}|^2)\right)^{-1} \sum_{\mathbf{j},\mathbf{p}=0}^{\infty} \frac{\mathbf{z}^{\mathbf{j}}}{\sqrt{\{\mathbf{j}\}!}} \frac{\bar{\mathbf{z}}^{\mathbf{p}}}{\sqrt{\{\mathbf{p}\}!}} |\Phi_{\mathbf{j}}^{[\mathbf{k}]}\rangle\langle\Phi_{\mathbf{p}}^{[\mathbf{k}]}| \right\}$$
$$= \sum_{\mathbf{j},\mathbf{p}=0}^{\infty} 2\pi \int_0^L r^{(\mathbf{j}+\mathbf{p})} d\lambda(r) P(r^2) \left(\mathscr{N}(r^2)\right)^{-1} \frac{|\Phi_{\mathbf{j}}^{[\mathbf{k}]}\rangle\langle\Phi_{\mathbf{p}}^{[\mathbf{k}]}|}{\sqrt{\{\mathbf{j}\}!\{\mathbf{p}\}!}} \delta_{\mathbf{jp}} \tag{5.28}$$

and using (5.1) and (5.24) providing

$$\langle\Phi_{\mathbf{n}}^{[\mathbf{k}]}|\rho_{[\mathbf{k}]}|\Phi_{\mathbf{m}}^{[\mathbf{k}]}\rangle := \rho_{[\mathbf{k}]}(\mathbf{n},\mathbf{m})$$
$$= \langle\Phi_{\mathbf{n}}^{[\mathbf{k}]}| \left\{ \sum_{\mathbf{j},\mathbf{p}=0}^{\infty} 2\pi \int_0^L r^{(\mathbf{j}+\mathbf{p})} d\lambda(r) P(r^2) \left(\mathscr{N}(r^2)\right)^{-1} \frac{|\Phi_{\mathbf{j}}^{[\mathbf{k}]}\rangle\langle\Phi_{\mathbf{p}}^{[\mathbf{k}]}|}{\sqrt{\{\mathbf{j}\}!\{\mathbf{p}\}!}} \delta_{\mathbf{jp}} \right\} |\Phi_{\mathbf{m}}^{[\mathbf{k}]}\rangle$$
$$= 2\pi \int_0^L r^{(\mathbf{n}+\mathbf{m})} d\lambda(r) P(r^2) \left(\mathscr{N}(r^2)\right)^{-1} \left[\frac{\delta_{\mathbf{nm}}}{\sqrt{\{\mathbf{n}\}!\{\mathbf{m}\}!}} \right], \tag{5.29}$$

the expression (5.25) can be rewritten as

$$\langle\Phi_{\mathbf{n}}^{[\mathbf{k}]}|\rho_{[\mathbf{k}]}|\Phi_{\mathbf{n}}^{[\mathbf{k}]}\rangle := \rho_{[\mathbf{k}]}(\mathbf{n},\mathbf{n}) = \int_{\mathscr{D}} d\mu(\mathbf{z},\bar{\mathbf{z}}) P(|\mathbf{z}|^2) \left(\mathscr{N}(|\mathbf{z}|^2)\right)^{-1} \frac{|\mathbf{z}|^{2\mathbf{n}}}{\{\mathbf{n}\}!}. \tag{5.30}$$

Therefore, by using (5.15), we obtain

$$\sum_{\mathbf{n}=0}^{\infty} \langle\Phi_{\mathbf{n}}^{[\mathbf{k}]}|\rho_{[\mathbf{k}]}|\Phi_{\mathbf{n}}^{[\mathbf{k}]}\rangle = \int_{\mathscr{D}} d\mu(\mathbf{z},\bar{\mathbf{z}}) P(|\mathbf{z}|^2) \left\{ \left(\mathscr{N}(|\mathbf{z}|^2)\right)^{-1} \sum_{\mathbf{n}=0}^{\infty} \frac{|\mathbf{z}|^{2\mathbf{n}}}{\{\mathbf{n}\}!} \right\} = 1 \tag{5.31}$$

which is in accordance with the density matrix trace condition $\mathrm{Tr}\rho_{[\mathbf{k}]} := \sum_{\mathbf{n}=0}^{\infty}\langle\Phi_{\mathbf{n}}^{[\mathbf{k}]}|\rho_{[\mathbf{k}]}|\Phi_{\mathbf{n}}^{[\mathbf{k}]}\rangle = 1$.

From the resolution of the identity and (5.12), we derive the reproducing kernel property of the density matrix (see [24]) by setting $\langle\eta_{\mathbf{z}'}|\rho_{[\mathbf{k}]}|\eta_{\mathbf{z}}\rangle := \rho_{[\mathbf{k}]}(\mathbf{z}',\mathbf{z})$, i.e.

$$\rho_{[\mathbf{k}]}(\mathbf{z}',\mathbf{z}) = \langle\eta_{\mathbf{z}'}| \int_{\mathscr{D}} d\mu(\mathbf{z}'',\bar{\mathbf{z}}'') |\eta_{\mathbf{z}''}\rangle\langle\eta_{\mathbf{z}''}|\rho_{[\mathbf{k}]}|\eta_{\mathbf{z}}\rangle$$
$$= \int_{\mathscr{D}} d\mu(\mathbf{z}'',\bar{\mathbf{z}}'') \mathscr{K}(\mathbf{z}'',\mathbf{z}') \rho_{[\mathbf{k}]}(\mathbf{z}'',\mathbf{z}). \tag{5.32}$$

By making use of the Proposition 4.1 and relation (5.32), we infer here that the density matrix also displays the self-reproducing kernel properties as provided below:

Proposition 5.1.

The following properties are satisfied by $\rho_{[\mathbf{k}]}$ on the Hilbert space $\mathfrak{H}$:

(i) Hermiticity

$$\rho_{[\mathbf{k}]}(\mathbf{z},\mathbf{z}') = \overline{\rho_{[\mathbf{k}]}(\mathbf{z}',\mathbf{z})}; \tag{5.33}$$

(ii) Positivity

$$\rho_{[\mathbf{k}]}(\mathbf{z},\mathbf{z}) > 0; \tag{5.34}$$

(iii) Idempotence

$$\int_{\mathscr{D}} d\mu(\mathbf{z}'',\bar{\mathbf{z}''})\rho_{[\mathbf{k}]}(\mathbf{z},\mathbf{z}'')\rho_{[\mathbf{k}]}(\mathbf{z}'',\mathbf{z}') = \rho_{[\mathbf{k}]}(\mathbf{z},\mathbf{z}'). \tag{5.35}$$

Proof. See Appendix.

□

Coming back to (5.19), setting $\mathbf{z}^{\mathbf{n}} := \mathbf{J}^{\mathbf{n}/2}e^{-\imath\varepsilon_{[\mathbf{n}]}\gamma}$, and with

$$|\eta_{\mathbf{z}_{[\mathbf{k}]}}\rangle\langle\eta_{\mathbf{z}_{[\mathbf{k}]}}| = \left(\mathscr{N}(|\mathbf{z}_{[\mathbf{k}]}|^2)\right)^{-1}\sum_{[\mathbf{m}],[\mathbf{n}]=0}^{\infty}\frac{\mathbf{z}_{[\mathbf{k}]}^{\mathbf{n}}}{\sqrt{\{\mathbf{n}\}!}}\frac{\bar{\mathbf{z}}_{[\mathbf{k}]}^{\mathbf{m}}}{\sqrt{\{\mathbf{m}\}!}}|\Psi_{[\mathbf{k}]}\otimes\Phi_{[\mathbf{n}]}^{[\mathbf{k}]}\rangle\langle\Psi_{[\mathbf{k}]}\otimes\Phi_{[\mathbf{m}]}^{[\mathbf{k}]}| \tag{5.36}$$

$d\mu(\mathfrak{Z},\bar{\mathfrak{Z}}) = d\mu(\gamma)d\nu(\mathfrak{J})$ where $d\mu(\gamma) = \prod_{[\mathbf{k}]\in\Gamma}d\mu(\gamma_{[\mathbf{k}]})$, $d\nu(\mathfrak{J}) = \prod_{[\mathbf{k}]\in\Gamma}d\nu(\mathbf{J}_{[\mathbf{k}]})$, and taking $\mathscr{D} = ([0,\infty)\times[0,2\pi))^{N_B}$, we get

$$\begin{aligned}\hat{\rho} = &\sum_{[\mathbf{k}]\in\Gamma}\sum_{[\mathbf{j}],[\mathbf{q}]=0}^{\infty}\int_{\mathscr{D}} d\mu(\gamma_{(0,\cdots,0,0)})d\nu_{(0,\cdots,0,0)}(\mathbf{J}_{(0,\cdots,0,0)})\int_{\mathscr{D}} d\mu(\gamma_{(1,\cdots,0,0)})d\nu_{(1,\cdots,0,0)}(\mathbf{J}_{(1,\cdots,0,0)})\cdots\\ &\int_{\mathscr{D}} d\mu(\gamma_{[\mathbf{k}]})d\nu_{[\mathbf{k}]}(\mathbf{J}_{[\mathbf{k}]})\cdots\int_{\mathscr{D}} d\mu(\gamma_{(1,\cdots,1,1)})d\nu_{(1,\cdots,1,1)}(\mathbf{J}_{(1,\cdots,1,1)})\times\\ &diag\left(\frac{\mathbf{J}_{(0,0,\ldots,0,0)}^{(\mathbf{j}+\mathbf{q})/2}e^{-\imath(\varepsilon_{[\mathbf{j}]}-\varepsilon_{[\mathbf{q}]})\gamma_{(0,0,\ldots,0,0)}}P(\mathbf{J}_{(0,0,\ldots,0,0)})}{\sqrt{\{\mathbf{j}\}!\{\mathbf{q}\}!}},\frac{\mathbf{J}_{(1,0,\ldots0,0)}^{(\mathbf{j}+\mathbf{q})/2}e^{-\imath(\varepsilon_{[\mathbf{j}]}-\varepsilon_{[\mathbf{q}]})\gamma_{(1,0,\ldots,0,0)}}P(\mathbf{J}_{(1,0,\ldots0,0)})}{\sqrt{\{\mathbf{j}\}!\{\mathbf{q}\}!}}\right.\\ &\left.\cdots,\frac{\mathbf{J}_{[\mathbf{k}]}^{(\mathbf{j}+\mathbf{q})/2}e^{-\imath(\varepsilon_{[\mathbf{j}]}-\varepsilon_{[\mathbf{q}]})\gamma_{[\mathbf{k}]}}P(\mathbf{J}_{[\mathbf{k}]})}{\sqrt{\{\mathbf{j}\}!\{\mathbf{q}\}!}},\cdots,\frac{\mathbf{J}_{(1,1,\ldots,1,1)}^{(\mathbf{j}+\mathbf{q})/2}e^{-\imath(\varepsilon_{[\mathbf{j}]}-\varepsilon_{[\mathbf{q}]})\gamma_{(1,1,\ldots,1,1)}}P(\mathbf{J}_{(1,1,\ldots,1,1)})}{\sqrt{\{\mathbf{j}\}!\{\mathbf{q}\}!}}\right)\\ &\times|\Psi_{[\mathbf{k}]}\otimes\Phi_{[\mathbf{j}]}^{[\mathbf{k}]}\rangle\langle\Psi_{[\mathbf{k}]}\otimes\Phi_{[\mathbf{q}]}^{[\mathbf{k}]}|,\end{aligned} \tag{5.37}$$

this latter being the density matrix representation in the NVCS basis. Then, (5.37) allows to derive the mean value of a diagonal operator $\mathscr{O}$ expressed in terms of 2^M operators, $\mathscr{O}(0,0,...,0,...,0,0),\mathscr{O}(1,0,...,0,...,0,0),...,\mathscr{O}(1,1,...,1,...,1,1)$, acting on corresponding subspaces of the Hilbert space $\mathbb{C}^{2^M}\otimes\mathfrak{H}$.

Therefore, the density operator matrix elements can be expressed in this case as follows:

$$\begin{aligned}&\langle\Psi_{[\mathbf{k}]}\otimes\Phi_{[\mathbf{n}]}^{[\mathbf{k}]}|\hat{\rho}|\Psi_{[\mathbf{k}]}\otimes\Phi_{[\mathbf{p}]}^{[\mathbf{k}]}\rangle\\ &= (2\pi)^{N_B}\int_0^{\infty}\int_0^{\infty}\cdots\int_0^{\infty}d\nu_{(0,\cdots,0,0)}(\mathbf{J}_{(0,\cdots,0,0)})\int_0^{\infty}\int_0^{\infty}\cdots\int_0^{\infty}d\nu_{(1,\cdots,0,0)}(\mathbf{J}_{(1,\cdots,0,0)})\cdots\\ &\int_0^{\infty}\int_0^{\infty}\cdots\int_0^{\infty}d\nu_{[\mathbf{k}]}(\mathbf{J}_{[\mathbf{k}]})\cdots\int_0^{\infty}\int_0^{\infty}\cdots\int_0^{\infty}d\nu_{(1,\cdots,1,1)}(\mathbf{J}_{(1,\cdots,1,1)})\times\\ &diag\left(\frac{\mathbf{J}_{(0,0,\ldots,0,0)}^{(\mathbf{n}+\mathbf{p})/2}}{\sqrt{\{\mathbf{n}\}!\{\mathbf{p}\}!}}P(\mathbf{J}_{(0,0,\ldots,0,0)}),\frac{\mathbf{J}_{(1,0,\ldots0,0)}^{(\mathbf{n}+\mathbf{p})/2}}{\sqrt{\{\mathbf{n}\}!\{\mathbf{p}\}!}}P(\mathbf{J}_{(1,0,\ldots0,0)})\cdots,\frac{\mathbf{J}_{[\mathbf{k}]}^{(\mathbf{n}+\mathbf{p})/2}}{\sqrt{\{\mathbf{n}\}!\{\mathbf{p}\}!}}P(\mathbf{J}_{[\mathbf{k}]}),\cdots,\right.\\ &\left.\frac{\mathbf{J}_{(1,1,\ldots,1,1)}^{(\mathbf{n}+\mathbf{p})/2}}{\sqrt{\{\mathbf{n}\}!\{\mathbf{p}\}!}}P(\mathbf{J}_{(1,1,\ldots,1,1)})\right)\delta_{[\mathbf{n}],[\mathbf{p}]}\end{aligned} \tag{5.38}$$

where the following identity

$$\int_0^{2\pi}\int_0^{2\pi}\cdots\int_0^{2\pi} d\mu(\gamma_{[\mathbf{k}]}) e^{-\imath(\varepsilon_{[\mathbf{n}]}-\varepsilon_{[\mathbf{p}]})\gamma_{[\mathbf{k}]}} = \begin{cases} 0 \quad \text{if} \quad [\mathbf{n}] \neq [\mathbf{p}], \\ (2\pi)^{N_B} \quad \text{if} \quad [\mathbf{n}] = [\mathbf{p}] \end{cases} \tag{5.39}$$

is used.

This is the matrix formulation of the density matrix in comparison with the quantity obtained in the case of bosonic states described in (5.26).

6. NCS quantization of the complex plane

Provided the resolution of the identity satisfied by the NCS, in this section we perform the NCS quantization of phase space classical observables. As a matter of illustration, the case of q-deformed CS is explicitly treated.

6.1. *General construction*

Such a quantization (see [13]) is realized through the mappings:

(i)

$$A_{\mathbf{z}} : \mathscr{D} \to \mathfrak{H}, \qquad \mathbf{z} \mapsto \int_{\mathscr{D}} \mathbf{z}|\eta_{\mathbf{z}}\rangle\langle\eta_{\mathbf{z}}|\mathscr{N}(|\mathbf{z}|^2) d\mu(\mathbf{z},\bar{\mathbf{z}}) = \sum_{\mathbf{n}=0}^{\infty} \sqrt{(\mathbf{n}+1)} f(\mathbf{n}+1) |\Phi_{\mathbf{n}}^{[\mathbf{k}]}\rangle\langle\Phi_{\mathbf{n}+1}^{[\mathbf{k}]}|, \tag{6.1}$$

(ii)

$$A_{\bar{\mathbf{z}}} : \mathscr{D} \to \mathfrak{H}, \qquad \mathbf{z} \mapsto \int_{\mathscr{D}} \bar{\mathbf{z}}|\eta_{\mathbf{z}}\rangle\langle\eta_{\mathbf{z}}|\mathscr{N}(|\mathbf{z}|^2) d\mu(\mathbf{z},\bar{\mathbf{z}}) = \sum_{\mathbf{n}=0}^{\infty} \sqrt{\mathbf{n}} f(\mathbf{n}) |\Phi_{\mathbf{n}}^{[\mathbf{k}]}\rangle\langle\Phi_{\mathbf{n}-1}^{[\mathbf{k}]}|, \tag{6.2}$$

(iii)

$$A'_{\mathbf{z}} : \tilde{\mathscr{D}} \to \mathfrak{H}, \qquad \mathbf{z} \mapsto \int_{\tilde{\mathscr{D}}} \mathbf{z}|\eta'_{\mathbf{z}}\rangle\langle\eta'_{\mathbf{z}}|\mathscr{N}'(|\mathbf{z}|^2) d\mu(\mathbf{z},\bar{\mathbf{z}}) = \sum_{\mathbf{n}=0}^{\infty} \frac{\sqrt{(\mathbf{n}+1)}}{f(\mathbf{n}+1)} |\Phi_{\mathbf{n}}^{[\mathbf{k}]}\rangle\langle\Phi_{\mathbf{n}+1}^{[\mathbf{k}]}|, \tag{6.3}$$

(iv)

$$A'_{\bar{\mathbf{z}}} : \tilde{\mathscr{D}} \to \mathfrak{H}, \qquad \mathbf{z} \mapsto \int_{\tilde{\mathscr{D}}} \bar{\mathbf{z}}|\eta'_{\mathbf{z}}\rangle\langle\eta'_{\mathbf{z}}|\mathscr{N}'(|\mathbf{z}|^2) d\mu(\mathbf{z},\bar{\mathbf{z}}) = \sum_{\mathbf{n}=0}^{\infty} \frac{\sqrt{\mathbf{n}}}{f(\mathbf{n})} |\Phi_{\mathbf{n}}^{[\mathbf{k}]}\rangle\langle\Phi_{\mathbf{n}-1}^{[\mathbf{k}]}|. \tag{6.4}$$

The defined NCS can be used to calculate the expectation (mean) values of quantized elementary classical observables. For instance, the following mean values are computed:

$$\langle \eta_{\mathbf{z}}|A_{\mathbf{z}}|\eta_{\mathbf{z}}\rangle = \mathbf{z}, \qquad \langle \eta_{\mathbf{z}}|A_{\bar{\mathbf{z}}}|\eta_{\bar{\mathbf{z}}}\rangle = \bar{\mathbf{z}}, \qquad \langle \eta_{\mathbf{z}}|A^2_{\mathbf{z}}|\eta_{\mathbf{z}}\rangle = \mathbf{z}^2, \qquad \langle \eta_{\mathbf{z}}|A^2_{\bar{\mathbf{z}}}|\eta_{\mathbf{z}}\rangle = \bar{\mathbf{z}}^2, \tag{6.5}$$

$$\langle \eta_{\mathbf{z}}|A_{\bar{\mathbf{z}}}A_{\mathbf{z}}|\eta_{\mathbf{z}}\rangle = |\mathbf{z}|^2 \tag{6.6}$$

$$\langle \eta_{\mathbf{z}}|A_{\mathbf{z}}A_{\bar{\mathbf{z}}}|\eta_{\mathbf{z}}\rangle = \left(\mathcal{N}(|\mathbf{z}|^2)\right)^{-1} \times \left[|\mathbf{z}|^2 \sum_{\mathbf{n}=1}^{\infty} \frac{[f(\mathbf{n}+1)]^2}{[f(\mathbf{n})]^2} \frac{|\mathbf{z}|^{2(\mathbf{n}-1)}}{(\mathbf{n}-1)![f(\mathbf{n}-1)!]^2} + \sum_{\mathbf{n}=0}^{\infty} \frac{[f(\mathbf{n}+1)]^2}{[f(\mathbf{n})!]^2} \frac{|\mathbf{z}|^{2\mathbf{n}}}{\mathbf{n}!}\right] \tag{6.7}$$

such that for $f(\mathbf{n}) \to 1$ with $f(\mathbf{n}+1) = 1 = f(\mathbf{n})$, $f(\mathbf{n}+1)! = 1 = f(\mathbf{n})!$, we get

$$\langle \eta_{\mathbf{z}}|A_{\mathbf{z}}A_{\bar{\mathbf{z}}}|\eta_{\mathbf{z}}\rangle = |\mathbf{z}|^2 + \langle \eta_{\mathbf{z}}|\mathbb{I}|\eta_{\mathbf{z}}\rangle \tag{6.8}$$

as in the case of the standard CS. Moreover, from the quantities:

$$\begin{aligned} A_{|\mathbf{z}|^2} &= \sum_{\mathbf{n}=0}^{\infty} (\mathbf{n}+1)[f(\mathbf{n}+1)]^2 |\Phi^{[\mathbf{k}]}_{\mathbf{n}}\rangle\langle\Phi^{[\mathbf{k}]}_{\mathbf{n}}|, \\ A_{\mathbf{z}^2} &= \sum_{\mathbf{n}=0}^{\infty} \sqrt{(\mathbf{n}+2)(\mathbf{n}+1)} f(\mathbf{n}+2) f(\mathbf{n}+1) |\Phi^{[\mathbf{k}]}_{\mathbf{n}}\rangle\langle\Phi^{[\mathbf{k}]}_{\mathbf{n}+2}|, \\ A_{\bar{\mathbf{z}}^2} &= \sum_{\mathbf{n}=0}^{\infty} \sqrt{(\mathbf{n}+2)(\mathbf{n}+1)} f(\mathbf{n}+2) f(\mathbf{n}+1) |\Phi^{[\mathbf{k}]}_{\mathbf{n}+2}\rangle\langle\Phi^{[\mathbf{k}]}_{\mathbf{n}}|, \end{aligned} \tag{6.9}$$

we derive the commutators:

- $$[A_{|\mathbf{z}|^2}, A_{\mathbf{z}}] = \sum_{\mathbf{n}=0}^{\infty} \sqrt{(\mathbf{n}+1)} f(\mathbf{n}+1) \times \left\{(\mathbf{n}+1)[f(\mathbf{n}+1)]^2 - (\mathbf{n}+2)[f(\mathbf{n}+2)]^2\right\} |\Phi^{[\mathbf{k}]}_{\mathbf{n}}\rangle\langle\Phi^{[\mathbf{k}]}_{\mathbf{n}+1}|; \tag{6.10}$$

- $$[A_{|\mathbf{z}|^2}, A_{\bar{\mathbf{z}}}] = \sum_{\mathbf{n}=0}^{\infty} \sqrt{(\mathbf{n}+1)} f(\mathbf{n}+1) \times \left\{(\mathbf{n}+2)[f(\mathbf{n}+2)]^2 - (\mathbf{n}+1)[f(\mathbf{n}+1)]^2\right\} |\Phi^{[\mathbf{k}]}_{\mathbf{n}+1}\rangle\langle\Phi^{[\mathbf{k}]}_{\mathbf{n}}|; \tag{6.11}$$

- $$[A_{|\mathbf{z}|^2}, A_{\mathbf{z}^2}] = \sqrt{(\mathbf{n}+2)(\mathbf{n}+1)} f(\mathbf{n}+2) f(\mathbf{n}+1) \times \left\{(\mathbf{n}+1)[f(\mathbf{n}+1)]^2 - (\mathbf{n}+3)[f(\mathbf{n}+3)]^2\right\} |\Phi^{[\mathbf{k}]}_{\mathbf{n}}\rangle\langle\Phi^{[\mathbf{k}]}_{\mathbf{n}+2}|; \tag{6.12}$$

•

$$[A_{|\mathbf{z}|^2}, A_{\bar{\mathbf{z}}^2}] = \sqrt{(\mathbf{n}+2)(\mathbf{n}+1)} f(\mathbf{n}+2) f(\mathbf{n}+1) \times \left\{ (\mathbf{n}+3)[f(\mathbf{n}+3)]^2 - (\mathbf{n}+1)[f(\mathbf{n}+1)]^2 \right\} |\Phi^{[\mathbf{k}]}_{\mathbf{n}+2}\rangle\langle\Phi^{[\mathbf{k}]}_{\mathbf{n}}|. \tag{6.13}$$

Considering the usual phase space conjugate coordinates $(\mathbf{q},\mathbf{p})$ through $\mathbf{z} = \frac{\mathbf{q}+i\mathbf{p}}{\sqrt{2}}$, we get for the classical *position* function $\mathbf{q}$:

$$Q := A_{\mathbf{q}} = \frac{1}{\sqrt{2}}(A_{\mathbf{z}} + A_{\bar{\mathbf{z}}}), \tag{6.14}$$

and for the classical *momentum* function $\mathbf{p}$:

$$P := A_{\mathbf{p}} = \frac{1}{i\sqrt{2}}(A_{\mathbf{z}} - A_{\bar{\mathbf{z}}}). \tag{6.15}$$

The mean values of the operators Q, P and Q^2, P^2 can be written as follows:

$$\begin{aligned} \langle\eta_{\mathbf{z}}|Q|\eta_{\mathbf{z}}\rangle &= \frac{1}{\sqrt{2}}[\langle\eta_{\mathbf{z}}|A_{\mathbf{z}}|\eta_{\mathbf{z}}\rangle + \langle\eta_{\mathbf{z}}|A_{\bar{\mathbf{z}}}|\eta_{\mathbf{z}}\rangle] = \frac{1}{\sqrt{2}}(\mathbf{z}+\bar{\mathbf{z}}) = \mathbf{q}, \\ \langle\eta_{\mathbf{z}}|P|\eta_{\mathbf{z}}\rangle &= \frac{1}{i\sqrt{2}}[\langle\eta_{\mathbf{z}}|A_{\mathbf{z}}|\eta_{\mathbf{z}}\rangle - \langle\eta_{\mathbf{z}}|A_{\bar{\mathbf{z}}}|\eta_{\mathbf{z}}\rangle] = \frac{1}{i\sqrt{2}}(\mathbf{z}-\bar{\mathbf{z}}) = \mathbf{p} \end{aligned} \tag{6.16}$$

$$\begin{aligned} \langle\eta_{\mathbf{z}}|Q^2|\eta_{\mathbf{z}}\rangle &= \frac{1}{2}\left[\langle\eta_{\mathbf{z}}|A^2_{\mathbf{z}}|\eta_{\mathbf{z}}\rangle + \langle\eta_{\mathbf{z}}|A^2_{\bar{\mathbf{z}}}|\eta_{\mathbf{z}}\rangle + \langle\eta_{\mathbf{z}}|\{A_{\mathbf{z}}, A_{\bar{\mathbf{z}}}\}|\eta_{\mathbf{z}}\rangle\right] \\ \langle\eta_{\mathbf{z}}|P^2|\eta_{\mathbf{z}}\rangle &= -\frac{1}{2}\left[\langle\eta_{\mathbf{z}}|A^2_{\mathbf{z}}|\eta_{\mathbf{z}}\rangle + \langle\eta_{\mathbf{z}}|A^2_{\bar{\mathbf{z}}}|\eta_{\mathbf{z}}\rangle - \langle\eta_{\mathbf{z}}|\{A_{\mathbf{z}}, A_{\bar{\mathbf{z}}}\}|\eta_{\mathbf{z}}\rangle\right] \end{aligned} \tag{6.17}$$

with the anti-commutator of two operators $\mathscr{A}$ and $\mathscr{B}$ given by $\{\mathscr{A},\mathscr{B}\} := \mathscr{A}\mathscr{B} + \mathscr{B}\mathscr{A}$, where

$$\langle\eta_{\mathbf{z}}|\{A_{\mathbf{z}}, A_{\bar{\mathbf{z}}}\}|\eta_{\mathbf{z}}\rangle = \left(\mathscr{N}(|\mathbf{z}|^2)\right)^{-1} \times \left[\sum_{\mathbf{m}=0}^{\infty} \left\{(\mathbf{m}+1)[f(\mathbf{m}+1)]^2 + \mathbf{m}[f(\mathbf{m})]^2\right\} \times \frac{|\mathbf{z}|^{2\mathbf{m}}}{\{\mathbf{m}\}!}\right]. \tag{6.18}$$

Using (6.5)-(6.7), the expressions (6.17) are obtained as

$$\langle\eta_{\mathbf{z}}|Q^2|\eta_{\mathbf{z}}\rangle = \frac{1}{2}\left[\mathbf{q}^2 - \mathbf{p}^2 + \langle\eta_{\mathbf{z}}|\{A_{\mathbf{z}}, A_{\bar{\mathbf{z}}}\}|\eta_{\mathbf{z}}\rangle\right] \tag{6.19}$$

$$\langle\eta_{\mathbf{z}}|P^2|\eta_{\mathbf{z}}\rangle = -\frac{1}{2}\left[\mathbf{q}^2 - \mathbf{p}^2 - \langle\eta_{\mathbf{z}}|\{A_{\mathbf{z}}, A_{\bar{\mathbf{z}}}\}|\eta_{\mathbf{z}}\rangle\right] \tag{6.20}$$

with $\langle\eta_{\mathbf{z}}|\{A_{\mathbf{z}}, A_{\bar{\mathbf{z}}}\}|\eta_{\mathbf{z}}\rangle$ given in (6.18).

Then, with $\{\mathbf{m}\} := \mathbf{m}[f(\mathbf{m})]^2$, we get the dispersions:

$$(\Delta Q)^2 = \langle Q^2\rangle - (\langle Q\rangle)^2 = -|\mathbf{z}|^2 + \frac{1}{2}\left(\mathscr{N}(|\mathbf{z}|^2)\right)^{-1}\left[\sum_{\mathbf{m}=0}^{\infty}[\{\mathbf{m}+1\}+\{\mathbf{m}\}]\times\frac{|\mathbf{z}|^{2\mathbf{m}}}{\{\mathbf{m}\}!}\right]$$
$$= -\frac{1}{2}|\mathbf{z}|^2 + \mathscr{G}(|\mathbf{z}|^2) \tag{6.21}$$

$$(\Delta P)^2 = \langle P^2\rangle - (\langle P\rangle)^2 = -|\mathbf{z}|^2 + \frac{1}{2}\left(\mathscr{N}(|\mathbf{z}|^2)\right)^{-1}\left[\sum_{\mathbf{m}=0}^{\infty}[\{\mathbf{m}+1\}+\{\mathbf{m}\}]\times\frac{|\mathbf{z}|^{2\mathbf{m}}}{\{\mathbf{m}\}!}\right]$$
$$= -\frac{1}{2}|\mathbf{z}|^2 + \mathscr{G}(|\mathbf{z}|^2) \tag{6.22}$$

where

$$\mathscr{G}(|\mathbf{z}|^2) := \frac{1}{2}\left(\mathscr{N}(|\mathbf{z}|^2)\right)^{-1}\left[\sum_{\mathbf{m}=0}^{\infty}\{\mathbf{m}+1\}\times\frac{|\mathbf{z}|^{2\mathbf{m}}}{\{\mathbf{m}\}!}\right]. \tag{6.23}$$

Now, taking into account both (6.14) and (6.15), and by defining $\langle[Q,P]\rangle := \langle\eta_{\mathbf{z}}|[Q,P]|\eta_{\mathbf{z}}\rangle$, we get

$$\langle[Q,P]\rangle = i\left(\mathscr{N}(|\mathbf{z}|^2)\right)^{-1}\left[\sum_{\mathbf{m}=0}^{\infty}[\{\mathbf{m}+1\}-\{\mathbf{m}\}]\times\frac{|\mathbf{z}|^{2\mathbf{m}}}{\{\mathbf{m}\}!}\right] \tag{6.24}$$

which leads, by using (6.21)-(6.23), to

$$(\Delta Q)^2 = (\Delta P)^2 = \frac{1}{2}|\langle[Q,P]\rangle| \tag{6.25}$$

and, therefore, to the relation

$$(\Delta Q)^2(\Delta P)^2 = \frac{1}{4}|\langle[Q,P]\rangle|^2 \tag{6.26}$$

attesting that the NCS $|\eta_{\mathbf{z}}\rangle$ are intelligent, as in the standard situation, for the operators Q and P obtained from the CS quantization of the classical observables **q** and **p**, respectively.

6.2. *Illustration: case of Quesne's q-deformed CS [27]*

The Quesne's q-deformed CS [27] can be retrieved from the above generalization by setting $f(\mathbf{n}) = \sqrt{\frac{[\mathbf{n}]_q}{\mathbf{n}}}$, with $0 < q < 1$ and $\mathbf{z}\in\mathbb{C}$, and

$$|\eta_{\mathbf{z}}\rangle_q = \left(\mathscr{N}_q(|\mathbf{z}|^2)\right)^{-1/2}\sum_{\mathbf{n}=0}^{\infty}\frac{\mathbf{z}^{\mathbf{n}}}{\sqrt{[\mathbf{n}]_q!}}|\Phi_{\mathbf{n}}^{[\mathbf{k}]}\rangle_q, \qquad \mathscr{N}_q(|\mathbf{z}|^2) = \sum_{\mathbf{n}=0}^{\infty}\frac{|\mathbf{z}|^{2\mathbf{n}}}{[\mathbf{n}]_q!} = E_q(|\mathbf{z}|^2) \tag{6.27}$$

where the q-factorial

$$[\mathbf{n}]_q! \equiv \begin{cases} 1 & \text{if} \quad \mathbf{n}=0 \\ [\mathbf{n}]_q[\mathbf{n}-1]_q \ldots [1]_q & \text{if} \quad \mathbf{n}=1,2,3,\ldots \end{cases} \tag{6.28}$$

is defined in terms of the q-numbers

$$[\mathbf{n}]_q \equiv \frac{q^{\mathbf{n}}-1}{q-1} = 1+q+q^2+\cdots+q^{\mathbf{n}-1} \tag{6.29}$$

such that in the limit $q \to 1$, $[\mathbf{n}]_q, [\mathbf{n}]_q!$ tend to $\mathbf{n}$ and $\mathbf{n}!$, respectively.

In this case the deformation (2.3), with $f_l(N_l) = \sqrt{\frac{[N_l]_q}{N_l}}$, is obtained through the following correspondences:

$$a_l \to b_l = a_l \sqrt{\frac{[N_l]_q}{N_l}}, \qquad a_l^\dagger \to b_l^\dagger = \sqrt{\frac{[N_l]_q}{N_l}} a_l^\dagger, \qquad 1 \le l \le N_B \tag{6.30}$$

where $N_l = a_l^\dagger a_l$ and $[N_l]_q$ defined as in (6.29). From (6.30), it comes that $b_l^\dagger b_l = [N_l]_q$ and $b_l b_l^\dagger = [N_l+1]_q$. The actions of the operators $b_l, b_l^\dagger$ on the states $|\Phi_{n_l}^{[\mathbf{k}]}\rangle_q$ are given for $1 \le l \le N_B$ by

$$b_l|\Phi_{n_l}^{[\mathbf{k}]}\rangle_q = \sqrt{[n_l]_q}|\Phi_{n_l-1}^{[\mathbf{k}]}\rangle_q, \quad b_l^\dagger|\Phi_{n_l}^{[\mathbf{k}]}\rangle_q = \sqrt{[n_l+1]_q}|\Phi_{n_l+1}^{[\mathbf{k}]}\rangle_q, \quad b_l b_l^\dagger - q b_l^\dagger b_l = \mathbb{I} \tag{6.31}$$

where in the limit $q \to 1$, $[b_l, b_l^\dagger] = \mathbb{I}$.

The states $|\Phi_{\mathbf{n}}^{[\mathbf{k}]}\rangle_q = \bigotimes_{l=1}^{N_B} |\Phi_{n_l}^{[\mathbf{k}]}\rangle_q$ are such that the states $|\Phi_{n_l}^{[\mathbf{k}]}\rangle_q$ are obtained with (2.34) as

$$|\Phi_{n_l}^{[\mathbf{k}]}\rangle_q = \frac{(\mathscr{A}_{[\mathbf{k}]_l}^\dagger)^{n_l}}{\sqrt{[n_l]_q!}}|\Phi_{0_l}^{[\mathbf{k}]}\rangle_q, \quad \mathscr{A}_{[\mathbf{k}]_l} = e^{i\sqrt{2}\frac{g_{[\mathbf{k}]}}{\omega_l}\mathscr{P}_l} b_l e^{-i\sqrt{2}\frac{g_{[\mathbf{k}]}}{\omega_l}\mathscr{P}_l} \tag{6.32}$$

with $|\Phi_{0_l}^{[\mathbf{k}]}\rangle_q = e^{i\sqrt{2}\frac{g_{[\mathbf{k}]}}{\omega_l}\mathscr{P}_l}|0\rangle_q$, where we assume $|0\rangle_q = |0\rangle, (b_l|0\rangle_q = 0)$, and $\mathscr{P}_l$ with $f_l(N_l) = \sqrt{\frac{[N_l]_q}{N_l}}$ is given as in (2.33). The $|\Phi_{\mathbf{n}}^{[\mathbf{k}]}\rangle_q$ are identified with the states $|n\rangle_q$ of the Hilbert space $\mathscr{F}_q$ [27] which is the q-deformed Fock space associated with the q-boson creation and annihilation operators.

Provided the resolution of the identity on $\mathscr{F}_q$ spanned by the states $|\Phi_{\mathbf{n}}^{[\mathbf{k}]}\rangle_q$

$$\int\int_{\mathbb{C}} d^2\mathbf{z} |\eta_{\mathbf{z}}\rangle_q W_q(|\mathbf{z}|^2)\, {}_q\langle\eta_{\mathbf{z}}| = \sum_{\mathbf{n}=0}^{\infty} |\Phi_{\mathbf{n}}^{[\mathbf{k}]}\rangle_q\, {}_q\langle\Phi_{\mathbf{n}}^{[\mathbf{k}]}| = I_{\mathscr{F}_q} \tag{6.33}$$

where

$$W_q(|\mathbf{z}|^2) = \frac{q-1}{\pi \ln q} \frac{E_q(|\mathbf{z}|^2)}{E_q(q|\mathbf{z}|^2)}, \tag{6.34}$$

the CS quantization is performed as in (6.1)-(6.2) and leads to the q-deformed analogue of (6.18):

$$_q\langle\eta_{\mathbf{z}}|\,\{A_{\mathbf{z}},A_{\bar{\mathbf{z}}}\}\,|\eta_{\mathbf{z}}\rangle_q = \left(\mathcal{N}_q(|\mathbf{z}|^2)\right)^{-1}\left[\sum_{\mathbf{m}=0}^{\infty}\left\{[\mathbf{m}+1]_q+[\mathbf{m}]_q\right\}\times\frac{|\mathbf{z}|^{2\mathbf{m}}}{[\mathbf{m}]_q!}\right]. \tag{6.35}$$

The dispersions are given by

$$(\Delta Q)_q^2 = (\Delta P)_q^2 = -\frac{1}{2}|\mathbf{z}|^2 + \mathcal{G}_q(|\mathbf{z}|^2) \tag{6.36}$$

where

$$\mathcal{G}_q(|\mathbf{z}|^2) := \frac{1}{2}\left(\mathcal{N}_q(|\mathbf{z}|^2)\right)^{-1}\left[\sum_{\mathbf{m}=0}^{\infty}[\mathbf{m}+1]_q\frac{|\mathbf{z}|^{2\mathbf{m}}}{[\mathbf{m}]_q!}\right]. \tag{6.37}$$

Proceeding as in (6.21)-(6.25) with (6.36) and (6.37), we arrive at

$$(\Delta Q)_q^2(\Delta P)_q^2 = \frac{1}{4}\left|\langle[Q,P]_q\rangle\right|^2. \tag{6.38}$$

7. Quantum optical features of the NCS

Now, we exploit the results issued from the above developed quantization procedure to inspect some quantum optical properties of the constructed NCS. Roy *et al* [28] pointed out interesting properties exhibited by NCS, such as squeezing and sub-Poissonian behavior. For more details see also [12]. In the following, we discuss some other relevant aspects like the signal-to-quantum-noise ratio and the Mandel parameter.

7.1. *Signal-to-quantum-noise ratio (SNR)*

Signal-to-quantum-noise ratio (SNR) is relevant when studying, for example, the exciton spin relaxation for dynamics of photoexcited excitons in an ensemble of InAs/GaAs self-assembled quantum dots [23]. For a normalized state $|\phi\rangle$, in terms of the self adjoint quadrature operator Q, the SNR is defined as [25]

$$\sigma_{|\phi\rangle} = \frac{\langle Q\rangle_{|\phi\rangle}^2}{(\Delta Q)_{|\phi\rangle}^2}. \tag{7.1}$$

In the NCS $|\eta_{\mathbf{z}}\rangle$, we obtain

$$\sigma_{\mathbf{z}} = \frac{\mathbf{q}^2}{(\Delta Q)^2} = \frac{2|\mathbf{z}|^2\cos^2\phi}{-\frac{1}{2}|\mathbf{z}|^2+\mathcal{G}(|\mathbf{z}|^2)}. \tag{7.2}$$

In the case where $f(\mathbf{n})\to 1$ with $f(\mathbf{n}+1)=1=f(\mathbf{n})$, $f(\mathbf{n}+1)! = 1 = f(\mathbf{n})!$, we have $\mathcal{G}(|\mathbf{z}|^2) = \frac{1}{2}+\frac{1}{2}|\mathbf{z}|^2$ providing $\sigma_{\mathbf{z}} = 4|\mathbf{z}|^2\cos^2\phi$, this latter being the SNR for the canonical CS (CCS).

For the q-deformed CS, the SNR is given by

$$\sigma_{\mathbf{z}_q} = \frac{\mathbf{q}^2}{(\Delta Q)_q^2} = \frac{2|\mathbf{z}|^2\cos^2\phi}{-\frac{1}{2}|\mathbf{z}|^2 + \mathscr{G}_q(|\mathbf{z}|^2)} \tag{7.3}$$

such that in the limit $q \to 1$, $\mathscr{G}_q(|\mathbf{z}|^2) \to \frac{1}{2} + \frac{1}{2}|\mathbf{z}|^2$ leading to $\sigma_{\mathbf{z}_q} \to 4|\mathbf{z}|^2\cos^2\phi$.

7.2. *Mandel parameter*

The Mandel parameter known as a convenient noise-indicator of a non-classical field and defined by [19]

$$\mathscr{Q} = \frac{(\Delta N)^2}{\langle N \rangle} - 1 \equiv \mathscr{F} - 1 \tag{7.4}$$

is closely related to the normalized variance also called the quantum Fano factor $\mathscr{F}$ [6], given by $\mathscr{F} = (\Delta N)^2/\langle N \rangle$, of the photon distribution. For $\mathscr{F} < 1(\mathscr{Q} \leq 0)$, the emitted light is referred to as sub-Poissonian, with $(\mathscr{F} = 1; \mathscr{Q} = 0)$, whereas for $\mathscr{F} > 1, (\mathscr{Q} > 0)$ the light is called super-Poissonian.

We get for $\mathscr{N} = A^\dagger A$, where A and $A^\dagger$ are given as in (2.32) with $A|\Phi_{\mathbf{n}}^{[\mathbf{k}]}\rangle = \sqrt{\{\mathbf{n}\}}|\Phi_{\mathbf{n-1}}^{[\mathbf{k}]}\rangle$ and $A^\dagger|\Phi_{\mathbf{n}}^{[\mathbf{k}]}\rangle = \sqrt{\{\mathbf{n}+1\}}|\Phi_{\mathbf{n+1}}^{[\mathbf{k}]}\rangle$, the following mean values

$$\langle \eta_{\mathbf{z}}|\mathscr{N}|\eta_{\mathbf{z}}\rangle = |\mathbf{z}|^2 \left(\mathscr{N}(|\mathbf{z}|^2)\right)^{-1} \sum_{\mathbf{n}=1}^{\infty} \frac{|\mathbf{z}|^{2(\mathbf{n}-1)}}{\{(\mathbf{n}-1)\}!} = |\mathbf{z}|^2 \tag{7.5}$$

and

$$\langle \eta_{\mathbf{z}}|\mathscr{N}^2|\eta_{\mathbf{z}}\rangle = \left(\mathscr{N}(|\mathbf{z}|^2)\right)^{-1} \sum_{\mathbf{n}=0}^{\infty} \frac{|\mathbf{z}|^{2\mathbf{n}}}{\{\mathbf{n}\}!} \{\mathbf{n}\}^2. \tag{7.6}$$

Thus, the dispersion is derived in the NCS $|\eta_{\mathbf{z}}\rangle$ as follows:

$$\begin{aligned}(\Delta\mathscr{N})_{\mathbf{z}}^2 &= \langle \mathscr{N}^2 \rangle_{\mathbf{z}} - (\langle \mathscr{N} \rangle_{\mathbf{z}})^2 \\ &= \left(\mathscr{N}(|\mathbf{z}|^2)\right)^{-1} \sum_{\mathbf{n}=0}^{\infty} \frac{|\mathbf{z}|^{2\mathbf{n}}}{\{\mathbf{n}\}!} \{\mathbf{n}\}^2 - (|\mathbf{z}|^2)^2. \end{aligned} \tag{7.7}$$

The Mandel parameter is therefore provided for a given f-deformed function of the number operator in the manner

$$\begin{aligned}\mathscr{Q} &= \frac{(\Delta\mathscr{N})_{\mathbf{z}}^2}{\langle \mathscr{N} \rangle_{\mathbf{z}}} - 1 \\ &= \left(\mathscr{N}(|\mathbf{z}|^2)\right)^{-1} \sum_{\mathbf{n}=0}^{\infty} \frac{|\mathbf{z}|^{2\mathbf{n}}}{\{\mathbf{n}\}!} \{\mathbf{n}+1\} - (|\mathbf{z}|^2 + 1). \end{aligned} \tag{7.8}$$

In the case where $f(\mathbf{n}) \to 1$ with $f(\mathbf{n}+1) = 1 = f(\mathbf{n})$, $f(\mathbf{n}+1)! = 1 = f(\mathbf{n})!$, we have the quantities related to the canonical coherent states (CCS) given by

$$\langle N^2\rangle_{\mathbf{z}} = |\mathbf{z}|^4 + |\mathbf{z}|^2, \qquad \langle N\rangle_{\mathbf{z}} = |\mathbf{z}|^2 \tag{7.9}$$

leading to the following Mandel parameter related to a Poissonian statistics:

$$\mathcal{Q}_{\rm CCS} = \frac{(|\mathbf{z}|^4 + |\mathbf{z}|^2) - (|\mathbf{z}|^2)^2}{|\mathbf{z}|^2} - 1 \\ = 0. \tag{7.10}$$

For the CS $|\eta_{\mathbf{z}}\rangle_q$, the Mandel parameter denoted by $\mathcal{Q}_q$ is obtained as

$$\mathcal{Q}_q = \frac{\left(\mathcal{N}_q(|\mathbf{z}|^2)\right)^{-1}\sum_{\mathbf{n}=0}^{\infty}\frac{|\mathbf{z}|^{2\mathbf{n}}}{[\mathbf{n}]_q!}[\mathbf{n}]_q^2 - (|\mathbf{z}|^2)^2}{|\mathbf{z}|^2} - 1 \\ = \left(\mathcal{N}_q(|\mathbf{z}|^2)\right)^{-1}\sum_{\mathbf{n}=0}^{\infty}\frac{|\mathbf{z}|^{2\mathbf{n}}}{[\mathbf{n}]_q!}[\mathbf{n}+1]_q - (|\mathbf{z}|^2+1) \tag{7.11}$$

such that in the limit $q \to 1$ we get $\mathcal{Q}_q \to \mathcal{Q}_{\rm CCS} = 0$.

Note that in [14], the quantum statistical properties of the deformed states are discussed in the context of conventional as well as deformed quantum optics.

Introducing the f-deformed $SU_f(1,1)$ algebra (see [18] and references therein), which consists of three generators

$$\mathcal{K}_- = A_l A_k, \quad \mathcal{K}_+ = A_l'^{\dagger} A_k'^{\dagger}, \quad \mathcal{K}_0 = \frac{1}{2}(\mathcal{N}_l + \mathcal{N}_k + \mathbb{I}), \quad \mathcal{N}_l = A_l'^{\dagger} A_l \quad \text{and} \quad \mathcal{N}_k = A_k'^{\dagger} A_k \tag{7.12}$$

where A_l, A_k and $A_l'^{\dagger}, A_k'^{\dagger}$ $(1 \le k,l \le N_B)$ are given by (2.30) and (2.31), we get the following commutation relations

$$[\mathcal{K}_+, \mathcal{K}_-] = -2\mathcal{K}_0, \qquad [\mathcal{K}_0, \mathcal{K}_\pm] = \pm\mathcal{K}_\pm. \tag{7.13}$$

This algebra is a generalization of the $SU(1,1)$ Lie algebra [26]. Indeed, when $\mathcal{K}_+$ and $\mathcal{K}_-$ are Hermitian conjugate to each other in the special case of $f(N_i) = 1$, i.e., $\mathcal{K}_-^{\dagger} = \mathcal{K}_+^{\dagger}$, the $SU_f(1,1)$ algebra contracts to the $SU(1,1)$ Lie algebra.

Let now

$$X = \frac{\mathcal{K}_-^{\dagger} + \mathcal{K}_-}{2}, \qquad Y = \frac{i(\mathcal{K}_-^{\dagger} - \mathcal{K}_-)}{2} \tag{7.14}$$

be the f-deformed quadrature operators satisfying the commutation relation

$$[X,Y] = \frac{i}{2}[\mathcal{K}_-, \mathcal{K}_-^{\dagger}] = \frac{i}{2}[(N_k+1)f_k^2(N_k+1)(N_l+1)f_l^2(N_l+1) - N_k f_k^2(N_k) N_l f_l^2(N_l)]. \tag{7.15}$$

The following uncertainty relation

$$\langle(\Delta X)^2\rangle\langle(\Delta Y)^2\rangle \geq \frac{1}{16}|\langle[\mathscr{K}_-,\mathscr{K}_-^\dagger]\rangle|^2 \tag{7.16}$$

holds and the $SU_f(1,1)$ squeezing is provided by the relation:

$$\langle(\Delta X_k)^2\rangle < \frac{1}{4}|\langle[\mathscr{K}_-,\mathscr{K}_-^\dagger]\rangle|, \qquad X_k = X,Y. \tag{7.17}$$

8. Concluding remarks

In this work we have provided a construction of a dual pair of nonlinear coherent states (NCS) in the context of changes of bases in the underlying Hilbert space for a Hamiltonian describing the electron-phonon dynamics in condensed matter physics, which obeys a f-deformed Heisenberg algebra. The existence and properties of reproducing kernel in the NCS Hilbert space have been studied and discussed; the probability density and its dynamics in the basis of constructed CS have been analyzed. Moreover, a NCS quantization of classical phase space observables has been performed and illustrated in the case of the Quesne's q-deformed CS corresponding to the situation where the deformation structure function $f(N) = \sqrt{\frac{[N]_q}{N}}$.

Another important result in this work consists in the generalization, in a NVCS basis, of the Glauber-Sudarshan P-representation of the density matrix for a diagonal operator, to complex systems such as electron-phonon models in condensed matter physics. The same approach can be applied to other quantum optical models including spin-orbit interactions systems [31] and the Jaynes-Cummings model [17]. The three main axioms characterizing the reproducing kernel property of the density matrix have been highlighted in the NCS. Besides, the analysis of the performed NCS quantization has permitted to investigate the signal-to-quantum-noise ratio and Mandel parameter measuring the deviation of a Poisson-like distribution from the Poisson distribution, pertaining to the considered deformed algebra. It is shown that the NCS also exhibit squeezing property under the $SU_f(1,1)$ algebra built from the constructed deformed operator algebra.

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Appendix A. Proof of the Proposition 4.1

(i) Hermiticity

From (4.1), we get

$$\mathscr{K}(\mathbf{z}',\mathbf{z}) = \langle \eta_{\mathbf{z}} | \eta_{\mathbf{z}'} \rangle = \frac{\mathscr{N}(\bar{\mathbf{z}}\mathbf{z}')}{\sqrt{\mathscr{N}(|\mathbf{z}'|^2)\mathscr{N}(|\mathbf{z}|^2)}} \tag{A.1}$$

and thereby

$$\begin{aligned}\overline{\mathscr{K}(\mathbf{z}',\mathbf{z})} &= \overline{\langle \eta_{\mathbf{z}} | \eta_{\mathbf{z}'} \rangle} \\ &= \frac{\mathscr{N}(\bar{\mathbf{z}}'\mathbf{z})}{\sqrt{\mathscr{N}(|\mathbf{z}'|^2)\mathscr{N}(|\mathbf{z}|^2)}} = \mathscr{K}(\mathbf{z},\mathbf{z}').\end{aligned} \tag{A.2}$$

(ii) Positivity

Using again (4.1), we get

$$\begin{aligned}\mathscr{K}(\mathbf{z},\mathbf{z}) = \langle \eta_{\mathbf{z}} | \eta_{\mathbf{z}} \rangle &= \frac{\mathscr{N}(\bar{\mathbf{z}}\mathbf{z})}{\sqrt{\mathscr{N}(|\mathbf{z}|^2)\mathscr{N}(|\mathbf{z}|^2)}} \\ &= \frac{\mathscr{N}(|\mathbf{z}|^2)}{\mathscr{N}(|\mathbf{z}|^2)} = 1 > 0\end{aligned} \tag{A.3}$$

implying that $\mathscr{K}(\mathbf{z},\mathbf{z}) > 0$.

(iii) Idempotence

The left-hand side of the expression (4.4) can be written, by use of the definition (4.1), as

$$\begin{aligned}\int_{\mathscr{D}} d\mu(\mathbf{z}'',\bar{\mathbf{z}}'')\mathscr{K}(\mathbf{z},\mathbf{z}'')\mathscr{K}(\mathbf{z}'',\mathbf{z}') \;=\; &\int_{\mathscr{D}} d\mu(\mathbf{z}'',\bar{\mathbf{z}}'')\left\{\frac{\mathscr{N}(\bar{\mathbf{z}}''\mathbf{z})}{\sqrt{\mathscr{N}(|\mathbf{z}|^2)\mathscr{N}(|\mathbf{z}''|^2)}}\right\} \\ &\times\left\{\frac{\mathscr{N}(\bar{\mathbf{z}}'\mathbf{z}'')}{\sqrt{\mathscr{N}(|\mathbf{z}'|^2)\mathscr{N}(|\mathbf{z}''|^2)}}\right\}\end{aligned}$$

$$
\begin{aligned}
&= \left\{ \frac{1}{\sqrt{\mathcal{N}(|\mathbf{z}|^2)\mathcal{N}(|\mathbf{z}'|^2)}} \sum_{\mathbf{n}=0}^{\infty} \frac{\bar{\mathbf{z}'}^{\mathbf{n}}\mathbf{z}^{\mathbf{n}}}{\{\mathbf{n}\}!} \right\} \\
&\quad \times \int_{\mathscr{D}} d\mu(\mathbf{z}'',\bar{\mathbf{z}''}) \frac{1}{\mathcal{N}(|\mathbf{z}''|^2)} \left\{ \sum_{\mathbf{n}=0}^{\infty} \frac{|\mathbf{z}''|^2}{\{\mathbf{n}\}!} \right\}
\end{aligned}
$$

$$
\begin{aligned}
\int_{\mathscr{D}} d\mu(\mathbf{z}'',\bar{\mathbf{z}''}) \mathscr{K}(\mathbf{z},\mathbf{z}'') \mathscr{K}(\mathbf{z}'',\mathbf{z}') &= \frac{1}{\sqrt{\mathcal{N}(|\mathbf{z}|^2)\mathcal{N}(|\mathbf{z}'|^2)}} \sum_{\mathbf{n}=0}^{\infty} \frac{\bar{\mathbf{z}'}^{\mathbf{n}}\mathbf{z}^{\mathbf{n}}}{\{\mathbf{n}\}!} \\
&=: \mathscr{K}(\mathbf{z},\mathbf{z}')
\end{aligned} \tag{A.4}
$$

where the following relations

$$
\frac{1}{\mathcal{N}(|\mathbf{z}|^2)} \sum_{\mathbf{n}=0}^{\infty} \frac{|\mathbf{z}|^2}{\{\mathbf{n}\}!} = 1, \qquad \int_{\mathscr{D}} d\mu(\mathbf{z},\bar{\mathbf{z}}) = 1 \tag{A.5}
$$

are used.

□

Appendix B. Proof of the Proposition 5.1

(i) Hermiticity

We have

$$
\rho_{[\mathbf{k}]}(\mathbf{z},\mathbf{z}') = \langle \eta_{\mathbf{z}} | \rho_{[\mathbf{k}]} | \eta_{\mathbf{z}'} \rangle \tag{B.1}
$$

and thereby

$$
\begin{aligned}
\overline{\rho_{[\mathbf{k}]}(\mathbf{z},\mathbf{z}')} &= \overline{\langle \eta_{\mathbf{z}} | \rho_{[\mathbf{k}]} | \eta_{\mathbf{z}'} \rangle} \\
&= \langle \eta_{\mathbf{z}'} | \rho_{[\mathbf{k}]} | \eta_{\mathbf{z}} \rangle = \rho_{[\mathbf{k}]}(\mathbf{z}',\mathbf{z}).
\end{aligned} \tag{B.2}
$$

(ii) Positivity

From (5.15) and (5.30), we get

$$
\begin{aligned}
\langle \eta_{\mathbf{z}} | \rho_{[\mathbf{k}]} | \eta_{\mathbf{z}} \rangle &= \langle \eta_{\mathbf{z}} | \rho_{[\mathbf{k}]} \int_{\mathscr{D}} d\mu(\mathbf{z},\bar{\mathbf{z}}) P(|\mathbf{z}|^2) \, | \eta_{\mathbf{z}} \rangle \\
&= \sum_{\mathbf{m},\mathbf{n}=0}^{\infty} \int_{\mathscr{D}} d\mu(\mathbf{z},\bar{\mathbf{z}}) \left(\mathcal{N}(|\mathbf{z}|^2) \right)^{-1} P(|\mathbf{z}|^2) \, \rho_{[\mathbf{k}]}(\mathbf{n},\mathbf{m}) \frac{r^{\mathbf{m}+\mathbf{n}} e^{-\iota(\mathbf{n}-\mathbf{m})\theta}}{\sqrt{\{\mathbf{n}\}!}\sqrt{\{\mathbf{m}\}!}} \\
&= \sum_{\mathbf{n}=0}^{\infty} \rho_{[\mathbf{k}]}(\mathbf{n},\mathbf{n}) \left\{ \int_{\mathscr{D}} d\mu(\mathbf{z},\bar{\mathbf{z}}) P(|\mathbf{z}|^2) \left(\mathcal{N}(|\mathbf{z}|^2) \right)^{-1} \frac{|\mathbf{z}|^{2\mathbf{n}}}{\{\mathbf{n}\}!} \right\} \\
&= \sum_{\mathbf{n}=0}^{\infty} [\rho_{[\mathbf{k}]}(\mathbf{n},\mathbf{n})]^2 \\
&= \sum_{\mathbf{n}=0}^{\infty} \left\{ 2\pi \int_0^L \frac{r^{2\mathbf{n}}}{\{\mathbf{n}\}!} \frac{P(r^2)}{\mathcal{N}(r^2)} d\lambda(r) \right\}^2 > 0
\end{aligned} \tag{B.3}
$$

implying that

$$
\langle \eta_{\mathbf{z}} | \rho_{[\mathbf{k}]} | \eta_{\mathbf{z}} \rangle = \rho_{[\mathbf{k}]}(\mathbf{z},\mathbf{z}) > 0. \tag{B.4}
$$

(iii) Idempotence

By setting $d\mu(\mathbf{z}'',\bar{\mathbf{z}''}) = \mathscr{N}(|\mathbf{z}''|^2)d\lambda(r'')d\theta$ and using the relation $\sum_{\mathbf{n}=0}^{\infty}\rho_{[\mathbf{k}]}(\mathbf{n},\mathbf{n}) = 1$, we get from (5.12)

$$
\begin{aligned}
&\int_{\mathscr{D}} d\mu(\mathbf{z}'',\bar{\mathbf{z}''})\rho_{[\mathbf{k}]}(\mathbf{z},\mathbf{z}'')\rho_{[\mathbf{k}]}(\mathbf{z}'',\mathbf{z}') \\
&= \int_{\mathscr{D}} d\mu(\mathbf{z}'',\bar{\mathbf{z}''}) \left[\left[\mathscr{N}(|\mathbf{z}|^2)\mathscr{N}(|\mathbf{z}''|^2)\right]^{-1/2} \sum_{\mathbf{n},\mathbf{m}=0}^{\infty} \rho_{[\mathbf{k}]}(\mathbf{n},\mathbf{m}) \frac{\bar{\mathbf{z}}^{\mathbf{n}}}{\sqrt{\{\mathbf{n}\}!}} \frac{\mathbf{z}''^{\mathbf{m}}}{\sqrt{\{\mathbf{m}\}!}} \right] \\
&\times \left[\left[\mathscr{N}(|\mathbf{z}''|^2)\mathscr{N}(|\mathbf{z}'|^2)\right]^{-1/2} \sum_{\mathbf{n},\mathbf{m}=0}^{\infty} \rho_{[\mathbf{k}]}(\mathbf{n},\mathbf{m}) \frac{\bar{\mathbf{z}''}^{\mathbf{n}}}{\sqrt{\{\mathbf{n}\}!}} \frac{\mathbf{z}'^{\mathbf{m}}}{\sqrt{\{\mathbf{m}\}!}} \right] \\
&= \left[\left[\mathscr{N}(|\mathbf{z}|^2)\mathscr{N}(|\mathbf{z}'|^2)\right]^{-1/2} \sum_{\mathbf{n},\mathbf{m}=0}^{\infty} \rho_{[\mathbf{k}]}(\mathbf{n},\mathbf{m}) \frac{\bar{\mathbf{z}}^{\mathbf{n}}}{\sqrt{\{\mathbf{n}\}!}} \frac{\mathbf{z}'^{\mathbf{m}}}{\sqrt{\{\mathbf{m}\}!}} \right] \\
&\times \left[\sum_{\mathbf{n}=0}^{\infty} \rho_{[\mathbf{k}]}(\mathbf{n},\mathbf{n}) \right] \\
&= \rho_{[\mathbf{k}]}(\mathbf{z},\mathbf{z}').
\end{aligned}
\tag{B.5}
$$

□