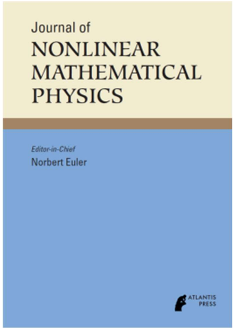

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Time-evolution-proof Scattering Data for the Focusing and Defocusing Zakharov-Shabat Systems

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In this article we give sufficient conditions on the scattering data of a defocusing or focusing Zakharov-Shabat system in order that its potential is square integrable. For a dense subset of integrable as well as square integrable potentials, we show that the scattering data actually satisfy these sufficient conditions.

Keywords: nonlinear Schrödinger equation; characterization problem; Zakharov-Shabat system.

2000 Mathematics Subject Classification: 35Q55, 35R30

1. Introduction

Nonlinear Schrödinger (NLS) equations have attracted the attention of the physical and mathematical community for over four decades. NLS equations arise in such diverse fields as deep water waves [4, 25], plasma physics [24], fiber optics [11], and Bose-Einstein condensation [17]. The basic method for solving the NLS initial-value problem is the inverse scattering transform (IST) method [2–4, 6, 10, 15, 21, 25], where the NLS time evolution is transcribed into the time evolution of the scattering data of the so-called Zakharov-Shabat system. In this article we characterize the scattering data for the *Zakharov-Shabat* (ZS) *system*

$$\psi_x = (-i\lambda\sigma_3 + Q)\psi, \tag{1.1}$$

where $\sigma_3 = \mathrm{diag}(1, -1)$ is the third Pauli matrix,

$$Q(x) = \begin{pmatrix} 0 & q(x) \\ \pm q(x)^* & 0 \end{pmatrix}$$

is the potential matrix, and λ is a spectral parameter. Further, $q \in L^2(\mathbb{R})$, the plus sign pertains to the *focusing case*, and the minus sign to the *defocusing case*. In this respect, we deviate from the usual practice of either having $q \in L^1(\mathbb{R})$ [3, 21] or having q belong to the Schwarz class [10].

To better understand the problem studied in this article, we begin by summarizing the scattering theory of the Zakharov-Shabat system under the assumption that the potential $q \in L^1(\mathbb{R})$ [3, 21]. In this case, for $\lambda \in \mathbb{R}$, we can define the *Jost matrices* $\Psi(x, \lambda)$ and $\Phi(x, \lambda)$ as those 2×2 matrix solutions to (1.1) satisfying the asymptotic conditions

$$\Psi(x,\lambda) = e^{-i\lambda x\sigma_3}[I_2 + o(1)], \qquad x \to +\infty, \tag{1.2a}$$
$$\Phi(x,\lambda) = e^{-i\lambda x\sigma_3}[I_2 + o(1)], \qquad x \to -\infty, \tag{1.2b}$$

where I_p is the $p \times p$ identity matrix. If we then define the *modified Jost matrices* by

$$F_\pm(x,\lambda) = \Phi(x,\lambda)E_\pm + \Psi(x,\lambda)E_\mp,$$

where $E_+ = \mathrm{diag}(1,0)$ and $E_- = \mathrm{diag}(0,1)$, it is known that $F_\pm(x,\lambda)e^{i\lambda x\sigma_3}$ extends to a matrix function which is analytic in $\lambda \in \mathbb{C}^\pm$, is continuous in $\lambda \in \overline{\mathbb{C}^\pm}$, and has a finite limit as $|\lambda| \to +\infty$ from within $\overline{\mathbb{C}^\pm}$.[a] We can then define the *scattering matrix*

$$S(\lambda) = \begin{pmatrix} T(\lambda) & L(\lambda) \\ R(\lambda) & T(\lambda) \end{pmatrix}$$

as the unique matrix satisfying

$$F_-(x,\lambda) = F_+(x,\lambda)\sigma_3 S(\lambda)\sigma_3, \tag{1.3}$$

where $T(\lambda)$, $R(\lambda)$, and $L(\lambda)$ are *transmission coefficient*, *reflection coefficient from the right*, and *reflection coefficient from the left*, respectively. It is then well-known [3, 7, 21] that the triangular representations

$$\Psi(x,\lambda) = e^{-i\lambda x\sigma_3} + \int_x^\infty dy\, \alpha_l(x,y)e^{-i\lambda y\sigma_3}, \tag{1.4a}$$
$$\Phi(x,\lambda) = e^{-i\lambda x\sigma_3} + \int_{-\infty}^x dy\, \alpha_r(x,y)e^{-i\lambda y\sigma_3}, \tag{1.4b}$$

are valid, where, for each $x \in \mathbb{R}$,

$$\int_x^\infty dy\, \|\alpha_l(x,y)\| + \int_{-\infty}^x dy\, \|\alpha_r(x,y)\| < +\infty.$$

Again for L^1 potentials, if we are in the *defocusing case*, the transmission coefficient is analytic in $\lambda \in \mathbb{C}^+$, is continuous in $\lambda \in \overline{\mathbb{C}^+}$, and tends to 1 as $|\lambda| \to +\infty$ from within $\overline{\mathbb{C}^+}$. Further, for $\lambda \in \mathbb{R}$ the matrix $S(\lambda)$ is unitary and $|R(\lambda)| = |L(\lambda)| < 1$. Then the scattering data consist of one of the reflection coefficients. On the other hand, if we are in the *focusing case* and assume $S(\lambda)$ to be continuous in $\lambda \in \mathbb{R}$,[b] the transmission coefficient $T(\lambda)$ is meromorphic in $\lambda \in \mathbb{C}^+$ having finitely many poles. The scattering data then consist of one reflection coefficient, the poles of the transmission coefficient, and, for each pole λ_j of multiplicity m_j, m_j parameters, the so-called norming constants c_{js} or d_{js}. In either case, the scattering data can be used to construct the

[a]Here $\mathbb{C}^+$ and $\mathbb{C}^-$ denote the open upper and lower complex half-planes, respectively.

[b]We assume the absence of *spectral singularities*, i.e., the absence of values $\lambda \in \mathbb{R}$ where $S(\lambda)$ is not defined or is discontinuous.

Marchenko kernels

$$\Omega_l(y) = \frac{1}{2\pi}\int_{-\infty}^{\infty} d\lambda\, e^{i\lambda y} R(\lambda) + \sum_{j=1}^{N}\sum_{s=0}^{m_j-1} c_{js}\frac{y^s}{s!}e^{i\lambda_j y}, \tag{1.5a}$$

$$\Omega_r(y) = \frac{1}{2\pi}\int_{-\infty}^{\infty} d\lambda\, e^{-i\lambda y} L(\lambda) + \sum_{j=1}^{N}\sum_{s=0}^{m_j-1} d_{js}\frac{y^s}{s!}e^{-i\lambda_j y}, \tag{1.5b}$$

where a $1,1$-correspondence exists between a Marchenko kernel and a set of "classical" scattering data. Solving uniquely one of the two Marchenko integral equations

$$\alpha_l(x,y) + \omega_l(x+y) + \int_x^{\infty} dz\, \alpha_l(x,z)\omega_l(z+y) = 0_{2\times 2}, \qquad y \geq x, \tag{1.6a}$$

$$\alpha_r(x,y) + \omega_r(x+y) + \int_{-\infty}^{x} dz\, \alpha_r(x,y)\omega_r(z+y) = 0_{2\times 2}, \qquad y \leq x, \tag{1.6b}$$

where $\omega_l = \begin{pmatrix} 0 & \mp\Omega_l^* \\ \Omega_l & 0 \end{pmatrix}$ and $\omega_r = \begin{pmatrix} 0 & \Omega_r \\ \mp\Omega_r^* & 0 \end{pmatrix}$, we can recover the potential $q(x)$ by using one of the equalities

$$q(x) = -2\begin{pmatrix} 1 & 0 \end{pmatrix}\alpha_l(x,x)\begin{pmatrix} 0 \\ 1 \end{pmatrix}, \qquad q(x) = 2\begin{pmatrix} 1 & 0 \end{pmatrix}\alpha_r(x,x)\begin{pmatrix} 0 \\ 1 \end{pmatrix}. \tag{1.7}$$

For later use and under the condition $q \in L^1(\mathbb{R}) \cap L^2(\mathbb{R})$, we mention the partial energy identities

$$\int_x^{\infty} dz\, |q(z)|^2 = \mp 2\begin{pmatrix} 1 & 0 \end{pmatrix}\alpha_l(x,x)\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \mp 2\begin{pmatrix} 0 & 1 \end{pmatrix}\alpha_l(x,x)\begin{pmatrix} 0 \\ 1 \end{pmatrix}, \tag{1.8a}$$

$$\int_{-\infty}^{x} dz\, |q(z)|^2 = \mp 2\begin{pmatrix} 1 & 0 \end{pmatrix}\alpha_r(x,x)\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \mp 2\begin{pmatrix} 0 & 1 \end{pmatrix}\alpha_r(x,x)\begin{pmatrix} 0 \\ 1 \end{pmatrix}. \tag{1.8b}$$

The *characterization problem* studied so far in the literature consists of proving a $1,1$-correspondence between (a) the potentials $q \in L^1(\mathbb{R})$ leading to scattering matrices without spectral singularities and (b) a class of scattering data. For the scattering data we can take either Marchenko kernel, in view of the $1,1$-correspondence mentioned above. There exist a full solution of the analogous characterization problem for the Schrödinger equation on the line [13, 14] and various partial solutions of the characterization problem for the Zakharov-Shabat system [5, 20, 22]. The above characterization problem has recently been solved completely [9] for the more general AKNS system with and without symmetries on the L^1 potential. In this solution, the scattering data comprise a reflection coefficient which is the Fourier transform of an L^1-function.

After this digression we return to the problem at hand and discuss the two principal defects of the existing characterization results. At first, the scattering theory of the Zakharov-Shabat system is usually applied to solve the initial-value problem of the defocusing or focusing nonlinear Schrödinger (NLS) equation

$$iq_t + q_{xx} \mp 2|q|^2 q = 0 \tag{1.9}$$

by the inverse scattering transform (IST) method [2, 3, 7, 10, 15, 21, 25]. In fact, by the direct and inverse scattering theory of (1.1), the NLS time evolution $q(x,0) \mapsto q(x,t)$ is converted into the following evolution of the Marchenko kernels:

$$\Omega_l(y,t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} d\lambda\, e^{i\lambda y} e^{4i\lambda^2 t} R(\lambda) + \sum_{j=1}^{N} c_{j0} e^{i\lambda_j y} e^{4i\lambda_j^2 t},$$

$$\Omega_r(y,t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} d\lambda\, e^{-i\lambda y} e^{-4i\lambda^2 t} L(\lambda) + \sum_{j=1}^{N} d_{j0} e^{-i\lambda_j y} e^{-4i\lambda_j^2 t},$$

where, for convenience, we have confined ourselves to the case of simple poles of the transmission coefficient. Under the time evolution, we have the energy conservation law

$$\frac{d}{dt}\int_{-\infty}^{\infty} dx\, |q(x,t)|^2 = 0. \tag{1.10}$$

It poses the question, though, of how to deal with the integral appearing in (1.10) [and likewise how to deal with the integrals appearing in (1.8)] in a preexisting scattering theory based on potentials $q \in L^1(\mathbb{R})$, especially since, to our knowledge, there does not exist a scattering theory of the Zakharov-Shabat system (1.1) under the *sole* assumption that $q \in L^2(\mathbb{R})$. The same question may be posed regarding other integrable equations associated with the Zakharov-Shabat system by means of the IST, such as the modified Korteweg-de Vries [23], sine-Gordon [1, 26], Hirota [12], and Sasa-Satsuma [19] equations.

Another inconvenience of the existing characterization results is that the class of scattering data is not closed under the time evolution according to any of the integrable equations mentioned above. In fact, if $R(\lambda)$ and $L(\lambda)$ are the Fourier transforms of L^1-functions (as is required in a scattering theory relying on L^1 potentials), then this property may fail to hold for the NLS time evolved reflection coefficients $e^{4i\lambda^2 t}R(\lambda)$ and $e^{-4i\lambda^2 t}L(\lambda)$. For this reason, in this article we shall modify the allowed scattering data as to make them time-evolution-proof, at the expense of reinterpreting the Marchenko integral equations (1.6) as Hankel operator equations. Using fundamental results on abstract Hankel operators [16, 18], we then prove that scattering data from the modified class of scattering data lead to potentials $q \in L^2(\mathbb{R})$. We shall therefore solve the inverse problem depicted in the following diagram:

$$\boxed{\begin{array}{c}\text{class of suitable}\\ \text{scattering data}\end{array}} \longrightarrow \boxed{q \in L^2(\mathbb{R})}$$

In the converse direction, we have only had partial success in treating the direct problem. We will prove that defocusing and focusing potentials $q \in L^1(\mathbb{R})$ having a distributional derivative $q_x \in L^1(\mathbb{R})$ lead to scattering data in the modified class of scattering data. In other words,

> *if we were to solve the NLS equation* (1.9) *with initial solution* $q(x,0)$ *satisfying* $q(\cdot,0), q_x(\cdot,0) \in L^1(\mathbb{R})$*, then the time evolved solution* $q(\cdot,t) \in L^2(\mathbb{R})$.

Let us briefly describe the contents of this article. In Sec. 2 we discuss Marchenko integral operators as operators which are unitarily equivalent to so-called abstract Hankel operators, proving their compactness. These properties are used in Sec. 3 to prove that certain (modified) scattering data lead to defocusing or focusing L^2-potentials. In Sec. 4 we revisit direct scattering theory for a class of L^2-potentials dense in $L^1(\mathbb{R})$ and derive for them the properties of the scattering data sufficient for L^2-inversion. We draw some conclusions in Sec. 5. Throughout this article the upper signs (in $\pm$ or $\mp$) pertain to the focusing case and the lower signs to the defocusing case.

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2. Marchenko Operators as Hankel Operators

Let us review some results involving the Hankel-Marchenko operator Γ defined by

$$(\Gamma\phi)(y) = \int_0^\infty dz\, \gamma(y+z)\phi(z), \qquad y \in \mathbb{R}^+, \tag{2.1}$$

for $\phi \in L^2(\mathbb{R}^+)$. Extending $\gamma(y)$ for $y \in \mathbb{R}$, we denote by its so-called *symbol* the Fourier transform

$$\hat{\gamma}(\lambda) = \int_{-\infty}^\infty dy\, e^{i\lambda y}\gamma(y).$$

Since this extension is not unique, various symbols correspond to the same Hankel-Marchenko operator Γ. We shall always deal with situations in which the symbol belongs to $L^\infty(\mathbb{R})$.

We have the following commutative diagram:

$$\begin{tikzcd}
L^2(\mathbb{R}^+) \arrow[r, "j_+"] \arrow[d, "\mathbb{F}"] & L^2(\mathbb{R}) \arrow[r, "\sigma"] \arrow[d, "\mathbb{F}"] & L^2(\mathbb{R}) \arrow[r, "C_\gamma"] \arrow[d, "\mathbb{F}"] & L^2(\mathbb{R}) \arrow[r, "\pi_+"] \arrow[d, "\mathbb{F}"] & L^2(\mathbb{R}^+) \arrow[d, "\mathbb{F}"] \\
H^2(\mathbb{C}^+) \arrow[r, "j_+"] & L^2(\mathbb{R}) \arrow[r, "\sigma"] & L^2(\mathbb{R}) \arrow[r, "M_{\hat{\gamma}}"] & L^2(\mathbb{R}) \arrow[r, "\Pi_+"] & H^2(\mathbb{C}^+)
\end{tikzcd}$$

where

- $H^2(\mathbb{C}^+)$ is the Hardy space of Fourier transforms of L^2-functions supported on the positive half-line,
- j_+ and j_+ are natural imbeddings,
- π_+ and Π_+ are orthogonal projections,
- σ is the sign reversal operator,
- C_γ is convolution with γ,
- $M_{\hat{\gamma}}$ is multiplication by $\hat{\gamma}$, and
- $\mathbb{F}$ is the Fourier transform.

Then the Hankel-Marchenko operator Γ defined on $L^2(\mathbb{R}^+)$ is unitarily equivalent to the so-called abstract Hankel operator

$$\Pi_+ M_{\hat{\gamma}} \sigma j_+,$$

provided the symbol $\hat{\gamma} \in L^\infty(\mathbb{R})$ ([16, Ch. 4]; [18, Ch. 1]). Two symbols correspond to the same abstract Hankel operator if their difference is the a.e. limit of an arbitrary bounded analytic function of $\lambda \in \mathbb{C}^-$ as λ approaches the real line in a nontangential way. As a result,

$$\|\Gamma\|_{L^2(\mathbb{R}^+)\to L^2(\mathbb{R}^+)} \leq \|\hat{\gamma}\|_\infty \leq \|\gamma\|_1. \tag{2.2}$$

According to Nehari's theorem [16, 18], the exact value of the norm of $\|\Gamma\|$ is obtained by choosing this bounded analytic function of $\lambda \in \mathbb{C}^-$ in such a way that the middle member of the inequality (2.2) assumes its minimum.

Let us establish the following useful property.

Proposition 2.1. *If $\gamma \in L^2(\mathbb{R}^+)$, then Γ maps $L^2(\mathbb{R}^+)$ into the Banach space $C_0[0,+\infty)$ of continuous functions vanishing at $+\infty$. Moreover,*

$$|(\Gamma\phi)(y)| \leq \left[\int_y^\infty dz\,|\gamma(z)|^2\right]^{1/2} \|\phi\|_2 \tag{2.3}$$

for each $y \geq 0$.

Proof. Indeed, if $\gamma, \phi \in L^2(\mathbb{R}^+)$, we obtain the estimate (2.3) by using the Cauchy-Schwarz inequality. On the other hand, the continuity of $(\Gamma\phi)(y)$ in $y \in [0,+\infty)$ follows from the estimate

$$|(\Gamma\phi)(y_1) - (\Gamma\phi)(y_2)| \leq \left[\int_0^\infty dz\,|\gamma(y_1+z) - \gamma(y_2+z)|^2\right]^{1/2} \|\phi\|_2,$$

where the right-hand side vanishes as $y_2 \to y_1$. $\square$

Let us write the Marchenko kernels in the form [8]

$$\Omega_l(y) = \frac{1}{2\pi}\int_{-\infty}^\infty d\lambda\, e^{i\lambda y} R(\lambda) + C_l e^{-yA_l} B_l, \tag{2.4a}$$

$$\Omega_r(y) = \frac{1}{2\pi}\int_{-\infty}^\infty d\lambda\, e^{-i\lambda y} L(\lambda) + C_r e^{yA_r} B_r, \tag{2.4b}$$

where the matrix triplets (A_l, B_l, C_l) and (A_r, B_r, C_r) are such that A_l and A_r only have eigenvalues with positive real parts. We have thus written the terms not involving a reflection coefficient in a concise way. Using the commutative diagrams

$$\begin{array}{ccc} L^2(\mathbb{R}^+) & \xrightarrow{\Gamma_l} & L^2(\mathbb{R}^+) \\ \downarrow \tau_+^x & & \downarrow \tau_+^x \\ L^2(x,+\infty) & \xrightarrow{\Omega_l} & L^2(x,+\infty) \end{array} \qquad \begin{array}{ccc} L^2(\mathbb{R}^+) & \xrightarrow{\Gamma_r} & L^2(\mathbb{R}^+) \\ \downarrow \tau_-^x & & \downarrow \tau_-^x \\ L^2(-\infty,x) & \xrightarrow{\Omega_r} & L^2(-\infty,x) \end{array}$$

where $(\tau_+^x\phi)(y) = \phi(x+y)$ and $(\tau_-^x\phi)(y) = \phi(x-y)$ are natural translations, we convert the Marchenko operators Ω_l and Ω_r defined by

$$(\Omega_l\phi)(y) = \int_x^\infty dz\,\Omega_l(y+z)\phi(z),$$

$$(\Omega_r\phi)(y) = \int_{-\infty}^x dz\,\Omega_r(y+z)\phi(z),$$

into Hankel-Marchenko operators which are finite rank perturbations of Hankel-Marchenko operators with respective symbols $e^{-2i\lambda x}R(-\lambda)$ and $e^{2i\lambda x}L(\lambda)$.

Apart from the compactness statements, we have proved the following:

Proposition 2.2. *Suppose the reflection coefficient from the right $R(\lambda)$ is continuous in $\lambda \in \mathbb{R}$, vanishes as $\lambda \to \pm\infty$, and belongs to $L^2(\mathbb{R})$. Then the Marchenko operator Ω_l is compact on $L^2(x,+\infty)$ and maps $L^2(x,+\infty)$ into $C_0[x,+\infty)$. Similarly, suppose $L(\lambda)$ is continuous in $\lambda \in \mathbb{R}$, vanishes as $\lambda \to \pm\infty$, and belongs to $L^2(\mathbb{R})$. Then the Marchenko operator Ω_r is compact on $L^2(-\infty,x)$ and maps $L^2(-\infty,x)$ into $C_0(-\infty,x]$.*

Proof. It remains to prove the compactness statements. Since, for each $x \in \mathbb{R}$, the symbols $e^{-2i\lambda x}R(-\lambda)$ and $e^{2i\lambda x}L(\lambda)$ are continuous in $\lambda \in \mathbb{R}$ and vanish as $\lambda \to \pm\infty$, it follows from Hartman's theorem ([16, Thm. 3.20]; [18, Thm. 4.1]) that the corresponding abstract Hankel operators are compact on $H^2(\mathbb{C}^+)$. Adding the discrete eigenvalue terms only adds a finite rank term to these operators, which does not affect their compactness. Using the above commutative diagram pair, we see that Ω_l and Ω_r are compact operators on $L^2(x,+\infty)$ and $L^2(-\infty,x)$, respectively. □

3. Constructing L^2 Potentials

In this section we introduce the modified scattering data and construct the potential $q \in L^2(\mathbb{R})$ by inverse scattering. We then go on to prove two characterization results, one for the defocusing case and the other for the focusing case.

We begin by defining the classes of scattering data.

- *Defocusing case*. The class $\mathscr{C}^-$ of scattering data is the set of those functions $G(\lambda)$ which are continuous in $\lambda \in \mathbb{R}$, vanish as $\lambda \to \pm\infty$, belong to $L^2(\mathbb{R})$, and satisfy $|G(\lambda)| < 1$ for $\lambda \in \mathbb{R}$.
- *Focusing case*. The class $\mathscr{C}_l^+$ of left scattering data is the set of functions $\Omega_l(y)$ of the type (2.4a), where (a) $R(\lambda)$ is continuous in $\lambda \in \mathbb{R}$, vanishes as $\lambda \to \pm\infty$, and belongs to $L^2(\mathbb{R})$, and (b) the matrix triplet (A_l,B_l,C_l) is such that A_l is a square matrix with only eigenvalues with positive real parts, B_l is a column vector, and C_l is a row vector. Similarly, the class $\mathscr{C}_r^+$ of right scattering data is the set of functions $\Omega_r(y)$ of the type (2.4b), where (a) $L(\lambda)$ is continuous in $\lambda \in \mathbb{R}$, vanishes as $\lambda \to \pm\infty$, and belongs to $L^2(\mathbb{R})$, and (b) the matrix triplet (A_r,B_r,C_r) is such that A_r is a square matrix with only eigenvalues with positive real parts, B_r is a column vector, and C_r is a row vector.

Let us now derive one-way [i.e., from scattering data to potential] characterization results for the defocusing and focusing cases.

Theorem 3.1 (defocusing case). *Suppose the reflection coefficient from the right $R(\lambda)$ and the reflection coefficient from the left $L(\lambda)$ both belong to the class of scattering data $\mathscr{C}^-$. Then inverse scattering leads to a unique potential $q \in L^2(\mathbb{R})$.*

Proof. As explained above, the Marchenko operators Ω_l and Ω_r are unitarily equivalent to Hankel-Marchenko operators with respective symbols $e^{-2i\lambda x}R(-\lambda)$ and $e^{2i\lambda x}L(\lambda)$. These two operators are strict contractions, because the L^∞-norms $\sup_{\lambda\in\mathbb{R}} |R(\lambda)|$ and $\sup_{\lambda\in\mathbb{R}} |L(\lambda)|$ of their symbols are strictly less than one. Thus the Marchenko equations (1.6) have unique solutions $\alpha_l(x,\cdot)$ with entries in $L^2(x,+\infty)$ and $\alpha_r(x,\cdot)$ with entries in $L^2(-\infty,x)$. Using (1.7) and (1.8), we now observe that

$$\alpha_l(x,x) = \tfrac{1}{2}\begin{pmatrix} \int_x^\infty dz\,|q(z)|^2 & -2\Omega_l(2x)^* - q(x) \\ -2\Omega_l(2x) - q(x)^* & \int_x^\infty dz\,|q(z)|^2 \end{pmatrix}, \tag{3.1a}$$

$$\alpha_r(x,x) = \tfrac{1}{2}\begin{pmatrix} \int_{-\infty}^x dz\,|q(z)|^2 & -2\Omega_r(2x) + q(x) \\ -2\Omega_r(2x)^* + q(x)^* & \int_{-\infty}^x dz\,|q(z)|^2 \end{pmatrix}. \tag{3.1b}$$

Since the entries of $\alpha_l(x,x)+\omega_l(2x) = -(\Omega_l\alpha_l(x,\cdot))(x)$ and $\alpha_r(x,x)+\omega_r(2x) = -(\Omega_r\alpha_r(x,\cdot))(x)$ are continuous in $x \in \mathbb{R}$ and vanish as $x \to +\infty$ and as $x \to -\infty$, respectively, we obtain $q \in L^2(\mathbb{R})$ by considering the diagonal entries in (3.1). □

Theorem 3.2 (focusing case). *Suppose the Marchenko kernel $\Omega_r(y)$ belongs to $\mathscr{C}_l^+$ and the Marchenko kernel $\Omega_l(y)$ belongs to $\mathscr{C}_r^+$. Then inverse scattering leads to a unique potential $q \in L^2(\mathbb{R})$.*

Proof. Since the Marchenko kernels $\omega_l(x+y)$ and $\omega_r(x+y)$ satisfy the symmetry relations $\omega_l = \begin{pmatrix} 0 & -\Omega_l^* \\ \Omega_l & 0 \end{pmatrix}$ and $\omega_r = \begin{pmatrix} 0 & \Omega_r \\ -\Omega_r^* & 0 \end{pmatrix}$, we can write the Marchenko integral equations (1.6) in the vector form

$$\begin{pmatrix} I & -\Omega_l^\dagger \\ \Omega_l & I \end{pmatrix} \alpha_l(x,\cdot) = -\omega_l(x+\cdot), \tag{3.2a}$$

$$\begin{pmatrix} I & \Omega_r \\ -\Omega_r^\dagger & I \end{pmatrix} \alpha_r(x,\cdot) = -\omega_r(x+\cdot). \tag{3.2b}$$

Here (3.2a) is a vector equation on the orthogonal direct sum of two copies of $L^2(x,+\infty)$ and (3.2b) is a vector equation on the orthogonal direct sum of two copies of $L^2(-\infty,x)$. Further, the dagger is the operator adjoint, and I stands for the identity operator. It is now easily seen that the two linear operators in the left-hand sides of (3.2a) and (3.2b) are boundedly invertible. In fact, writing $K = -\Omega_l^\dagger$ in (3.2a) and $K = \Omega_r$ in (3.2b), we have

$$\begin{pmatrix} I & K \\ -K^\dagger & I \end{pmatrix}^{-1} = \begin{pmatrix} (I+KK^\dagger)^{-1} & -K(I+K^\dagger K)^{-1} \\ K^\dagger(I+KK^\dagger)^{-1} & (I+K^\dagger K)^{-1} \end{pmatrix},$$

which proves the unique solvability of the Marchenko equations (1.6). Using (1.7) and (1.8), we now observe that

$$\alpha_l(x,x) = \tfrac{1}{2} \begin{pmatrix} -\int_x^\infty dz\,|q(z)|^2 & 2\Omega_l(2x)^* - q(x) \\ -2\Omega_l(2x) + q(x)^* & -\int_x^\infty dz\,|q(z)|^2 \end{pmatrix}, \tag{3.3a}$$

$$\alpha_r(x,x) = \tfrac{1}{2} \begin{pmatrix} -\int_{-\infty}^x dz\,|q(z)|^2 & -2\Omega_r(2x) + q(x) \\ 2\Omega_r(2x)^* - q(x)^* & -\int_{-\infty}^x dz\,|q(z)|^2 \end{pmatrix}. \tag{3.3b}$$

Since the entries of $\alpha_l(x,x) + \omega_l(2x) = -(\Omega_l \alpha_l(x,\cdot))(x)$ and $\alpha_r(x,x) + \omega_r(2x) = -(\Omega_r \alpha_r(x,\cdot))(x)$ are continuous in $x \in \mathbb{R}$ and vanish as $x \to +\infty$ and as $x \to -\infty$, respectively, we obtain $q \in L^2(\mathbb{R})$ by considering the diagonal entries in (3.1). $\square$

4. Direct Scattering

In this section we summarize direct scattering theory and fine-tune it to prove that potentials from a suitable dense linear subspace of L^2-potentials have scattering data where the reflection coefficients belong to $L^2(\mathbb{R})$.

For $\lambda \in \mathbb{R}$ and potential $q \in L^1(\mathbb{R})$ the existence of unique Jost matrices $\Psi(x,\lambda)$ and $\Phi(x,\lambda)$ satisfying the asymptotic conditions (1.2) follows by constructing them as the unique solutions to the Volterra integral equations

$$\Psi(x,\lambda) = e^{-i\lambda x\sigma_3} - \int_x^\infty dy\, e^{i\lambda(y-x)\sigma_3} \sigma_3 Q(y) \Psi(y,\lambda), \tag{4.1a}$$

$$\Phi(x,\lambda) = e^{-i\lambda x\sigma_3} + \int_{-\infty}^x dy\, e^{-i\lambda(x-y)\sigma_3} \sigma_3 Q(y) \Phi(y,\lambda). \tag{4.1b}$$

Here the L^1-property is used in an essential way to solve these equations by iteration [3, 7, 21], leading to the upper bounds

$$\|\Psi(x,\lambda)\| \leq \exp\left(\int_x^\infty dy\,|q(y)|\right), \ \|\Phi(x,\lambda)\| \leq \exp\left(\int_{-\infty}^x dy\,|q(y)|\right). \tag{4.2}$$

As a result, we obtain the proportionality relations

$$\Phi(x,\lambda) = \Psi(x,\lambda)a_r(\lambda), \qquad \Psi(x,\lambda) = \Phi(x,\lambda)a_l(\lambda), \tag{4.3}$$

where

$$a_l(\lambda) = \begin{pmatrix} T(\lambda)^* & [L(\lambda)/T(\lambda)] \\ -[R(\lambda)/T(\lambda)] & T(\lambda) \end{pmatrix} = a_r(\lambda)^{-1}.$$

Letting $x \to -\infty$ in (4.1a) and $x \to +\infty$ in (4.1b) and using the proportionality relations (4.3), we obtain

$$a_l(\lambda) = I_2 - \int_{-\infty}^\infty dy\, e^{i\lambda y\sigma_3}\sigma_3 Q(y)\Psi(y,\lambda), \tag{4.4a}$$

$$a_r(\lambda) = I_2 + \int_{-\infty}^\infty dy\, e^{i\lambda y\sigma_3}\sigma_3 Q(y)\Phi(y,\lambda). \tag{4.4b}$$

Writing $\alpha_l = \begin{pmatrix} \overline{K}^{\text{up}} & K^{\text{up}} \\ \overline{K}^{\text{dn}} & K^{\text{dn}} \end{pmatrix}$ and $\alpha_r = \begin{pmatrix} M^{\text{up}} & \overline{M}^{\text{up}} \\ M^{\text{dn}} & \overline{M}^{\text{dn}} \end{pmatrix}$, where $\alpha_l(x,y)$ and $\alpha_r(x,y)$ are connected to the Jost matrices by means of (1.4), we can convert the Volterra integral equations for the Jost matrices (4.1) into the following Volterra integral equations for the kernel functions:

$$\overline{K}^{\text{up}}(x,y) = -\int_x^\infty dz\, q(z)\overline{K}^{\text{dn}}(z,z+y-x), \tag{4.5a}$$

$$\overline{K}^{\text{dn}}(x,y) = \pm\tfrac{1}{2}q(\tfrac{1}{2}(x+y))^* \pm \int_x^{\frac{1}{2}(x+y)} dz\, q(z)^*\overline{K}^{\text{up}}(z,x+y-z), \tag{4.5b}$$

$$K^{\text{up}}(x,y) = -\tfrac{1}{2}q(\tfrac{1}{2}(x+y)) - \int_x^{\frac{1}{2}(x+y)} dz\, q(z)K^{\text{dn}}(z,x+y-z), \tag{4.5c}$$

$$K^{\text{dn}}(x,y) = \pm\int_x^\infty dz\, q(z)^*K^{\text{up}}(z,z+y-x), \tag{4.5d}$$

as well as

$$M^{\text{up}}(x,y) = \int_{-\infty}^x dz\, q(z)M^{\text{dn}}(z,z+y-x), \tag{4.6a}$$

$$M^{\text{dn}}(x,y) = \mp\tfrac{1}{2}q(\tfrac{1}{2}(x+y))^* \mp \int_{\frac{1}{2}(x+y)}^x dz\, q(z)^*M^{\text{up}}(z,x+y-z), \tag{4.6b}$$

$$\overline{M}^{\text{up}}(x,y) = \tfrac{1}{2}q(\tfrac{1}{2}(x+y)) + \int_{\frac{1}{2}(x+y)}^x dz\, q(z)\overline{M}^{\text{dn}}(z,z+y-x), \tag{4.6c}$$

$$\overline{M}^{\text{dn}}(x,y) = \mp\int_{-\infty}^x dz\, q(z)^*\overline{M}^{\text{up}}(z,z+y-x). \tag{4.6d}$$

Theorem 4.1 (defocusing and focusing cases). *Let the potential $q(x)$ and its distributional derivative q_x belong to $L^1(\mathbb{R})$ and let there be no spectral singularities. Then the reflection coefficients $R(\lambda)$ and $L(\lambda)$ are continuous in $\lambda \in \mathbb{R}$, are of the order of $o(1/\lambda)$ as $\lambda \to \pm\infty$, and hence belong to $L^2(\mathbb{R})$.*

Proof. Let us now assume that $q, q_x \in L^1(\mathbb{R})$, where q_x is the distributional derivative of q. Then $q(x)$ is continuous in $x \in \mathbb{R}$ and vanishes as $x \to \pm\infty$. Since $\|q\|_2 \le \|q\|_\infty \|q\|_1 \le \|q_x\|_1 \|q\|_1$, we see that $q \in L^2(\mathbb{R})$ as well. Differentiating (4.5) and (4.6) with respect to y and utilizing (1.7) and (1.8), we obtain from (4.5) and (4.6) the Volterra integral equations

$$\overline{K}_y^{\mathrm{up}}(x,y) = -\int_x^\infty dz\, q(z) \overline{K}_y^{\mathrm{dn}}(z, z+y-x), \tag{4.7a}$$

$$\begin{aligned}\overline{K}_y^{\mathrm{dn}}(x,y) &= \pm\tfrac{1}{4} q_x(\tfrac{1}{2}(x+y))^* - \tfrac{1}{4} q(\tfrac{1}{2}(x+y))^* \int_{\frac{1}{2}(x+y)}^\infty dz\, |q(z)|^2 \\ &\quad \pm \int_x^{\frac{1}{2}(x+y)} dz\, q(z)^* \overline{K}_y^{\mathrm{up}}(z, x+y-z),\end{aligned} \tag{4.7b}$$

$$\begin{aligned}K_y^{\mathrm{up}}(x,y) &= -\tfrac{1}{4} q_x(\tfrac{1}{2}(x+y)) \pm \tfrac{1}{4} q(\tfrac{1}{2}(x+y)) \int_{\frac{1}{2}(x+y)}^\infty dz\, |q(z)|^2 \\ &\quad - \int_x^{\frac{1}{2}(x+y)} dz\, q(z) K_y^{\mathrm{dn}}(z, x+y-z),\end{aligned} \tag{4.7c}$$

$$K_y^{\mathrm{dn}}(x,y) = \pm \int_x^\infty dz\, q(z)^* K_y^{\mathrm{up}}(z, z+y-x), \tag{4.7d}$$

as well as

$$M_y^{\mathrm{up}}(x,y) = \int_{-\infty}^x dz\, q(z) M_y^{\mathrm{dn}}(z, z+y-x), \tag{4.8a}$$

$$\begin{aligned}M_y^{\mathrm{dn}}(x,y) &= \mp\tfrac{1}{4} q_x(\tfrac{1}{2}(x+y))^* - \tfrac{1}{4} q(\tfrac{1}{2}(x+y))^* \int_{-\infty}^{\frac{1}{2}(x+y)} dz\, |q(z)|^2 \\ &\quad \mp \int_{\frac{1}{2}(x+y)}^x dz\, q(z)^* M_y^{\mathrm{up}}(z, x+y-z),\end{aligned} \tag{4.8b}$$

$$\begin{aligned}\overline{M}_y^{\mathrm{up}}(x,y) &= \tfrac{1}{4} q_x(\tfrac{1}{2}(x+y)) \pm \tfrac{1}{4} q(\tfrac{1}{2}(x+y)) \int_{-\infty}^{\frac{1}{2}(x+y)} dz\, |q(z)|^2 \\ &\quad + \int_{\frac{1}{2}(x+y)}^x dz\, q(z) \overline{M}_y^{\mathrm{dn}}(z, x+y-z),\end{aligned} \tag{4.8c}$$

$$\overline{M}_y^{\mathrm{dn}}(x,y) = \mp \int_{-\infty}^x dz\, q(z)^* \overline{M}_y^{\mathrm{up}}(z, z+y-x). \tag{4.8d}$$

In (4.7b), (4.7c), (4.8c), and (4.8c) the inhomogeneous terms belong to $L^1(\mathbb{R})$ as functions of y.

Put $\mu(K,x) = \int_x^\infty dy\, |K(x,y)|$ for a kernel functions in (4.5) and (4.7) and $\mu(M,x) = \int_{-\infty}^x dy\, |M(x,y)|$ for a kernel function in (4.6) and (4.8). Then iterating (4.7) and (4.8), we obtain the estimates

$$\begin{aligned}\mu(\overline{K}_y^{\mathrm{up}};x) &\le \int_x^\infty dz\, |q(z)|\, \mu(\overline{K}_y^{\mathrm{dn}};z), \\ \mu(\overline{K}_y^{\mathrm{dn}};x) &\le \tfrac{1}{2}\int_x^\infty dz\, |q_x(z)| + \tfrac{1}{2}\|q\|_2^2 \int_x^\infty dz\, |q(z)| + \int_x^\infty dz\, |q(z)|\, \mu(\overline{K}_y^{\mathrm{up}};z), \\ \mu(K_y^{\mathrm{up}};x) &\le \tfrac{1}{2}\int_x^\infty dz\, |q_x(z)| + \tfrac{1}{2}\|q\|_2^2 \int_x^\infty dz\, |q(z)| + \int_x^\infty dz\, |q(z)|\, \mu(K_y^{\mathrm{dn}};z), \\ \mu(K_y^{\mathrm{dn}};x) &\le \int_x^\infty dz\, |q(z)|\, \mu(K_y^{\mathrm{up}};z),\end{aligned}$$

as well as

$$\begin{aligned}
\mu(M_y^{\rm up};x) &\le \int_{-\infty}^{x} dz\,|q(z)|\,\mu(M_y^{\rm dn};z),\\
\mu(M_y^{\rm dn};x) &\le \tfrac12\int_{-\infty}^{x} dz\,|q_x(z)|+\tfrac12\|q\|_2^2\int_{-\infty}^{x} dz\,|q(z)|+\int_{-\infty}^{x} dz\,|r(z)|\,\mu(M_y^{\rm up};z),\\
\mu(\overline{M}_y^{\rm up};x) &\le \tfrac12\int_{-\infty}^{x} dz\,|q_x(z)|+\tfrac12\|q\|_2^2\int_{-\infty}^{x} dz\,|q(z)|+\int_{-\infty}^{x} dz\,|q(z)|\,\mu(\overline{M}_y^{\rm dn};z),\\
\mu(\overline{M}_y^{\rm dn};x) &\le \int_{-\infty}^{x} dz\,|q(z)|\,\mu(\overline{M}_y^{\rm up};z).
\end{aligned}$$

Adding the two inequalities of each of the four successive pairs of inequalities and applying Gronwall's inequality to the resulting relations we see that $(\partial/\partial y)\alpha_l(x,y)$ has its entries in $L^1(x,+\infty)$ and that $(\partial/\partial y)\alpha_r(x,y)$ has its entries in $L^1(-\infty,x)$, uniformly in $x\in\mathbb{R}$. Thus, for $x\in\mathbb{R}$, the Fourier transforms $\Psi(x,\lambda)-e^{-i\lambda x\sigma_3}$ and $\Phi(x,\lambda)-e^{-i\lambda x\sigma_3}$ have entries of the form $o(1/\lambda)$ as $\lambda\to\pm\infty$ and hence have their entries belonging to $L^2(\mathbb{R},d\lambda)$ [cf. (1.4)].

Let us return to the integral representations (4.4). Since $Q(y)$ is an off-diagonal matrix and $e^{i\lambda y\sigma_3}\sigma_3$ a diagonal matrix, we can decompose the two 2×2 identities (4.4) into eight scalar identities, where the diagonal (off-diagonal) entries of $a_l(\lambda)$ and $a_r(\lambda)$ appear in the same equality as an off-diagonal (diagonal) entry of a Jost matrix. Using that the Jost matrices are additive perturbations of $e^{-i\lambda x\sigma_3}$ which are $o(1/\lambda)$ uniformly in $x\in\mathbb{R}$, we see, upon reconstitution of the 2×2 matrices $a_l(\lambda)$ and $a_r(\lambda)$, that $a_l(\lambda)=I_2+o(1/\lambda)$ and $a_r(\lambda)=I_2+o(1/\lambda)$ as $\lambda\to\pm\infty$.

Now recall that, apart from a sign, the reflection coefficients are ratios with an off-diagonal entry of $a_l(\lambda)$ in the numerator and a diagonal entry in the denominator. Under the assumption that there are no spectral singularities, the denominator (which is the reciprocal of the transmission coefficient) does not vanish for $\lambda\in\mathbb{R}$, while the numerators are $o(1/\lambda)$ as $\lambda\to\pm\infty$. Consequently, if there are no spectral singularities, the reflection coefficients are L^2, which completes the proof. □

5. Conclusions

In [9] we have established a $1,1$-correspondence between (a) L^1-potentials leading to scattering data without spectral singularities and (b) scattering data where the reflection coefficients are Fourier transforms of L^1-functions. In [9] the major technical problem has been to prove the potentials reconstructed by inverse scattering to belong to $L^{1,\text{loc}}(\mathbb{R})$, whereas the direct scattering part had been fully developed at the time [3, 7, 21]. This had allowed us to formulate the characterization theory for the more general AKNS system, also if the potentials do not have any symmetries that make them focusing or defocusing. As explained in the introduction, this type of characterization is not invariant under time evolution according to the NLS, mKdV, sine-Gordon, Hirota, Sasa-Satsuma, and other integrable equations associated with the Zakharov-Shabat system by means of the IST. In particular, we have not established the convergence of the iteration method for the Volterra integral equation (4.1) for L^2 potentials.

In this article we have reconstructed an L^2-potential from scattering data which remain in the same class when evolving them in time while the potential evolves according to the an integrable equation associated with the Zakharov-Shabat system by means of the IST. Here the inverse scattering problem has been easy to solve by relying on abstract Hankel operator theory. The main difficulty has been the apparent lack of success in developing direct scattering theory under the *sole* assumption of having an L^2-potential.

The present article can be generalized to the focusing and defocusing $1+n$ and $m+1$ AKNS systems without any problem. The direct scattering theory goes though for $m+n$ AKNS systems without being in the focusing or defocusing case. The inverse scattering theory of Sec. 3 requires us to remain in the focusing and defocusing cases and to limit ourselves to $1+n$ or $m+1$ AKNS systems, but only when analyzing (3.3). Without such limitations, we would arrive at potentials for which we can only prove that they have their entries in $L^{2,\text{loc}}(\mathbb{R})$.

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