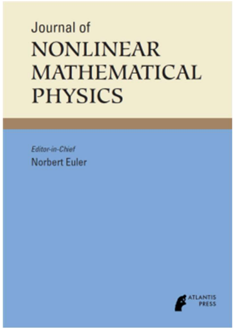

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Algebro-geometric solutions for the two-component Hunter-Saxton hierarchy

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This paper is dedicated to provide theta function representations of algebro-geometric solutions and related crucial quantities for the two-component Hunter-Saxton (HS2) hierarchy through an algebro-geometric initial value problem. Our main tools include the polynomial recursive formalism, the hyperelliptic curve with finite number of genus, the Baker-Akhiezer functions, the meromorphic function, the Dubrovin-type equations for auxiliary divisors, and the associated trace formulas. With the help of these tools, the explicit representations of the algebro-geometric solutions are obtained for the entire HS2 hierarchy.

Keywords: Hyperelliptic curve; Baker-Akhiezer function; Algebro-geometric solutions; Trace formulas.

2000 Mathematics Subject Classification: 35Q53, 58F07, 53C35

1. Introduction

In this paper, we consider the following two-component Hunter-Saxton (HS2) system

$$\begin{cases} m_t + 2u_x m + u m_x + \sigma \rho \rho_x = 0, \\ \rho_t + (u\rho)_x = 0, \end{cases} \tag{1.1}$$

where $m = -u_{xx}$, $\sigma = \pm 1$. The system (1.1) was recently introduced by Constantin and Ivanov in [9]. The variable $u(x,t)$ describes the horizontal velocity of the fluid and the variable $\rho(x,t)$ is in connection with the horizontal deviation of the surface from equilibrium, all measured in dimensionless units [9]. The HS2 system (1.1) is the short-wave (or high-frequency) limit of the two-component Camassa-Holm shallow water system [9, 17, 39, 43]. This system, reading as (1.1) with m replaced by $(1-\partial_{xx}^2)u$, has recently been the object of intensive study (see [8, 9, 25, 26, 45]). The HS2 system (1.1) is integrable, since it has a Lax pair [9] and a bi-Hamiltonian structure [9, 39]. It is also a particular case of the Gurevich-Zybin system describing the dynamics in a model of non-dissipative dark matter [24, 40, 44]. The mathematical properties of the system (1.1) have been studied further in a series of works (see [9, 23, 24, 36, 42–44]).

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For $\rho = 0$, the system (1.1) becomes the Hunter-Saxton (HS) equation [30], which models the propagation of weakly nonlinear orientation waves in a massive nematic liquid crystal director field. Here, $u(x,t)$ describes the director field of a nematic liquid crystal, x is a space variable in a reference frame moving with the linearized wave velocity, and t is a slow time variable. The field of unit vectors $(\cos u(x,t), \sin u(x,t))$ describes the orientation of the molecules [23, 30, 44].

The HS equation also arises in a different physical context as the high-frequency limit [13, 31] of the Camassa-Holm (CH) equation –a model equation for shallow water waves [7, 11, 32] and a re-expression of the geodesic flow on the diffeomorphism group of the circle [10] with a bi-Hamiltonian structure [18], which is completely integrable [12]. The HS equation is also a completely integrable system with a bi-Hamiltonian structure, and hence, it possesses a Lax pair, an infinite family of commuting Hamiltonian flows, as well as an associated sequence of conservation laws [5, 30, 31, 39, 41]. It also describes the geodesic flow on the homogeneous space related to the Virasoro group [33–35]. Recently, algebro-geometric solutions for the HS hierarchy was investigated in [28]. Moreover, peakon solutions, global weak solutions and the Cauchy problem of the system (1.1) were discussed in [9, 23, 36, 44]. However, within the knowledge of the authors, the algebro-geometric solutions of the entire HS2 hierarchy are not studied yet.

This paper concerns algebro-geometric quasi-periodic solutions of the whole HS2 hierarchy associated with (1.1). The algebro-geometric solution, as an important feature of integrable system, is a kind of explicit solution closely related to the inverse spectral theory [1, 6, 14–16]. In a degenerated case of the algebro-geometric solution, the multi-soliton solution and periodic solution in elliptic function type may be obtained [6, 37, 38]. A systematic approach, proposed by Gesztesy and Holden to construct algebro-geometric solutions for integrable equations, has been extended to the whole (1+1) dimensional integrable hierarchy, such as the AKNS hierarchy, the CH hierarchy, etc. [19–22]. Recently, we investigated algebro-geometric solutions for the modified CH hierarchy and the Degasperis-Procesi hierarchy [27, 29].

The outline of the present paper is as follows.

In section 2, based on the polynomial recursion formalism, we derive the HS2 hierarchy associated with the 2×2 spectral problem. A hyperelliptic curve $\mathscr{K}_n$ of arithmetic genus n is introduced with the help of the characteristic polynomial of Lax matrix V_n for the stationary HS2 hierarchy.

In Section 3, we decompose the stationary HS2 equations into a system of Dubrovin-type equations. Moreover, we obtain the stationary trace formulas for the HS2 hierarchy.

In Section 4, we present the first set of our results, the explicit theta function representations of the potentials u, ρ for the entire stationary HS2 hierarchy. Furthermore, we study the initial value problem on an algebro-geometric curve for the stationary HS2 hierarchy.

In Sections 5 and 6, we extend the analyses of Sections 3 and 4, respectively, to the time-dependent case. Each equation in the HS2 hierarchy is permitted to evolve in terms of an independent time parameter t_r. As initial data, we use a stationary solution of the nth equation and then construct a time-dependent solution of the rth equation of the HS2 hierarchy. The Baker-Akhiezer function, the analogs of the Dubrovin-type equations, the trace formulas, and the theta function representations in Section 4 are all extended to the time-dependent case.

Finally, we remark that although our focus in this paper is on the case $\sigma = 1$ in (1.1), all of the arguments presented here can be adapted to study the corresponding case $\sigma = -1$.

2. The HS2 hierarchy

In this section, we provide the construction of HS2 hierarchy and derive the corresponding sequence of zero-curvature pairs using a polynomial recursion formalism. Moreover, we introduce the underlying hyperelliptic curve in connection with the stationary HS2 hierarchy.

Throughout this section, we make the following hypothesis.

Hypothesis 2.1. In the stationary case, we assume that

$$u,\rho \in C^\infty(\mathbb{R}),\ \partial_x^k u, \partial_x^k \rho \in L^\infty(\mathbb{R}),\ k \in \mathbb{N}_0. \tag{2.1}$$

In the time-dependent case, we suppose

$$\begin{aligned} & u(\cdot,t),\rho(\cdot,t) \in C^\infty(\mathbb{R}),\ \ \partial_x^k u(\cdot,t), \partial_x^k \rho(\cdot,t) \in L^\infty(\mathbb{R}),\ \ k \in \mathbb{N}_0,\ t \in \mathbb{R}, \\ & u(x,\cdot), u_{xx}(x,\cdot), \rho(x,\cdot), \rho_x(x,\cdot) \in C^1(\mathbb{R}), \quad x \in \mathbb{R}. \end{aligned} \tag{2.2}$$

We first introduce the basic polynomial recursion formalism. Define $\{f_l\}_{l\in\mathbb{N}_0}$, $\{g_l\}_{l\in\mathbb{N}_0}$, and $\{h_l\}_{l\in\mathbb{N}_0}$ recursively by

$$\begin{aligned} & f_0 = \frac{1}{2}, \\ & f_{l,x} = -2\mathscr{G}\left(2\rho^2 f_{l-2,x} + 2u_{xx} f_{l-1,x} + 2\rho\rho_x f_{l-2} + u_{xxx} f_{l-1}\right), \quad l \in \mathbb{N}, \\ & g_l = \frac{1}{2} f_{l+1,x}, \quad l \in \mathbb{N}_0, \\ & h_l = -g_{l+1,x} - \rho^2 f_l - u_{xx} f_{l+1}, \quad l \in \mathbb{N}_0, \end{aligned} \tag{2.3}$$

where $\mathscr{G}$ is given by

$$\begin{aligned} & \mathscr{G} : C_0^\infty(\mathbb{R}) \to C^\infty(\mathbb{R}), \\ & (\mathscr{G}v)(x) = \int_{-\infty}^{x} \int_{-\infty}^{x_1} v(y)\, dy dx_1, \quad x \in \mathbb{R},\ v \in C_0^\infty(\mathbb{R}). \end{aligned} \tag{2.4}$$

One observes that $\mathscr{G}$ satisfies

$$\frac{d^2}{dx^2} \circ \mathscr{G} = v. \tag{2.5}$$

Explicitly, one computes

$$\begin{aligned} & f_0 = \frac{1}{2}, \\ & f_1 = -u + c_1, \\ & f_2 = \mathscr{G}(u_x^2 - \rho^2 + 2uu_{xx}) + 2c_1(-u) + c_2, \\ & g_0 = -\frac{1}{2} u_x, \\ & g_1 = \frac{1}{2}\mathscr{G}(4u_x u_{xx} + 2uu_{xxx} - 2\rho\rho_x) + 2c_1(-\frac{1}{2}u_x), \\ & h_0 = -\frac{1}{2} f_{2,xx} - \rho^2 f_0 - u_{xx} f_1, \text{ etc.}, \end{aligned} \tag{2.6}$$

where $\{c_l\}_{l\in\mathbb{N}} \subset \mathbb{C}$ are integration constants.

Next, it is convenient to introduce the corresponding homogeneous coefficients $\hat{f}_l, \hat{g}_l$, and $\hat{h}_l$, defined by the vanishing of the integration constants $c_k, k = 1, \ldots, l$,

$$\begin{aligned} &\hat{f}_0 = f_0 = \frac{1}{2}, \quad \hat{f}_l = f_l|_{c_k=0,\ k=1,\ldots,l}, \quad l \in \mathbb{N}, \\ &\hat{g}_0 = g_0 = -\frac{1}{2}u_x, \quad \hat{g}_l = g_l|_{c_k=0,\ k=1,\ldots,l}, \quad l \in \mathbb{N}, \\ &\hat{h}_0 = h_0, \qquad \hat{h}_l = h_l|_{c_k=0,\ k=1,\ldots,l}, \quad l \in \mathbb{N}. \end{aligned} \tag{2.7}$$

Hence,

$$f_l = \sum_{k=0}^{l} 2c_{l-k}\hat{f}_k, \quad g_l = \sum_{k=0}^{l} 2c_{l-k}\hat{g}_k, \quad h_l = \sum_{k=0}^{l} 2c_{l-k}\hat{h}_k, \quad l \in \mathbb{N}_0, \tag{2.8}$$

defining

$$c_0 = \frac{1}{2}. \tag{2.9}$$

Now, given Hypothesis 2.1, one introduces the following 2×2 matrix U by

$$\psi_x = U(z,x)\psi = \begin{pmatrix} 0 & 1 \\ -z^{-2}\rho^2 - z^{-1}u_{xx} & 0 \end{pmatrix} \psi, \tag{2.10}$$

and for each $n \in \mathbb{N}_0$, the following 2×2 matrix V_n by

$$\psi_{t_n} = V_n(z)\psi, \tag{2.11}$$

with

$$V_n(z) = \begin{pmatrix} -G_n(z) & F_n(z) \\ z^{-2}H_n(z) & G_n(z) \end{pmatrix}, \quad z \in \mathbb{C} \setminus \{0\}, \quad n \in \mathbb{N}_0, \tag{2.12}$$

assuming F_n, G_n, and H_n to be polynomials[a] with respect to z and C^∞ in x. The compatibility condition of linear system (2.10) and (2.11) yields the stationary zero-curvature equation

$$-V_{n,x} + [U, V_n] = 0, \tag{2.13}$$

which is equivalent to

$$F_{n,x} = 2G_n, \tag{2.14}$$

$$H_{n,x} = 2(\rho^2 + zu_{xx})G_n, \tag{2.15}$$

$$z^2 G_{n,x} = -H_n - (\rho^2 + zu_{xx})F_n. \tag{2.16}$$

[a] F_n, G_n, H_n are polynomials of degree $n+1, n, n+1$, respectively.

From (2.14)-(2.16), one infers that

$$\frac{d}{dx}\det(V_n(z,x)) = -\frac{1}{z^2}\frac{d}{dx}\Big(z^2 G_n(z,x)^2 + F_n(z,x)H_n(z,x)\Big) = 0, \tag{2.17}$$

and hence

$$z^2 G_n(z,x)^2 + F_n(z,x)H_n(z,x) = R_{2n+2}(z), \tag{2.18}$$

where the polynomial R_{2n+2} of degree $2n+2$ is x-independent. In another way, one can write R_{2n+2} as

$$R_{2n+2}(z) = \left(\frac{1}{4}u_x^2 + \frac{1}{2}h_0\right)\prod_{m=0}^{2n+1}(z - E_m), \quad \{E_m\}_{m=0,\dots,2n+1} \in \mathbb{C}. \tag{2.19}$$

Here, we emphasize that the coefficient $(\frac{1}{4}u_x^2 + \frac{1}{2}h_0)$ is a constant. In fact, equation (2.17) is equivalent to

$$2z^2 G_n G_{n,x} + F_n H_{n,x} + H_n F_{n,x} = 0. \tag{2.20}$$

Then comparing the coefficient of powers z^{2n+2} yields

$$2g_0 g_{0,x} + f_0 h_{0,x} + h_0 f_{0,x} = 0, \tag{2.21}$$

which indicates

$$\frac{1}{2}u_x u_{xx} + \frac{1}{2}h_{0,x} = 0. \tag{2.22}$$

Therefore,

$$\frac{1}{4}u_x^2 + \frac{1}{2}h_0 = \partial^{-1}\left(\frac{1}{2}u_x u_{xx} + \frac{1}{2}h_{0,x}\right) = \text{constant}. \tag{2.23}$$

For simplicity, we denote it by a^2, $a \in \mathbb{C}$. Then, $R_{2n+2}(z)$ can be rewritten as

$$R_{2n+2}(z) = a^2 \prod_{m=0}^{2n+1}(z - E_m), \quad \{E_m\}_{m=0,\dots,2n+1} \in \mathbb{C}. \tag{2.24}$$

Next, we compute the characteristic polynomial $\det(yI - zV_n)$ of Lax matrix zV_n,

$$\begin{aligned}\det(yI - zV_n) &= y^2 - z^2 G_n(z)^2 - F_n(z)H_n(z)\\ &= y^2 - R_{2n+2}(z) = 0,\end{aligned} \tag{2.25}$$

and then introduce the (possibly singular) hyperelliptic curve $\mathcal{K}_n$ of arithmetic genus n defined by

$$\mathcal{K}_n : \mathcal{F}_n(z,y) = y^2 - R_{2n+2}(z) = 0. \tag{2.26}$$

In the following, we will occasionally impose further constraints on the zeros E_m of R_{2n+2} introduced in (2.24) and assume that

$$E_m \in \mathbb{C}, \quad E_m \neq E_{m'}, \quad \forall m \neq m', \quad m, m' = 0, \dots, 2n+1. \tag{2.27}$$

The stationary zero-curvature equation (2.13) implies polynomial recursion relations (2.3). Introducing the following polynomials $F_n(z), G_n(z)$, and $H_n(z)$ with respect to the spectral parameter

z,

$$F_n(z) = \sum_{l=0}^{n+1} f_l z^{n+1-l}, \tag{2.28}$$

$$G_n(z) = \sum_{l=0}^{n} g_l z^{n-l}, \tag{2.29}$$

$$H_n(z) = \sum_{l=0}^{n+1} h_l z^{n+1-l}. \tag{2.30}$$

Inserting (2.28)-(2.30) into (2.14)-(2.16) then yields the recursion relations (2.3) for f_l, $l = 0,\ldots,n+1$, and g_l, $l = 0,\ldots,n$. For fixed $n \in \mathbb{N}_0$, we obtain the recursion relations for h_l, $l = 0,\ldots,n-1$ in (2.3) and

$$h_n = -\rho^2 f_n - u_{xx} f_{n+1}, \quad h_{n+1} = -\rho^2 f_{n+1}. \tag{2.31}$$

Moreover, from (2.15), one infers that

$$\begin{aligned} &-h_{n,x} + \rho^2 f_{n,x} + u_{xx} f_{n+1,x} = 0, \quad n \in \mathbb{N}_0, \\ &-h_{n+1,x} + \rho^2 f_{n+1,x} = 0, \quad n \in \mathbb{N}_0. \end{aligned} \tag{2.32}$$

Then using (2.31) and (2.32) permits one to write the stationary HS2 hierarchy as

$$\text{s-HS2}_n(u,\rho) = \begin{pmatrix} 2u_{xx} f_{n+1,x} + u_{xxx} f_{n+1} + 2\rho\rho_x f_n + 2\rho^2 f_{n,x} \\ -2\rho\rho_x f_{n+1} - 2\rho^2 f_{n+1,x} \end{pmatrix} = 0, \quad n \in \mathbb{N}_0. \tag{2.33}$$

We record the first equation explicitly,

$$\text{s-HS2}_0(u,\rho) = \begin{pmatrix} -2u_x u_{xx} - u u_{xxx} + \rho\rho_x + c_1 u_{xxx} \\ \rho_x u + \rho u_x - c_1 \rho_x \end{pmatrix} = 0. \tag{2.34}$$

By definition, the set of solutions of (2.33) represents the class of algebro-geometric HS2 solutions, with n ranging in $\mathbb{N}_0$ and c_l in $\mathbb{C}$, $l \in \mathbb{N}$. We call the stationary algebro-geometric HS2 solutions u, ρ as HS2 potentials at times.

Remark 2.2. Here, we emphasize that if u, ρ satisfy one of the stationary HS2 equations in (2.33) for a particular value of n, then they satisfy infinitely many such equations of order higher than n for certain choices of integration constants c_l. This is a common characteristic of the general integrable soliton equations such as the KdV, AKNS, and CH hierarchies [21].

Next, we introduce the corresponding homogeneous polynomials $\widehat{F}_l, \widehat{G}_l, \widehat{H}_l$ by

$$\widehat{F}_l(z) = F_l(z)|_{c_k=0,\ k=1,\ldots,l} = \sum_{k=0}^{l} \hat{f}_k z^{l-k}, \ \ l = 0,\ldots,n+1, \tag{2.35}$$

$$\widehat{G}_l(z) = G_l(z)|_{c_k=0,\ k=1,\ldots,l} = \sum_{k=0}^{l} \hat{g}_k z^{l-k}, \ \ l = 0,\ldots,n, \tag{2.36}$$

$$\widehat{H}_l(z) = H_l(z)|_{c_k=0,\ k=1,\ldots,l} = \sum_{k=0}^{l} \hat{h}_k z^{l-k}, \ \ l = 0,\ldots,n-1, \tag{2.37}$$

$$\widehat{H}_n(z) = -\rho^2 \hat{f}_n - u_{xx}\hat{f}_{n+1} + \sum_{k=0}^{n-1} \hat{h}_k z^{n-k}, \tag{2.38}$$

$$\widehat{H}_{n+1}(z) = -\rho^2 \hat{f}_{n+1} + z\widehat{H}_n(z). \tag{2.39}$$

In accordance with our notation introduced in (2.7) and (2.35)-(2.39), the corresponding homogeneous stationary HS2 equations are then defined by

$$\text{s-}\widehat{\text{HS2}}_n(u,\rho) = \text{s-HS2}_n(u,\rho)|_{c_l=0,\ l=1,\dots,n} = 0, \quad n \in \mathbb{N}_0. \tag{2.40}$$

At the end of this section, we turn to the time-dependent HS2 hierarchy. In this case, u, ρ are considered as functions of both space and time. We introduce a deformation parameter $t_n \in \mathbb{R}$ in u and ρ, replacing $u(x), \rho(x)$ by $u(x,t_n), \rho(x,t_n)$, for each equation in the hierarchy. In addition, the definitions (2.10), (2.12), and (2.28)-(2.30) of U, V_n and F_n, G_n, and H_n, respectively, still apply. The corresponding zero-curvature equation reads

$$U_{t_n} - V_{n,x} + [U,V_n] = 0, \quad n \in \mathbb{N}_0, \tag{2.41}$$

which results in the following set of equations

$$F_{n,x} = 2G_n, \tag{2.42}$$

$$z^2 G_{n,x} = -H_n - (\rho^2 + zu_{xx})F_n, \tag{2.43}$$

$$-2\rho\rho_{t_n} - zu_{xxt_n} - H_{n,x} + 2(\rho^2 + zu_{xx})G_n = 0. \tag{2.44}$$

For fixed $n \in \mathbb{N}_0$, inserting the polynomial expressions for F_n, G_n, and H_n into (2.42)-(2.44), respectively, first yields recursion relations (2.3) for $f_l|_{l=0,\dots,n+1}$, $g_l|_{l=0,\dots,n}$, $h_l|_{l=0,\dots,n-1}$ and

$$h_n = -\rho^2 f_n - u_{xx}f_{n+1}, \quad h_{n+1} = -\rho^2 f_{n+1}. \tag{2.45}$$

Moreover, using (2.44), one finds

$$\begin{aligned} &-u_{xxt_n} - h_{n,x} + \rho^2 f_{n,x} + u_{xx}f_{n+1,x} = 0, \quad n \in \mathbb{N}_0, \\ &2\rho\rho_{t_n} + h_{n+1,x} - \rho^2 f_{n+1,x} = 0, \quad n \in \mathbb{N}_0. \end{aligned} \tag{2.46}$$

Hence, using (2.45) and (2.46) permits one to write the time-dependent HS2 hierarchy as

$$\text{HS2}_n(u,\rho) = \begin{pmatrix} -u_{xxt_n} + 2u_{xx}f_{n+1,x} + u_{xxx}f_{n+1} + 2\rho\rho_x f_n + 2\rho^2 f_{n,x} \\ 2\rho\rho_{t_n} - 2\rho\rho_x f_{n+1} - 2\rho^2 f_{n+1,x} \end{pmatrix} = 0, \quad n \in \mathbb{N}_0. \tag{2.47}$$

For convenience, we record the first equation in this hierarchy explicitly,

$$\text{HS2}_0(u,\rho) = \begin{pmatrix} -u_{xxt_0} - 2u_x u_{xx} - uu_{xxx} + \rho\rho_x + c_1 u_{xxx} \\ \rho_{t_0} + \rho_x u + \rho u_x - c_1\rho_x \end{pmatrix} = 0. \tag{2.48}$$

The first equation $\text{HS2}_0(u,\rho) = 0$ (with $c_1 = 0$) in the hierarchy represents the HS2 system as discussed in section 1. Similarly, one can introduce the corresponding homogeneous HS2 hierarchy by

$$\widehat{\text{HS2}}_n(u,\rho) = \text{HS2}_n(u,\rho)|_{c_l=0,\ l=1,\dots,n} = 0, \quad n \in \mathbb{N}_0. \tag{2.49}$$

In fact, since the Lenard recursion formalism is almost universally adopted in the contemporary literature, we thought it might be worthwhile to use the Gesztesy and Holden's method, the polynomial recursion formalism, to construct the HS2 hierarchy.

3. The stationary HS2 formalism

This section is devoted to a detailed study of the stationary HS2 hierarchy. We first define a fundamental meromorphic function $\phi(P,x)$ on the hyperelliptic curve $\mathcal{K}_n$, using the polynomial recursion formalism described in section 2, and then study the properties of the Baker-Akhiezer function $\psi(P,x,x_0)$, Dubrovin-type equations, and trace formulas.

For major parts of this section, we assume (2.1), (2.3), (2.6), (2.10)-(2.16), (2.26)-(2.30), and (2.33), keeping $n \in \mathbb{N}_0$ fixed.

Recall the hyperelliptic curve $\mathcal{K}_n$

$$\begin{aligned} &\mathcal{K}_n : \mathcal{F}_n(z,y) = y^2 - R_{2n+2}(z) = 0, \\ &R_{2n+2}(z) = a^2 \prod_{m=0}^{2n+1} (z - E_m), \quad \{E_m\}_{m=0,\dots,2n+1} \in \mathbb{C}, \end{aligned} \tag{3.1}$$

which is compactified by joining two points at infinity $P_{\infty_\pm}$, with $P_{\infty_+} \neq P_{\infty_-}$. But for notational simplicity, the compactification is also denoted by $\mathcal{K}_n$. Hence, $\mathcal{K}_n$ becomes a two-sheeted Riemann surface of arithmetic genus n. Points P on $\mathcal{K}_n \backslash \{P_{\infty\pm}\}$ are denoted by $P = (z, y(P))$, where $y(\cdot)$ is the meromorphic function on $\mathcal{K}_n$ satisfying $\mathcal{F}_n(z, y(P)) = 0$.

The complex structure on $\mathcal{K}_n$ is defined in the usual way by introducing local coordinates

$$\zeta_{Q_0} : P \to (z - z_0)$$

near points $Q_0 = (z_0, y(Q_0)) \in \mathcal{K}_n$, which are neither branch nor singular points of $\mathcal{K}_n$; near the branch and singular points $Q_1 = (z_1, y(Q_1)) \in \mathcal{K}_n$, the local coordinates are

$$\zeta_{Q_1} : P \to (z - z_1)^{1/2};$$

near the points $P_{\infty_\pm} \in \mathcal{K}_n$, the local coordinates are

$$\zeta_{P_{\infty_\pm}} : P \to z^{-1}.$$

The holomorphic map $*$, changing sheets, is defined by

$$\begin{aligned} &* : \begin{cases} \mathcal{K}_n \to \mathcal{K}_n, \\ P = (z, y_j(z)) \to P^* = (z, y_{j+1(\text{mod } 2)}(z)), \quad j = 0, 1, \end{cases} \\ &P^{**} := (P^*)^*, \quad \text{etc.}, \end{aligned} \tag{3.2}$$

where $y_j(z)$, $j = 0, 1$ denote the two branches of $y(P)$ satisfying $\mathcal{F}_n(z, y) = 0$, namely,

$$(y - y_0(z))(y - y_1(z)) = y^2 - R_{2n+2}(z) = 0. \tag{3.3}$$

Taking into account (3.3), one easily finds

$$\begin{aligned} &y_0 + y_1 = 0, \\ &y_0 y_1 = -R_{2n+2}(z), \\ &y_0^2 + y_1^2 = 2R_{2n+2}(z). \end{aligned} \tag{3.4}$$

Moreover, positive divisors on $\mathcal{K}_n$ of degree n are denoted by

$$\mathcal{D}_{P_1,\ldots,P_n}:\begin{cases}\mathcal{K}_n \to \mathbb{N}_0,\\ P \to \mathcal{D}_{P_1,\ldots,P_n} = \begin{cases} k & \text{if } P \text{ occurs } k \text{ times in } \{P_1,\ldots,P_n\},\\ 0 & \text{if } P \notin \{P_1,\ldots,P_n\}.\end{cases}\end{cases} \tag{3.5}$$

Next, we define the stationary Baker-Akhiezer function $\psi(P,x,x_0)$ on $\mathcal{K}_n \setminus \{P_{\infty_+}, P_{\infty_-}, P_0 = (0, i\rho f_{n+1})\}$ by

$$\begin{aligned}
&\psi(P,x,x_0) = \begin{pmatrix}\psi_1(P,x,x_0)\\ \psi_2(P,x,x_0)\end{pmatrix},\\
&\psi_x(P,x,x_0) = U(u(x),\rho(x),z(P))\psi(P,x,x_0),\\
&zV_n(u(x),\rho(x),z(P))\psi(P,x,x_0) = y(P)\psi(P,x,x_0),\\
&\psi_1(P,x_0,x_0) = 1;\\
&P = (z,y) \in \mathcal{K}_n \setminus \{P_{\infty_+}, P_{\infty_-}, P_0 = (0, i\rho f_{n+1})\},\ (x,x_0) \in \mathbb{R}^2.
\end{aligned} \tag{3.6}$$

Closely related to $\psi(P,x,x_0)$ is the following meromorphic function $\phi(P,x)$ on $\mathcal{K}_n$ defined by

$$\phi(P,x) = z\frac{\psi_{1,x}(P,x,x_0)}{\psi_1(P,x,x_0)}, \quad P \in \mathcal{K}_n,\ x \in \mathbb{R} \tag{3.7}$$

such that

$$\psi_1(P,x,x_0) = \exp\left(z^{-1}\int_{x_0}^{x} \phi(P,x')\,dx'\right), \quad P \in \mathcal{K}_n \setminus \{P_{\infty_+}, P_{\infty_-}, P_0\}. \tag{3.8}$$

Then, based on (3.6) and (3.7), a direct calculation shows that

$$\begin{aligned}\phi(P,x) &= \frac{y + zG_n(z,x)}{F_n(z,x)}\\ &= \frac{H_n(z,x)}{y - zG_n(z,x)},\end{aligned} \tag{3.9}$$

and

$$\psi_2(P,x,x_0) = \psi_1(P,x,x_0)\phi(P,x)/z. \tag{3.10}$$

In the following, the roots of polynomials F_n and H_n will play a special role, and hence, we introduce on $\mathbb{C} \times \mathbb{R}$

$$F_n(z,x) = \frac{1}{2}\prod_{j=0}^{n}(z - \mu_j(x)), \quad H_n(z,x) = h_0\prod_{l=0}^{n}(z - \nu_l(x)). \tag{3.11}$$

Moreover, we introduce

$$\hat{\mu}_j(x) = (\mu_j(x), -\mu_j(x)G_n(\mu_j(x),x)) \in \mathcal{K}_n,\ j = 0,\ldots,n,\ x \in \mathbb{R}, \tag{3.12}$$

and

$$\hat{\nu}_l(x) = (\nu_l(x), \nu_l(x)G_n(\nu_l(x),x)) \in \mathcal{K}_n,\ l = 0,\ldots,n,\ x \in \mathbb{R}. \tag{3.13}$$

Due to assumption (2.1), u and ρ are smooth and bounded, and hence, $F_n(z,x)$ and $H_n(z,x)$ share the same property. Thus, one concludes

$$\mu_j, \nu_l \in C(\mathbb{R}), \quad j,l = 0,\dots,n, \tag{3.14}$$

taking multiplicities (and appropriate reordering) of the zeros of F_n and H_n into account. From (3.9), the divisor $(\phi(P,x))$ of $\phi(P,x)$ is given by

$$(\phi(P,x)) = \mathscr{D}_{\hat{\nu}_0(x)\underline{\hat{\nu}}(x)}(P) - \mathscr{D}_{\hat{\mu}_0(x)\underline{\hat{\mu}}(x)}(P). \tag{3.15}$$

Here, we abbreviated

$$\underline{\hat{\mu}} = \{\hat{\mu}_1,\dots,\hat{\mu}_n\}, \quad \underline{\hat{\nu}} = \{\hat{\nu}_1,\dots,\hat{\nu}_n\} \in \mathrm{Sym}^n(\mathscr{K}_n). \tag{3.16}$$

Further properties of $\phi(P,x)$ are summarized as follows.

Lemma 3.1. *Suppose* (2.1)*, assume the nth stationary HS*2 *equation* (2.33) *holds, and let* $P = (z,y) \in \mathscr{K}_n \setminus \{P_{\infty_+}, P_{\infty_-}, P_0\}$, $(x,x_0) \in \mathbb{R}^2$. *Then* ϕ *satisfies the Riccati-type equation*

$$\phi_x(P) + z^{-1}\phi(P)^2 = -z^{-1}\rho^2 - u_{xx}, \tag{3.17}$$

as well as

$$\phi(P)\phi(P^*) = -\frac{H_n(z)}{F_n(z)}, \tag{3.18}$$

$$\phi(P) + \phi(P^*) = \frac{2zG_n(z)}{F_n(z)}, \tag{3.19}$$

$$\phi(P) - \phi(P^*) = \frac{2y}{F_n(z)}. \tag{3.20}$$

Proof. Equation (3.17) follows using the definition (3.9) of ϕ as well as relations (2.14)-(2.16). Relations (3.18)-(3.20) are clear from (2.18), (3.4), and (3.9). □

The properties of $\psi(P,x,x_0)$ are summarized in the following lemma.

Lemma 3.2. *Suppose* (2.1)*, assume the nth stationary HS*2 *equation* (2.33) *holds, and let* $P = (z,y) \in \mathscr{K}_n \setminus \{P_{\infty_+}, P_{\infty_-}, P_0\}$, $(x,x_0) \in \mathbb{R}^2$. *Then* $\psi_1(P,x,x_0), \psi_2(P,x,x_0)$ *satisfy*

$$\psi_1(P,x,x_0) = \left(\frac{F_n(z,x)}{F_n(z,x_0)}\right)^{1/2} \exp\left(\frac{y}{z}\int_{x_0}^{x} F_n(z,x')^{-1}dx'\right), \tag{3.21}$$

$$\psi_1(P,x,x_0)\psi_1(P^*,x,x_0) = \frac{F_n(z,x)}{F_n(z,x_0)}, \tag{3.22}$$

$$\psi_2(P,x,x_0)\psi_2(P^*,x,x_0) = -\frac{H_n(z,x)}{z^2F_n(z,x_0)}, \tag{3.23}$$

$$\psi_1(P,x,x_0)\psi_2(P^*,x,x_0) + \psi_1(P^*,x,x_0)\psi_2(P,x,x_0) = 2\frac{G_n(z,x)}{F_n(z,x_0)}, \tag{3.24}$$

$$\psi_1(P,x,x_0)\psi_2(P^*,x,x_0) - \psi_1(P^*,x,x_0)\psi_2(P,x,x_0) = \frac{-2y}{zF_n(z,x_0)}. \tag{3.25}$$

Proof. Equation (3.21) is a consequence of (2.14), (3.8), and (3.9). Equation (3.22) is clear from (3.21) and (3.23) is a consequence of (3.10), (3.18), and (3.22). Equation (3.24) follows using (3.10), (3.19), and (3.22). Finally, (3.25) follows from (3.10), (3.20), and (3.22). □

In Lemma 3.2, we denote by

$$\psi_1(P) = \psi_{1,+},\ \psi_1(P^*) = \psi_{1,-},\ \psi_2(P) = \psi_{2,+},\ \psi_2(P^*) = \psi_{2,-},$$

and then (3.22)-(3.25) imply

$$(\psi_{1,+}\psi_{2,-} - \psi_{1,-}\psi_{2,+})^2 = (\psi_{1,+}\psi_{2,-} + \psi_{1,-}\psi_{2,+})^2 - 4\psi_{1,+}\psi_{2,-}\psi_{1,-}\psi_{2,+}, \tag{3.26}$$

which is equivalent to the basic identity (2.18), $z^2G_n^2 + F_nH_n = R_{2n+2}$. This fact reveals the relations between our approach and the algebro-geometric solutions of the HS2 hierarchy.

Remark 3.3. The Baker-Akhiezer function ψ of the stationary $HS2$ hierarchy is formally analogous to that defined in the context of KdV or AKNS hierarchies. However, its actual properties in a neighborhood of its essential singularity will feature characteristic differences to standard Baker-Akhiezer functions (cf. Remark 4.2).

Next, we derive Dubrovin-type equations, that is, first-order coupled systems of differential equations that govern the dynamics of $\mu_j(x)$ and $\nu_l(x)$ with respect to variations of x.

Lemma 3.4. *Assume* (2.1) *and the nth stationary HS*2 *equation* (2.33) *holds subject to the constraint* (2.27).

(i) *Suppose that the zeros $\{\mu_j(x)\}_{j=0,\dots,n}$ of $F_n(z,x)$ remain distinct for $x \in \Omega_\mu$, where $\Omega_\mu \subseteq \mathbb{R}$ is an open interval, then $\{\mu_j(x)\}_{j=0,\dots,n}$ satisfy the system of differential equations,*

$$\mu_{j,x} = 4\frac{y(\hat{\mu}_j)}{\mu_j}\prod_{\substack{k=0\\k\neq j}}^{n}(\mu_j(x)-\mu_k(x))^{-1}, \quad j=0,\dots,n, \tag{3.27}$$

with initial conditions

$$\{\hat{\mu}_j(x_0)\}_{j=0,\dots,n} \in \mathscr{K}_n, \tag{3.28}$$

for some fixed $x_0 \in \Omega_\mu$. The initial value problem (3.27), (3.28) *has a unique solution satisfying*

$$\hat{\mu}_j \in C^\infty(\Omega_\mu, \mathscr{K}_n), \quad j=0,\dots,n. \tag{3.29}$$

(ii) *Suppose that the zeros $\{\nu_l(x)\}_{l=0,\dots,n}$ of $H_n(z,x)$ remain distinct for $x \in \Omega_\nu$, where $\Omega_\nu \subseteq \mathbb{R}$ is an open interval, then $\{\nu_l(x)\}_{l=0,\dots,n}$ satisfy the system of differential equations,*

$$\nu_{l,x} = -2\frac{(\rho^2+u_{xx}\nu_l)y(\hat{\nu}_l)}{h_0\,\nu_l}\prod_{\substack{k=0\\k\neq l}}^{n}(\nu_l(x)-\nu_k(x))^{-1}, \quad l=0,\dots,n, \tag{3.30}$$

with initial conditions

$$\{\hat{\nu}_l(x_0)\}_{l=0,\dots,n} \in \mathscr{K}_n, \tag{3.31}$$

for some fixed $x_0 \in \Omega_\nu$. The initial value problem (3.30), (3.31) *has a unique solution satisfying*

$$\hat{\nu}_l \in C^\infty(\Omega_\nu, \mathscr{K}_n), \quad l=0,\dots,n. \tag{3.32}$$

Proof. It suffices to prove (3.27) and (3.29) since the proof of (3.30) and (3.32) follow in an identical manner. Differentiating (3.11) with respect to x then yields

$$F_{n,x}(\mu_j) = -\frac{1}{2}\mu_{j,x}\prod_{\substack{k=0\\k\neq j}}^{n}(\mu_j(x)-\mu_k(x)). \tag{3.33}$$

On the other hand, taking into account equation (2.14), one finds

$$F_{n,x}(\mu_j) = 2G_n(\mu_j) = 2\frac{y(\hat{\mu}_j)}{-\mu_j}. \tag{3.34}$$

Then combining equation (3.33) with (3.34) leads to (3.27). The proof of smoothness assertion (3.29) is analogous to the KdV case in [21]. □

Next, we turn to the trace formulas of the HS2 invariants, that is, expressions of f_l and h_l in terms of symmetric functions of the zeros μ_j and ν_l of F_n and H_n, respectively. For simplicity, we just record the simplest case.

Lemma 3.5. *Suppose* (2.1)*, assume the nth stationary HS*2 *equation* (2.33) *holds, and let* $x \in \mathbb{R}$. *Then*

$$u(x) = \frac{1}{2}\sum_{j=0}^{n}\mu_j(x) - \frac{1}{2}\sum_{m=0}^{2n+1}E_m. \tag{3.35}$$

Proof. Equation (3.35) follows by considering the coefficient of z^n in F_n in (2.28) and (3.11), which yields

$$-u + c_1 = -\frac{1}{2}\sum_{j=0}^{n}\mu_j. \tag{3.36}$$

The constant c_1 can be determined by a long straightforward calculation considering the coefficient of z^{2n+1} in (2.18), which results in

$$c_1 = -\frac{1}{2}\sum_{m=0}^{2n+1}E_m. \tag{3.37}$$

4. Stationary algebro-geometric solutions of HS2 hierarchy

In this section, we obtain explicit Riemann theta function representations for the meromorphic function ϕ, and especially, for the solutions u,ρ of the stationary HS2 hierarchy.

We begin with the asymptotic properties of ϕ and ψ_j, $j = 1,2$.

Lemma 4.1. *Suppose* (2.1)*, assume the nth stationary HS*2 *equation* (2.33) *holds, and let* $P = (z,y) \in \mathcal{K}_n \setminus \{P_{\infty_+}, P_{\infty_-}, P_0\}$, $(x,x_0) \in \mathbb{R}^2$. *Then*

$$\phi(P) \underset{\zeta\to 0}{=} -u_x(x) + O(\zeta), \quad P \to P_{\infty_\pm}, \quad \zeta = z^{-1}, \tag{4.1}$$

$$\phi(P) \underset{\zeta\to 0}{=} i\rho(x) + \frac{iu_{xx}(x)-\rho_x(x)}{2\rho(x)}\zeta + O(\zeta^2), \quad P \to P_0, \quad \zeta = z, \tag{4.2}$$

and

$$\psi_1(P,x,x_0) \underset{\zeta\to 0}{=} \exp\left((u(x_0)-u(x))\zeta + O(\zeta^2)\right), \quad P\to P_{\infty_\pm}, \quad \zeta = z^{-1}, \tag{4.3}$$

$$\psi_2(P,x,x_0) \underset{\zeta\to 0}{=} O(\zeta)\exp\left((u(x_0)-u(x))\zeta + O(\zeta^2)\right), \quad P\to P_{\infty_\pm}, \quad \zeta = z^{-1}, \tag{4.4}$$

$$\psi_1(P,x,x_0) \underset{\zeta\to 0}{=} \exp\left(\frac{i}{\zeta}\int_{x_0}^{x} dx'\, \rho(x') + O(1)\right), \quad P\to P_0, \quad \zeta = z, \tag{4.5}$$

$$\psi_2(P,x,x_0) \underset{\zeta\to 0}{=} O(\zeta^{-1})\exp\left(\frac{i}{\zeta}\int_{x_0}^{x} dx'\, \rho(x') + O(1)\right), \quad P\to P_0, \quad \zeta = z. \tag{4.6}$$

Proof. The existence of the asymptotic expansions of ϕ in terms of the appropriate local coordinates $\zeta = z^{-1}$ near $P_{\infty_\pm}$ and $\zeta = z$ near P_0 is clear from its explicit expression in (3.9). Next, we compute the coefficients of these expansions utilizing the Riccati-type equation (3.17). Indeed, inserting the ansatz

$$\phi \underset{z\to\infty}{=} \phi_0 + \phi_1 z^{-1} + O(z^{-2}) \tag{4.7}$$

into (3.17) and comparing the same powers of z then yields (4.1). Similarly, inserting the ansatz

$$\phi \underset{z\to 0}{=} \phi_0 + \phi_1 z + O(z^2) \tag{4.8}$$

into (3.17) and comparing the same powers of z then yields (4.2). Finally, expansions (4.3)-(4.6) follow from (3.8), (3.10), (4.1), and (4.2). $\square$

Remark 4.2. We note the unusual fact that P_0, as opposed to $P_{\infty\pm}$, is the essential singularity of ψ_j, $j = 1,2$. In addition, one easily finds the leading-order exponential term in ψ_j, $j = 1,2$, near P_0 is x-dependent, which makes matters worse. This is in sharp contrast to standard Baker-Akhiezer functions that typically feature a linear behavior with respect to x in connection with their essential singularities of the type $\exp(c(x-x_0)\zeta^{-1})$ near $\zeta = 0$.

Next, we introduce the holomorphic differentials $\eta_l(P)$ on $\mathscr{K}_n$

$$\eta_l(P) = \frac{a\, z^{l-1}}{y(P)}dz, \quad l = 1,\ldots,n, \tag{4.9}$$

and choose a homology basis $\{a_j, b_j\}_{j=1}^n$ on $\mathscr{K}_n$ in such a way that the intersection matrix of the cycles satisfies

$$a_j \circ b_k = \delta_{j,k}, \quad a_j \circ a_k = 0, \quad b_j \circ b_k = 0, \quad j,k = 1,\ldots,n.$$

Associated with $\mathscr{K}_n$, one introduces an invertible matrix $E \in GL(n,\mathbb{C})$

$$\begin{aligned} &E = (E_{j,k})_{n\times n}, \quad E_{j,k} = \int_{a_k}\eta_j, \\ &\underline{c}(k) = (c_1(k),\ldots,c_n(k)), \quad c_j(k) = (E^{-1})_{j,k}, \end{aligned} \tag{4.10}$$

and the normalized holomorphic differentials

$$\omega_j = \sum_{l=1}^{n} c_j(l)\eta_l, \quad \int_{a_k}\omega_j = \delta_{j,k}, \quad \int_{b_k}\omega_j = \tau_{j,k}, \quad j,k = 1,\ldots,n. \tag{4.11}$$

Apparently, the matrix τ is symmetric and has a positive-definite imaginary part.

We choose a fixed base point $Q_0 \in \mathscr{K}_n \setminus \{\hat{\mu}_0(x), \hat{\nu}_0(x)\}$. The Abel maps $\underline{A}_{Q_0}(\cdot)$ and $\underline{\alpha}_{Q_0}(\cdot)$ are defined by

$$\begin{aligned}&\underline{A}_{Q_0} : \mathscr{K}_n \to J(\mathscr{K}_n) = \mathbb{C}^n/L_n,\\ &P \mapsto \underline{A}_{Q_0}(P) = (A_{Q_0,1}(P), \ldots, A_{Q_0,n}(P)) = \left(\int_{Q_0}^P \omega_1, \ldots, \int_{Q_0}^P \omega_n\right) (\mathrm{mod}\ L_n)\end{aligned} \tag{4.12}$$

and

$$\begin{aligned}&\underline{\alpha}_{Q_0} : \mathrm{Div}(\mathscr{K}_n) \to J(\mathscr{K}_n),\\ &\mathscr{D} \mapsto \underline{\alpha}_{Q_0}(\mathscr{D}) = \sum_{P\in\mathscr{K}_n} \mathscr{D}(P)\underline{A}_{Q_0}(P),\end{aligned} \tag{4.13}$$

where $L_n = \{\underline{z} \in \mathbb{C}^n |\ \underline{z} = \underline{N} + \tau\underline{M},\ \underline{N},\ \underline{M} \in \mathbb{Z}^n\}$.

Let $\Phi_n^{(j)}(\bar{\mu})$, $\Psi_{n+1}(\bar{\mu})$ be the symmetric functions of $\{\mu_j\}_{j=0,\ldots,n}$,

$$\Phi_n^{(j)}(\bar{\mu}) = (-1)^n \prod_{\substack{p=0\\ p\neq j}}^n \mu_p, \quad \Psi_{n+1}(\bar{\mu}) = (-1)^{n+1} \prod_{p=0}^n \mu_p. \tag{4.14}$$

Here, $\bar{\mu}(x) = (\mu_0(x), \mu_1(x), \ldots, \mu_n(x)) = \mu_0(x)\underline{\mu}(x)$.

The following result shows the nonlinearity of the Abel map with respect to the variable x, which indicates a characteristic difference between the HS2 hierarchy and other completely integrable systems such as the KdV and AKNS hierarchies.

Theorem 4.3. *Assume* (2.27) *and suppose that* $\{\hat{\mu}_j(x)\}_{j=0,\ldots,n}$ *satisfies the stationary Dubrovin equations* (3.27) *on an open interval* $\Omega_\mu \subseteq \mathbb{R}$ *such that* $\mu_j(x), j = 0, \ldots, n$, *remain distinct and nonzero for* $x \in \Omega_\mu$. *Introducing the associated divisor* $\mathscr{D}_{\hat{\mu}_0(x)\underline{\hat{\mu}}(x)}$, *one computes*

$$\partial_x \underline{\alpha}_{Q_0}(\mathscr{D}_{\hat{\mu}_0(x)\underline{\hat{\mu}}(x)}) = -\frac{4a}{\Psi_{n+1}(\bar{\mu}(x))}\underline{c}(1), \quad x \in \Omega_\mu. \tag{4.15}$$

In particular, the Abel map does not linearize the divisor $\mathscr{D}_{\hat{\mu}_0(x)\underline{\hat{\mu}}(x)}$ *on* Ω_μ.

Proof. Let $x \in \Omega_\mu$. Then, using

$$\frac{1}{\mu_j} = \frac{\prod_{\substack{p=0\\ p\neq j}}^n \mu_p}{\prod_{p=0}^n \mu_p} = -\frac{\Phi_n^{(j)}(\bar{\mu})}{\Psi_{n+1}(\bar{\mu})}, \quad j = 0, \ldots, n, \tag{4.16}$$

one obtains

$$\begin{aligned}\partial_x \underline{\alpha}_{Q_0}(\mathscr{D}_{\hat{\mu}_0(x)\underline{\hat{\mu}}(x)}) &= \partial_x \left(\sum_{j=0}^n \int_{Q_0}^{\hat{\mu}_j} \underline{\omega}\right) = \sum_{j=0}^n \mu_{j,x} \sum_{k=1}^n \underline{c}(k) \frac{a\,\mu_j^{k-1}}{y(\hat{\mu}_j)}\\ &= \sum_{j=0}^n \sum_{k=1}^n \frac{4a\,\mu_j^{k-1}}{\mu_j} \frac{1}{\prod_{\substack{l=0\\ l\neq j}}^n (\mu_j - \mu_l)} \underline{c}(k)\\ &= -\frac{4a}{\Psi_{n+1}(\bar{\mu})} \sum_{j=0}^n \sum_{k=1}^n \underline{c}(k) \frac{\mu_j^{k-1}}{\prod_{\substack{l=0\\ l\neq j}}^n (\mu_j - \mu_l)} \Phi_n^{(j)}(\bar{\mu})\end{aligned}$$

$$
\begin{aligned}
&= -\frac{4a}{\Psi_{n+1}(\bar{\mu})}\sum_{j=0}^{n}\sum_{k=1}^{n}\underline{c}(k)(U_{n+1}(\bar{\mu}))_{k,j}(U_{n+1}(\bar{\mu}))^{-1}_{j,1}\\
&= -\frac{4a}{\Psi_{n+1}(\bar{\mu})}\sum_{k=1}^{n}\underline{c}(k)\delta_{k,1}\\
&= -\frac{4a}{\Psi_{n+1}(\bar{\mu})}\underline{c}(1),
\end{aligned}
\tag{4.17}
$$

where we used the notation $\underline{\omega} = (\omega_1,\ldots,\omega_n)$, and the relations (cf.(E.13), (E.14) [21]),

$$
U_{n+1}(\bar{\mu}) = \left(\frac{\mu_j^{k-1}}{\prod_{\substack{l=0\\ l\neq j}}^{n}(\mu_j-\mu_l)}\right)_{\substack{j=0\\ k=1}}^{n}, \quad U_{n+1}(\bar{\mu})^{-1} = \left(\Phi_n^{(j)}(\bar{\mu})\right)_{j=0}^{n}. \tag{4.18}
$$

The analogous results hold for the corresponding divisor $\mathscr{D}_{\hat{\nu}_0(x)\underline{\hat{\nu}}(x)}$ associated with $\phi(P,x)$.

Next, we introduce

$$
\begin{aligned}
&\underline{\widehat{B}}_{Q_0} : \mathscr{K}_n\setminus\{P_{\infty_+},P_{\infty_-}\}\to\mathbb{C}^n,\\
&P\mapsto \underline{\widehat{B}}_{Q_0}(P) = (\widehat{B}_{Q_0,1},\ldots,\widehat{B}_{Q_0,n})\\
&\qquad = \begin{cases} \int_{Q_0}^{P}\tilde{\omega}^{(3)}_{P_{\infty_+},P_{\infty_-}}, & n=1,\\ \left(\int_{Q_0}^{P}\eta_2,\ldots,\int_{Q_0}^{P}\eta_n,\int_{Q_0}^{P}\tilde{\omega}^{(3)}_{P_{\infty_+},P_{\infty_-}}\right), & n\geq 2,\end{cases}
\end{aligned}
\tag{4.19}
$$

where $\tilde{\omega}^{(3)}_{P_{\infty_+},P_{\infty_-}} = a\, z^n dz/y(P)$ (cf.(F.53) [21]) and

$$
\begin{aligned}
&\underline{\hat{\beta}}_{Q_0} : \mathrm{Sym}^n(\mathscr{K}_n\setminus\{P_{\infty_+},P_{\infty_-}\})\to\mathbb{C}^n,\\
&\mathscr{D}_{\underline{Q}}\mapsto \underline{\hat{\beta}}_{Q_0}(\mathscr{D}_{\underline{Q}}) = \sum_{j=1}^{n}\underline{\widehat{B}}_{Q_0}(Q_j), \quad \underline{Q} = \{Q_1,\ldots,Q_n\}\in\mathrm{Sym}^n(\mathscr{K}_n\setminus\{P_{\infty_+},P_{\infty_-}\}),
\end{aligned}
\tag{4.20}
$$

choosing identical paths of integration from Q_0 to P in all integrals in (4.19) and (4.20). Then, one obtains the following result.

Corollary 4.4. *Assume* (2.27) *and suppose that* $\{\hat{\mu}_j(x)\}_{j=0,\ldots,n}$ *satisfies the stationary Dubrovin equations* (3.27) *on an open interval* $\Omega_\mu\subseteq\mathbb{R}$ *such that* $\mu_j(x), j=0,\ldots,n$, *remain distinct and nonzero for* $x\in\Omega_\mu$. *Then one computes*

$$
\partial_x\sum_{j=0}^{n}\int_{Q_0}^{\hat{\mu}_j(x)}\eta_1 = -\frac{4a}{\Psi_{n+1}(\bar{\mu}(x))}, \quad x\in\Omega_\mu, \tag{4.21}
$$

$$
\partial_x\underline{\hat{\beta}}(\mathscr{D}_{\underline{\hat{\mu}}(x)}) = \begin{cases} 4a, & n=1,\\ 4a\,(0,\ldots,0,1), & n\geq 2,\end{cases} \quad x\in\Omega_\mu. \tag{4.22}
$$

Proof. Equation (4.21) is a special case of (4.15). Equation (4.22) follows from (4.17). □

The fact that the Abel map does not provide the proper change of variables to linearize the divisor $\mathscr{D}_{\hat{\mu}_0(x)\underline{\hat{\mu}}(x)}$ in the HS2 context is in sharp contrast to standard integrable soliton equations such as the KdV and AKNS hierarchies. However, the change of variables

$$x \mapsto \tilde{x} = \int^x dx' \Big(\frac{4a}{\Psi_{n+1}(\bar{\mu}(x'))} \Big) \tag{4.23}$$

linearizes the Abel map $\underline{A}_{Q_0}(\mathscr{D}_{\hat{\mu}_0(\tilde{x})\underline{\hat{\mu}}(\tilde{x})})$, $\tilde{\mu}_j(\tilde{x}) = \mu_j(x), j = 0, \dots, n$. The intricate relation between the variable x and $\tilde{x}$ is detailed in (4.34).

Next, given the Riemann surface $\mathscr{K}_n$ and the homology basis $\{a_j, b_j\}_{j=1,\dots,n}$, one introduces the Riemann theta function by

$$\theta(\underline{z}) = \sum_{\underline{n}\in\mathbb{Z}^n} \exp\Big(2\pi i(\underline{n},\underline{z}) + \pi i(\underline{n},\tau\underline{n})\Big), \ \ \underline{z} \in \mathbb{C}^n,$$

where $(\underline{A},\underline{B}) = \sum_{j=1}^n \overline{A}_j B_j$ denotes the scalar product in $\mathbb{C}^n$.

Let

$$\omega^{(3)}_{\hat{\mu}_0(x),\hat{\nu}_0(x)}(P) = \left(\frac{y - \mu_0 G_n(x)}{z - \mu_0} - \frac{y + \nu_0 G_n(x)}{z - \nu_0} \right) \frac{dz}{2y} + \frac{\lambda_n}{y} \prod_{j=1}^{n-1} (z - \lambda_j) dz \tag{4.24}$$

be the normalized differential of the third kind holomorphic on $\mathscr{K}_n \setminus \{\hat{\mu}_0(x), \hat{\nu}_0(x)\}$ with simple poles at $\hat{\mu}_0(x)$ and $\hat{\nu}_0(x)$ and residues 1 and -1, respectively,

$$\omega^{(3)}_{\hat{\mu}_0(x),\hat{\nu}_0(x)}(P) \underset{\zeta\to 0}{=} (\zeta^{-1} + O(1))d\zeta, \quad \text{as } P \to \hat{\mu}_0(x), \tag{4.25}$$

$$\omega^{(3)}_{\hat{\mu}_0(x),\hat{\nu}_0(x)}(P) \underset{\zeta\to 0}{=} (-\zeta^{-1} + O(1))d\zeta, \quad \text{as } P \to \hat{\nu}_0(x), \tag{4.26}$$

where ζ in (4.25) and (4.26) denotes the corresponding local coordinate near $\hat{\mu}_0(x)$ and $\hat{\nu}_0(x)$ (cf. Appendix C [21]). The constants $\{\lambda_j\}_{j=1,\dots,n}$ in (4.24) are determined by the normalization condition

$$\int_{a_k} \omega^{(3)}_{\hat{\mu}_0(x),\hat{\nu}_0(x)} = 0, \quad k = 1, \dots, n.$$

Then

$$\int_{Q_0}^{P} \omega^{(3)}_{\hat{\mu}_0(x),\hat{\nu}_0(x)}(P) \underset{\zeta\to 0}{=} \ln\zeta + e_0 + O(\zeta), \quad \text{as } P \to \hat{\mu}_0(x), \tag{4.27}$$

$$\int_{Q_0}^{P} \omega^{(3)}_{\hat{\mu}_0(x),\hat{\nu}_0(x)}(P) \underset{\zeta\to 0}{=} -\ln\zeta + d_0 + O(\zeta), \quad \text{as } P \to \hat{\nu}_0(x), \tag{4.28}$$

for some constants $e_0, d_0 \in \mathbb{C}$. We also record

$$\underline{A}_{Q_0}(P) - \underline{A}_{Q_0}(P_{\infty_\pm}) \underset{\zeta\to 0}{=} \pm\underline{U}\zeta + O(\zeta^2), \quad \text{as } P \to P_{\infty_\pm}, \quad \underline{U} = \underline{c}(n). \tag{4.29}$$

In the following, it will be convenient to introduce the abbreviations

$$\begin{aligned} &\underline{z}(P,\underline{Q}) = \underline{\Xi}_{Q_0} - \underline{A}_{Q_0}(P) + \underline{\alpha}_{Q_0}(\mathscr{D}_{\underline{Q}}), \\ &P \in \mathscr{K}_n, \ \underline{Q} = (Q_1, \dots, Q_n) \in \mathrm{Sym}^n(\mathscr{K}_n), \end{aligned} \tag{4.30}$$

where $\underline{\Xi}_{Q_0}$ is the vector of Riemann constants (cf.(A.45) [21]). It turns out that $\underline{z}(\cdot,\underline{Q})$ is independent of the choice of base point Q_0 (cf.(A.52), (A.53) [21]).

Based on above preparations, we will give explicit representations for the meromorphic function ϕ and the stationary HS2 solutions u,ρ in terms of the Riemann theta function associated with $\mathcal{K}_n$.

Theorem 4.5. *Suppose* (2.1)*, and assume the nth stationary HS*2 *equation* (2.33) *holds on* Ω *subject to the constraint* (2.27)*. Moreover, let* $P=(z,y)\in\mathcal{K}_n\setminus\{P_0\}$ *and* $x\in\Omega$*, where* $\Omega\subseteq\mathbb{R}$ *is an open interval. In addition, suppose that* $\mathcal{D}_{\underline{\hat{\mu}}(x)}$*, or equivalently,* $\mathcal{D}_{\underline{\hat{\nu}}(x)}$ *is nonspecial for* $x\in\Omega$*. Then,* ϕ*,* u*, and* ρ *admit the following representations*

$$\phi(P,x)=i\rho(x)\frac{\theta(\underline{z}(P,\underline{\hat{\nu}}(x)))\theta(\underline{z}(P_0,\underline{\hat{\mu}}(x)))}{\theta(\underline{z}(P_0,\underline{\hat{\nu}}(x)))\theta(\underline{z}(P,\underline{\hat{\mu}}(x)))}\exp\left(e_0-\int_{Q_0}^{P}\omega^{(3)}_{\hat{\mu}_0(x),\hat{\nu}_0(x)}\right), \tag{4.31}$$

$$u(x)=-\frac{1}{2}\sum_{m=0}^{2n+1}E_m+\frac{1}{2}\sum_{j=1}^{n}\lambda_j-\frac{1}{2}\sum_{j=1}^{n}U_j\partial_{\omega_j}\ln\left(\frac{\theta(\underline{z}(P_{\infty_+},\underline{\hat{\mu}}(x))+\underline{\omega})}{\theta(\underline{z}(P_{\infty_-},\underline{\hat{\mu}}(x))+\underline{\omega})}\right)\Big|_{\underline{\omega}=0}, \tag{4.32}$$

$$\rho(x)=iu_x(x)\frac{\theta(\underline{z}(P_0,\underline{\hat{\nu}}(x)))\theta(\underline{z}(P_{\infty_+},\underline{\hat{\mu}}(x)))}{\theta(\underline{z}(P_{\infty_+},\underline{\hat{\nu}}(x)))\theta(\underline{z}(P_0,\underline{\hat{\mu}}(x)))}. \tag{4.33}$$

Moreover, let $\widetilde{\Omega}\subseteq\Omega$ *be such that* μ_j*,* $j=0,\dots,n$*, are nonvanishing on* $\widetilde{\Omega}$*. Then, the constraint*

$$\begin{aligned}4a(x-x_0)=&-4a\int_{x_0}^{x}\frac{dx'}{\prod_{k=0}^{n}\mu_k(x')}\sum_{j=1}^{n}\Big(\int_{a_j}\tilde{\omega}^{(3)}_{P_{\infty_+}P_{\infty_-}}\Big)c_j(1)\\&+\ln\left(\frac{\theta(\underline{z}(P_{\infty_-},\underline{\hat{\mu}}(x_0)))\theta(\underline{z}(P_{\infty_+},\underline{\hat{\mu}}(x)))}{\theta(\underline{z}(P_{\infty_+},\underline{\hat{\mu}}(x_0)))\theta(\underline{z}(P_{\infty_-},\underline{\hat{\mu}}(x)))}\right),\quad x,x_0\in\widetilde{\Omega}\end{aligned} \tag{4.34}$$

holds, with

$$\begin{aligned}\underline{\hat{\alpha}}_{Q_0}(\mathcal{D}_{\hat{\mu}_0(x)\underline{\hat{\mu}}(x)})&=\underline{\hat{\alpha}}_{Q_0}(\mathcal{D}_{\hat{\mu}_0(x_0)\underline{\hat{\mu}}(x_0)})-4a\int_{x_0}^{x}\frac{dx'}{\Psi_{n+1}(\bar{\mu}(x'))}\underline{c}(1)\\&=\underline{\hat{\alpha}}_{Q_0}(\mathcal{D}_{\hat{\mu}_0(x)\underline{\hat{\mu}}(x_0)})-\underline{c}(1)(\tilde{x}-\tilde{x}_0),\quad x\in\widetilde{\Omega}.\end{aligned} \tag{4.35}$$

Proof. First, we temporarily assume that

$$\mu_j(x)\neq\mu_{j'}(x),\quad \nu_k(x)\neq\nu_{k'}(x)\quad\text{for } j\neq j',k\neq k' \text{ and } x\in\widetilde{\Omega}, \tag{4.36}$$

for appropriate $\widetilde{\Omega}\subseteq\Omega$. Since by (3.15), $\mathcal{D}_{\hat{\nu}_0\underline{\hat{\nu}}}\sim\mathcal{D}_{\hat{\mu}_0\underline{\hat{\mu}}}$, and $(\hat{\mu}_0)^*\notin\{\hat{\mu}_1,\dots,\hat{\mu}_n\}$ by hypothesis, one can use Theorem A.31 [21] to conclude that $\mathcal{D}_{\underline{\hat{\nu}}}\in\operatorname{Sym}^n(\mathcal{K}_n)$ is nonspecial. This argument is of course symmetric with respect to $\underline{\hat{\mu}}$ and $\underline{\hat{\nu}}$. Thus, $\mathcal{D}_{\underline{\hat{\mu}}}$ is nonspecial if and only if $\mathcal{D}_{\underline{\hat{\nu}}}$ is.

Next, we derive the representations of ϕ, u, and ρ in terms of the Riemann theta function. A special case of Riemann's vanishing theorem (cf. Theorem A.26 [21]) yields

$$\theta(\underline{\Xi}_{Q_0}-\underline{A}_{Q_0}(P)+\underline{\alpha}_{Q_0}(\mathcal{D}_{\underline{Q}}))=0\quad\text{if and only if } P\in\{Q_1,\dots,Q_n\}. \tag{4.37}$$

Therefore, the divisor (3.15) shows that $\phi(P,x)$ has expression of the type

$$C(x)\frac{\theta(\underline{\Xi}_{Q_0}-\underline{A}_{Q_0}(P)+\underline{\alpha}_{Q_0}(\mathcal{D}_{\underline{\hat{\nu}}(x)}))}{\theta(\underline{\Xi}_{Q_0}-\underline{A}_{Q_0}(P)+\underline{\alpha}_{Q_0}(\mathcal{D}_{\underline{\hat{\mu}}(x)}))}\exp\Big(e_0-\int_{Q_0}^{P}\omega^{(3)}_{\hat{\mu}_0(x),\hat{\nu}_0(x)}\Big), \tag{4.38}$$

where $C(x)$ is independent of $P\in\mathcal{K}_n$. Then taking into account the asymptotic expansion of $\phi(P,x)$ near P_0 in (4.2), we obtain (4.31). The representation (4.32) for u on $\widetilde{\Omega}$ follows from trace formula (3.35) and the expression (F.59) [21] for $\sum_{j=0}^{n}\mu_j$. The representation (4.33) for ρ on $\widetilde{\Omega}$ is clear from

(4.1) and (4.31). By continuity, (4.31), (4.32), and (4.33) extend from $\widetilde{\Omega}$ to Ω. Assuming $\mu_j \neq 0$, $j = 0,\ldots,n$, the constraint (4.34) follows by combining (4.21), (4.22), and (F.58) [21]. Equation (4.35) is clear from (4.15). Finally, (4.34) and (4.35) extend to $\widetilde{\Omega}$ by continuity. □

Remark 4.6. We note that the stationary $HS2$ solutions u,ρ in (4.32) and (4.33) are meromorphic quasi-periodic functions with respect to the new variable $\tilde{x}$ in (4.23). In addition, the Abel map in (4.35) linearizes the divisor $\mathscr{D}_{\hat{\mu}_0(x)\underline{\hat{\mu}}(x)}$ with respect to $\tilde{x}$ under the constraint (4.34).

Remark 4.7. Since by (3.15) $\mathscr{D}_{\hat{\nu}_0\underline{\hat{\nu}}}$ and $\mathscr{D}_{\hat{\mu}_0\underline{\hat{\mu}}}$ are linearly equivalent, that is

$$\underline{A}_{Q_0}(\hat{\mu}_0(x)) + \underline{\alpha}_{Q_0}(\mathscr{D}_{\underline{\hat{\mu}}(x)}) = \underline{A}_{Q_0}(\hat{\nu}_0(x)) + \underline{\alpha}_{Q_0}(\mathscr{D}_{\underline{\hat{\nu}}(x)}). \tag{4.39}$$

Then one infers

$$\underline{\alpha}_{Q_0}(\mathscr{D}_{\underline{\hat{\nu}}(x)}) = \underline{\Delta} + \underline{\alpha}_{Q_0}(\mathscr{D}_{\underline{\hat{\mu}}(x)}), \quad \underline{\Delta} = \underline{A}_{\hat{\nu}_0(x)}(\hat{\mu}_0(x)). \tag{4.40}$$

Hence, one can eliminate $\mathscr{D}_{\underline{\hat{\nu}}(x)}$ in (4.31), in terms of $\mathscr{D}_{\underline{\hat{\mu}}(x)}$ using

$$\underline{z}(P,\underline{\hat{\nu}}) = \underline{z}(P,\underline{\hat{\mu}}) + \underline{\Delta}, \quad P \in \mathscr{K}_n. \tag{4.41}$$

Remark 4.8. We emphasized in Remark 4.2 that ψ in (3.8) and (3.10) differs from standard Baker-Akhiezer functions. Hence, one can not expect the usual theta function representation of ψ_j, $j = 1,2$, in terms of ratios of theta functions times an exponential term containing a meromorphic differential with a pole at the essential singularity of ψ_j multiplied by $(x - x_0)$. However, using $(E.3)$ and $(F.59)$ [21], one computes

$$\begin{aligned} F_n(z) &= \frac{1}{2}z^{n+1} + \frac{1}{2}\sum_{k=0}^{n}\Psi_{n+1-k}(\bar{\mu}(x))z^k \\ &= \frac{1}{2}z^{n+1} + \frac{1}{2}\sum_{k=1}^{n}\left(\Psi_{n+1-k}(\underline{\lambda}) - \sum_{j=1}^{n}c_j(k)\partial_{\omega_j}\ln\left(\frac{\theta(\underline{z}(P_{\infty_+},\underline{\hat{\mu}}(x)) + \underline{\omega})}{\theta(\underline{z}(P_{\infty_-},\underline{\hat{\mu}}(x)) + \underline{\omega})}\right)\Big|_{\underline{\omega}=0}\right)z^k \\ &= \frac{1}{2}\prod_{j=0}^{n}(z-\lambda_j) - \frac{1}{2}\sum_{j=1}^{n}\sum_{k=1}^{n}c_j(k)\partial_{\omega_j}\ln\left(\frac{\theta(\underline{z}(P_{\infty_+},\underline{\hat{\mu}}(x)) + \underline{\omega})}{\theta(\underline{z}(P_{\infty_-},\underline{\hat{\mu}}(x)) + \underline{\omega})}\right)\Big|_{\underline{\omega}=0}z^k, \end{aligned} \tag{4.42}$$

and hence obtains the theta function representation of ψ_1 upon inserting (4.42) into (3.21). Then, the corresponding theta function representation of ψ_2 follows by (3.10) and (4.31).

At the end of this section, we turn to the initial value problem for the stationary HS2 hierarchy. We will show that the solvability of the Dubrovin equations (3.27) on $\Omega_\mu \subseteq \mathbb{R}$ in fact implies the stationary HS2 equation (2.33) on Ω_μ.

Theorem 4.9. *Fix $n \in \mathbb{N}_0$, assume (2.27), and suppose that $\{\hat{\mu}_j\}_{j=0,\ldots,n}$ satisfies the stationary Dubrovin equations (3.27) on Ω_μ such that μ_j, $j = 0,\ldots,n$, remain distinct and nonzero on Ω_μ,* *where $\Omega_\mu \subseteq \mathbb{R}$ is an open interval. Then, $u, \rho \in C^\infty(\Omega_\mu)$, defined by*

$$u = -\frac{1}{2}\sum_{m=0}^{2n+1} E_m + \frac{1}{2}\sum_{j=0}^{n} \mu_j, \tag{4.43}$$

and

$$\rho^2 = u_x^2 + 2uu_{xx} - \frac{d^2}{dx^2}\left(\Psi_2(\bar{\mu}) - u\sum_{m=0}^{2n+1} E_m\right), \tag{4.44}$$

*satisfy the nth stationary HS*2 *equation* (2.33), *that is,*

$$\text{s-HS2}_n(u,\rho) = 0 \text{ on } \Omega_\mu. \tag{4.45}$$

Proof. Given the solutions $\hat{\mu}_j = (\mu_j, y(\hat{\mu}_j)) \in C^\infty(\Omega_\mu, \mathscr{K}_n)$, $j = 0, \ldots, n$ of (3.27), we introduce

$$\begin{aligned} F_n(z) &= \frac{1}{2}\prod_{j=0}^{n}(z-\mu_j), \\ G_n(z) &= \frac{1}{2}F_{n,x}(z), \end{aligned} \tag{4.46}$$

on $\mathbb{C} \times \Omega_\mu$. Taking into account (4.46), the Dubrovin equations (3.27) imply

$$y(\hat{\mu}_j) = \frac{1}{4}\mu_j\mu_{j,x}\prod_{\substack{k=0\\k\neq j}}^{n}(\mu_j - \mu_k) = -\frac{1}{2}\mu_j F_{n,x}(\mu_j) = -\mu_j G_n(\mu_j). \tag{4.47}$$

Hence

$$R_{2n+2}(\mu_j)^2 - \mu_j^2 G_n(\mu_j)^2 = y(\hat{\mu}_j)^2 - \mu_j^2 G_n(\mu_j)^2 = 0, \quad j = 0, \ldots, n. \tag{4.48}$$

Next, we define a polynomial H_n on $\mathbb{C} \times \Omega_\mu$ such that

$$R_{2n+2}(z) - z^2 G_n(z)^2 = F_n(z) H_n(z). \tag{4.49}$$

Such a polynomial H_n exists since the left-hand side of (4.49) vanishes at $z = \mu_j,\ j = 0, \cdots, n$, by (4.48). To determine the degree of H_n, using (4.46), one computes

$$R_{2n+2}(z) - z^2 G_n(z)^2 \underset{|z|\to\infty}{=} \frac{1}{2}h_0 z^{2n+2} + O(z^{2n+1}). \tag{4.50}$$

Then combining (4.46), (4.49), and (4.50), one infers that H_n has degree $n+1$ with respect to z. Hence, we may write

$$H_n(z) = h_0\prod_{l=0}^{n}(z - \nu_l), \quad \text{on } \mathbb{C} \times \Omega_\mu. \tag{4.51}$$

Next, one defines the polynomial P_n by

$$P_n(z) = H_n(z) + (\rho^2 + u_{xx}z)F_n(z) + z^2 G_{n,x}(z). \tag{4.52}$$

Using (4.46) and (4.51), one infers that indeed P_n has degree at most n. Differentiating (4.49) with respect to x yields

$$2z^2 G_n(z) G_{n,x}(z) + F_{n,x}(z) H_n(z) + F_n(z) H_{n,x}(z) = 0. \tag{4.53}$$

Then multiplying (4.52) by G_n and replacing the term $G_nG_{n,x}$ with (4.53) leads to

$$G_n(z)P_n(z) = F_n(z)((\rho^2 + u_{xx}z)G_n(z) - \frac{1}{2}H_{n,x}(z)) + (G_n(z) - \frac{1}{2}F_{n,x}(z))H_n(z), \tag{4.54}$$

and hence

$$G_n(\mu_j)P_n(\mu_j) = 0, \quad j = 0, \dots, n \tag{4.55}$$

on Ω_μ. Restricting $x \in \Omega_\mu$ temporarily to $x \in \widetilde{\Omega}_\mu$, where

$$\begin{aligned} \widetilde{\Omega}_\mu &= \{x \in \Omega_\mu \mid G(\mu_j(x), x) = -\frac{y(\hat{\mu}_j(x))}{\mu_j(x)} \neq 0,\, j = 0, \dots, n\} \\ &= \{x \in \Omega_\mu \mid \mu_j(x) \notin \{E_m\}_{m=0,\dots,2n+1},\, j = 0, \dots, n\}, \end{aligned} \tag{4.56}$$

one infers that

$$P_n(\mu_j(x), x) = 0, \quad j = 0, \dots, n,\ x \in \widetilde{\Omega}_\mu. \tag{4.57}$$

Since $P_n(z)$ has degree at most n, (4.57) implies

$$P_n = 0 \quad \text{on } \mathbb{C} \times \widetilde{\Omega}_\mu, \tag{4.58}$$

and hence (2.16) holds, that is,

$$z^2 G_{n,x}(z) = -H_n(z) - (\rho^2 + u_{xx}z)F_n(z) \tag{4.59}$$

on $\mathbb{C} \times \widetilde{\Omega}_\mu$. Inserting (4.59) and (4.46) into (4.53) yields

$$F_n(z)(H_{n,x}(z) - 2(\rho^2 + u_{xx}z)G_n(z)) = 0, \tag{4.60}$$

namely

$$H_{n,x}(z) = 2(\rho^2 + u_{xx}z)G_n(z) \tag{4.61}$$

on $\mathbb{C} \times \widetilde{\Omega}_\mu$. Thus, we obtain the fundamental equations (2.14)-(2.16) and (2.18) on $\mathbb{C} \times \widetilde{\Omega}_\mu$. In order to extend these results to Ω_μ, we next investigate the case where $\hat{\mu}_j$ hits a branch point $(E_{m_0}, 0)$. Hence, we suppose

$$\mu_{j_1}(x) \to E_{m_0} \quad \text{as } x \to x_0 \in \Omega_\mu \tag{4.62}$$

for some $j_1 \in \{0, \dots, n\}$, $m_0 \in \{0, \dots, 2n+1\}$. Introducing

$$\begin{aligned} &\zeta_{j_1}(x) = \sigma(\mu_{j_1}(x) - E_{m_0})^{1/2}, \quad \sigma = \pm 1, \\ &\mu_{j_1}(x) = E_{m_0} + \zeta_{j_1}(x)^2, \end{aligned} \tag{4.63}$$

for some x in an open interval centered near x_0, the Dubrovin equation (3.27) for μ_{j_1} becomes

$$\begin{aligned} \zeta_{j_1,x}(x) = c(\sigma)\frac{2a}{E_{m_0}}\Big(\prod_{\substack{m=0\\ m\neq m_0}}^{2n+1}(E_{m_0} - E_m)\Big)^{1/2} \\ \times \prod_{\substack{k=0\\ k\neq j_1}}^{n}(E_{m_0} - \mu_k(x))^{-1}(1 + O(\zeta_{j_1}(x)^2)) \end{aligned} \tag{4.64}$$

for some $|c(\sigma)| = 1$. Hence, (4.58)-(4.61) extend to Ω_μ by continuity. We have now established relations (2.14)-(2.16) on $\mathbb{C} \times \Omega_\mu$, and one can proceed as in Section 2 to obtain (4.45). $\square$

Remark 4.10. Although we formulated Theorem 4.9 in terms of $\{\mu_j\}_{j=0,\ldots,n}$ only, the analogous result (and strategy of proof) obviously works in terms of $\{\nu_j\}_{j=0,\ldots,n}$.

Remark 4.11. A closer look at Theorem 4.9 reveals that u, ρ are uniquely determined in an open neighborhood Ω of x_0 by $\mathcal{K}_n$ and the initial condition $(\hat{\mu}_0(x_0), \hat{\mu}_1(x_0), \ldots, \hat{\mu}_n(x_0))$, or equivalently, by the auxiliary divisor $\mathcal{D}_{\hat{\mu}_0(x_0)\underline{\hat{\mu}}(x_0)}$ at $x = x_0$. Conversely, given $\mathcal{K}_n$ and u, ρ in an open neighborhood Ω of x_0, one can construct the corresponding polynomial $F_n(z,x)$, $G_n(z,x)$ and $H_n(z,x)$ for $x \in \Omega$, and then recover the auxiliary divisor $\mathcal{D}_{\hat{\mu}_0(x)\underline{\hat{\mu}}(x)}$ for $x \in \Omega$ from the zeros of $F_n(z,x)$ and from (3.12). In this sense, once the curve $\mathcal{K}_n$ is fixed, elements of the isospectral class of the $HS2$ potentials u, ρ can be characterized by nonspecial auxiliary divisors $\mathcal{D}_{\hat{\mu}_0(x)\underline{\hat{\mu}}(x)}$.

5. The time-dependent HS2 formalism

In this section, we extend the algebro-geometric analysis of Section 3 to the time-dependent HS2 hierarchy.

Throughout this section, we assume (2.2) holds.

The time-dependent algebro-geometric initial value problem of the HS2 hierarchy is to solve the time-dependent rth HS2 flow with a stationary solution of the nth equation as initial data in the hierarchy. More precisely, given $n \in \mathbb{N}_0$, based on the solution $u^{(0)}, \rho^{(0)}$ of the nth stationary HS2 equation s-HS2$_n(u^{(0)}, \rho^{(0)}) = 0$ associated with $\mathcal{K}_n$ and a set of integration constants $\{c_l\}_{l=1,\ldots,n} \subset \mathbb{C}$, we want to construct a solution u, ρ of the rth HS2 flow HS2$_r(u, \rho) = 0$ such that $u(t_{0,r}) = u^{(0)}$, $\rho(t_{0,r}) = \rho^{(0)}$, for some $t_{0,r} \in \mathbb{R}$, $r \in \mathbb{N}_0$.

To emphasize that the integration constants in the definitions of the stationary and the time-dependent HS2 equations are independent of each other, we indicate this by adding a tilde on all the time-dependent quantities. Hence, we employ the notation $\widetilde{V}_r$, $\widetilde{F}_r$, $\widetilde{G}_r$, $\widetilde{H}_r$, $\tilde{f}_s$, $\tilde{g}_s$, $\tilde{h}_s$, $\tilde{c}_s$ in order to distinguish them from V_n, F_n, G_n, H_n, f_l, g_l, h_l, c_l in the following. In addition, we mark the individual rth HS2 flow by a separate time variable $t_r \in \mathbb{R}$.

Summing up, we are seeking a solution u, ρ of the time-dependent algebro-geometric initial value problem

$$\text{HS2}_r(u,\rho) = \begin{pmatrix} -u_{xxt_r} + 2u_{xx}\tilde{f}_{r+1,x} + u_{xxx}\tilde{f}_{r+1} + 2\rho\rho_x\tilde{f}_r + 2\rho^2\tilde{f}_{r,x} \\ 2\rho\rho_{t_r} - 2\rho\rho_x\tilde{f}_{r+1} - 2\rho^2\tilde{f}_{r+1,x} \end{pmatrix} = 0, \tag{5.1}$$

$$(u,\rho)|_{t_r=t_{0,r}} = (u^{(0)}, \rho^{(0)}),$$

$$\text{s-HS2}_n(u^{(0)},\rho^{(0)}) = \begin{pmatrix} 2u_{xx}f_{n+1,x} + u_{xxx}f_{n+1} + 2\rho\rho_x f_n + 2\rho^2 f_{n,x} \\ -2\rho\rho_x f_{n+1} - 2\rho^2 f_{n+1,x} \end{pmatrix} = 0, \tag{5.2}$$

for some $t_{0,r} \in \mathbb{R}$, $n, r \in \mathbb{N}_0$, where $u = u(x,t_r)$, $\rho = \rho(x,t_r)$ satisfy (2.2), and the curve $\mathcal{K}_n$ is associated with the initial data $(u^{(0)}, \rho^{(0)})$ in (5.2). Noticing that the HS2 flows are isospectral, we further assume that (5.2) holds not only for $t_r = t_{0,r}$, but also for all $t_r \in \mathbb{R}$. Hence, we start with the zero-curvature equations

$$U_{t_r} - \widetilde{V}_{r,x} + [U, \widetilde{V}_r] = 0, \tag{5.3}$$

$$-V_{n,x} + [U, V_n] = 0, \tag{5.4}$$

where

$$U(z)=\begin{pmatrix}0 & 1\\ -z^{-2}\rho^2-z^{-1}u_{xx} & 0\end{pmatrix},$$

$$V_n(z)=\begin{pmatrix}-G_n(z) & F_n(z)\\ z^{-2}H_n(z) & G_n(z)\end{pmatrix}, \tag{5.5}$$

$$\widetilde{V}_r(z)=\begin{pmatrix}-\widetilde{G}_r(z) & \widetilde{F}_r(z)\\ z^{-2}\widetilde{H}_r(z) & \widetilde{G}_r(z)\end{pmatrix},$$

and

$$F_n(z)=\sum_{l=0}^{n+1} f_l z^{n+1-l}=f_0\prod_{j=0}^{n}(z-\mu_j), \tag{5.6}$$

$$G_n(z)=\sum_{l=0}^{n} g_l z^{n-l}, \tag{5.7}$$

$$H_n(z)=\sum_{l=0}^{n+1} h_l z^{n+1-l}=h_0\prod_{l=0}^{n}(z-\nu_l), \tag{5.8}$$

$$\widetilde{F}_r(z)=\sum_{s=0}^{r+1}\tilde{f}_s z^{r+1-s}, \tag{5.9}$$

$$\widetilde{G}_r(z)=\sum_{s=0}^{r}\tilde{g}_s z^{r-s}, \tag{5.10}$$

$$\widetilde{H}_r(z)=\sum_{s=0}^{r+1}\tilde{h}_s z^{r+1-s}, \tag{5.11}$$

for fixed $n,r\in\mathbb{N}_0$. Here, $\{f_l\}_{l=0,\dots,n+1}$, $\{g_l\}_{l=0,\dots,n}$, $\{h_l\}_{l=0,\dots,n+1}$, $\{\tilde{f}_s\}_{s=0,\dots,r+1}$, $\{\tilde{g}_s\}_{s=0,\dots,r}$, and $\{\tilde{h}_s\}_{s=0,\dots,r+1}$ are defined as in (2.3), with $u(x),\rho(x)$ replaced by $u(x,t_r),\rho(x,t_r)$ etc., and with appropriate integration constants. Explicitly, (5.3) and (5.4) are equivalent to

$$-2\rho\rho_{t_r}-zu_{xxt_r}-\widetilde{H}_{r,x}+2(\rho^2+zu_{xx})\widetilde{G}_r=0, \tag{5.12}$$

$$\widetilde{F}_{r,x}=2\widetilde{G}_r, \tag{5.13}$$

$$z^2\widetilde{G}_{r,x}=-\widetilde{H}_r-(\rho^2+zu_{xx})\widetilde{F}_r \tag{5.14}$$

and

$$F_{n,x}=2G_n, \tag{5.15}$$

$$H_{n,x}=2(\rho^2+zu_{xx})G_n, \tag{5.16}$$

$$z^2G_{n,x}=-H_n-(\rho^2+zu_{xx})F_n. \tag{5.17}$$

From (5.15)-(5.17), one finds

$$\frac{d}{dx}\det(V_n(z))=-\frac{1}{z^2}\frac{d}{dx}\Big(z^2G_n(z)^2+F_n(z)H_n(z)\Big)=0, \tag{5.18}$$

and meanwhile (see Lemma 5.2)

$$\frac{d}{dt_r}\det(V_n(z))=-\frac{1}{z^2}\frac{d}{dt_r}\Big(z^2G_n(z)^2+F_n(z)H_n(z)\Big)=0. \tag{5.19}$$

Hence, $z^2G_n(z)^2+F_n(z)H_n(z)$ is independent of variables both x and t_r, which implies the fundamental identity (2.18) holds,

$$z^2G_n(z)^2+F_n(z)H_n(z)=R_{2n+2}(z), \tag{5.20}$$

and the hyperelliptic curve $\mathcal{K}_n$ is still given by (2.26).

Next, we define the time-dependent Baker-Akhiezer function $\psi(P,x,x_0,t_r,t_{0,r})$ on $\mathcal{K}_n \setminus \{P_{\infty_\pm},P_0\}$ by

$$\begin{aligned}
&\psi(P,x,x_0,t_r,t_{0,r})=\begin{pmatrix}\psi_1(P,x,x_0,t_r,t_{0,r})\\ \psi_2(P,x,x_0,t_r,t_{0,r})\end{pmatrix},\\
&\psi_x(P,x,x_0,t_r,t_{0,r})=U(u(x,t_r),\rho(x,t_r),z(P))\psi(P,x,x_0,t_r,t_{0,r}),\\
&\psi_{t_r}(P,x,x_0,t_r,t_{0,r})=\widetilde{V}_r(u(x,t_r),\rho(x,t_r),z(P))\psi(P,x,x_0,t_r,t_{0,r}),\\
&zV_n(u(x,t_r),\rho(x,t_r),z(P))\psi(P,x,x_0,t_r,t_{0,r})=y(P)\psi(P,x,x_0,t_r,t_{0,r}),\\
&\psi_1(P,x_0,x_0,t_{0,r},t_{0,r})=1;\\
&P=(z,y)\in\mathcal{K}_n\setminus\{P_{\infty_\pm},P_0\},\ (x,t_r)\in\mathbb{R}^2.
\end{aligned} \tag{5.21}$$

Closely related to $\psi(P,x,x_0,t_r,t_{0,r})$ is the following meromorphic function $\phi(P,x,t_r)$ on $\mathcal{K}_n$ defined by

$$\phi(P,x,t_r)=z\frac{\psi_{1,x}(P,x,x_0,t_r,t_{0,r})}{\psi_1(P,x,x_0,t_r,t_{0,r})},\quad P\in\mathcal{K}_n\setminus\{P_{\infty_\pm},P_0\},\ (x,t_r)\in\mathbb{R}^2 \tag{5.22}$$

such that

$$\begin{aligned}
\psi_1(P,x,x_0,t_r,t_{0,r})=\exp\Big(\int_{t_{0,r}}^{t_r} ds(z^{-1}\widetilde{F}_r(z,x_0,s)\phi(P,x_0,s)\\
-\widetilde{G}_r(z,x_0,s))+z^{-1}\int_{x_0}^{x}dx'\phi(P,x',t_r)\Big),\\
P=(z,y)\in\mathcal{K}_n\setminus\{P_{\infty_\pm},P_0\}.
\end{aligned} \tag{5.23}$$

Then, using (5.21) and (5.22), one infers that

$$\begin{aligned}
\phi(P,x,t_r)&=\frac{y+zG_n(z,x,t_r)}{F_n(z,x,t_r)}\\
&=\frac{H_n(z,x,t_r)}{y-zG_n(z,x,t_r)},
\end{aligned} \tag{5.24}$$

and

$$\psi_2(P,x,x_0,t_r,t_{0,r})=\psi_1(P,x,x_0,t_r,t_{0,r})\phi(P,x,t_r)/z. \tag{5.25}$$

In analogy to (3.12) and (3.13), we introduce

$$\begin{aligned}
\hat{\mu}_j(x,t_r)=(\mu_j(x,t_r),-\mu_j(x,t_r)G_n(\mu_j(x,t_r),x,t_r))\in\mathcal{K}_n,\\
j=0,\dots,n,\ (x,t_r)\in\mathbb{R}^2,
\end{aligned} \tag{5.26}$$

$$\begin{aligned}
\hat{\nu}_l(x,t_r)=(\nu_l(x,t_r),\nu_l(x,t_r)G_n(\nu_l(x,t_r),x,t_r))\in\mathcal{K}_n,\\
l=0,\dots,n,\ (x,t_r)\in\mathbb{R}^2.
\end{aligned} \tag{5.27}$$

The regularity properties of F_n, H_n, μ_j, and ν_l are analogous to those in Section 3 due to assumptions (2.2). Similar to (3.15), the divisor $(\phi(P,x,t_r))$ of $\phi(P,x,t_r)$ reads

$$(\phi(P,x,t_r)) = \mathscr{D}_{\hat{\nu}_0(x,t_r)\underline{\hat{\nu}}(x,t_r)}(P) - \mathscr{D}_{\hat{\mu}_0(x,t_r)\underline{\hat{\mu}}(x,t_r)}(P) \tag{5.28}$$

with

$$\underline{\hat{\mu}} = \{\hat{\mu}_1,\ldots,\hat{\mu}_n\}, \quad \underline{\hat{\nu}} = \{\hat{\nu}_1,\ldots,\hat{\nu}_n\} \in \mathrm{Sym}^n(\mathscr{K}_n). \tag{5.29}$$

The properties of $\phi(P,x,t_r)$ are summarized as follows.

Lemma 5.1. *Assume* (2.2) *and suppose that* (5.3), (5.4) *hold. Moreover, let* $P=(z,y)\in\mathscr{K}_n\setminus\{P_{\infty_\pm},P_0\}$ *and* $(x,t_r)\in\mathbb{R}^2$. *Then* ϕ *satisfies*

$$\phi_x(P) + z^{-1}\phi(P)^2 = -z^{-1}\rho^2 - u_{xx}, \tag{5.30}$$

$$\begin{aligned}\phi_{t_r}(P) &= (-z\widetilde{G}_r(z) + \widetilde{F}_r(z)\phi(P))_x \\ &= z^{-1}\widetilde{H}_r(z) + (z^{-1}\rho^2 + u_{xx})\widetilde{F}_r(z) + (\widetilde{F}_r(z)\phi(P))_x,\end{aligned} \tag{5.31}$$

$$\phi_{t_r}(P) = z^{-1}\widetilde{H}_r(z) + 2\widetilde{G}_r(z)\phi(P) - z^{-1}\widetilde{F}_r(z)\phi(P)^2, \tag{5.32}$$

$$\phi(P)\phi(P^*) = -\frac{H_n(z)}{F_n(z)}, \tag{5.33}$$

$$\phi(P) + \phi(P^*) = 2\frac{zG_n(z)}{F_n(z)}, \tag{5.34}$$

$$\phi(P) - \phi(P^*) = \frac{2y}{F_n(z)}. \tag{5.35}$$

Proof. Equations (5.30) and (5.33)-(5.35) can be proved as in Lemma 3.1. Using (5.21) and (5.22), one infers that

$$\begin{aligned}\phi_{t_r} &= z(\ln\psi_1)_{xt_r} = z(\ln\psi_1)_{t_r x} = z\Big(\frac{\psi_{1,t_r}}{\psi_1}\Big)_x \\ &= z\Big(\frac{-\widetilde{G}_r\psi_1 + \widetilde{F}_r\psi_2}{\psi_1}\Big)_x = (-z\widetilde{G}_r + \widetilde{F}_r\phi)_x.\end{aligned} \tag{5.36}$$

Insertion of (5.14) into (5.36) then yields (5.31). To prove (5.32), one observes that

$$\begin{aligned}\phi_{t_r} =& z\Big(\frac{\psi_2}{\psi_1}\Big)_{t_r} = z\Big(\frac{\psi_{2,t_r}}{\psi_1} - \frac{\psi_2\psi_{1,t_r}}{\psi_1^2}\Big) \\ =& z\Big(\frac{z^{-2}\widetilde{H}_r\psi_1 + \widetilde{G}_r\psi_2}{\psi_1} - z^{-1}\phi\frac{-\widetilde{G}_r\psi_1 + \widetilde{F}_r\psi_2}{\psi_1}\Big) \\ =& z^{-1}\widetilde{H}_r + 2\widetilde{G}_r\phi - z^{-1}\widetilde{F}_r\phi^2,\end{aligned} \tag{5.37}$$

which leads to (5.32). Alternatively, one can also insert (5.12)-(5.14) into (5.31) to obtain (5.32). □

Next, we determine the time evolution of F_n, G_n, and H_n, using relations (5.12)-(5.14) and (5.15)-(5.17).

Lemma 5.2. *Assume* (2.2) *and suppose that* (5.3), (5.4) *hold. Then*

$$F_{n,t_r} = 2(G_n\widetilde{F}_r - \widetilde{G}_r F_n), \tag{5.38}$$

$$z^2 G_{n,t_r} = \widetilde{H}_r F_n - H_n \widetilde{F}_r, \tag{5.39}$$

$$H_{n,t_r} = 2(H_n \widetilde{G}_r - G_n \widetilde{H}_r). \tag{5.40}$$

Equations (5.38)–(5.40) *are equivalent to*

$$-V_{n,t_r} + [\widetilde{V}_r, V_n] = 0. \tag{5.41}$$

Proof. Differentiating (5.35) with respect to t_r naturally yields

$$(\phi(P) - \phi(P^*))_{t_r} = -2yF_{n,t_r}F_n^{-2}. \tag{5.42}$$

On the other hand, using (5.32), (5.34), and (5.35), the left-hand side of (5.42) can be expressed as

$$\begin{aligned}\phi(P)_{t_r} - \phi(P^*)_{t_r} =& 2\widetilde{G}_r(\phi(P) - \phi(P^*)) - z^{-1}\widetilde{F}_r(\phi(P)^2 - \phi(P^*)^2)\\ =& 4y(\widetilde{G}_r F_n - \widetilde{F}_r G_n)F_n^{-2}.\end{aligned} \tag{5.43}$$

Combining (5.42) and (5.43) then proves (5.38). Similarly, differentiating (5.34) with respect to t_r, one finds

$$(\phi(P) + \phi(P^*))_{t_r} = 2z(G_{n,t_r}F_n - G_n F_{n,t_r})F_n^{-2}. \tag{5.44}$$

Meanwhile, the left-hand side of (5.44) also equals

$$\begin{aligned}\phi(P)_{t_r} + \phi(P^*)_{t_r} =& 2\widetilde{G}_r(\phi(P) + \phi(P^*)) - z^{-1}\widetilde{F}_r(\phi(P)^2 + \phi(P^*)^2) + 2z^{-1}\widetilde{H}_r\\ =& -2zG_nF_n^{-2}F_{n,t_r} + 2z^{-1}F_n^{-1}(\widetilde{H}_r F_n - \widetilde{F}_r H_n),\end{aligned} \tag{5.45}$$

using (5.32), (5.33), and (5.34). Equation (5.39) is clear from (5.44) and (5.45). Then, (5.40) follows by differentiating (2.18), that is, $z^2G_n^2 + F_nH_n = R_{2n+2}(z)$, with respect to t_r, and using (5.38) and (5.39). Finally, a direct calculation shows (5.41) holds. □

Basic properties of $\psi(P,x,x_0,t_r,t_{0,r})$ are summarized as follows.

Lemma 5.3. *Assume* (2.2) *and suppose that* (5.3), (5.4) *hold. Moreover, let* $P = (z,y) \in \mathscr{K}_n \setminus \{P_{\infty_\pm}, P_0\}$ *and* $(x,x_0,t_r,t_{0,r}) \in \mathbb{R}^4$. *Then, the Baker-Akhiezer function* ψ *satisfies*

$$\begin{aligned}\psi_1(P,x,x_0,t_r,t_{0,r}) = \Big(\frac{F_n(z,x,t_r)}{F_n(z,x_0,t_{0,r})}\Big)^{1/2}\exp\Bigg(&\frac{y}{z}\int_{t_{0,r}}^{t_r} ds\widetilde{F}_r(z,x_0,s)F_n(z,x_0,s)^{-1}\\ &+\frac{y}{z}\int_{x_0}^{x} dx' F_n(z,x',t_r)^{-1}\Bigg),\end{aligned} \tag{5.46}$$

$$\psi_1(P,x,x_0,t_r,t_{0,r})\psi_1(P^*,x,x_0,t_r,t_{0,r}) = \frac{F_n(z,x,t_r)}{F_n(z,x_0,t_{0,r})}, \tag{5.47}$$

$$\psi_2(P,x,x_0,t_r,t_{0,r})\psi_2(P^*,x,x_0,t_r,t_{0,r}) = -\frac{H_n(z,x,t_r)}{z^2F_n(z,x_0,t_{0,r})}, \tag{5.48}$$

$$\psi_1(P,x,x_0,t_r,t_{0,r})\psi_2(P^*,x,x_0,t_r,t_{0,r}) + \psi_1(P^*,x,x_0,t_r,t_{0,r})\psi_2(P,x,x_0,t_r,t_{0,r}) = 2\frac{G_n(z,x,t_r)}{F_n(z,x_0,t_{0,r})}, \tag{5.49}$$

$$\psi_1(P,x,x_0,t_r,t_{0,r})\psi_2(P^*,x,x_0,t_r,t_{0,r}) - \psi_1(P^*,x,x_0,t_r,t_{0,r})\psi_2(P,x,x_0,t_r,t_{0,r}) = -\frac{2y}{zF_n(z,x_0,t_{0,r})}. \tag{5.50}$$

Proof. To prove (5.46), we first consider the part of time variable in the definition (5.23), that is,

$$\exp\left(\int_{t_{0,r}}^{t_r} ds\ (z^{-1}\widetilde{F}_r(z,x_0,s)\phi(P,x_0,s)-\widetilde{G}_r(z,x_0,s))\right). \tag{5.51}$$

The integrand in the above integral equals

$$\begin{aligned}
&z^{-1}\widetilde{F}_r(z,x_0,s)\phi(P,x_0,s)-\widetilde{G}_r(z,x_0,s)\\
&\quad=z^{-1}\widetilde{F}_r(z,x_0,s)\frac{y+zG_n(z,x_0,s)}{F_n(z,x_0,s)}-\widetilde{G}_r(z,x_0,s)\\
&\quad=\frac{y}{z}\widetilde{F}_r(z,x_0,s)F_n(z,x_0,s)^{-1}+(\widetilde{F}_r(z,x_0,s)G_n(z,x_0,s)\\
&\quad\ -\widetilde{G}_r(z,x_0,s)F_n(z,x_0,s))F_n(z,x_0,s)^{-1}\\
&\quad=\frac{y}{z}\widetilde{F}_r(z,x_0,s)F_n(z,x_0,s)^{-1}+\frac{1}{2}\frac{F_{n,s}(z,x_0,s)}{F_n(z,x_0,s)},
\end{aligned} \tag{5.52}$$

using (5.24) and (5.38). Hence, (5.51) can be expressed as

$$\left(\frac{F_n(z,x_0,t_r)}{F_n(z,x_0,t_{0,r})}\right)^{1/2}\exp\left(\frac{y}{z}\int_{t_{0,r}}^{t_r} ds\widetilde{F}_r(z,x_0,s)F_n(z,x_0,s)^{-1}\right). \tag{5.53}$$

On the other hand, the part of space variable in (5.23) can be written as

$$\left(\frac{F_n(z,x,t_r)}{F_n(z,x_0,t_r)}\right)^{1/2}\exp\left(\frac{y}{z}\int_{x_0}^{x} dx'F_n(z,x',t_r)^{-1}\right), \tag{5.54}$$

using the similar procedure in Lemma 3.2. Then combining (5.53) and (5.54) readily leads to (5.46). Evaluating (5.46) at the points P and P^* and multiplying the resulting expressions yields (5.47). The remaining statements are direct consequences of (5.25), (5.33)-(5.35), and (5.47). □

In analogy to Lemma 3.4, the dynamics of the zeros $\{\mu_j(x,t_r)\}_{j=0,\dots,n}$ and $\{\nu_l(x,t_r)\}_{l=0,\dots,n}$ of $F_n(z,x,t_r)$ and $H_n(z,x,t_r)$ with respect to x and t_r are described in terms of the following Dubrovin-type equations.

Lemma 5.4. *Assume* (2.2) *and suppose that* (5.3), (5.4) *hold subject to the constraint* (2.27).

(i) *Suppose that the zeros $\{\mu_j(x,t_r)\}_{j=0,\dots,n}$ of $F_n(z,x,t_r)$ remain distinct for $(x,t_r)\in\Omega_\mu$, where $\Omega_\mu\subseteq\mathbb{R}^2$ is open and connected, then $\{\mu_j(x,t_r)\}_{j=0,\dots,n}$ satisfy the system of differential equations,*

$$\mu_{j,x}=4\frac{y(\hat{\mu}_j)}{\mu_j}\prod_{\substack{k=0\\k\neq j}}^{n}(\mu_j-\mu_k)^{-1},\quad j=0,\dots,n, \tag{5.55}$$

$$\mu_{j,t_r}=4\frac{\widetilde{F}_r(\mu_j)y(\hat{\mu}_j)}{\mu_j}\prod_{\substack{k=0\\k\neq j}}^{n}(\mu_j-\mu_k)^{-1},\quad j=0,\dots,n, \tag{5.56}$$

with initial conditions

$$\{\hat{\mu}_j(x_0,t_{0,r})\}_{j=0,\ldots,n} \in \mathscr{K}_n, \tag{5.57}$$

for some fixed $(x_0,t_{0,r}) \in \Omega_\mu$. *The initial value problem* (5.56), (5.57) *has a unique solution satisfying*

$$\hat{\mu}_j \in C^\infty(\Omega_\mu, \mathscr{K}_n), \quad j=0,\ldots,n. \tag{5.58}$$

(ii) *Suppose that the zeros* $\{\nu_l(x,t_r)\}_{l=0,\ldots,n}$ *of* $H_n(z,x,t_r)$ *remain distinct for* $(x,t_r) \in \Omega_\nu$, *where* $\Omega_\nu \subseteq \mathbb{R}^2$ *is open and connected, then* $\{\nu_l(x,t_r)\}_{l=0,\ldots,n}$ *satisfy the system of differential equations,*

$$\nu_{l,x} = -2\frac{(\rho^2+u_{xx}\nu_l)y(\hat{\nu}_l)}{h_0\ \nu_l}\prod_{\substack{k=0\\k\neq l}}^{n}(\nu_l-\nu_k)^{-1}, \quad l=0,\ldots,n, \tag{5.59}$$

$$\nu_{l,t_r} = 2\frac{\widetilde{H}_r(\nu_l)y(\hat{\nu}_l)}{h_0\ \nu_l}\prod_{\substack{k=0\\k\neq l}}^{n}(\nu_l-\nu_k)^{-1}, \quad l=0,\ldots,n, \tag{5.60}$$

with initial conditions

$$\{\hat{\nu}_l(x_0,t_{0,r})\}_{l=0,\ldots,n} \in \mathscr{K}_n, \tag{5.61}$$

for some fixed $(x_0,t_{0,r}) \in \Omega_\nu$. *The initial value problem* (5.60), (5.61) *has a unique solution satisfying*

$$\hat{\nu}_l \in C^\infty(\Omega_\nu, \mathscr{K}_n), \quad l=0,\ldots,n. \tag{5.62}$$

Proof. It suffices to prove (5.56) since the argument for (5.60) is analogous and that for (5.55) and (5.59) has been given in the proof of Lemma 3.4. Differentiating (5.6) with respect to t_r yields

$$F_{n,t_r}(\mu_j) = -\frac{1}{2}\mu_{j,t_r}\prod_{\substack{k=0\\k\neq j}}^{n}(\mu_j-\mu_k). \tag{5.63}$$

On the other hand, inserting $z=\mu_j$ into (5.38) and using (5.26), one finds

$$F_{n,t_r}(\mu_j) = 2G_n(\mu_j)\widetilde{F}_r(\mu_j) = 2\frac{y(\hat{\mu}_j)}{-\mu_j}\widetilde{F}_r(\mu_j). \tag{5.64}$$

Combining (5.63) and (5.64) then yields (5.56). The rest is analogous to the proof of Lemma 3.4. □

Since the stationary trace formulas for HS2 invariants in terms of symmetric functions of μ_j in Lemma 3.5 extend line by line to the corresponding time-dependent setting, we next record the t_r-dependent trace formulas without proof. For simplicity, we confine ourselves to the simplest one only.

Lemma 5.5. *Assume* (2.2), *suppose that* (5.3), (5.4) *hold, and let* $(x,t_r) \in \mathbb{R}^2$. *Then,*

$$u(x,t_r) = \frac{1}{2}\sum_{j=0}^{n}\mu_j(x,t_r) - \frac{1}{2}\sum_{m=0}^{2n+1}E_m. \tag{5.65}$$

6. Time-dependent algebro-geometric solutions of HS2 hierarchy

In our final section, we extend the results of section 4 from the stationary HS2 hierarchy, to the time-dependent case. We obtain Riemann theta function representations for the meromorphic function ϕ, and especially, for the algebro-geometric solutions u,ρ of the whole HS2 hierarchy.

We first record the asymptotic properties of ϕ in the time-dependent case.

Lemma 6.1. *Assume* (2.2) *and suppose that* (5.3), (5.4) *hold. Moreover, let* $P=(z,y)\in\mathcal{K}_n\setminus\{P_{\infty_\pm},P_0\}$, $(x,t_r)\in\mathbb{R}^2$. *Then,*

$$\phi(P)\underset{\zeta\to 0}{=}-u_x(x,t_r)+O(\zeta),\quad P\to P_{\infty_\pm},\quad \zeta=z^{-1}, \tag{6.1}$$

$$\phi(P)\underset{\zeta\to 0}{=}i\rho(x,t_r)+\frac{iu_{xx}(x,t_r)-\rho_x(x,t_r)}{2\rho(x,t_r)}\zeta+O(\zeta^2),\quad P\to P_0,\quad \zeta=z. \tag{6.2}$$

Since the proof of Lemma 6.1 is identical to the corresponding stationary results in Lemma 4.1, we omit the corresponding details.

Next, we investigate the properties of the Abel map. To do this, let $\bar{\mu}=(\mu_0,\ldots,\mu_n)\in\mathbb{C}^{n+1}$, we define the following symmetric functions by

$$\Psi_{k+1}(\bar{\mu})=(-1)^{k+1}\sum_{\underline{l}\in\mathcal{S}_{k+1}}\mu_{l_1}\ldots\mu_{l_{k+1}},\quad k=0,\ldots,n, \tag{6.3}$$

where $\mathcal{S}_{k+1}=\{\underline{l}=(l_1,\ldots,l_{k+1})\in\mathbb{N}_0^{k+1}\mid l_1<\cdots<l_{k+1}\le n\}$;

$$\Phi^{(j)}_{k+1}(\bar{\mu})=(-1)^{k+1}\sum_{\underline{l}\in\mathcal{T}^{(j)}_{k+1}}\mu_{l_1}\ldots\mu_{l_{k+1}},\quad k=0,\ldots,n-1, \tag{6.4}$$

where $\mathcal{T}^{(j)}_{k+1}=\{\underline{l}=(l_1,\ldots,l_{k+1})\in\mathcal{S}_{k+1}\mid l_m\ne j\},\ \ j=0,\ldots,n$. For the properties of $\Psi_{k+1}(\bar{\mu})$ and $\Phi^{(j)}_{k+1}(\bar{\mu})$, we refer to Appendix E [21].

Introducing

$$\tilde{d}_{r+1,k}(\underline{E})=\sum_{s=0}^{r+1-k}\tilde{c}_{r+1-k-s}\hat{c}_s(\underline{E}),\quad k=0,\ldots,r+1\wedge n+1, \tag{6.5}$$

for a given set of constants $\{\tilde{c}_l\}_{l=1,\ldots,r+1}\subset\mathbb{C}$, the corresponding homogeneous and nonhomogeneous quantities $\widehat{F}_r(\mu_j)$ and $\widetilde{F}_r(\mu_j)$ in the HS2 case are then given by [b]

$$\begin{aligned}
\widehat{F}_r(\mu_j)&=\sum_{s=(r-n)\vee 0}^{r+1}\hat{c}_s(\underline{E})\Phi^{(j)}_{r+1-s}(\bar{\mu}),\\
\widetilde{F}_r(\mu_j)&=\sum_{s=0}^{r+1}\tilde{c}_{r+1-s}\widehat{F}_s(\mu_j)=\sum_{k=0}^{(r+1)\wedge(n+1)}\tilde{d}_{r+1,k}(\underline{E})\Phi^{(j)}_k(\bar{\mu}),\quad r\in\mathbb{N}_0,\ \tilde{c}_0=1,
\end{aligned} \tag{6.6}$$

using (D.59) and (D.60) [21]. Here, $\hat{c}_s(\underline{E})$, $s\in\mathbb{N}_0$, is defined by (D.2) [21].

[b] $m\wedge n=\min\{m,n\}$, $m\vee n=\max\{m,n\}$

We now state the analog of Theorem 4.3, which indicates marked differences between the HS2 hierarchy and other completely integrable systems such as the KdV and AKNS hierarchies.

Theorem 6.2. *Assume* (2.27) *and suppose that* $\{\hat{\mu}_j\}_{j=0,\dots,n}$ *satisfies the Dubrovin equations* (5.55), (5.56) *on an open set* $\Omega_\mu \subseteq \mathbb{R}^2$ *such that* $\mu_j,\ j=0,\dots,n$, *remain distinct and nonzero on* Ω_μ *and that* $\widetilde{F}_r(\mu_j)\neq 0$ *on* Ω_μ, $j=0,\dots,n$. *Introducing the associated divisor* $\mathscr{D}_{\hat{\mu}_0(x,t_r)\underline{\hat{\mu}}(x,t_r)}$, *one computes,*

$$\partial_x \underline{\alpha}_{Q_0}(\mathscr{D}_{\hat{\mu}_0(x,t_r)\underline{\hat{\mu}}(x,t_r)}) = -\frac{4a}{\Psi_{n+1}(\bar{\mu}(x,t_r))}\underline{c}(1), \quad (x,t_r)\in\Omega_\mu, \tag{6.7}$$

$$\begin{aligned}\partial_{t_r} \underline{\alpha}_{Q_0}(\mathscr{D}_{\hat{\mu}_0(x,t_r)\underline{\hat{\mu}}(x,t_r)}) &= -\frac{4a}{\Psi_{n+1}(\bar{\mu}(x,t_r))}\left(\sum_{k=0}^{(r+1)\wedge(n+1)} \tilde{d}_{r+1,k}(\underline{E})\Psi_k(\bar{\mu}(x,t_r))\right)\underline{c}(1)\\ &\quad + 4a\left(\sum_{\ell=1\vee(n+1-r)}^{n+1} \tilde{d}_{r+1,n+2-\ell}(\underline{E})\underline{c}(\ell)\right), \quad (x,t_r)\in\Omega_\mu.\end{aligned} \tag{6.8}$$

In particular, the Abel map dose not linearize the divisor $\mathscr{D}_{\hat{\mu}_0(x,t_r)\underline{\hat{\mu}}(x,t_r)}$ *on* Ω_μ.

Proof. Let $(x,t_r)\in\Omega_\mu$. It suffices to prove (6.8), since (6.7) is proved as in the stationary context of Theorem 4.3. We first recall a fundamental identity (E.10) [21], that is,

$$\Phi_{k+1}^{(j)}(\bar{\mu}) = \mu_j \Phi_k^{(j)}(\bar{\mu}) + \Psi_{k+1}(\bar{\mu}), \quad k=0,\dots,n,\ j=0,\dots,n. \tag{6.9}$$

Then, applying (4.16), (6.6), and (6.9), one finds

$$\begin{aligned}\frac{\widetilde{F}_r(\mu_j)}{\mu_j} &= \mu_j^{-1}\sum_{m=0}^{(r+1)\wedge(n+1)} \tilde{d}_{r+1,m}(\underline{E})\Phi_m^{(j)}(\bar{\mu})\\ &= \mu_j^{-1}\sum_{m=0}^{(r+1)\wedge(n+1)} \tilde{d}_{r+1,m}(\underline{E})\Big(\mu_j\Phi_{m-1}^{(j)}(\bar{\mu})+\Psi_m(\bar{\mu})\Big)\\ &= \sum_{m=1}^{(r+1)\wedge(n+1)} \tilde{d}_{r+1,m}(\underline{E})\Phi_{m-1}^{(j)}(\bar{\mu}) - \sum_{m=0}^{(r+1)\wedge(n+1)} \tilde{d}_{r+1,m}(\underline{E})\Psi_m(\bar{\mu})\frac{\Phi_n^{(j)}(\bar{\mu})}{\Psi_{n+1}(\bar{\mu})}.\end{aligned} \tag{6.10}$$

Hence, using (5.56), (6.10), (E.4), (E.13), and (E.14) [21], one infers that

$$\begin{aligned}&\partial_{t_r}\Big(\sum_{j=0}^{n}\int_{Q_0}^{\hat{\mu}_j}\underline{\omega}\Big) = \sum_{j=0}^{n}\mu_{j,t_r}\sum_{k=1}^{n}\underline{c}(k)\frac{a\,\mu_j^{k-1}}{y(\hat{\mu}_j)}\\ &= 4a\sum_{j=0}^{n}\sum_{k=1}^{n}\underline{c}(k)\frac{\mu_j^{k-1}}{\prod_{\substack{l=0\\ l\neq j}}^{n}(\mu_j-\mu_l)}\frac{\widetilde{F}_r(\mu_j)}{\mu_j}\\ &= 4a\sum_{j=0}^{n}\sum_{k=1}^{n}\underline{c}(k)\frac{\mu_j^{k-1}}{\prod_{\substack{l=0\\ l\neq j}}^{n}(\mu_j-\mu_l)}\\ &\times\Big(-\sum_{m=0}^{(r+1)\wedge(n+1)}\tilde{d}_{r+1,m}(\underline{E})\Psi_m(\bar{\mu})\frac{\Phi_n^{(j)}(\bar{\mu})}{\Psi_{n+1}(\bar{\mu})}+\sum_{m=1}^{(r+1)\wedge(n+1)}\tilde{d}_{r+1,m}(\underline{E})\Phi_{m-1}^{(j)}(\bar{\mu})\Big)\\ &= -4a\sum_{m=0}^{(r+1)\wedge(n+1)}\tilde{d}_{r+1,m}(\underline{E})\frac{\Psi_m(\bar{\mu})}{\Psi_{n+1}(\bar{\mu})}\sum_{k=1}^{n}\sum_{j=0}^{n}\underline{c}(k)(U_{n+1}(\bar{\mu}))_{k,j}(U_{n+1}(\bar{\mu}))_{j,1}^{-1}\end{aligned}$$

$$+4a\sum_{m=1}^{(r+1)\wedge(n+1)}\tilde{d}_{r+1,m}(\underline{E})\sum_{k=1}^{n}\sum_{j=0}^{n}\underline{c}(k)(U_{n+1}(\bar{\mu}))_{k,j}(U_{n+1}(\bar{\mu}))^{-1}_{j,n-m+2}$$
$$=-\frac{4a}{\Psi_{n+1}(\bar{\mu})}\sum_{m=0}^{(r+1)\wedge(n+1)}\tilde{d}_{r+1,m}(\underline{E})\Psi_m(\bar{\mu})\underline{c}(1)+4a\sum_{m=1}^{(r+1)\wedge(n+1)}\tilde{d}_{r+1,m}(\underline{E})\underline{c}(n-m+2)$$
$$=-\frac{4a}{\Psi_{n+1}(\bar{\mu})}\sum_{m=0}^{(r+1)\wedge(n+1)}\tilde{d}_{r+1,m}(\underline{E})\Psi_m(\bar{\mu})\underline{c}(1)+4a\sum_{m=1\vee(n+1-r)}^{n+1}\tilde{d}_{r+1,n+2-m}(\underline{E})\underline{c}(m), \qquad (6.11)$$

which is equivalent to (6.8). □

The analogous results hold for the corresponding divisor $\mathscr{D}_{\hat{\nu}_0(x,t_r)\underline{\hat{\nu}}(x,t_r)}$ associated with $\phi(P,x,t_r)$.

Next, recalling the definition of $\underline{\widehat{B}}_{Q_0}$ and $\underline{\hat{\beta}}_{Q_0}$ in (4.19) and (4.20), one obtains the following result.

Corollary 6.3. *Assume* (2.27) *and suppose that* $\{\hat{\mu}_j\}_{j=0,\dots,n}$ *satisfies the Dubrovin equations* (5.55), (5.56) *on an open set* $\Omega_\mu \subseteq \mathbb{R}^2$ *such that* μ_j, $j=0,\dots,n$, *remain distinct and nonzero on* Ω_μ *and that* $\widetilde{F}_r(\mu_j)\neq 0$ *on* Ω_μ, $j=0,\dots,n$. *Then, one computes*

$$\partial_x\sum_{j=0}^{n}\int_{Q_0}^{\hat{\mu}_j(x,t_r)}\eta_1=-\frac{4a}{\Psi_{n+1}(\bar{\mu}(x,t_r))},\quad (x,t_r)\in\Omega_\mu, \qquad (6.12)$$

$$\partial_x\underline{\hat{\beta}}(\mathscr{D}_{\underline{\hat{\mu}}(x,t_r)})=\begin{cases}4a, & n=1,\\ 4a(0,\dots,0,1), & n\geq 2,\end{cases}\qquad (x,t_r)\in\Omega_\mu, \qquad (6.13)$$

$$\partial_{t_r}\sum_{j=0}^{n}\int_{Q_0}^{\hat{\mu}_j(x,t_r)}\eta_1=-\frac{4a}{\Psi_{n+1}(\bar{\mu}(x,t_r))}\sum_{k=0}^{(r+1)\wedge(n+1)}\tilde{d}_{r+1,k}(\underline{E})\Psi_k(\bar{\mu}(x,t_r))$$
$$+4a\tilde{d}_{r+1,n+1}(\underline{E})\delta_{n+1,r+1\wedge n+1},\quad (x,t_r)\in\Omega_\mu, \qquad (6.14)$$

$$\partial_{t_r}\underline{\hat{\beta}}(\mathscr{D}_{\underline{\hat{\mu}}(x,t_r)})=4a\Big(\sum_{s=0}^{r+1}\tilde{c}_{r+1-s}\hat{c}_{s+1-n}(\underline{E}),\dots,\sum_{s=0}^{r+1}\tilde{c}_{r+1-s}\hat{c}_{s+1}(\underline{E}),\sum_{s=0}^{r+1}\tilde{c}_{r+1-s}\hat{c}_{s}(\underline{E})\Big),$$
$$\hat{c}_{-l}(\underline{E})=0,\ l\in\mathbb{N},\quad (x,t_r)\in\Omega_\mu. \qquad (6.15)$$

Proof. Equations (6.12) and (6.13) are proved as in the stationary context of Corollary 4.4. Equation (6.14) is a special case of (6.8), and (6.15) follows by (6.11), taking into account (E.4) [21]. □

The fact that the Abel map does not effect a linearization of the divisor $\mathscr{D}_{\hat{\mu}_0(x,t_r)\underline{\hat{\mu}}(x,t_r)}$ in the time-dependent HS2 context, which is well known and discussed (using different approaches) by Constantin and McKean [12], Alber, Camassa, Fedorov, Holm, and Marsden [2], Alber and Fedorov [3,4]. The change of variables

$$x\mapsto\tilde{x}=\int^{x}dx'\left(\frac{4a}{\Psi_{n+1}(\bar{\mu}(x',t_r))}\right) \qquad (6.16)$$

and

$$t_r\mapsto\tilde{t}_r=\int^{t_r}ds\Bigg(\frac{4a}{\Psi_{n+1}(\bar{\mu}(x,s))}\sum_{k=0}^{(r+1)\wedge(n+1)}\tilde{d}_{r+1,k}(\underline{E})\Psi_k(\bar{\mu}(x,s))$$

$$-4a\sum_{\ell=1\vee(n+1-r)}^{n+1}\tilde{d}_{r+1,n+2-\ell}(\underline{E})\frac{\underline{c}(\ell)}{\underline{c}(1)}\Bigg) \tag{6.17}$$

linearizes the Abel map $\underline{A}_{Q_0}(\mathscr{D}_{\hat{\mu}_0(\tilde{x},\tilde{t}_r)\underline{\hat{\mu}}(\tilde{x},\tilde{t}_r)})$, $\tilde{\mu}_j(\tilde{x},\tilde{t}_r)=\mu_j(x,t_r)$, $j=0,\ldots,n$. The intricate relation between the variables (x,t_r) and $(\tilde{x},\tilde{t}_r)$ is detailed in (6.21). Our approach follows a route similar to Gesztesy and Holden's treatment of the CH hierarchy [21].

Next, we shall provide the explicit representations of ϕ and u,ρ in terms of the Riemann theta function associated with $\mathscr{K}_n$, assuming the affine part of $\mathscr{K}_n$ to be nonsingular. Recalling (4.24)-(4.30), the analog of Theorem 4.5 in the stationary case then reads as follows.

Theorem 6.4. *Assume* (2.2) *and suppose that* (5.3), (5.4) *hold on* Ω *subject to the constraint* (2.27). *In addition, let* $P=(z,y)\in\mathscr{K}_n\setminus\{P_0\}$ *and* $(x,t_r),(x_0,t_{0,r})\in\Omega$, *where* $\Omega\subseteq\mathbb{R}^2$ *is open and connected. Moreover, suppose that* $\mathscr{D}_{\underline{\hat{\mu}}(x,t_r)}$, *or equivalently,* $\mathscr{D}_{\underline{\hat{\nu}}(x,t_r)}$ *is nonspecial for* $(x,t_r)\in\Omega$. *Then,* ϕ, u, *and* ρ *admit the representations*

$$\phi(P,x,t_r)=i\rho(x,t_r)\frac{\theta(\underline{z}(P,\underline{\hat{\nu}}(x,t_r)))\theta(\underline{z}(P_0,\underline{\hat{\mu}}(x,t_r)))}{\theta(\underline{z}(P_0,\underline{\hat{\nu}}(x,t_r)))\theta(\underline{z}(P,\underline{\hat{\mu}}(x,t_r)))}\exp\left(e_0-\int_{Q_0}^{P}\omega^{(3)}_{\hat{\mu}_0(x,t_r),\hat{\nu}_0(x,t_r)}\right), \tag{6.18}$$

$$u(x,t_r)=-\frac{1}{2}\sum_{m=0}^{2n+1}E_m+\frac{1}{2}\sum_{j=1}^{n}\lambda_j-\frac{1}{2}\sum_{j=1}^{n}U_j\partial_{\omega_j}\ln\left(\frac{\theta(\underline{z}(P_{\infty_+},\underline{\hat{\mu}}(x,t_r))+\underline{\omega})}{\theta(\underline{z}(P_{\infty_-},\underline{\hat{\mu}}(x,t_r))+\underline{\omega})}\right)\Big|_{\underline{\omega}=0}, \tag{6.19}$$

$$\rho(x,t_r)=iu_x(x,t_r)\frac{\theta(\underline{z}(P_0,\underline{\hat{\nu}}(x,t_r)))\theta(\underline{z}(P_{\infty_+},\underline{\hat{\mu}}(x,t_r)))}{\theta(\underline{z}(P_{\infty_+},\underline{\hat{\nu}}(x,t_r)))\theta(\underline{z}(P_0,\underline{\hat{\mu}}(x,t_r)))}. \tag{6.20}$$

Moreover, let $\widetilde{\Omega}\subseteq\Omega$ *be such that* μ_j, $j=0,\ldots,n$, *are nonvanishing on* $\widetilde{\Omega}$. *Then, the constraint*

$$\begin{aligned}
&4a(x-x_0)+4a(t_r-t_{0,r})\sum_{s=0}^{r+1}\tilde{c}_{r+1-s}\hat{c}_s(\underline{E})=\sum_{j=1}^{n}\left(\int_{a_j}\tilde{\omega}^{(3)}_{P_{\infty_+}P_{\infty_-}}\right)c_j(1)\\
&\times\left(-4a\int_{x_0}^{x}\frac{dx'}{\prod_{k=0}^{n}\mu_k(x',t_r)}-4a\sum_{k=0}^{(r+1)\wedge(n+1)}\tilde{d}_{r+1,k}(\underline{E})\int_{t_{0,r}}^{t_r}\frac{\Psi_k(\bar{\mu}(x_0,s))}{\Psi_{n+1}(\bar{\mu}(x_0,s))}ds\right)\\
&+4a(t_r-t_{0,r})\sum_{\ell=1\vee(n+1-r)}^{n+1}\tilde{d}_{r+1,n+2-\ell}(\underline{E})\sum_{j=1}^{n}\left(\int_{a_j}\tilde{\omega}^{(3)}_{P_{\infty_+}P_{\infty_-}}\right)c_j(\ell)\\
&+\ln\left(\frac{\theta(\underline{z}(P_{\infty_+},\underline{\hat{\mu}}(x,t_r)))\theta(\underline{z}(P_{\infty_-},\underline{\hat{\mu}}(x_0,t_{0,r})))}{\theta(\underline{z}(P_{\infty_-},\underline{\hat{\mu}}(x,t_r)))\theta(\underline{z}(P_{\infty_+},\underline{\hat{\mu}}(x_0,t_{0,r})))}\right),\\
&\qquad (x,t_r),(x_0,t_{0,r})\in\widetilde{\Omega}
\end{aligned} \tag{6.21}$$

holds, with

$$\underline{\hat{\alpha}}_{Q_0}(\mathscr{D}_{\hat{\mu}_0(x,t_r)\underline{\hat{\mu}}(x,t_r)})=\underline{\hat{\alpha}}_{Q_0}(\mathscr{D}_{\hat{\mu}_0(x_0,t_r)\underline{\hat{\mu}}(x_0,t_r)})-4a\left(\int_{x_0}^{x}\frac{dx'}{\Psi_{n+1}(\bar{\mu}(x',t_r))}\right)\underline{c}(1) \tag{6.22}$$

$$\begin{aligned}
&=\underline{\hat{\alpha}}_{Q_0}(\mathscr{D}_{\hat{\mu}_0(x,t_{0,r})\underline{\hat{\mu}}(x,t_{0,r})})-4a\left(\sum_{k=0}^{(r+1)\wedge(n+1)}\tilde{d}_{r+1,k}(\underline{E})\int_{t_{0,r}}^{t_r}\frac{\Psi_k(\bar{\mu}(x,s))}{\Psi_{n+1}(\bar{\mu}(x,s))}ds\right)\underline{c}(1)\\
&+4a(t_r-t_{0,r})\left(\sum_{\ell=1\vee(n+1-r)}^{n+1}\tilde{d}_{r+1,n+2-\ell}(\underline{E})\underline{c}(\ell)\right),
\end{aligned} \tag{6.23}$$

$$(x,t_r),(x_0,t_{0,r}) \in \widetilde{\Omega}.$$

Proof. We first assume that μ_j, $j=0,\dots,n$, are distinct and nonvanishing on $\widetilde{\widetilde{\Omega}}$ and $\widetilde{F}_r(\mu_j)\neq 0$ on $\widetilde{\widetilde{\Omega}}$, $j=0,\dots,n$, where $\widetilde{\widetilde{\Omega}} \subseteq \Omega$. Then, the representation (6.18) for ϕ on $\widetilde{\widetilde{\Omega}}$ follows by combining (5.28), (6.1), (6.2), and Theorem A.26 [21]. The representation (6.19) for u on $\widetilde{\widetilde{\Omega}}$ follows from the trace formulas (5.65) and (F.59) [21]. The representation (6.20) for ρ on $\widetilde{\widetilde{\Omega}}$ is clear from (6.18) and (6.1). In fact, since the proofs of (6.18), (6.19), and (6.20) are identical to the corresponding stationary results in Theorem 4.5, which can be extended line by line to the time-dependent setting, here we omit the corresponding details. By continuity, (6.18), (6.19), and (6.20) extend from $\widetilde{\widetilde{\Omega}}$ to Ω. The constraint (6.21) then holds on $\widetilde{\widetilde{\Omega}}$ by combining (6.12)-(6.15) and (F.58) [21]. Equations (6.22) and (6.23) are clear from (6.7) and (6.8). Again by continuity, (6.21)-(6.23) extend from $\widetilde{\widetilde{\Omega}}$ to $\widetilde{\Omega}$. $\square$

Remark 6.5. One observes that (6.22) and (6.23) are equivalent to

$$\underline{\hat{\alpha}}_{Q_0}(\mathscr{D}_{\hat{\mu}_0(x,t_r)\underline{\hat{\mu}}(x,t_r)}) = \underline{\hat{\alpha}}_{Q_0}(\mathscr{D}_{\hat{\mu}_0(x_0,t_r)\underline{\hat{\mu}}(x_0,t_r)}) - \underline{c}(1)(\tilde{x}-\tilde{x}_0) \tag{6.24}$$

$$= \underline{\hat{\alpha}}_{Q_0}(\mathscr{D}_{\hat{\mu}_0(x,t_{0,r})\underline{\hat{\mu}}(x,t_{0,r})}) - \underline{c}(1)(\tilde{t}_r-\tilde{t}_{0,r}), \tag{6.25}$$

under the change of variables $x \mapsto \tilde{x}$ and $t_r \mapsto \tilde{t}_r$ in (6.16) and (6.17). Hence, the Abel map linearizes the divisor $\mathscr{D}_{\hat{\mu}_0(x,t_r)\underline{\hat{\mu}}(x,t_r)}$ on Ω with respect to $\tilde{x},\tilde{t}_r$.

Remark 6.6. Remark 4.7 applies in the present time-dependent context. Moreover, to obtain the theta function representation of ψ_j, $j=1,2,$, one can write $\widetilde{F}_r$ in terms of $\Psi_k(\bar{\mu})$ and use (5.46), in analogy to the stationary case discussed in Remark 4.8. Here we omit further details.

At the end of this section, we turn to the time-dependent algebro-geometric initial value problem of HS2 hierarchy. We will show that the solvability of the Dubrovin equations (5.55) and (5.56) on $\Omega_\mu \subseteq \mathbb{R}^2$ in fact implies equations (5.3) and (5.4) on Ω_μ.

Theorem 6.7. *Fix $n\in\mathbb{N}_0$, assume (2.27), and suppose that $\{\hat{\mu}_j\}_{j=0,\dots,n}$ satisfies the Dubrovin equations (5.55), (5.56) on an open and connected set $\Omega_\mu \subseteq \mathbb{R}^2$, with $\widetilde{F}_r(\mu_j)$ in (5.56) expressed in terms of μ_k, $k=0,\dots,n$, by (6.6). Moreover, assume that μ_j, $j=0,\dots,n$, remain distinct and nonzero on Ω_μ. Then, $u,\rho\in C^\infty(\Omega_\mu)$, defined by*

$$u = -\frac{1}{2}\sum_{m=0}^{2n+1} E_m + \frac{1}{2}\sum_{j=0}^{n}\mu_j, \tag{6.26}$$

and

$$\rho^2 = u_x^2 + 2uu_{xx} - \frac{d^2}{dx^2}\left(\Psi_2(\bar{\mu}) - u\sum_{m=0}^{2n+1} E_m\right), \tag{6.27}$$

satisfy the rth HS2 equation (5.1), that is,

$$\mathrm{HS2}_r(u,\rho)=0 \quad on\ \Omega_\mu, \tag{6.28}$$

with initial values satisfying the nth stationary HS2 equation (5.2).

Proof. Given the solutions $\hat{\mu}_j = (\mu_j, y(\hat{\mu}_j)) \in C^\infty(\Omega_\mu, \mathscr{K}_n),\ j = 0, \ldots, n$ of (5.55) and (5.56), we define polynomials F_n, G_n, and H_n on Ω_μ as in the stationary case (cf. Theorem 4.9) with properties

$$F_n(z) = \frac{1}{2}\prod_{j=0}^{n}(z-\mu_j), \tag{6.29}$$

$$G_n(z) = \frac{1}{2}F_{n,x}(z), \tag{6.30}$$

$$z^2 G_{n,x}(z) = -H_n(z) - (\rho^2 + u_{xx}z)F_n(z), \tag{6.31}$$

$$H_{n,x}(z) = 2(\rho^2 + u_{xx}z)G_n(z), \tag{6.32}$$

$$R_{2n+2}(z) = z^2 G_n^2(z) + F_n(z)H_n(z), \tag{6.33}$$

treating t_r as a parameter. Define the polynomials $\widetilde{G}_r$ and $\widetilde{H}_r$ by

$$\widetilde{G}_r(z) = \frac{1}{2}\widetilde{F}_{r,x}(z) \quad \text{on } \mathbb{C}\times\Omega_\mu, \tag{6.34}$$

$$\widetilde{H}_r(z) = -z^2\widetilde{G}_{r,x}(z) - (\rho^2 + u_{xx}z)\widetilde{F}_r(z) \quad \text{on } \mathbb{C}\times\Omega_\mu, \tag{6.35}$$

respectively. Next, we claim that

$$F_{n,t_r}(z) = 2(G_n(z)\widetilde{F}_r(z) - F_n(z)\widetilde{G}_r(z)) \quad \text{on } \mathbb{C}\times\Omega_\mu. \tag{6.36}$$

To prove (6.36), one computes from (5.55) and (5.56) that

$$F_{n,x}(z) = -F_n(z)\sum_{j=0}^{n}\mu_{j,x}(z-\mu_j)^{-1}, \tag{6.37}$$

$$F_{n,t_r}(z) = -F_n(z)\sum_{j=0}^{n}\widetilde{F}_r(\mu_j)\mu_{j,x}(z-\mu_j)^{-1}. \tag{6.38}$$

Using (6.30) and (6.34), one concludes that (6.36) is equivalent to

$$\widetilde{F}_{r,x}(z) = \sum_{j=0}^{n}(\widetilde{F}_r(\mu_j) - \widetilde{F}_r(z))\mu_{j,x}(z-\mu_j)^{-1}. \tag{6.39}$$

Equation (6.39) is proved in Lemma F.9 [21]. This in turn proves (6.36).

Next, differentiating (6.30) with respect to t_r yields

$$F_{n,xt_r} = 2G_{n,t_r}. \tag{6.40}$$

On the other hand, taking the derivative of (6.36) with respect to x, and using (6.30), (6.31), (6.34), one obtains

$$\begin{aligned} F_{n,t_r x} = &-2z^{-2}H_n\widetilde{F}_r - 2(z^{-2}\rho^2 + u_{xx}z^{-1})F_n\widetilde{F}_r + 2G_n\widetilde{F}_{r,x} \\ &-2\widetilde{G}_{r,x}F_n - 4\widetilde{G}_r G_n. \end{aligned} \tag{6.41}$$

Combining (6.30), (6.34), (6.40), and (6.41), one concludes

$$z^2 G_{n,t_r}(z) = \widetilde{H}_r(z)F_n(z) - H_n(z)\widetilde{F}_r(z) \quad \text{on } \mathbb{C}\times\Omega_\mu. \tag{6.42}$$

Next, differentiating (6.33) with respect to t_r, and using expressions (6.36) and (6.42) for F_{n,t_r} and G_{n,t_r}, respectively, one obtains

$$H_{n,t_r}(z) = 2(H_n(z)\widetilde{G}_r(z) - G_n(z)\widetilde{H}_r(z)) \quad \text{on } \mathbb{C}\times\Omega_\mu. \tag{6.43}$$

Finally, taking the derivative of (6.42) with respect to x, and using expressions (6.30), (6.32), and (6.34) for $F_{n,x}$, $H_{n,x}$, and $\widetilde{F}_{r,x}$, respectively, one infers that

$$z^2 G_{n,t_r x} = F_n\widetilde{H}_{r,x} + 2G_n\widetilde{H}_r - 2(\rho^2 + u_{xx}z)G_n\widetilde{F}_r - 2H_n\widetilde{G}_r. \tag{6.44}$$

On the other hand, differentiating (6.31) with respect to t_r, using (6.36) and (6.43) for F_{n,t_r} and H_{n,t_r}, respectively, leads to

$$z^2 G_{n,xt_r} = 2G_n\widetilde{H}_r - 2H_n\widetilde{G}_r - (2\rho\rho_{t_r} + u_{xxt_r}z)F_n - 2(\rho^2 + u_{xx}z)(G_n\widetilde{F}_r - \widetilde{G}_r F_n). \tag{6.45}$$

Combining (6.44) and (6.45) then yields

$$-2\rho\rho_{t_r} - u_{xxt_r}z - \widetilde{H}_{r,x} + 2(\rho^2 + u_{xx}z)\widetilde{G}_r = 0. \tag{6.46}$$

Thus, we have proved (5.12)-(5.17) and (5.38)-(5.40) on $\mathbb{C}\times\Omega_\mu$ and hence conclude that (6.28) holds on $\mathbb{C}\times\Omega_\mu$. $\square$

Remark 6.8. Again we formulated Theorem 6.7 in terms of $\{\mu_j\}_{j=0,\dots,n}$ only. Obviously, the analogous result (and strategy proof) works in terms of $\{\nu_j\}_{j=0,\dots,n}$.

The analog of Remark 4.11 directly extends to the current time-dependent setting.

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References

[1] M. J. Ablowitz, D. J. Kaup, A. C. Newell, and H. Segur, The inverse scattering transform–Fourier analysis for nonlinear problems, *Stud. Appl. Math.* **53** (1974) 249–315.

[2] M. S. Alber, R. Camassa, Y. N. Fedorov, D. D. Holm, and J. E. Marsden, The complex geometry of weak piecewise smooth solutions of integrable nonlinear PDEs of shallow water and Dym type, *Commun. Math. Phys.* **221** (2001) 197–227.

[3] M. S. Alber, and Y. N. Fedorov, Wave solutions of evolution equations and Hamiltonian flows on nonlinear subvarieties of generalized Jacobians, *J. Phys. A.* **33** (2000) 8409–8425.

[4] M. S. Alber, and Y. N. Fedorov, Algebraic geometrical solutions for certain evolution equations and Hamiltonian flows on nonlinear subvarieties of generalized Jacobians, *Inverse Problems.* **17** (2001) 1017–1042.

[5] R. Beals, D. Sattinger, and J. Szmigielski, Inverse scattering solutions of the Hunter-Saxton equations, *Appl. Anal.* **78** (2001) 255–269.

[6] E. D. Belokolos, A. I. Bobenko, V. Z. Enol'skii, A. R. Its, and V. B. Matveev, *Algebro-Geometric Approach to Nolinear Integrable Equations,* (Springer, Berlin, 1994).

[7] R. Camassa, and D. D. Holm, An integrable shallow water equation with peaked solitons, *Phys. Rev. Lett.* **71** (1993) 1661–1664.

[8] M. Chen, S. Q. Liu, and Y. Zhang, A two-component generalization of the Camassa-Holm equation and its solutions, *Lett. Math. Phys.* **75** (2006) 1–15.

[9] A. Constantin, and R. I. Ivanov, On an integrable two-component Camassa-Holm shallow water system, *Phys. Lett. A* **372** (2008) 7129–7132.

[10] A. Constantin and B. Kolev, On the geometric approach to the motion of inertial mechanical systems, *J. Phys. A* **35** (2002) R51–R79.

[11] A. Constantin, and D. Lannes, The hydrodynamical relevance of the Camassa-Holm and Degasperis-Procesi equations, *Arch. Ration. Mech. Anal.* **192** (2009) 165–186.

[12] A. Constantin, and H. P. McKean, A shallow water equation on the circle, *Commun. Pure Appl. Math.* **52** (1999) 949–982.

[13] H. H. Dai, and M. Pavlov, Transformations for the Camassa-Holm equation, its high-frequency limit and the Sinh-Gordon equation, *J. P. Soc. Japan,* **67** (1998) 3655–3657.

[14] B. A. Dubrovin, Completely integrable Hamiltonian systems associated with matrix operators and Abelian varieties, *Funct. Anal. Appl.* **11** (1977) 265–277.

[15] B. A. Dubrovin, Theta functions and nonlinear equations, *Russian Math. Surv.* **36** (1981) 11–92.

[16] B. A. Dubrovin, Matrix finite-zone operators, *Revs. Sci. Technol.* **23** (1983) 20–50.

[17] J. Escher, O. Lechtenfeld and Z. Yin, Well-posedness and blow-up phenomena for the 2-component Camassa- Holm equation, *Discrete Contin. Dyn. Syst.* **19** (2007) 493–513.

[18] A. S. Fokas, and B. Fuchssteiner, Symplectic structures, their Bäcklund transformation and hereditary symmetries, *Phys. D* **4** (1981) 47–66.

[19] F. Gesztesy, and H. Holden, Algebro-geometric solutions of the Camassa-Holm hierarchy, *Rev. Mat. Iberoam.* **19** (2003) 73–142.

[20] F. Gesztesy, and H. Holden, Real-valued algebro-geometric solutions of the Camassa-Holm hierarchy, *Philos. Trans. R. Soc. Lond. Ser. A.* **366** (2008) 1025–1054.

[21] F. Gesztesy, and H. Holden, *Soliton Equations and Their Algebro- Geometric Solutions. vol. I: (1+1)-Dimensional Continuous Models.* (Cambridge Studies in Advanced Mathematics, Cambridge University Press, Cambridge, 2003).

[22] F. Gesztesy, and R. Ratneseelan, An alternative approach to algebro-geometric solutions of the AKNS hierarchy, *Rev. Math. Phys.* **10** (1998) 345–391.

[23] C. Guan and Z. Yin, Global weak solutions and smooth solutions for a two-component Hunter-Saxton system, *J. Math. Phys.* **52** (2011) 103707.

[24] C. Guan and Z. Yin, Global weak solutions for a periodic two-component Hunter-Saxton system, *Q. Appl. math.* **2** (2012) 285–297.

[25] G. L. Gui, and Y. Liu, On the global existence and wave-breaking criteria for the two-component Camassa-Holm system, *J. Funct. Anal.* **258** (2010) 4251–4278.

[26] D. D. Holm, and L. Ó Náraigh and C. Tronci, Singular solutions of a modified two-component Camassa-Holm equation, *Phys. Rev. E* **79** (2009) 016601.

[27] Y. Hou, E. G. Fan, and Z. J. Qiao, The algebro-geometric solutions for the modified Camassa-Holm hierarchy, preprint, (2012) arXiv: 1205.6062.

[28] Y. Hou, E. G. Fan, and P. Zhao, Algebro-geometirc solutions for the Hunter-Saxton hierarchy, *Z. Angew. Math. Phys.* (2013) Springer Basel DOI 10.1007/s00033-013-0339-8.

[29] Y. Hou, P. Zhao, E. G. Fan, and Z. J. Qiao, Algebro-geometric solutions for Degasperis-Procesi hierarchy, *SIAM J. Math. Anal.* **45** (2013) 1216–1266.

[30] J. K. Hunter, and R. Saxton, Dynamics of director fields, *SIAM J. Appl. Math.* **51** (1991) 1498–1521.

[31] J. K. Hunter, and Y. X. Zheng, On a completely integrable nonlinear hyperbolic variational equation, *Phys. D* **79** (1994) 361–386.

[32] R. S. Johnson, Camassa-Holm, Korteweg-de Vries and related models for water waves, *J. Fluid Mech.* **457** (2002) 63–82.

[33] B. Khesin, and G. Misiołek, Euler equations on homogeneous spaces and Virasoro orbits, *Adv. Math.* **176** (2003) 116–144.
[34] J. Lenells, Weak geodesic flow and global solutions of the Hunter-Saxton equation, *Discret. Contin. Dyn. Syst.* **18** (2007) 643–656.
[35] J. Lenells, The Hunter-Saxton equation describes the geodesic flow on a sphere, *J. Geom. Phys.* **57** (2007) 2049–2064.
[36] J. Liu and Z. Yin, Blow-up phenomena and global existence for a periodic Hunter-Saxton system, Preprint, (2011) arXiv:1012.5448v3.
[37] V. B. Matveev, and M. I. Yavor, Solutions presque périodiques et à *N*-solitons de l'équation hydrodynamique non linéaire de Kaup. *Ann. Inst. H. Poincaré Sect. A* **31** (1979) 25–41.
[38] S. P. Novikov, S. V. Manakov, L. P. Pitaevskii, and V. E. Zakharov, *Theory of Solitons, the Inverse Scattering Methods.* (Concultants Bureau, New York, 1984).
[39] P. Olver, and P. Rosenau, Tri-Hamiltonian duality between solitons and solitary-wave solutions having compact support, *Phys. Rev. E* **53** (1996) 1900–1906.
[40] M. V. Pavlov, The Gurevich-Zybin system, *J. Phys. A* **38** (2005) 3823–3840.
[41] E. G. Reyes, The soliton content of the Camassa-Holm and Hunter-Saxton equations. In: Nikitin, A.G., Boyko, V.M., Popovych, R.O. (eds.), Proceedings of the Fourth International Conference on Symmetry in Nonlinear Mathematical Physics, Proceedings of the Institute of Mathematics of the NAS of Ukraine, vol. 43, pp. 201–208. Kyiv (2002)
[42] J. F. Song, and C. Z. Qu, Geometric integrability of two-component Camassa-Holm and Hunter-Saxton systems, *Commun. Theor. Phys.* **55** (2011) 955–959.
[43] H. Wu, and M. Wunsch, Global existence for the generalized two-component Hunter-Saxton system, *J. Math. Fluid Mech.* **14** (2012) 455–469.
[44] M. Wunsch, On the Hunter-Saxton system, *Discret Contin. Dyn. Syst. Ser. B* **12** (2009) 647–656.
[45] P. Zhang, and Y. Liu, Stability of solitary waves and wave-breaking phenomena for the two-component Camassa-Holm system, *Int. Math. Res. Not.* **2010** (2010) 1981–2021.