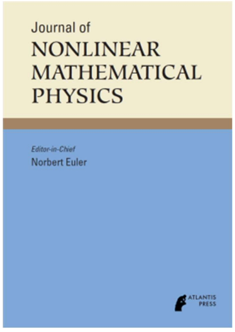

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Kh. Goodarzi, M. Nadjafikhah

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μ-symmetry and μ-conservation law for the extended mKdV equation

Kh. Goodarzi
Department of Mathematics, College of Basic Sciences, Karaj, Islamic Azad University, Alborz, Iran
kh.goodarzi@kiau.ac.ir

M. Nadjafikhah*
Department of Mathematics, College of Basic Sciences, Karaj, Islamic Azad University, Alborz, Iran
m_nadjafikhah@iust.ac.ir

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In this paper, we obtain μ-symmetry and μ-conservation law of the extended mKdV equation. The extended mKdV equation dose not admit a variational problem since it is of odd order. First we obtain μ-conservation law of the extended mKdV equation in potential form because it admits a variational problem, using it, we can obtain μ-conservation law of the extended mKdV equation.

Keywords: Symmetry; μ-symmetry; μ-conservation law; variational problem; order reduction.

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1. Introduction

In 2001, Muriel and Romero introduced λ-symmetries method to order reduction of ODEs. In 2004, Gaeta and Morando expanded this approach to the PDE frame with p independent variables $x = (x^1, ..., x^p)$ and q dependent variables $u = (u^1, ..., u^q)$, in order to do this, the central object is a horizontal one-form $\mu = \lambda_i dx^i$ on first order jet space $(J^{(1)}M, \pi, M)$, where μ is a compatible, i.e. $D_i\lambda_j - D_j\lambda_i = 0$, and thus one speaks of μ-symmetries.

In 2006, Muriel, Romero and Olver generalized the concept of variational problem and conservation law, based on λ-symmetries, and presented an adapted formulation of the Nother's theorem for λ-symmetry of ODEs. In 2007, Cicogna and Gaeta extended the results obtained by Muriel, Romero and Olver, in the case of λ-symmetries to the case of μ-symmetries. They called, conservation law in the case of μ-symmetry of the Lagrangian, μ-conservation law. The Korteweg-de Vries (KdV) equation $u_t + u_{xxx} + uu_x = 0$, is one of the most popular equations by Korteweg and de Vries in the 19^{th} century as water waves equations. The KdV equation is a nonlinear partial differential equation arising in the study of a number of different physical systems, e.g., water waves, plasma physics, harmonic lattices, elastic rods and nonlinear long dynamo waves observed in the Sun. The modified Korteweg-de Vries (mKdV) equation is one of the most important nonlinear wave equation in physics and mechanic. For example, in the study of plasma physics, nonlinear optics, solid state physics and fluid mechanics, whose general form is $u_t + au_{xxx} + bu^2u_x = 0$.

*Corresponding author.

In the paper [8] H. Liu and J. Li studied the nonlinear evolution equation in the form of

$$u_t + a_1 u_{xxx} + a_2 u_x + a_3 u u_x + a_4 u^2 u_x = 0.$$

They called this equation "extended form of the mKdV equation" and the parameters $a_i \in \mathbb{R}$. We know that the basis of mKdV equation rather than KdV type equation is obtained in terms of the basis of extended form of the mKdV equation. In view of this, we would rather name this equation, *the extended mKdV equation*. All properties of the extended mKdV equation can be obtained from the well-known properties of the mKdV equation taking transformation mentioned in the paper [7] into account. So, solutions of the extended mKdV equation can be expressed via the Painleve' transcendents as well. In partial cases solutions of the extended mKdV equation can be expressed using rational function and the Airy functions [1].

The outline of this paper is as follows. Firstly, μ-symmetry and reduced equations for the extended mKdV equation. Secondly, μ-symmetry and reduced equations for particular cases - the mKdV equation, the KdV equation and the Euler equation - of the extended mKdV equation. Finally, μ-conservation law for the extended mKdV equation.

2. μ-prolongation and μ-symmetry

In this section, the starting point will be a discussion of some of the foundational results about μ-prolongation and μ-symmetry rather briefly. Let $\mu = \lambda_i dx^i$ be horizontal one-form on first order jet space $(J^{(1)}M, \pi, M)$ and compatible with contact structure $\mathscr{E}$ on $J^{(k)}M$ for $k \geq 2$, i.e. $d\mu \in J(\mathscr{E})$, where $J(\mathscr{E})$ is Cartan ideal generated by contact structure $\mathscr{E}$ and $\lambda_i : J^{(1)}M \longrightarrow \mathbb{R}$. In the paper [5], condition $d\mu \in J(\mathscr{E})$ is equivalent to $D_i\lambda_j - D_j\lambda_i = 0$, where D_i is total derivative x^i. For given the vector bundle $(M, \pi, \mathbb{R})$, the horizontal one-form $\mu \in \bigwedge^1(J^1M)$, i.e. the one-form $\mu = \lambda(x, u, u_x)dx$, where $\lambda(x, u, u_x) : J^1M \longrightarrow \mathbb{R}$ is smooth real function.

Suppose $\Delta(x, u^{(n)}) = 0$ is a scalar PDEs involving p independent variables $x = (x^1, ..., x^p)$ and one dependent variable. Let $X = \xi^i \partial_{x^i} + \varphi \partial_u$ be a vector field on M. We define $Y = X + \sum_{J=1}^{k} \Psi_J \, \partial_{u_J}$ on k-th order jet space J^kM as μ-prolongation of X if its coefficient (with $\Psi_0 = \varphi$) satisfy the μ-prolongation formula

$$\Psi_{J,i} = (D_i + \lambda_i)\Psi_J - u_{J,m}(D_i + \lambda_i)\xi^m. \tag{2.1}$$

Let us observe that, if $\mu = 0$ in (2.1), then we gain ordinary prolongation of X. So we can assume ordinary prolongation as 0-prolongation in μ-prolongation framework. Let X be a vector field on M, and Y be its μ -prolongation of order k. Let Δ be a differential equation (PDE) of order k in M, $\Delta(x, u^{(k)}) = 0$, and $\mathscr{S} \subset J^{(k)}M$ be the solution manifold for Δ. If $Y : \mathscr{S} \longrightarrow T\mathscr{S}$, we say that X is a μ-symmetry for Δ.

Suppose $\mu = \lambda_i dx^i$ is a horizontal 1-form and $V = \exp\left(\int \mu\right)X$ is an exponential vector field, where X is a vector field on M. For μ, consider an equation Δ such that $D_i\lambda_j - D_j\lambda_i = 0$ is satisfied on $\mathscr{S}_\Delta$. Then V is a general symmetry for Δ if and only if X is a μ-symmetry for Δ.

In the paper [5], we observe reduction of PDEs under μ-symmetries in the following theorem.

Theorem 2.1. *Let Δ be a scalar PDE of order k for $u = u(x^1, ..., x^p)$. Let $X = \xi^i(\frac{\partial}{\partial x^i}) + \varphi(\frac{\partial}{\partial u})$ be a vector field on M, with characteristic $Q = \varphi - u_i\xi^i$, and let Y be the μ-prolong of order k of X. If X is a μ-symmetry for Δ, then $Y : \mathscr{S}_X \longrightarrow T\mathscr{S}_X$, where $\mathscr{S}_X \subset J^{(k)}M$ is the solution manifold for the system Δ_X made of Δ and of $E_J := D_J Q = 0$ for all J with $\mid J \mid = 0, 1, ..., k-1$.*

3. μ-symmetry for the extended mKdV equation

In the extended mKdV equation we will consider PDEs in two independent variables, (x,t). In this case we will also write $X = \xi\partial_x + \tau\partial_t + \varphi\partial_u$ and $\mu = \lambda_1 dx + \lambda_2 dt$.

3.1. *μ-symmetry of given equations*

In order to determine μ-symmetry of a given PDE Δ of order n, one can proceed in the same way as for ordinary symmetries. That is, consider a generic vector field X acting in M, and its μ-prolongation Y of order n for a generic $\mu = \lambda_i dx^i$, acting in $J^{(n)}M$. One then applies Y to Δ, and restricts the obtained expression to the solution manifold $\mathscr{S}_\Delta \subset J^{(n)}M$. The equation Δ_* resulting by requiring this is zero is the determining equation for μ-symmetries of Δ; this is an equation for ξ, τ, φ and λ_i, and as such is nonlinear.

If we require λ_i are functions on $J^{(k)}M$, all the dependences on u_J with $| J |> k$ will be explicit, and one obtains a system of determining equation. This system should be complemented with the compatibility conditions between the λ_i. If we determine a priori the form μ, we are left with a system of linear equation for ξ, τ, φ; similarly, if we fix a vector field X and try to find the μ for which it is a μ-symmetry of the given equation Δ, we have a system of quasilinear equation for the λ_i.

3.2. *μ-symmetry for the extended mKdV equation*

Let us consider the extended mKdV equation

$$u_t + a_1 u_{xxx} + a_2 u_x + a_3 u u_x + a_4 u^2 u_x = 0. \tag{3.1}$$

Suppose $X = \xi\partial_x + \tau\partial_t + \varphi\partial_u$ is a vector field and $\mu = \lambda_1 dx + \lambda_2 dt$ is a horizontal one-form. For this μ, we should have the compatibility condition $D_t\lambda_1 = D_x\lambda_2$ when $u_t + a_1 u_{xxx} + a_2 u_x + a_3 u u_x + a_4 u^2 u_x = 0$. Let us come to the third μ-prolongation of X. For this computation we can use the Eq. (2.1), hence, we show μ-prolongation of X as

$$Y = X + \Psi^x \partial_{u_x} + \Psi^t \partial_{u_t} + \Psi^{xx} \partial_{u_{xx}} + \ldots + \Psi^{ttt} \partial_{u_{ttt}},$$

where

$$\begin{aligned}
\Psi^x &= (D_x + \lambda_1)\varphi - u_x(D_x + \lambda_1)\xi - u_t(D_x + \lambda_1)\tau, \\
\Psi^t &= (D_t + \lambda_2)\varphi - u_x(D_t + \lambda_2)\xi - u_t(D_t + \lambda_2)\tau, \\
\Psi^{xx} &= (D_x + \lambda_1)\Psi^x - u_{xx}(D_x + \lambda_1)\xi - u_{xt}(D_x + \lambda_1)\tau, \\
\Psi^{xt} &= (D_t + \lambda_2)\Psi^x - u_{xx}(D_t + \lambda_2)\xi - u_{xt}(D_t + \lambda_2)\tau, \\
\Psi^{tt} &= (D_t + \lambda_2)\Psi^t - u_{tx}(D_t + \lambda_2)\xi - u_{tt}(D_t + \lambda_2)\tau, \\
\Psi^{xxx} &= (D_x + \lambda_1)\Psi^{xx} - u_{xxx}(D_x + \lambda_1)\xi - u_{xxt}(D_x + \lambda_1)\tau, \\
\Psi^{xxt} &= (D_t + \lambda_2)\Psi^{xx} - u_{xxx}(D_t + \lambda_2)\xi - u_{xxt}(D_t + \lambda_2)\tau, \\
\Psi^{xtt} &= (D_t + \lambda_2)\Psi^{xt} - u_{xtx}(D_t + \lambda_2)\xi - u_{xtt}(D_t + \lambda_2)\tau, \\
\Psi^{ttt} &= (D_t + \lambda_2)\Psi^{tt} - u_{ttx}(D_t + \lambda_2)\xi - u_{ttt}(D_t + \lambda_2)\tau.
\end{aligned} \tag{3.2}$$

In this case, the μ-prolongation Y acts on the Eq. (3.1) and substituting $(u_t - a_2u_x - a_3uu_x - a_4u^2u_x)/a_1$ for u_{xxx}, we obtain the following system

$$\begin{gathered}
a_1\tau_u = 0, \qquad a_1\tau_{uu} = 0, \qquad a_1\tau_{uuu} = 0, \qquad a_1\xi_u = 0, \qquad a_1\xi_{uu} = 0,\\
a_1\xi_{uuu} = 0, \qquad a_1\lambda_1\tau + a_1\tau_x = 0, \qquad -2a_1\lambda_1\tau_u + a_1(\lambda_1)_u\tau + 2a_1\tau_{ux} = 0\,,\\
3a_1\tau_{xu} + 3a_1\lambda_1\tau_u + a_1(\lambda_1)_u\tau = 0, \qquad 9a_1\lambda_1\xi_u + 9a_1\xi_{ux} + 4a_1(\lambda_1)_u\xi - 3a_1\phi_{uu} = 0\,,\\
3a_1\lambda_1\tau_{uu} + a_1(\lambda_1)_{uu}\tau + 3a_1\tau_{xuu} + 3a_1(\lambda_1)_u\tau_u = 0,\\
3a_1(\lambda_1)_x\tau + 3a_1\tau_{xx} + 3a_1\lambda_1^2\tau + 6a_1\lambda_1\tau_u = 0\,,\\
a_1(\lambda_1)_{uu}\xi + 3a_1(\lambda_1)_u\xi_u + 3a_1\xi_{xuu} - a_1\phi_{uuu} + 3a_1\lambda_1\xi_{uu} = 0\,,\\
-3a_1(\lambda_1)_x\xi + 3a_1\phi_{xu} + 3a_1\lambda_1\phi_u - 3a_1\xi_{xx} - 3a_1\lambda_1^2\xi + a_1(\lambda_1)_u\phi - 6a_1\lambda_1\xi_x = 0,\\
6a_1\lambda_1\tau_{xu} + 3a_1(\lambda_1)_x\tau_u - 3\xi_u + 2a_1(\lambda_1)_{xu}\xi + 3a_1(\lambda_1)_u\lambda_1\tau\\
+3a_1(\lambda_1)_u\tau_x + 3a_1\lambda_1^2\tau_u + 3a_1\tau_{xxu} = 0\,,\\
3a_1\lambda_1^2\phi_x + a_1(\lambda_1)_{xx}\phi + \lambda_2\phi + \phi_t + a_1\phi_{xxx} + a_3u\lambda_1\phi + 3a_1(\lambda_1)_x\phi_x + 3a_1\lambda_1\phi_{xx}\\
+3a_1(\lambda_1)_x\lambda_1\phi + a_3u\phi_x + a_1\lambda_1^3\phi + a_2\varphi_x + a_2\lambda_1\varphi + a_4u^2\varphi_x + a_4u^2\lambda_1\varphi = 0,\\
-3a_1(\lambda_1)_u\lambda_1\xi - 3a_1(\lambda_1)_u\xi_x + 3a_3u\xi_u + 3a_1\phi_{xuu} - 3a_1\lambda_1^2\xi_u - 2a_1(\lambda_1)_{xu}\xi + 3a_2\xi_u\\
-6a_1\lambda_1\xi_{xu} + 3a_1\lambda_1\phi_{uu} - 3a_1\xi_{xxu} - 3a_1(\lambda_1)_x\xi_u + a_1(\lambda_1)_{uu}\phi + 3a_1(\lambda_1)_u\phi_u\\
+3a_4u^2\xi_u + 3a_1(\lambda_1)_u\phi u = 0,\\
-a_1\lambda_1^3\tau - \lambda_2\tau - \tau_t - a_1(\lambda_1)_{xx}\tau - a_3u\lambda_1\tau + 3\xi_x - a_1\tau_{xxx} - 3a_1\lambda_1\tau_{xx} - 3a_1\lambda_1^2\tau_x\\
-a_2\tau_x - 3a_1(\lambda_1)_x\lambda_1\tau - 3a_1(\lambda_1)_x\tau_x + 3\lambda_1\xi - a_3u\tau_x - a_2\lambda_1\tau - a_4u^2\tau_x - a_4u^2\lambda_1\tau = 0\,,\\
-a_1\lambda_1^3\xi - \xi_t + 2a_3u\xi_x - 3a_1\lambda_1\xi_{xx} - \lambda_2\xi - 3a_1(\lambda_1)_x\lambda_1\xi - 3a_1\lambda_1^2\xi_x + 3a_1\lambda_1^2\phi_u + 2a_2\xi_x\\
+3a_1(\lambda_1)_x\phi_u - a_1(\lambda_1)_{xx}\xi + 3a_1\phi_{xxu} + 2a_1(\lambda_1)_{ux}\phi + 6a_1\lambda_1\phi_{xu} + 3a_1(\lambda_1)_u\lambda_1\phi + a_3\phi\\
-a_1\xi_{xxx} + 2a_3u\lambda_1\xi - 3a_1(\lambda_1)_x\xi_x + 3a_1(\lambda_1)_u\phi_x + +2a_4u\varphi + 2a_2\lambda_1\xi\\
+2a_4u^2\lambda_1\xi + 2a_4u^2\xi_x = 0\,.
\end{gathered} \tag{3.3}$$

For any choice of the type

$$\lambda_1 = D_x[f(x,t)] + g(x), \qquad \lambda_2 = D_t[f(x,t)] + h(t), \tag{3.4}$$

where $f(x,t)$, $g(x)$ and $h(t)$ are arbitrary functions, we have the compatibility condition, i.e. $D_t\lambda_1 = D_x\lambda_2$ (on solutions to the Eq. (3.1)). For instance, we consider two cases to obtain μ-symmetry of the Eq. (3.1) as the following:

1. When $g(x) = 0$ and $h(t) = 0$, then substituting the functions $\lambda_1 = D_xf(x,t)$ and $\lambda_2 = D_tf(x,t)$ into the system of (3.3) and solving them, we obtain

$$\xi = F(x,t), \qquad \tau = 0, \qquad \varphi = 0,$$

   where $f(x,t) = -\ln(F(x,t))$ and $F(x,t)$ is an arbitrary positive function. Then $X = F(x,t)\partial_x$ is μ-symmetry of the Eq. (3.1) and corresponds to an ordinary symmetry $V = \exp\Big(\int D_xf(x,t)dx + D_tf(x,t)dt\Big)X$ of exponential type. In this case, reduction of the Eq. (3.1) is

$$Q = \varphi - \xi u_x - \tau u_t = -F(x,t)u_x. \tag{3.5}$$

2. When $g(x) = 0$ and $h(t) = c_1/(c_1t + 12a_4^2)$ where c_1 is an arbitrary constant, then substituting the functions $\lambda_1 = D_x f(x,t)$ and $\lambda_2 = D_t f(x,t) + c_1/(c_1t + 12a_4^2)$ into the system of (3.3) and solving them, we obtain

$$\xi = \frac{(4a_2a_4 - a_2^3)(c_1t + 12a_4^2) + 2a_4(c_1x + c_2)}{6a_4(c_1t + 12a_4^2)} F(x,t),$$
$$\tau = F(x,t), \qquad \varphi = -\frac{c_1(a_3 + 2a_4u)}{6a_4(c_1t + 12a_4^2)} F(x,t),$$

where $f(x,t) = -\ln(F(x,t))$, $F(x,t)$ is an arbitrary positive function and c_2 is an arbitrary constant. Then

$$X = \Big(\frac{(4a_2a_4 - a_2^3)(c_1t + 12a_4^2) + 2a_4(c_1x + c_2)}{6a_4(c_1t + 12a_4^2)} \partial_x + \partial_t - \frac{c_1(a_3 + 2a_4u)}{6a_4(c_1t + 12a_4^2)} \partial_u\Big) F(x,t),$$

is μ-symmetry of the Eq. (3.1) and corresponds to an ordinary symmetry $V = \exp\Big(\int D_x f(x,t)dx + (D_t f(x,t) + c_1/(c_1t + 12a_4^2))dt\Big)X$ of exponential type. In this case, reduction of the Eq. (3.1) is

$$\begin{aligned} Q &= \varphi - \xi u_x - \tau u_t \qquad (3.6)\\ &= -\Big(\frac{c_1(a_3 + 2a_4u)}{6a_4(c_1t + 12a_4^2)} + \frac{(4a_2a_4 - a_2^3)(c_1t + 12a_4^2) + 2a_4(c_1x + c_2)}{6a_4(c_1t + 12a_4^2)} u_x + u_t\Big) F(x,t). \end{aligned}$$

4. μ-symmetry for Particular cases of the extended mKdV equation

In this section, we obtain μ-symmetry for Particular cases of the extended mKdV equation $u_t + a_1u_{xxx} + a_2u_x + a_3uu_x + a_4u^2u_x = 0$. Clearly, when $a_2 = a_3 = 0$, the extended mKdV equation is the mKdV equation $u_t + a_1u_{xxx} + a_4u^2u_x = 0$. When $a_1 = a_3 = 1$ and $a_2 = a_4 = 0$, the extended mKdV equation is the KdV equation $u_t + u_{xxx} + uu_x = 0$. When $a_3 = -1$ and $a_1 = a_2 = a_4 = 0$, the extended mKdV equation is the Euler equation $u_t - uu_x = 0$.

4.1. *μ-symmetry for the mKdV equation*

When $a_2 = a_3 = 0$, the Eq. (3.1) is the mKdV equation

$$u_t + a_1u_{xxx} + a_4u^2u_x = 0. \qquad (4.1)$$

Similar to the extended mKdV equation, for instance, we consider two cases to obtain μ-symmetry of the mKdV equation as the following:

1. When $g(x) = 0$ and $h(t) = 0$ in the functions of (3.4), then substituting the functions $\lambda_1 = D_x f(x,t)$ and $\lambda_2 = D_t f(x,t)$ into the system of (3.3) and solving them, we obtain

$$\xi = F(x,t), \qquad \tau = 0, \qquad \varphi = 0,$$

where $f(x,t) = -\ln(F(x,t))$, $F(x,t)$ is an arbitrary positive function. Then $X = F(x,t)\partial_x$ is μ-symmetry of the Eq. (3.1) and corresponds to an ordinary symmetry $V = \exp\Big(\int D_x f(x,t)dx + D_t f(x,t)dt\Big)X$ of exponential type. In this case, reduction of the Eq. (3.1) is

$$Q = \varphi - \xi u_x - \tau u_t = -F(x,t)u_x. \qquad (4.2)$$

2. When $g(x) = 0$ and $h(t) = 3c_1/(3c_1t+1)$ in the functions of (3.4), where c_1 is an arbitrary constant, then substituting the functions $\lambda_1 = D_x f(x,t)$ and $\lambda_2 = D_t f(x,t) + 3c_1/(3c_1t+1)$ into the system of (3.3) and solving them, we obtain

$$\xi = \frac{c_1x+c_2}{3c_1t+1}F(x,t), \qquad \tau = F(x,t), \qquad \varphi = -\frac{c_1u}{3c_1t+1}F(x,t),$$

where $f(x,t) = -\ln(F(x,t))$, $F(x,t)$ is an arbitrary positive function and c_2 is an arbitrary constant. Then

$$X = F(x,t)\left(\frac{c_1x+c_2}{3c_1t+1}\partial_x + \partial_t - \frac{c_1u}{3c_1t+1}\partial_u\right),$$

is μ-symmetry of the Eq. (4.1) and corresponds to an ordinary symmetry $V = \exp\left(\int D_x f(x,t)dx + (D_t f(x,t) + 3c_1/(3c_1t+1))dt\right)X$ of exponential type. In this case, reduction of the mKdV equation $u_t + a_1u_{xxx} + a_4u^2u_x = 0$ is

$$Q = \varphi - \xi u_x - \tau u_t = -\left(\frac{c_1u}{3c_1t+1} + \frac{c_1x+c_2}{3c_1t+1}u_x + u_t\right)F(x,t). \tag{4.3}$$

4.2. *μ-symmetry for the KdV equation*

When $a_1 = a_3 = 1$ and $a_2 = a_4 = 0$, the Eq. (3.1) is the KdV equation

$$u_t + u_{xxx} + uu_x = 0. \tag{4.4}$$

Similar to the extended mKdV equation, for instance, we consider two cases to obtain μ-symmetry of the KdV equation as the following:

1. When $g(x) = 0$ and $h(t) = 1/(t+c)$ in the functions of (3.4), where c is an arbitrary constant, then substituting the functions $\lambda_1 = D_x f(x,t)$ and $\lambda_2 = D_t f(x,t) + 1/(t+c)$ into the system of (3.3) and solving them, we obtain

$$\xi = F(x,t), \quad \tau = 0, \quad \varphi = \frac{1}{t+c}F(x,t),$$

where $f(x,t) = -\ln(F(x,t))$, $F(x,t)$ is an arbitrary positive function. Then $X = F(x,t)\partial_x + (1/(t+c))F(x,t)\partial_u$ is a μ-symmetry of the KdV equation and corresponds to an ordinary symmetry $V = \exp\left(\int D_x f(x,t)dx + (D_t f(x,t) + 1/(t+c))dt\right)X$ of exponential type. In this case, reduction of the KdV equation is

$$Q = \varphi - \xi u_x - \tau u_t = \left(\frac{1}{t+c} - u_x\right)F(x,t). \tag{4.5}$$

2. When $g(x) = 0$ and $h(t) = 3/(3t+c_1)$ in the functions of (3.4), where c_1 is an arbitrary constant, then substituting the functions $\lambda_1 = D_x f(x,t)$ and $\lambda_2 = D_t f(x,t) + 3/(3t+c_1)$

into the system of (3.3) and solving them, we obtain

$$\xi = (c_2 + \frac{x+3c_3}{3t+c_1})F(x,t), \qquad \tau = F(x,t), \qquad \varphi = \frac{3(-2u+3c_2)}{3t+c_1}F(x,t),$$

where $f(x,t) = -\ln(F(x,t))$, $F(x,t)$ is an arbitrary positive function, c_2 and c_3 are arbitrary constants. Then

$$X = F(x,t)\left((c_2 + \frac{x+3c_3}{3t+c_1})\partial_x + \partial_t + \frac{3(-2u+3c_2)}{3t+c_1}\partial_u\right),$$

is a μ-symmetry of the KdV equation and corresponds to an ordinary symmetry $V = \exp\left(\int D_x f(x,t)dx + (D_t f(x,t) + 3/(3t+c_1))dt\right)X$ of exponential type. In this case, reduction of the KdV equation under μ-symmetry is

$$Q = \varphi - \xi u_x - \tau u_t = F(x,t)\Big(\frac{3(-2u+3c_2)}{3t+c_1} - (c_2 + \frac{x+3c_3}{3t+c_1})u_x - u_t\Big). \tag{4.6}$$

4.3. *μ-symmetry for the Euler equation*

When $a_3 = -1$ and $a_1 = a_2 = a_4 = 0$, the Eq. (3.1) is the Euler equation

$$u_t - uu_x = 0. \tag{4.7}$$

If $a_3 = -1$ and $a_1 = a_2 = a_4 = 0$ in the system of (3.3), then we obtain

$$(u_x+\lambda_1 u+\lambda_2)\varphi+uu_x(u\lambda_1+\lambda_2)\tau-u_x(u\lambda_1-\lambda_2)\xi+\varphi_t+uu_x\tau_t-u_x\xi_t+u\varphi_x+u^2u_x\tau_x-uu_x\xi_x = 0.$$

With the ansatz $\lambda_1(x,t,u) = \lambda_1$ and $\lambda_2 = \lambda_2(x,t,u)$ the dependence of the equations above in u_x is explicit, and it splits into two equations:

$$(\lambda_1 u + \lambda_2)\varphi + \varphi_t + u\varphi_x = 0,$$
$$\varphi + (\lambda_1 u^2 + \lambda_2 u)\tau - (\lambda_1 u + \lambda_2)\xi + u\tau_t - \xi_t + u^2\tau_x - u\xi_x = 0.$$

A special solution is given by

$$\xi = \frac{x^2+tu^2+txu+ux}{u^2}e^{-ux/2}, \qquad \tau = 0, \qquad \varphi = \frac{x+tu}{u}e^{-ux/2}, \qquad \lambda_1 = u, \qquad \lambda_2 = \frac{u^2}{2},$$

and $D_t\lambda_1 = D_x\lambda_2$ when $u_t - uu_x = 0$. Hence, vector field

$$X = \Big(\frac{x^2+tu^2+txu+ux}{u^2}e^{-ux/2}\Big)\partial_x + \Big(\frac{x+tu}{u}e^{-ux/2}\Big)\partial_u \tag{4.8}$$

is a μ-symmetry for the Euler equation $u_t = uu_x$. This μ-symmetry corresponds to an ordinary symmetry V of exponential type, i.e. $V = e^{\int \mu}X$, or $V = \exp\left(\int udx + \frac{1}{2}u^2dt\right)X$. Also, reduction of the Euler equation $u_t = uu_x$ under μ-symmetry is

$$Q = \varphi - \xi u_x - \tau u_t = \frac{x+tu}{u}e^{-ux/2} - \frac{x^2+tu^2+txu+ux}{u^2}e^{-ux/2}u_x. \tag{4.9}$$

5. Lagrangian of the extended mKdV equation in potential form

In this section, we show that *the extended mKdV equation* does not admit a variational problem since it is of odd order, but *the extended mKdV equation in potential form* admits a variational problem. In the book [13], a system of a variational formulation if and only if its Frechet derivative is self-adjoint. In fact, we have the following theorem.

Theorem 5.1. *Let $\Delta = 0$ be a system of differential equation. Then Δ is the Euler-Lagrange expression for some variational problem $\mathfrak{L} = \int L dx$, i.e. $\Delta = E(L)$, if and only if the Frechet derivative D_Δ is self-adjoint: $D^*_\Delta = D_\Delta$. In this case, a Lagrangian for Δ can be explicitly constructed using the homotopy formula $L[u] = \int_0^1 u.\Delta[\lambda u] d\lambda$.*

We consider the extended mKdV equation as

$$\Delta : u_t + a_1 u_{xxx} + a_2 u_x + a_3 u u_x + a_4 u^2 u_x = 0. \tag{5.1}$$

The Frechet derivative of Δ is

$$D_\Delta = D_t + a_1 D_x^3 + (a_2 + a_3 u + a_4 u^2) D_x + a_3 u_x.$$

Obviously it does not admit a variational problem since $D^*_\Delta \neq D_\Delta$. But the well-known differential substitution $u = v_x$ yields the related transformed the extended mKdV equation as the following

$$\Delta_v : v_{xt} + a_1 v_{xxxx} + a_2 v_{xx} + a_3 v_x v_{xx} + a_4 v_x^2 v_{xx} = 0\,. \tag{5.2}$$

We called this equation "the extended mKdV equation in potential form " and the Frechet derivative it is

$$D_{\Delta_v} = D_x D_t + a_1 D_x^4 + (a_2 + a_3 v_x + a_4 v_x^2) D_x^2 + (a_3 v_{xx} + 2 a_4 v_x v_{xx}) D_x\,,$$

which is self-adjoint: $D^*_{\Delta_v} = D_{\Delta_v}$. By the Theorem (5.1), the extended mKdV equation in potential form Δ_v has a Lagrangian of the form

$$L[v] = \int_0^1 v.\Delta_v[\lambda v] d\lambda = -\frac{1}{12}\Big(6 v_x v_t - 6 a_1 v_{xx}^2 + 6 a_2 v_x^2 + 2 a_3 v_x^3 + a_4 v_x^4\Big) + \mathrm{Div} P.$$

Hence, Lagrangian of Δ_v equation, up to Div-equivalence is

$$\mathscr{L}[v] = -\frac{1}{12}\Big(6 v_x v_t - 6 a_1 v_{xx}^2 + 6 a_2 v_x^2 + 2 a_3 v_x^3 + a_4 v_x^4\Big), \tag{5.3}$$

6. μ-conservation laws

A (standard) *conservation law* is a relation $\mathrm{Div}\,\mathbf{P} := \sum_{i=1}^p D_i P^i = 0$, where $\mathbf{P} = (P^1, \cdots, P^p)$ is a $p-$dimensional vector. Suppose $\mu = \lambda_i dx^i$ is a horizontal one-form, such that $D_i \lambda_j = D_j \lambda_i$. We define a *$\mu$-conservation law* as a relation

$$(D_i + \lambda_i) P^i = 0, \tag{6.1}$$

where P^i is a (Matrix-valued) vector and the $M-$vector P^i is called a μ-conserved vector. In the paper [4], we observe the following theorem.

Theorem 6.1. *Consider the $n-th$ order Lagrangian $\mathscr{L} = \mathscr{L}(x, u^{(n)})$, and vector field X, then X is a μ-symmetry for $\mathscr{L}$, i.e. $Y[\mathscr{L}] = 0$ if and only if there exists $M-$vector P^i satisfying the μ-conservation law $(D_i + \lambda_i)P^i = 0$.*

Using the other theorems in [4] and the theorem (6.1), the $M-$vector P^i is obtained For first and second order Lagrangian, as the following:

- For first order Lagrangian $\mathscr{L}(x,t,u,u_x,u_t)$ and the vector field $X = \varphi(\partial/\partial u)$ is a μ-symmetry for $\mathscr{L}$, then the $M-$vector $P^i := \varphi(\partial\mathscr{L}/\partial u_i)$, is a μ-conserved vector.
- For second order Lagrangian $\mathscr{L}$ and the vector field $X = \varphi(\partial/\partial u)$ is a μ-symmetry for $\mathscr{L}$, then the $M-$vector

$$P^i := \varphi\frac{\partial\mathscr{L}}{\partial u_i} + ((D_j + \lambda_j)\varphi)\frac{\partial\mathscr{L}}{\partial u_{ij}} - \varphi D_j\frac{\partial\mathscr{L}}{\partial u_{ij}}, \tag{6.2}$$

  is a μ-conserved vector.

6.1. *μ-conservation laws of the extended mKdV equation in potential form*

In this section, we want to compute μ-conservation law for *the extended mKdV equation in potential form* and using it we compute μ-conservation law for *the extended mKdV equation* in section (6.2). Consider the second order Lagrangian (5.3) for the extended mKdV equation in potential form

$$\Delta_v = v_{xt} + a_1 v_{xxxx} + a_2 v_{xx} + a_3 v_x v_{xx} + a_4 v_x^2 v_{xx} = E(\mathscr{L}). \tag{6.3}$$

Suppose $X = \varphi\partial_v$ is a vector field and $\mu = \lambda_1 dx + \lambda_2 dt$ is a horizontal one-form. For this μ, we should have the compatibility condition $D_t\lambda_1 = D_x\lambda_2$ when $v_{xt} + a_1 v_{xxxx} + a_2 v_{xx} + a_3 v_x v_{xx} + a_4 v_x^2 v_{xx} = 0$. Let us come to the second μ-prolongation of X. For this computation we can use the Eq. (2.1), hence, we show μ-prolongation of X as

$$Y = \varphi\partial_v + \Psi^x\partial_{v_x} + \Psi^t\partial_{v_t} + \Psi^{xx}\partial_{v_{xx}} + \Psi^{xt}\partial_{v_{xt}} + \Psi^{tt}\partial_{v_{tt}},$$

where

$$\begin{gathered}\Psi^x = (D_x + \lambda_1)\varphi, \qquad \Psi^t = (D_t + \lambda_2)\varphi, \qquad \Psi^{xx} = (D_x + \lambda_1)\Psi^x,\\ \Psi^{xt} = (D_t + \lambda_2)\Psi^x, \qquad \Psi^{tt} = (D_t + \lambda_2)\Psi^t.\end{gathered} \tag{6.4}$$

In this case, the μ-prolongation Y acts on the Eq. (6.3) and substituting $(a_1 v_{xx}^2 - a_2 v_x^2 - a_3 v_x^3/3 - a_4 v_x^4)/v_x$ for v_t, we obtain

$$\begin{gathered}a_1\varphi_{vv} = 0, \qquad a_4\varphi_v = 0, \qquad a_1(\lambda_1\varphi + \varphi_x) = 0, \qquad a_3(\lambda_1\varphi - \varphi_x) = 0,\\ \varphi_t + \lambda_2\varphi + a_2\lambda_1\varphi + a_2\varphi_x = 0, \qquad a_1(\lambda_1^2\varphi + 2\lambda_1\varphi_x + \varphi_{xx} + \lambda_{1x}\varphi) = 0,\\ a_1(2\lambda_1\varphi_v + 2\varphi_{vx} + \lambda_{1v}\varphi) = 0, \qquad 3a_4\varphi_x + 2a_3\varphi_v + 3a_4\lambda_1\varphi = 0.\end{gathered} \tag{6.5}$$

Suppose $\varphi = F(x,t)$, where $F(x,t)$ is an arbitrary positive function satisfying $\mathscr{L}[v] = 0$, where $\mathscr{L}[v]$ is from (5.3), then a special solution them is given by

$$\lambda_1 = -\frac{F_x(x,t)}{F(x,t)}, \qquad \lambda_2 = -\frac{F_t(x,t)}{F(x,t)}, \tag{6.6}$$

where λ_1 and λ_2 are satisfying to $D_t\lambda_1 = D_x\lambda_2$. Hence, $X = F(x,t)\partial_v$ is a μ-symmetry for $\mathscr{L}$, in this case by the Theorem (6.1) there exists $M-$vector P^i satisfying the μ-conservation law $(D_i + \lambda_i)P^i =$

0, by the Eq. (6.2), we have

$$P^1 = -\frac{1}{6}\Big(3v_t + a_1 v_{xxx} + 6a_2 v_x + 3a_3 v_x^2 + 2a_4 v_x^3\Big) F(x,t), \quad P^2 = -\frac{v_x}{2} F(x,t). \tag{6.7}$$

Hence, μ-conservation law for second order Lagrangian $\mathscr{L}[v]$ is $(D_x + \lambda_1)P^1 + (D_t + \lambda_2)P^2 = 0$, or corresponds to

$$D_x P^1 + D_t P^2 + \lambda_1 P^1 + \lambda_2 P^2 = 0. \tag{6.8}$$

Hence, $X = F(x,t)\partial_v$ is a μ-symmetry and $D_x P^1 + D_t P^2 + \lambda_1 P^1 + \lambda_2 P^2 = 0$ is μ-conservation law for the extended mKdV equation in potential form Δ_v.

Remark 6.1. By the Noether's Theorem for μ-symmetry (for the extended mKdV equation in potential form), we have

$$\begin{aligned}(D_i + \lambda_i)P^i &= (D_x + \lambda_1)P^1 + (D_t + \lambda_2)P^2 \\ &= F(x,t)(v_{xt} + a_1 v_{xxxx} + a_2 v_{xx} + a_3 v_x v_{xx} + a_4 v_x^2 v_{xx}) \\ &= QE(\mathscr{L}).\end{aligned} \tag{6.9}$$

6.2. *μ-conservation laws of the extended mKdV equation*

We want to compute μ-conservation law for the extended mKdV equation. Consider the extended mKdV equation in potential form $\Delta_v = v_{xt} + a_1 v_{xxxx} + a_2 v_{xx} + a_3 v_x v_{xx} + a_4 v_x^2 v_{xx} = 0$, or equivalently $D_x(v_t + a_1 v_{xxx} + a_2 v_x + a_3 v_x^2/2 + a_4 v_x^3/3) = 0$. If we substitute u for v_x, then, we have $v_t + a_1 u_{xx} + a_2 u + a_3 u^2/2 + a_4 u^3/3 = F_1(t)$ where $F_1(t)$ is an arbitrary function. Hence P^1 and P^2 in the Eq. (6.7) are as the following

$$P^1 = -\frac{1}{12}\Big(6a_1 u_{xx} + 6a_2 u + 3a_3 u^2 + 2a_4 u^3 + 6F_1(t)\Big) F(x,t), \qquad P^2 = -\frac{u}{2} F(x,t). \tag{6.10}$$

In doing so, μ-conservation law for the extended mKdV equation is as

$$D_x P^1 + D_t P^2 + \lambda_1 P^1 + \lambda_2 P^2 = 0, \tag{6.11}$$

Remark 6.2. By the characteristic form for the extended mKdV equation, we have

$$\begin{aligned}(D_i + \lambda_i)P^i &= (D_x + \lambda_1)P^1 + (D_t + \lambda_2)P^2 \\ &= F(x,t)(u_t + a_1 u_{xxx} + a_2 u_x + a_3 u u_x + a_4 u^2 u_x) \\ &= Q\Delta.\end{aligned} \tag{6.12}$$

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