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Traces on the Algebra of Observables of the Rational Calogero Model Based on the Root System

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It is shown that $H_{W(\mathcal{R})}(\eta)$, the algebra of observables of the rational Calogero model based on the root system $\mathcal{R} \subset \mathbb{R}^N$, has $T_{\mathcal{R}}$ independent traces, where $T_{\mathcal{R}}$ is the number of conjugacy classes of elements without eigenvalue 1 belonging to the Coxeter group $W(\mathcal{R}) \subset \text{End}(\mathbb{R}^N)$ generated by the root system \mathcal{R} .

Simultaneously, we reproduce a known result: the algebra $H_{W(\mathcal{R})}(\eta)$, considered as a superalgebra with a natural parity, has $ST_{\mathcal{R}}$ independent supertraces, where $ST_{\mathcal{R}}$ is the number of conjugacy classes of elements without eigenvalue -1 belonging to $W(\mathcal{R})$.

Keywords: Calogero model; Cherednik algebra; trace, supertrace.

2000 Mathematics Subject Classification: 17B80, 16W55

1. Introduction

It was shown in [6] and [8] that, for every associative superalgebra $H_{W(\mathcal{R})}(\eta)$ of observables of the rational Calogero model based on the root system \mathcal{R} , the space of supertraces is nonzero. The dimensions of these spaces for every root system are listed in [9].

Here we consider these superalgebras as algebras (parity forgotten) and find the conditions for existence of traces and the dimensions of the spaces of traces on these algebras.

Astonishingly, the proof differs from the one in [6] and [8] in several signs only, and we provide it here indicating change of signs by means of a parameter \varkappa with $\varkappa = -1$ for the supertraces and $\varkappa = +1$ for the traces. As a result, some parts of this text are almost copypasted from [6] and [8], especially Subsection 4.3 and Appendix.

1.1. Preliminaries

1.1.1. Traces

Let \mathcal{A} be an associative superalgebra with parity π . All expressions of linear algebra are given for homogenous elements only and are supposed to be extended to inhomogeneous elements via linearity.

A linear function str on \mathcal{A} is called a *supertrace* if

$$str(fg) = (-1)^{\pi(f)\pi(g)} str(gf) \text{ for all } f, g \in \mathcal{A}.$$

A linear function tr on \mathcal{A} is called a *trace* if

$$tr(fg) = tr(gf) \text{ for all } f, g \in \mathcal{A}.$$

Let $\varkappa = \pm 1$. We can unify the definitions of trace and supertrace by introducing a \varkappa -trace. We say that a linear function^a sp on \mathcal{A} is a \varkappa -trace if

$$sp(fg) = \varkappa^{\pi(f)\pi(g)} sp(gf) \text{ for all } f, g \in \mathcal{A}. \tag{1.1}$$

A linear function L is *even* (resp. *odd*) if $L(f) = 0$ for any odd (resp even) $f \in \mathcal{A}$.

Let \mathcal{A}_1 and \mathcal{A}_2 be associative superalgebras with parities π_1 and π_2 , respectively. Define the tensor product $\mathcal{A} = \mathcal{A}_1 \otimes \mathcal{A}_2$ as a superalgebra with the product $(a_1 \otimes a_2)(b_1 \otimes b_2) = (a_1 b_1) \otimes (a_2 b_2)$ (no sign factors in this formula) and the parity π defined by the formula $\pi(a \otimes b) = \pi_1(a) + \pi_2(b)$.

Let T_i be a trace on \mathcal{A}_i . Clearly, the function T such that $T(a \otimes b) = T_1(a)T_2(b)$ is a trace on \mathcal{A} .

Let S_i be an *even* supertrace on \mathcal{A}_i . Clearly, the function S such that $S(a \otimes b) = S_1(a)S_2(b)$ is an even supertrace on \mathcal{A} .

In what follows, we use three types of brackets:

$$\begin{aligned} [f, g] &= fg - gf, \\ \{f, g\} &= fg + gf, \\ [f, g]_{\varkappa} &= fg - \varkappa^{\pi(f)\pi(g)} gf. \end{aligned}$$

1.1.2. The superalgebra of observables

The superalgebra $H_{W(\mathcal{R})}(\eta)$ of observables of the rational Calogero model based on the root system \mathcal{R} is a deform of the skew product^b of the Weyl algebra and the group algebra of a finite group generated by reflections. We will define it by Definition 1.1 (see below); now let us describe the necessary ingredients.

Let $V = \mathbb{R}^N$ be endowed with a non-degenerate symmetric bilinear form (\cdot, \cdot) and the vectors \vec{a}_i constitute an orthonormal basis in V , i.e.,

$$(\vec{a}_i, \vec{a}_j) = \delta_{ij}.$$

Let x^i be the coordinates of $\vec{x} \in V$, i.e., $\vec{x} = \vec{a}_i x^i$. Then $(\vec{x}, \vec{y}) = \sum_{i=1}^N x^i y^i$ for any $\vec{x}, \vec{y} \in V$. The indices i are raised and lowered by means of the forms δ_{ij} and δ^{ij} .

For any nonzero $\vec{v} \in V$, define the *reflections* $R_{\vec{v}}$ as follows:

$$R_{\vec{v}}(\vec{x}) = \vec{x} - 2 \frac{(\vec{x}, \vec{v})}{(\vec{v}, \vec{v})} \vec{v} \text{ for any } \vec{x} \in V. \tag{1.2}$$

The reflections (1.2) have the following properties

$$R_{\vec{v}}(\vec{v}) = -\vec{v}, \quad R_{\vec{v}}^2 = 1, \quad (R_{\vec{v}}(\vec{x}), \vec{u}) = (\vec{x}, R_{\vec{v}}(\vec{u})) \text{ for any } \vec{v}, \vec{x}, \vec{u} \in V. \tag{1.3}$$

A finite set of vectors $\mathcal{R} \subset V$ is said to be a *root system* if the following conditions hold:

- i) \mathcal{R} is $R_{\vec{v}}$ -invariant for any $\vec{v} \in \mathcal{R}$,

^aFrom the German word *Spur*.

^bLet \mathcal{A} and \mathcal{B} be the superalgebras, and \mathcal{A} is a \mathcal{B} -module. We say that the superalgebra $\mathcal{A} * \mathcal{B}$ is a *skew product* of \mathcal{A} and \mathcal{B} if $\mathcal{A} * \mathcal{B} = \mathcal{A} \otimes \mathcal{B}$ as a superspace and $(a_1 \otimes b_1) * (a_2 \otimes b_2) = a_1 b_1 (a_2) \otimes b_1 b_2$.

ii) if $\vec{v}_1, \vec{v}_2 \in \mathcal{R}$ are collinear, then either $\vec{v}_1 = \vec{v}_2$ or $\vec{v}_1 = -\vec{v}_2$.

The group $W(\mathcal{R}) \subset O(N, \mathbb{R}) \subset \text{End}(V)$ generated by all reflections $R_{\vec{v}}$ with $\vec{v} \in \mathcal{R}$ is finite.

As it follows from this definition of a root system, we consider both crystallographic and non-crystallographic root systems. We consider also the empty root system denoted by A_0 , assuming that it generates the trivial group consisting of the unity element only.

Let \mathcal{H}^α , where $\alpha = 0, 1$, be two copies of V with orthonormal bases $a_{\alpha i} \in \mathcal{H}^\alpha$, where $i = 1, \dots, N$. For every vector $\vec{v} = \sum_{i=1}^N \vec{a}_i v^i \in V$, let $v_\alpha \in \mathcal{H}^\alpha$ be the vectors $v_\alpha = \sum_{i=1}^N a_{\alpha i} v^i$, so four bilinear forms on $\mathcal{H}^0 \oplus \mathcal{H}^1$ can be defined by the expression

$$(x_\alpha, y_\beta) = (\vec{x}, \vec{y}) \text{ for } \alpha, \beta = 0, 1, \tag{1.4}$$

where $\vec{x}, \vec{y} \in V$ and $x_\alpha, y_\alpha \in \mathcal{H}^\alpha$ are their copies. The reflections $R_{\vec{v}}$ act on \mathcal{H}^α as follows:

$$R_{\vec{v}}(h_\alpha) = h_\alpha - 2 \frac{(h_\alpha, v_\alpha)}{(\vec{v}, \vec{v})} v_\alpha \quad \text{for any } h_\alpha \in \mathcal{H}^\alpha. \tag{1.5}$$

So the $W(\mathcal{R})$ -action on the spaces \mathcal{H}^α is defined.

Let $\mathbb{C}[W(\mathcal{R})]$ be the group algebra of $W(\mathcal{R})$, i.e., the set of all linear combinations $\sum_{g \in W(\mathcal{R})} \alpha_g \bar{g}$, where $\alpha_g \in \mathbb{C}$ and we temporarily use the notation \bar{g} to distinguish g considered as an element of $W(\mathcal{R}) \subset \text{End}(V)$ from the same element $\bar{g} \in \mathbb{C}[W(\mathcal{R})]$ of the group considered as an element of the group algebra. The addition in $\mathbb{C}[W(\mathcal{R})]$ is defined as follows:

$$\sum_{g \in W(\mathcal{R})} \alpha_g \bar{g} + \sum_{g \in W(\mathcal{R})} \beta_g \bar{g} = \sum_{g \in W(\mathcal{R})} (\alpha_g + \beta_g) \bar{g}$$

and the multiplication is defined by setting $\overline{g_1 g_2} = \overline{g_1} \overline{g_2}$.

Note that the additions in $\mathbb{C}[W(\mathcal{R})]$ and in $\text{End}(V)$ differ. For example, if $I \in W(\mathcal{R})$ is unity and the matrix $K = -I$ from $\text{End}(V)$ belongs to $W(\mathcal{R})$, then $I + K = 0$ in $\text{End}(V)$ while $\bar{I} + \bar{K} \neq 0$ in $\mathbb{C}[W(\mathcal{R})]$. In what follows, the element $\bar{K} \in H_{W(\mathcal{R})}(\eta)$ is a *Klein operator*.^c

Let η be a set of constants $\eta_{\vec{v}}$ with $\vec{v} \in \mathcal{R}$ such that $\eta_{\vec{v}} = \eta_{\vec{w}}$ if $R_{\vec{v}}$ and $R_{\vec{w}}$ belong to one conjugacy class of $W(\mathcal{R})$.

Definition 1.1. The superalgebra $H_{W(\mathcal{R})}(\eta)$ is an associative superalgebra with unity **1**; it is the superalgebra of polynomials in the $a_{\alpha i}$ with coefficients in the group algebra $\mathbb{C}[W(\mathcal{R})]$ subject to

^cLet \mathcal{A} be an associative superalgebra with parity π . Following M. Vasiliev, see, e.g. [15], we say that an element $K \in \mathcal{A}$ is a *Klein operator* if $\pi(K) = 0$, $Kf = (-1)^{\pi(f)} fK$ for any $f \in \mathcal{A}$ and $K^2 = 1$. Every Klein operator belongs to the *anticenter* of the superalgebra \mathcal{A} , see [10], p.41. (Recall that the *anticenter* $AC(\mathcal{A})$ of an associative superalgebra \mathcal{A} is defined by the formula

$$AC(\mathcal{A}) = \{a \in \mathcal{A} \mid ax - (-1)^{\pi(x)(\pi(a)+1)} xa = 0 \text{ for any } x \in \mathcal{A}\}.$$

Any Klein operator, if exists, establishes an isomorphism between the space of even traces and the space of even supertraces on \mathcal{A} . Namely, if $f \mapsto T(f)$ is an even trace, then $f \mapsto T(fK)$ is a supertrace, and if $f \mapsto S(f)$ is an even supertrace, then $f \mapsto S(fK)$ is a trace.

It is proved in [9] that if $H_{W(\mathcal{R})}(\eta)$ has isomorphic spaces of the traces and supertraces, then $H_{W(\mathcal{R})}(\eta)$ contains a Klein operator.

the relations

$$\bar{g}h_\alpha = g(h_\alpha)h_\alpha\bar{g} \text{ for any } g \in W(\mathcal{R}) \text{ and } h_\alpha \in \mathcal{H}^\alpha, \tag{1.6}$$

$$[x_\alpha\bar{I}, y_\beta\bar{I}] = \varepsilon_{\alpha\beta} \left((\vec{x}, \vec{y})\bar{I}\bar{I} + \sum_{\vec{v} \in \mathcal{R}} \eta_{\vec{v}} \frac{(\vec{x}, \vec{v})(\vec{y}, \vec{v})}{(\vec{v}, \vec{v})} \bar{I}R_{\vec{v}} \right) \text{ for any } x_\alpha \in \mathcal{H}^\alpha \text{ and } y_\beta \in \mathcal{H}^\beta, \tag{1.7}$$

where $\varepsilon_{\alpha\beta}$ is the antisymmetric tensor, $\varepsilon_{01} = 1$, and \bar{I} is the unity in $\mathbb{C}[a_{\alpha i}]$. The element $\mathbf{1} = \bar{I} \cdot \bar{I}$ is the unity of $H_{W(\mathcal{R})}(\eta)$.^d The action of any operator $g \in \text{End}(V)$ is given by a matrix (g^i_j) :

$$g(a_{\alpha i}h^i) = a_{\alpha i}g^i_j h^j, \quad g_1(g_2(h_\alpha)) = (g_1g_2)(h_\alpha) \text{ for any } h_\alpha = a_{\alpha i}h^i \in \mathcal{H}^\alpha, \tag{1.8}$$

$$g(\bar{I}) = \bar{I}. \tag{1.9}$$

The commutation relations (1.7) suggest to define the *parity* π by setting:

$$\pi(a_{\alpha i}\bar{g}) = 1 \text{ for any } \alpha, i \text{ and } g \in \mathbb{C}[W(\mathcal{R})]; \quad \pi(\bar{I}\bar{g}) = 0 \text{ for any } g \in \mathbb{C}[W(\mathcal{R})]. \tag{1.10}$$

We say that $H_{W(R)}(\eta)$ is a *the superalgebra of observables of the Calogero model based on the root system* \mathcal{R} .

These algebras $H_{W(R)}(\eta)$ (with parity forgotten) are particular cases of *Symplectic Reflection Algebras* [4] and are also known as *rational Cherednik algebras* (see, for example, [5]).

Below we will usually designate $\mathbf{1}, \bar{I}, I$ and \bar{I} by 1, and $F\bar{I} = \bar{I}F$ by F for any $F \in \mathbb{C}[a_{\alpha i}]$, and $\bar{I}G = G\bar{I}$ by G for any $G \in \mathbb{C}[W(\mathcal{R})]$. Besides, we will just write g instead of \bar{g} because it will always be clear, whether $g \in W(\mathcal{R})$ or $g \in \mathbb{C}[W(\mathcal{R})]$.

The associative algebra $H_{W(R)}(\eta)$ has a faithful representation via Dunkl differential-difference operators D_i , see [3], acting on the space of smooth functions on V . Namely, let $v_i = \delta_{ij}v^j, x_i = \delta_{ij}x^j$, and

$$D_i = \frac{\partial}{\partial x^i} + \frac{1}{2} \sum_{\vec{v} \in \mathcal{R}} \eta_{\vec{v}} \frac{v_i}{(\vec{x}, \vec{v})} (1 - R_{\vec{v}}), \tag{1.11}$$

where $(1 - R_{\vec{v}})f(x) = f(x) - f(R_{\vec{v}}(x))$ for every smooth function f . Let (see [14,1])

$$a_{\alpha i} = \frac{1}{\sqrt{2}}(x_i + (-1)^\alpha D_i) \text{ for } \alpha = 0, 1. \tag{1.12}$$

The reflections $R_{\vec{v}}$ transform the deformed creation (a_{1i}) and annihilation (a_{0i}) operators (1.12) as vectors:

$$R_{\vec{v}}a_{\alpha i} = \sum_{j=1}^N \left(\delta_{ij} - 2 \frac{v_i v_j}{(\vec{v}, \vec{v})} \right) a_{\alpha j} R_{\vec{v}}. \tag{1.13}$$

Since $[D_i, D_j] = 0$, see [3], it follows that

$$[a_{\alpha i}, a_{\beta j}] = \varepsilon_{\alpha\beta} \left(\delta_{ij} + \sum_{\vec{v} \in \mathcal{R}} \eta_{\vec{v}} \frac{v_i v_j}{(\vec{v}, \vec{v})} R_{\vec{v}} \right), \tag{1.14}$$

which manifestly coincides with (1.7).

^dClearly, $H_{W(\mathcal{R})}$ does not contain either $\bar{I} \in \mathbb{C}[a_{\alpha i}]$ or \bar{I} but does contain their product.

Observe an important property of superalgebra $H_{W(R)}(\eta)$: The Lie (super)algebra of its inner derivations contains \mathfrak{sl}_2 generated by the operators

$$T_{\alpha\beta} = \frac{1}{2} \sum_{i=1}^N \{a_{\alpha i}, a_{\beta i}\} \tag{1.15}$$

which commute with $\mathbb{C}[W(\mathcal{R})]$, i.e., $[T_{\alpha\beta}, R_{\bar{v}}] = 0$, and act on $a_{\alpha i}$ as on vectors of the irreducible 2-dimensional \mathfrak{sl}_2 -modules:

$$[T_{\alpha\beta}, a_{\gamma i}] = \varepsilon_{\alpha\gamma} a_{\beta i} + \varepsilon_{\beta\gamma} a_{\alpha i}, \quad \text{where } i = 1, \dots, N. \tag{1.16}$$

The restriction of the operator T_{01} in the representation (1.12) on the subspace of $W(\mathcal{R})$ -invariant functions on V is a second-degree differential operator which is the well-known Hamiltonian of the rational Calogero model, see [2], based on the root system \mathcal{R} , see [12]. One of the relations (1.15), namely, $[T_{01}, a_{\alpha i}] = -(-1)^\alpha a_{\alpha i}$, allows one to find the solutions of the equation $T_{01}\psi = \varepsilon\psi$ and eigenvalues ε via usual Fock procedure with the vacuum $|0\rangle$ such that $a_{0i}|0\rangle=0$ for any i , see [1]. After $W(\mathcal{R})$ -symmetrization these eigenfunctions become the eigenfunctions of the Calogero Hamiltonian.

2. The \varkappa -traces on $H_{W(R)}(\eta)$

Every \varkappa -trace $sp(\cdot)$ on \mathcal{A} generates the following bilinear form on \mathcal{A} :

$$B_{sp}(f, g) = sp(f \cdot g) \text{ for any } f, g \in \mathcal{A}. \tag{2.1}$$

It is obvious that if such a bilinear form B_{sp} is degenerate, then the null-vectors of this form (i.e., $v \in \mathcal{A}$ such that $B(v, x) = 0$ for any $x \in \mathcal{A}$) constitute the two-sided ideal $\mathcal{I} \subset \mathcal{A}$. If the \varkappa -trace generating degenerate bilinear form is homogeneous (even or odd), then the corresponding ideal is a superalgebra.

If $\varkappa = -1$, the ideals of this sort are present, for example, in the superalgebras $H_{W(A_1)}(\eta)$ (corresponding to the two-particle Calogero model) at $\eta = k + \frac{1}{2}$, see [15], and in the superalgebras $H_{W(A_2)}(\eta)$ (corresponding to three-particle Calogero model) at $\eta = k + \frac{1}{2}$ and $\eta = k \pm \frac{1}{3}$, see [7], for every integer k . For all other values of η all supertraces on these superalgebras generate nondegenerate bilinear forms (2.1).

The general case of $H_{W(A_{n-1})}(\eta)$ for arbitrary n is considered in [11]. Theorem 5.8.1 of [11] states that the associative algebra $H_{W(A_{n-1})}(\eta)$ is not simple if and only if $\eta = \frac{q}{m}$, where q, m are mutually prime integers such that $1 < m \leq n$, and presents the structure of corresponding ideals.

Conjecture: Each of the ideals found in [11] is the set of null-vectors of the degenerate bilinear form (2.1) for some \varkappa -trace sp on $H_{W(A_{n-1})}(\eta)$.^e

2.1. Main results

Theorem 2.1. *Each nonzero \varkappa -trace on $H_{W(\mathcal{R})}(\eta)$ is even.*

^eThe dimension of the space of supertraces on $H_{W(A_{n-1})}(\eta)$ is the number of the partition of $n \geq 1$ into the sum of different positive integers, see [6], and the space of the traces on $H_{W(A_{n-1})}(\eta)$ is one-dimensional for $n \geq 2$ due to Theorem 2.3, see also [9].

Proof. The space of superalgebra $H_{W(\mathcal{R})}(\eta)$ can be decomposed into the direct sum of irreducible \mathfrak{sl}_2 -modules, where \mathfrak{sl}_2 is defined by eq. (1.15). Clearly, each \varkappa -trace should vanish on all these irreducible modules except singlets, and can take nonzero value only on singlets, i.e., on elements $f \in H_{W(\mathcal{R})}(\eta)$ such that $[T_{\alpha\beta}, f] = 0$ for $\alpha, \beta = 0, 1$. So, if $sp(f) \neq 0$, then $[T_{01}, f] = 0$, which implies $\pi(f) = 0$. \square

Theorem 2.2. *The dimension of the space of \varkappa -traces on the superalgebra $H_{W(\mathcal{R})}(\eta)$ is equal to the number of conjugacy classes of elements without eigenvalue \varkappa belonging to the Coxeter group $W(\mathcal{R}) \subset \text{End}(\mathbb{R}^N)$ generated by the finite root system $\mathcal{R} \subset \mathbb{R}^N$.*

Proof. This Theorem follows from Theorem 4.1 and Theorem 3.2. \square

Clearly, Theorem 2.2 is equivalent to the following theorem

Theorem 2.3. *Let the Coxeter group $W(\mathcal{R}) \subset \text{End}(\mathbb{R}^N)$ generated by the finite root system $\mathcal{R} \subset \mathbb{R}^N$ have $T_{\mathcal{R}}$ conjugacy classes without eigenvalue 1 and $ST_{\mathcal{R}}$ conjugacy classes without eigenvalue -1 . Then the superalgebra $H_{W(\mathcal{R})}(\eta)$ possesses $T_{\mathcal{R}}$ independent traces and $ST_{\mathcal{R}}$ independent super-traces.*

3. Ground Level Conditions

Clearly, $\bar{1} \cdot \mathbb{C}[W(\mathcal{R})]$ is an isomorphic to $\mathbb{C}[W(\mathcal{R})]$ subalgebra of $H_{W(\mathcal{R})}(\eta)$.

It is easy to describe all \varkappa -traces on $\mathbb{C}[W(\mathcal{R})]$. Every \varkappa -trace on $\mathbb{C}[W(\mathcal{R})]$ is completely determined by its values on $W(\mathcal{R})$ and is a central function on $W(\mathcal{R})$, i.e., the function constant on the conjugacy classes due to $W(\mathcal{R})$ -invariance. Thus, the number of \varkappa -traces on $\mathbb{C}[W(\mathcal{R})]$ is equal to the number of conjugacy classes in $W(\mathcal{R})$.

Since $\mathbb{C}[W(\mathcal{R})] \subset H_{W(\mathcal{R})}(\eta)$, some additional restrictions on these functions follow from the definition (1.1) of \varkappa -trace and the defining relations (1.7) for $H_{W(\mathcal{R})}(\eta)$. Namely, for any $g \in W(\mathcal{R})$ consider elements $c_i^\alpha \in \mathcal{H}^\alpha$ such that

$$gc_i^\alpha = \varkappa c_i^\alpha g. \tag{3.1}$$

Then, eqs. (1.1) and (3.1) imply that

$$sp(c_i^0 c_j^1 g) = \varkappa sp(c_j^1 g c_i^0) = sp(c_j^1 c_i^0 g),$$

and therefore

$$sp([c_i^0, c_j^1]g) = 0. \tag{3.2}$$

Since $[c_i^0, c_j^1]g \in \mathbb{C}[W(\mathcal{R})]$, the conditions (3.2) single out the central functions on $\mathbb{C}[W(\mathcal{R})]$ which can in principle be extended to \varkappa -traces on $H_{W(\mathcal{R})}(\eta)$, and Theorem 4.1 states that each central function on $\mathbb{C}[W(\mathcal{R})]$ satisfying conditions (3.2) can be extended to a \varkappa -trace on $H_{W(\mathcal{R})}(\eta)$. In [6], the conditions (3.2) are called *Ground Level Conditions*.

Ground Level Conditions (3.2) is an overdetermined system of linear equations for the central functions on $\mathbb{C}[W(\mathcal{R})]$. The dimension of the space of its solution is given in Theorem 3.2.

3.1. The number of independent solutions of Ground Level Conditions

For any $g \in W(\mathcal{R})$, consider the subspaces $\mathcal{E}^\alpha(g) \subset \mathcal{H}^\alpha$:

$$\mathcal{E}^\alpha(g) = \{h \in \mathcal{H}^\alpha \mid g(h) = \varkappa h\}. \quad (3.3)$$

Clearly, $\dim \mathcal{E}^0(g) = \dim \mathcal{E}^1(g)$.

In the vector space $\mathbb{C}[W(\mathcal{R})]$, introduce the grading E :[†]

$$E(g) = \dim \mathcal{E}^\alpha(g). \quad (3.4)$$

For any $g \in W(\mathcal{R})$, the number $E(g)$ is an integer such that $0 \leq E(g) \leq N$; recall that $\dim \mathcal{H}^\alpha = N$.

Let $W_l = \{g \in W(\mathcal{R}) \mid E(g) = l\}$. Clearly,

$$W(\mathcal{R}) = \bigcup_{l=0}^N W_l. \quad (3.5)$$

The set W_l is $W(\mathcal{R})$ -invariant, i.e., $hW_lh^{-1} = W_l$ for any $h \in W(\mathcal{R})$, and we can introduce the space W_l^* of $W(\mathcal{R})$ -invariant functions on W_l .

Theorem 3.1. *Each function $S \in W_0^*$ can be uniquely extended to a central function on $W(\mathcal{R})$ satisfying the Ground Level Conditions.*

The following theorem follows from Theorem 3.1:

Theorem 3.2. *The dimension of the space of solutions of Ground Level Conditions (3.2) is equal to the number of conjugacy classes in $W(\mathcal{R})$ with $E(g) = 0$.*

Theorems 3.1 and 3.2 are simultaneously proved below.

The following lemmas are needed to prove these theorems.

Lemma 3.1. *Let g be an orthogonal $N \times N$ real matrix without eigenvalue \varkappa , i.e., the matrix $g - \varkappa$ is invertible. Then the matrix $R_{\vec{v}}g$ has exactly one eigenvalue equal to \varkappa .*

Proof. Consider the equation $R_{\vec{v}}g\vec{x} - \varkappa\vec{x} = 0$ or, equivalently, $g\vec{x} - \varkappa R_{\vec{v}}\vec{x} = 0$ for the eigenvectors \vec{x} corresponding to eigenvalue \varkappa . Using the definition of $R_{\vec{v}}$ this equation can be written as

$$g\vec{x} - \varkappa\left(\vec{x} - 2\frac{(\vec{v}, \vec{x})}{|\vec{v}|^2}\vec{v}\right) = 0;$$

hence,

$$\vec{x} = -2\varkappa\frac{(\vec{v}, \vec{x})}{|\vec{v}|^2}(g - \varkappa)^{-1}\vec{v}. \quad (3.6)$$

[†]It follows from Lemma 3.2 formulated below that if $\varkappa = -1$, then $\rho(g) := E(g)|_{\text{mod}2}$ is a parity on the group algebra $\mathbb{C}[W(\mathcal{R})]$. It is a well known parity of elements of the Coxeter group $W(\mathcal{R})$. Besides, $(E(g)|_{\varkappa=+1} - E(g)|_{\varkappa=-1})|_{\text{mod}2} = N|_{\text{mod}2}$.

It remains to show that this equation has a nonzero solution. Let $\vec{v} = (g - \varkappa)\vec{w}$; it follows from eq. (3.6) that $\vec{x} = \mu\vec{w}$, where $\mu \in \mathbb{R}$. Then

$$\begin{aligned} |\vec{v}|^2 &= 2(|\vec{w}|^2 - \varkappa(\vec{w}, g\vec{w})), \\ -2\varkappa(\vec{v}, \vec{x}) &= 2(|\vec{w}|^2 - \varkappa(\vec{w}, g\vec{w}))\mu, \end{aligned}$$

and eq. (3.6) becomes an identity $\mu\vec{w} = \mu\vec{w}$. So the vector $\vec{x}_1 = (g - \varkappa)^{-1}\vec{v}$ is the only solution, up to a factor. \square

Lemma 3.2. *Let g be an orthogonal $N \times N$ real matrix and \vec{c}_i , where $i = 1, \dots, E(g)$, the complete orthonormal set of its eigenvectors corresponding to eigenvalue \varkappa . Then*

- i) $E(R_{\vec{v}}g) = E(g) + 1$ if $(\vec{v}, \vec{c}_i) = 0$ for all i ;
- ii) if there exists an i such that $(\vec{v}, \vec{c}_i) \neq 0$, then $E(R_{\vec{v}}g) = E(g) - 1$ and the space of eigenvectors of $R_{\vec{v}}g$ corresponding to eigenvalue \varkappa is the subspace of $\text{span}\{\vec{c}_1, \dots, \vec{c}_{E(g)}\}$ orthogonal to \vec{v} .

Proof. Let $C \stackrel{\text{def}}{=} \text{span}\{\vec{c}_1, \dots, \vec{c}_{E(g)}\}$ and $V = C \oplus B$ the orthogonal direct sum. Clearly, $gB = B$ and $g - \varkappa$ is invertible on B .

Let us seek a null-vector \vec{z} of the operator $R_{\vec{v}}g - \varkappa$, i.e., the solution of the equation

$$R_{\vec{v}}g\vec{z} - \varkappa\vec{z} = 0, \tag{3.7}$$

in the form $\vec{z} = \vec{c} + \vec{b}$, where $\vec{c} \in C$ and $\vec{b} \in B$. The definition of $R_{\vec{v}}$ and (3.7) yield

$$-\frac{2}{(\vec{v}, \vec{v})} \left(\varkappa(\vec{c}, \vec{v}) + (g\vec{b}, \vec{v}) \right) \vec{v} + (g - \varkappa)\vec{b} = 0. \tag{3.8}$$

Represent \vec{v} in the form $\vec{v} = \vec{v}_c + \vec{v}_b$, where $\vec{v}_c \in C$, $\vec{v}_b \in B$. Let $\vec{v}_b = (g - \varkappa)\vec{w}$, where $\vec{w} \in B$. Then eq. (3.7) is equivalent to the system

$$-\frac{2}{(\vec{v}, \vec{v})} \left(\varkappa(\vec{c}, \vec{v}_c) + (g\vec{b}, (g - \varkappa)\vec{w}) \right) \vec{v}_c = 0, \tag{3.9}$$

$$-\frac{2}{(\vec{v}, \vec{v})} \left(\varkappa(\vec{c}, \vec{v}_c) + (g\vec{b}, (g - \varkappa)\vec{w}) \right) \vec{w} + \vec{b} = 0. \tag{3.10}$$

Consider the two cases:

1) Let $(\vec{v}, \vec{c}_i) = 0$ for all $i = 1, \dots, E(g)$. So, $\vec{v}_c = 0$, and hence $\vec{v} \in B$. Then (3.10) acquires the form

$$-\frac{2}{(\vec{v}, \vec{v})} (g\vec{b}, (g - \varkappa)\vec{w})\vec{w} + \vec{b} = 0. \tag{3.11}$$

It is easy to check that $\vec{b} = \vec{w}$ is the only nonzero solution of (3.11) orthogonal to C .

So, all the solutions of eq. (3.7) are linear combinations of the vectors $\vec{z}_i = \vec{c}_i$, where $i = 1, \dots, E(g)$, and $\vec{z}_{E(g)+1} = \vec{w}$.

2) Let $\vec{v}_c \neq 0$. Then eq. (3.9) gives

$$\varkappa(\vec{c}, \vec{v}_c) + (g\vec{b}, (g - \varkappa)\vec{w}) = 0 \tag{3.12}$$

which reduces eq. (3.10) to $\vec{b} = 0$ which, in its turn, reduces eq. (3.12) to $(\vec{c}, \vec{v}) = 0$. \square

Let \mathcal{P} be the projection $\mathbb{C}[W(\mathcal{R})] \rightarrow \mathbb{C}[W(\mathcal{R})]$ defined as

$$\mathcal{P}\left(\sum_i \alpha_i g_i\right) = \sum_{i: g_i \neq 1} \alpha_i g_i \quad \text{for any } g_i \in W(\mathcal{R}), \alpha_i \in \mathbb{C}. \quad (3.13)$$

Lemma 3.3. Let $g \in W(\mathcal{R})$. Let $c_1^\alpha, c_2^\alpha \in \mathcal{E}^\alpha(g) \subset H_{W(\mathcal{R})}(\eta)$ (i.e., $gc_1^\alpha = \varkappa c_1^\alpha g$, $gc_2^\alpha = \varkappa c_2^\alpha g$). Then

$$E(\mathcal{P}([c_1^\alpha, c_2^\beta])g) = E(g) - 1 \quad \text{for any } g \in W(\mathcal{R}). \quad (3.14)$$

Proof. Proof easily follows from the formula

$$\mathcal{P}([c_1^\alpha, c_2^\beta]) = \varepsilon^{\alpha\beta} \sum_{\vec{v} \in \mathcal{R}} \eta_{\vec{v}} \frac{(\vec{c}_1, \vec{v})(\vec{c}_2, \vec{v})}{(\vec{v}, \vec{v})} R_{\vec{v}}. \quad (3.15)$$

Indeed, if $(\vec{c}_1, \vec{v})(\vec{c}_2, \vec{v}) \neq 0$, then Lemma 3.2 implies that $E(R_{\vec{v}}g) = E(g) - 1$. □

3.2. Proof of Theorems 3.1 and 3.2

Due to Lemma 3.3 some of the Ground Level Conditions express the \varkappa -trace of elements g with $E(g) = l$ via the \varkappa -traces of elements $R_{\vec{v}}g$ with $E(R_{\vec{v}}g) = l - 1$:

$$sp(g) = -sp([c_i^0, c_i^1] - 1)g \quad \text{if } (\vec{c}_i, \vec{c}_i) = 1. \quad (3.16)$$

We prove Theorems 3.1 and 3.2 using induction on $E(g)$.

The first step is simple: if $E(g) = 0$, then $sp(g)$ is an arbitrary central function. The next step is also simple: if $E(g) = 1$, then there exists a unique element $c_1^0 \in \mathcal{E}^0(g)$ and a unique element $c_1^1 \in \mathcal{E}^1(g)$ such that $|c_1^\alpha| = 1$ and $gc_1^\alpha = \varkappa c_1^\alpha g$. Since $([c_1^0, c_1^1] - 1)g \in \mathbb{C}[W(\mathcal{R})]$ and $E(([c_1^0, c_1^1] - 1)g) = 0$, then

$$sp(g) = -sp([c_1^0, c_1^1] - 1)g \quad (3.17)$$

is the unique possible value for $sp(g)$ with $E(g) = 1$. In such a way, W_0^* is extended to W_1^* .

A priori these values can be not consistent with other Ground Level Conditions.

Suppose that the Ground Level Conditions (3.2) which are equivalent to the conditions considered for all g with $E(g) \leq l$ and for all $c_1^\alpha, c_2^\alpha \in \mathcal{E}^\alpha(g)$ have Q_l independent solutions.

Proposition 3.1. The value Q_l does not depend on l .

Proof. It was shown above that $Q_1 = Q_0$. Let $l \geq 1$. Consider $g \in W(\mathcal{R})$ with $E(g) = l + 1$. Let $c_i^\alpha \in \mathcal{E}^\alpha(g)$, where $i = 1, 2$, be such that $(c_i^\alpha, c_j^\beta) = \delta_{ij}$. These elements c_i^α give the conditions:

$$sp(g) = -sp([c_1^0, c_1^1] - 1)g, \quad (3.18)$$

$$sp(g) = -sp([c_2^0, c_2^1] - 1)g, \quad (3.19)$$

$$sp([c_1^0, c_2^1]g) = 0. \quad (3.20)$$

Below we prove that eqs. (3.18) and (3.19) are equivalent and eq. (3.20) follows from them. So, we will prove that eq. (3.18) considered for all $g \in W_s$, where $0 < s \leq l + 1$, realizes the extension of W_0^* to W_{l+1}^* .

Let us transform (3.18):

$$sp(g) = sp(S_1) - sp(S_{12}), \tag{3.21}$$

where

$$\begin{aligned} S_1 &= - \left([c_1^0, c_1^1] - 1 - \sum_{\vec{v} \in \mathcal{R}: (\vec{v}, \vec{c}_1)(\vec{v}, \vec{c}_2) \neq 0} \eta_{\vec{v}} \frac{(\vec{v}, \vec{c}_1)^2}{|\vec{v}|^2} R_{\vec{v}} \right) g = \\ &= - \left(\sum_{\vec{v} \in \mathcal{R}: (\vec{v}, \vec{c}_2) = 0} \eta_{\vec{v}} \frac{(\vec{v}, \vec{c}_1)^2}{|\vec{v}|^2} R_{\vec{v}} \right) g, \end{aligned} \tag{3.22}$$

$$S_{12} = \left(\sum_{\vec{v} \in \mathcal{R}: (\vec{v}, \vec{c}_1)(\vec{v}, \vec{c}_2) \neq 0} \eta_{\vec{v}} \frac{(\vec{v}, \vec{c}_1)^2}{|\vec{v}|^2} R_{\vec{v}} \right) g. \tag{3.23}$$

It is clear from eq. (3.22) and Lemma 3.2 that $E(S_1) = l$ and $S_1 c_2^0 = \varkappa c_2^0 S_1$. Hence, due to eq. (3.16) and inductive hypothesis

$$sp(S_1) = -sp([c_2^0, c_2^1] - 1)S_1 = sp([c_2^0, c_2^1] - 1)([c_1^0, c_1^1] - 1)g - S_{12}) \tag{3.24}$$

and as a result

$$sp(S_1) = sp([c_2^0, c_2^1] - 1)([c_1^0, c_1^1] - 1)g - sp([c_2^0, c_2^1]S_{12}) + sp(S_{12}). \tag{3.25}$$

Finally, eq. (3.18) is equivalent under inductive hypothesis to

$$sp(g) = sp([c_2^0, c_2^1] - 1)([c_1^0, c_1^1] - 1)g - sp([c_2^0, c_2^1]S_{12}). \tag{3.26}$$

Analogously, eq. (3.19) is equivalent under inductive hypothesis to

$$sp(g) = sp([c_1^0, c_1^1] - 1)([c_2^0, c_2^1] - 1)g - sp([c_1^0, c_1^1]S_{21}), \tag{3.27}$$

where

$$S_{21} = \left(\sum_{\vec{v} \in \mathcal{R}: (\vec{v}, \vec{c}_1)(\vec{v}, \vec{c}_2) \neq 0} \eta_{\vec{v}} \frac{(\vec{v}, \vec{c}_2)^2}{|\vec{v}|^2} R_{\vec{v}} \right) g. \tag{3.28}$$

Now, let us compare the corresponding terms in eqs. (3.26) and (3.27). First, the relation

$$sp([c_1^0, c_1^1] - 1)([c_2^0, c_2^1] - 1)g = sp([c_2^0, c_2^1] - 1)([c_1^0, c_1^1] - 1)g \tag{3.29}$$

is identically true for every \varkappa -trace on $\mathbb{C}[W(\mathcal{R})]$ since $[c_1^0, c_1^1]$ commutes with g . Second,

$$sp([c_1^0, c_1^1]S_{21}) = sp([c_2^0, c_2^1]S_{12}) \tag{3.30}$$

since

$$sp([c_1^0, c_1^1](\vec{v}, \vec{c}_2)^2 R_{\vec{v}} g) = sp([c_2^0, c_2^1](\vec{v}, \vec{c}_1)^2 R_{\vec{v}} g) \tag{3.31}$$

for every $\vec{v} \in \mathcal{R}$ such that $(\vec{v}, \vec{c}_1)(\vec{v}, \vec{c}_2) \neq 0$. Indeed, the element

$$\vec{c} = \alpha \vec{c}_1 + \beta \vec{c}_2, \text{ where } \alpha = -(\vec{v}, \vec{c}_2) \neq 0 \text{ and } \beta = (\vec{v}, \vec{c}_1) \neq 0, \tag{3.32}$$

is orthogonal to \vec{v} :

$$(\vec{v}, \vec{c}) = 0 \tag{3.33}$$

and satisfies the relation

$$R_{\vec{v}}g c^\alpha = \varkappa c^\alpha R_{\vec{v}}g \tag{3.34}$$

due to Lemma 3.2. This fact together with the fact that

$$E(\mathcal{P}([c_i^0, c^1])R_{\vec{v}}g) = l - 1 \text{ for } i = 1, 2 \tag{3.35}$$

(this also follows from Lemma 3.2) and inductive hypothesis imply

$$sp([c_i^0, c^1]R_{\vec{v}}g) = sp([c_i^0, c_i^1]R_{\vec{v}}g) = 0 \text{ for } i = 1, 2. \tag{3.36}$$

Substituting $\vec{c}_1 = \frac{1}{\alpha}(\vec{c} - \beta\vec{c}_2)$ and $\vec{c}_2 = \frac{1}{\beta}(\vec{c} - \alpha\vec{c}_1)$ in the left-hand side of eq. (3.31) and using eqs. (3.33) and (3.36) one obtains the right-hand side of eq. (3.31). Thus, eq. (3.18) is equivalent to eq. (3.19); hence

$$sp([c_1^0, c_1^1] - 1)g - sp([c_2^0, c_2^1] - 1)g = 0 \tag{3.37}$$

for every orthonormal pair $c_1, c_2 \in \mathcal{E}(g)$. Consequently,

$$sp([c_1^0, c_2^1]g) = 0 \tag{3.38}$$

which finishes the proof of Proposition 3.1 and Theorem 3.2. □

4. The number of independent \varkappa -traces on $H_{W(\mathcal{R})}(\eta)$

For proof of the following theorem, see this and subsequent sections.

Theorem 4.1. *Every \varkappa -trace on the algebra $\mathbb{C}[W(\mathcal{R})]$ satisfying the equation*

$$sp([h_0, h_1]g) = 0 \quad \text{for any } g \in W(\mathcal{R}) \text{ with } E(g) \neq 0 \text{ and } h_\alpha \in \mathcal{E}^\alpha(g), \tag{4.1}$$

can be uniquely extended to a \varkappa -trace on $H_{W(\mathcal{R})}(\eta)$.

4.1. Notation

For each $g \in W(\mathcal{R})$, introduce eigenbases $b_{\alpha i}$ in $\mathbb{C}\mathcal{H}^\alpha$ ($i = 1, \dots, N$, $\alpha = 0, 1$) such that

$$gb_{0i} = \lambda_i b_{0i}g, \tag{4.2}$$

$$gb_{1i} = \frac{1}{\lambda_i} b_{1i}g, \tag{4.3}$$

$$(b_{0i}, b_{1j}) = \delta_{ij}. \tag{4.4}$$

Let \mathbf{B}_g be the set of all these $b_{\alpha i}$ for a fixed g .

In what follows we use the generalized indices I, J, \dots instead of pairs (α, i) and sometimes write $i(I), \lambda_I, \alpha(I)$ meaning that

$$b_I = b_{\alpha(I)i(I)}, \quad gb_I = \lambda_I b_I g. \tag{4.5}$$

Introduce also a symplectic form

$$\mathcal{C}_{IJ} = [b_I, b_J]|_{\eta=0} \tag{4.6}$$

and let f_{IJ} be the η -dependent part of the commutator $[b_I, b_J]$:

$$F_{IJ} \stackrel{def}{=} [b_I, b_J] = \mathcal{C}_{IJ} + f_{IJ}. \tag{4.7}$$

The indices I, J are raised and lowered with the help of the symplectic forms \mathcal{C}^{IJ} and \mathcal{C}_{IJ} :

$$\mu_I = \sum_J \mathcal{C}_{IJ} \mu^J, \quad \mu^I = \sum_J \mu_J \mathcal{C}^{JI}; \quad \sum_M \mathcal{C}_{IM} \mathcal{C}^{MJ} = -\delta_I^J. \tag{4.8}$$

Let $\mathbf{M}(g)$ be the matrix of the map $\mathbf{B}_1 \rightarrow \mathbf{B}_g$, such that

$$b_I = \sum_{i,\alpha} \mathbf{M}_I^{\alpha i}(g) a_{\alpha i}. \tag{4.9}$$

Obviously this map is invertible. Using the matrix notation one can rewrite (4.5) as

$$gb_I = \sum_{J=1}^{2N} \Lambda_I^J(g) b_J g, \tag{4.10}$$

where the matrix (Λ_I^J) is diagonal, i.e., $\Lambda_I^J = \delta_I^J \lambda_I$.

We say that the monomial $b_{I_1} b_{I_2} \dots b_{I_k} g$ is *regular* if $b_{I_s} \in \mathbf{B}_g$ for all $s = 1, \dots, k$ and at least one of λ_{I_s} is not equal to \varkappa .

We say that the monomial $b_{I_1} b_{I_2} \dots b_{I_k} g$ is *special* if $b_{I_s} \in \mathbf{B}_g$ for all $s = 1, \dots, k$ and $\lambda_{I_s} = \varkappa$ for all s . Clearly, in this case $E(g) > 0$.

Introduce a lexicographical partial ordering on $H_{W(\mathcal{R})}(\eta)$: For any $P_1, P_2 \in \mathbb{C}[a_{\alpha i}]$ and $g_1, g_2 \in W(\mathcal{R})$, we say

$$P_1 g_1 > P_2 g_2 \text{ if either } \deg P_1 > \deg P_2 \text{ or } \deg P_1 = \deg P_2 \text{ and } E(g_1) > E(g_2). \tag{4.11}$$

4.2. The \varkappa -trace of General Elements

To find the \varkappa -trace, we consider the defining relations (1.1) as a system of linear equations for the linear function sp .

Clearly, this system can be reduced to the following two equations

$$sp([b_I, P(a)g]_{\varkappa}) = 0, \tag{4.12}$$

$$sp(\tau^{-1} P(a)g\tau) = sp(P(a)g), \tag{4.13}$$

where polynomials P and $g, \tau \in W(\mathcal{R})$ are arbitrary.

Since each \varkappa -trace is even, eq. (4.12) can be rewritten in the form

$$sp(b_I P(a)g - \varkappa P(a)gb_I) = 0. \tag{4.14}$$

Eq. (4.14) enables us to express a \varkappa -trace of any monomial in $H_{W(\mathcal{R})}(\eta)$ in terms of \varkappa -trace on $\mathbb{C}[W(\mathcal{R})]$. Indeed, this can be done in a finite number of the following step operations.

Regular step operation. Let $b_{I_1} b_{I_2} \dots b_{I_k} g$ be a regular monomial. Up to a polynomial of lesser degree, this monomial can be expressed in a form such that $\lambda_{I_1} \neq \varkappa$.

Then

$$sp(b_{I_1} b_{I_2} \dots b_{I_k} g) = \varkappa sp(b_{I_2} \dots b_{I_k} g b_{I_1}) = \varkappa \lambda_{I_1} sp(b_{I_2} \dots b_{I_k} b_{I_1} g),$$

which implies

$$sp(b_{I_1} b_{I_2} \dots b_{I_k} g) - \varkappa \lambda_{I_1} sp(b_{I_2} \dots b_{I_k} g) = \varkappa \lambda_{I_1} sp([b_{I_2} \dots b_{I_k}, b_{I_1}] g).$$

Thus,

$$sp(b_{I_1} b_{I_2} \dots b_{I_k} g) = \frac{\varkappa \lambda_{I_1}}{1 - \varkappa \lambda_{I_1}} sp([b_{I_2} \dots b_{I_k}, b_{I_1}] g). \quad (4.15)$$

This step operation expresses the \varkappa -trace of any regular degree k monomial in terms of the \varkappa -trace of degree $k - 2$ polynomials.

Special step operation. Let $M \stackrel{def}{=} b_{I_1} b_{I_2} \dots b_{I_k} g$ be a special monomial and $E(g) = l > 0$. We can choose a basis b_I in $\mathcal{E}^0 \oplus \mathcal{E}^1$ such that $\mathcal{C}_{IJ}|_{\mathcal{E}^0 \oplus \mathcal{E}^1}$ has the normal form:

$$\mathcal{C}_{IJ}|_{\mathcal{E}^0 \oplus \mathcal{E}^1} = \begin{pmatrix} 0 & I_{E(g)} \\ -I_{E(g)} & 0 \end{pmatrix}$$

Up to a polynomial of lesser degree, the monomial M can be expressed in the form

$$M = b_I^p b_J^q b_{L_1} \dots b_{L_{k-p-q}} g + \text{a lesser_degree_polynomial},$$

where

$$\begin{aligned} 0 \leq p, q \leq k, \quad p + q \leq k, \\ \lambda_I = \lambda_J = \lambda_{L_s} = \varkappa \quad \text{for any } s, \\ \mathcal{C}_{IJ} = 1, \quad \mathcal{C}_{IL_s} = 0, \quad \mathcal{C}_{JL_s} = 0 \quad \text{for any } s. \end{aligned} \quad (4.16)$$

Let $M' \stackrel{def}{=} b_I^p b_J^q b_{L_1} \dots b_{L_{k-p-q}}$ and derive the equation for $sp(M' g)$. Since

$$sp(b_J b_I M' g) = \varkappa sp(b_I M' g b_J) = sp(b_I M' b_J g),$$

it follows that

$$sp([b_I M', b_J] g) = 0. \quad (4.17)$$

Since $[b_I M', b_J]$ can be expressed as follows:

$$\begin{aligned} [b_I^{p+1} b_J^q b_{L_1} \dots b_{L_{k-p-q}}, b_J] &= \sum_{t=0}^p b_I^t (1 + f_{IJ}) b_I^{p-t} b_J^q b_{L_1} \dots b_{L_{k-p-q}} + \\ &+ \sum_{t=1}^{k-p-q} b_I^{p+1} b_J^q b_{L_1} \dots b_{L_{t-1}} f_{L_t J} b_{L_{t+1}} \dots b_{L_{k-p-q}} \end{aligned} \quad (4.18)$$

eq. (4.17) can be rewritten in the form

$$(p+1)sp(M'g) = -sp\left(\sum_{t=0}^p b_I^t f_{IJ} b_I^{p-t} b_J^q b_{L_1} \dots b_{L_{k-p-q}} g + \sum_{t=1}^{k-p-q} b_I^{p+1} b_J^q b_{L_1} \dots b_{L_{t-1}} f_{L_t J} b_{L_{t+1}} \dots b_{L_{k-p-q}} g\right), \quad (4.19)$$

which is the desired equation for $sp(M'g)$.

Due to Lemma 3.2 it is easy to see that eq. (4.19) can be rewritten in the form

$$sp(M'g) = \sum_{\tilde{g} \in W(\mathcal{R}): E(\tilde{g})=E(g)-1} sp(P_{\tilde{g}}(a_{\alpha i})\tilde{g}), \quad (4.20)$$

where the $P_{\tilde{g}}$ are some polynomials such that $\deg P_{\tilde{g}} \leq \deg M'$.

So, the special step operation expresses the \varkappa -trace of a special polynomial in terms of the \varkappa -trace of polynomials lesser in the sense of the ordering (4.11).

Thus, we showed that it is possible to express the \varkappa -trace of any polynomial in terms of the \varkappa -trace on $\mathbb{C}[W(\mathcal{R})]$ using a finite number of regular and special step operations. Since each step operation is manifestly $W(\mathcal{R})$ -invariant, and the \varkappa -trace on $\mathbb{C}[W(\mathcal{R})]$ is also $W(\mathcal{R})$ -invariant, the resulting \varkappa -trace is $W(\mathcal{R})$ -invariant.

This does not prove Theorem 4.1 yet because the resulting values of \varkappa -traces may a priori depend on the sequence of step operations used and impose additional constraints on the values of \varkappa -trace on $\mathbb{C}[W(\mathcal{R})]$.

Below we prove that the value of \varkappa -trace does not depend on the sequence of step operations used. We use the following inductive procedure:

(\star) Let $F \stackrel{\text{def}}{=} P(a_{\alpha i})g \in H_{W(\mathcal{R})}(\eta)$, where P is a polynomial such that $\deg P = 2k$ and $g \in W(\mathcal{R})$. Assuming that \varkappa -trace is correctly defined for all elements of $H_{W(\mathcal{R})}(\eta)$ lesser than F relative to the ordering (4.11), we prove that $sp(F)$ is defined also without imposing an additional constraints on the solution of the Ground Level Conditions.

The central point of the proof is consistency conditions (4.33), (4.34) and (4.50) proved in Appendices A.1 and A.2.

Assume that the Ground Level Conditions hold. The proof of Theorem 4.1 will be given in a constructive way by the following double induction procedure, equivalent to (\star):

(i) Assume that

$$sp([b_I, P_p(a)g]_{\varkappa}) = 0 \quad \text{for any } P_p(a), g \text{ and } I \text{ provided } b_I \in \mathbf{B}_g$$

and

$$\begin{aligned} \lambda(I) &\neq \varkappa; p \leq k \text{ or} \\ \lambda(I) &= \varkappa, E(g) \leq l, p \leq k \text{ or} \\ \lambda(I) &= \varkappa; p \leq k-2, \end{aligned}$$

where $P_p(a)$ is an arbitrary degree p polynomial in $a_{\alpha i}$ and p is odd. This implies that there exists a unique extension of the \varkappa -trace such that the same is true for l replaced with $l+1$.

(ii) Assuming that $sp(b_I P_p(a)g - \varkappa P_p(a)g b_I) = 0$ for any $P_p(a), g$ and $b_I \in \mathbf{B}_g$, where $p \leq k$, one proves that there exists a unique extension of the \varkappa -trace such that the assumption (i) is true for k replaced with $k+2$ and $l=0$.

As a result, this inductive procedure uniquely extends any solution of the Ground Level Conditions to a \varkappa -trace on the whole $H_{W(\mathcal{R})}(\eta)$. (Recall that the \varkappa -trace of any odd element of $H_{W(\mathcal{R})}(\eta)$ vanishes because the \varkappa -trace is even.)

It is convenient to work with the exponential generating functions

$$\Psi_g(\mu) = sp(e^S g), \text{ where } S = \sum_{L=1}^{2N} (\mu^L b_L), \quad (4.21)$$

where g is a fixed element of $W(\mathcal{R})$, $b_L \in \mathbf{B}_g$ and $\mu^L \in \mathbb{C}$ are independent parameters. By differentiating eq. (4.21) n times with respect to μ^L at $\mu = 0$ one can obtain an arbitrary polynomial of n -th degree in b_L as a coefficient of g up to polynomials of lesser degrees. In these terms, the induction on the degree of polynomials is equivalent to the induction on the homogeneity degree in μ of the power series expansions of $\Psi_g(\mu)$.

As a consequence of general properties of the \varkappa -trace, the generating function $\Psi_g(\mu)$ must be $W(\mathcal{R})$ -invariant:

$$\Psi_{\tau g \tau^{-1}}(\mu) = \Psi_g(\tilde{\mu}), \quad (4.22)$$

where the $W(\mathcal{R})$ -transformed parameters are of the form

$$\tilde{\mu}^I = \sum_J (\mathbf{M}(\tau g \tau^{-1}) \mathbf{M}^{-1}(\tau) \Lambda^{-1}(\tau) \mathbf{M}(\tau) \mathbf{M}^{-1}(g))_J^I \mu^J \quad (4.23)$$

and matrices $\mathbf{M}(g)$ and $\Lambda(g)$ are defined in eqs. (4.9) and (4.10).

The necessary and sufficient conditions for the existence of an even \varkappa -trace are the $W(\mathcal{R})$ -covariance conditions (4.22) and the condition

$$sp([b_L, e^S g]_{\varkappa}) = 0 \quad \text{for any } g \text{ and } L, \quad (4.24)$$

or, equivalently,

$$sp(b_L e^S g - \varkappa e^S g b_L) = 0 \quad \text{for any } g \text{ and } L. \quad (4.25)$$

4.3. General relations

To transform eq. (4.25) to a form convenient for the proof, we use the following two general relations true for arbitrary operators X and Y and parameter $\mu \in \mathbb{C}$:

$$X \exp(Y + \mu X) = \frac{\partial}{\partial \mu} \exp(Y + \mu X) + \int t_2 \exp(t_1(Y + \mu X)) [X, Y] \exp(t_2(Y + \mu X)) D^1 t, \quad (4.26)$$

$$\exp(Y + \mu X) X = \frac{\partial}{\partial \mu} \exp(Y + \mu X) - \int t_1 \exp(t_1(Y + \mu X)) [X, Y] \exp(t_2(Y + \mu X)) D^1 t \quad (4.27)$$

with the convention that

$$D^{n-1} t = \delta(t_1 + \dots + t_n - 1) \theta(t_1) \dots \theta(t_n) dt_1 \dots dt_n. \quad (4.28)$$

The relations (4.26) and (4.27) can be derived with the help of partial integration (e.g., over t_1) and the following formula

$$\frac{\partial}{\partial \mu} \exp(Y + \mu X) = \int \exp(t_1(Y + \mu X)) X \exp(t_2(Y + \mu X)) D^1 t \quad (4.29)$$

which can be proven by expanding in power series. The well-known formula

$$[X, \exp(Y)] = \int \exp(t_1 Y) [X, Y] \exp(t_2 Y) D^1 t \tag{4.30}$$

is a consequence of eqs. (4.26) and (4.27).[§]

With the help of eqs. (4.26), (4.27) and (4.5) one rewrites eq. (4.25) as

$$(1 - \varkappa \lambda_L) \frac{\partial}{\partial \mu^L} \Psi_g(\mu) = \int (-\varkappa \lambda_L t_1 - t_2) sp \left(\exp(t_1 S) [b_L, S] \exp(t_2 S) g \right) D^1 t. \tag{4.31}$$

This condition should be true for any g and L and plays the central role in the analysis in this section. Eq. (4.31) is an overdetermined system of linear equations for sp ; we show below that it has the only solution extending any fixed solution of the Ground Level Conditions.

There are two essentially distinct cases, $\lambda_L \neq \varkappa$ and $\lambda_L = \varkappa$. In the latter case, the eq. (4.31) takes the form

$$0 = \int sp \left(\exp(t_1 S) [b_L, S] \exp(t_2 S) g \right) D^1 t, \quad \lambda_L = \varkappa. \tag{4.32}$$

In Appendix A.1 we prove by induction that eqs. (4.31) and (4.32) are consistent in the following sense

$$(1 - \varkappa \lambda_K) \frac{\partial}{\partial \mu^K} \int (-\varkappa \lambda_L t_1 - t_2) sp \left(\exp(t_1 S) [b_L, S] \exp(t_2 S) g \right) D^1 t - (L \leftrightarrow K) = 0 \tag{4.33}$$

for $\lambda_L \neq \varkappa, \lambda_K \neq \varkappa$

and

$$(1 - \varkappa \lambda_K) \frac{\partial}{\partial \mu^K} \int sp \left(\exp(t_1 S) [b_L, S] \exp(t_2 S) g \right) D^1 t = 0 \text{ for } \lambda_L = \varkappa. \tag{4.34}$$

Note that this part of the proof is quite general and does not depend on a concrete form of the commutation relations between the $a_{\alpha i}$ in eq. (1.7).

By expanding the exponential e^S in eq. (4.21) into power series in μ^K (equivalently b_K) we conclude that eq. (4.31) uniquely reconstructs the \varkappa -trace of monomials containing b_K with $\lambda_K \neq \varkappa$ (i.e., *regular monomials*) in terms of \varkappa -traces of some lower degree polynomials. Then the consistency conditions (4.33) and (4.34) guarantee that eq. (4.31) does not impose any additional conditions on the \varkappa -traces of lower degree polynomials and allow one to represent the generating function

[§]The independent proof of eq. (4.30) follows from the equalities:

$$[X, \exp(Y)] = \lim_{n \rightarrow \infty} [X, (\exp(Y/n))^n] = \lim_{n \rightarrow \infty} \sum_{k=0}^{n-1} (\exp(Y/n))^k [X, (1 + \frac{1}{n} Y)] (\exp(Y/n))^{n-k-1}.$$

The same trick can be used for the proof of eq. (4.29).

in the form

$$\begin{aligned} \Psi_g &= \Phi_g(\mu) + \\ &+ \sum_{L: \lambda_L \neq \varkappa} \int_0^1 \frac{\mu_L d\tau}{1 - \varkappa \lambda_L} \int D^1 t (-\varkappa \lambda_L t_1 - t_2) sp \left(e^{t_1(\tau S'' + S')} [b_L, (\tau S'' + S')] e^{t_2(\tau S'' + S')} g \right), \end{aligned} \quad (4.35)$$

where we introduced the generating functions Φ_g for the \varkappa -trace of *special polynomials*, i.e., the polynomials depending only on b_L with $\lambda_L = \varkappa$,

$$\Phi_g(\mu) \stackrel{def}{=} sp \left(e^{S'} g \right) \Big|_{(\mu^l = 0 \text{ if } \lambda_l \neq \varkappa)} \quad (4.36)$$

and

$$S' = \sum_{L: b_L \in \mathbf{B}_g, \lambda_L = \varkappa} (\mu^L b_L); \quad S'' = S - S'. \quad (4.37)$$

The relation (4.35) successively expresses the \varkappa -trace of higher degree regular polynomials via the \varkappa -traces of lower degree polynomials.

One can see that the arguments above prove the inductive hypotheses **(i)** and **(ii)** for the particular case where the polynomials $P_p(a)$ are regular and/or $\lambda_l \neq \varkappa$. Note that for this case the induction **(i)** on the grading E is trivial: one simply proves that the degree of the polynomial can be increased by two.

Let us now turn to a less trivial case of the special polynomials:

$$sp \left(b_I e^{S'} g - \varkappa e^{S'} g b_I \right) = 0, \text{ where } \lambda_I = \varkappa. \quad (4.38)$$

This equation implies

$$sp \left([b_I, e^{S'}] g \right) = 0, \text{ where } \lambda_I = \varkappa. \quad (4.39)$$

Consider the part of $sp([b_I, \exp S']g)$ which is of degree k in μ and let $E(g) = l + 1$. By eq. (4.32) the conditions (4.39) give

$$0 = \int sp \left(\exp(t_1 S') [b_I, S'] \exp(t_2 S') g \right) D^1 t. \quad (4.40)$$

Substituting $[b_I, S'] = \mu_I + \sum_M f_{IM} \mu^M$, where the quantities f_{IJ} and μ_I are defined in eqs. (4.7)-(4.8), one can rewrite eq. (4.40) in the form

$$\mu_I \Phi_g(\mu) = - \int sp \left(\exp(t_1 S') \sum_M f_{IM} \mu^M \exp(t_2 S') g \right) D^1 t. \quad (4.41)$$

Now we use the inductive hypothesis **(i)**. The right hand side of eq. (4.41) is a \varkappa -trace of a polynomial of degree $\leq k - 1$ in the a_{α_i} in the sector of degree k polynomials in μ , and $E(f_{IM}g) = l$. Therefore one can use the inductive hypothesis **(i)** to obtain the equality

$$\int sp \left(\exp(t_1 S') \sum_M f_{IM} \mu^M \exp(t_2 S') g \right) D^1 t = \int sp \left(\exp(t_2 S') \exp(t_1 S') \sum_M f_{IM} \mu^M g \right) D^1 t,$$

where we used that $sp(S' F g) = \varkappa sp(F g S') = sp(F S' g)$ by definition of S' .

As a result, the inductive hypothesis allows one to transform eq. (4.38) to the following form:

$$X_I = 0, \text{ where } X_I \stackrel{def}{=} \mu_I \Phi_g(\mu) + sp \left(\exp(S') \sum_M f_{IM} \mu^M g \right). \quad (4.42)$$

By differentiating this equation with respect to μ^J one obtains after symmetrization

$$\frac{\partial}{\partial \mu^J} (\mu_I \Phi_g(\mu)) + (I \leftrightarrow J) = - \int sp \left(e^{t_1 S'} b_J e^{t_2 S'} \sum_M f_{IM} \mu^M g \right) D^1 t + (I \leftrightarrow J). \quad (4.43)$$

An important point is that the system of equations (4.43) is equivalent to the original equations (4.42) except for the ground level part $\Phi_g(0)$. This can be easily seen from the simple fact that the general solution of the system of equations for entire functions $X_I(\mu)$

$$\frac{\partial}{\partial \mu^J} X_I(\mu) + \frac{\partial}{\partial \mu^I} X_J(\mu) = 0$$

is of the form

$$X_I(\mu) = X_I(0) + \sum_J c_{IJ} \mu^J$$

where $X_I(0)$ and $c_{IJ} = -c_{JI}$ are some constants.

The part of eq. (4.42) linear in μ is however equivalent to the Ground Level Conditions analyzed in Section 3. Thus, eq. (4.43) contains all information of eq. (3.2) additional to the Ground Level Conditions. For this reason, we will from now on analyze equation (4.43).

Using again the inductive hypothesis we move b_I to the left and to the right of the right hand side of eq. (4.43) with equal weights equal to $\frac{1}{2}$ to get

$$\begin{aligned} \frac{\partial}{\partial \mu^J} \mu_I \Phi_g(\mu) + (I \leftrightarrow J) &= -\frac{1}{2} \sum_M sp \left(\exp(S') \{b_J, f_{IM}\} \mu^M g \right) - \\ &- \frac{1}{2} \int \sum_{L,M} (t_1 - t_2) sp \left(\exp(t_1 S') F_{JL} \mu^L \exp(t_2 S') f_{IM} \mu^M g \right) D^1 t + (I \leftrightarrow J). \end{aligned} \quad (4.44)$$

The last terms with the factor $t_1 - t_2$ vanish as is not difficult to show, so eq. (4.44) reduces to

$$L_{IJ} \Phi_g(\mu) = -\frac{1}{2} R_{IJ}(\mu), \quad (4.45)$$

where

$$R_{IJ}(\mu) = \sum_M sp \left(\exp(S') \{b_J, f_{IM}\} \mu^M g \right) + (I \leftrightarrow J) \quad (4.46)$$

and

$$L_{IJ} = \frac{\partial}{\partial \mu^J} \mu_I + \frac{\partial}{\partial \mu^I} \mu_J, \quad (4.47)$$

or, equivalently,

$$L_{IJ} = \mu_I \frac{\partial}{\partial \mu^J} + \mu_J \frac{\partial}{\partial \mu^I}. \quad (4.48)$$

The differential operators L_{IJ} satisfy the standard commutation relations of the Lie algebra $\mathfrak{sp}(2E(g))$

$$[L_{IJ}, L_{KL}] = -(\mathcal{C}_{IK}L_{JL} + \mathcal{C}_{IL}L_{JK} + \mathcal{C}_{JK}L_{IL} + \mathcal{C}_{JL}L_{IK}). \quad (4.49)$$

In Appendix A.2 we show by induction that this Lie algebra $\mathfrak{sp}(2E(g))$ realized by differential operators is consistent with the right-hand side of the basic relation (4.45), i.e., that

$$[L_{IJ}, R_{KL}] - [L_{KL}, R_{IJ}] = -(\mathcal{C}_{IK}R_{JL} + \mathcal{C}_{JL}R_{IK} + \mathcal{C}_{JK}R_{IL} + \mathcal{C}_{IL}R_{JK}). \quad (4.50)$$

Generally, these consistency conditions guarantee that eqs. (4.45) express $\Phi_g(\mu)$ in terms of R^{IJ} in the following way

$$\Phi_g(\mu) = \Phi_g(0) + \frac{1}{8E(g)} \sum_{I,J=1}^{2E(g)} \int_0^1 \frac{dt}{t} (1 - t^{2E(g)})(L_{IJ}R^{IJ})(t\mu), \quad (4.51)$$

provided

$$R^{IJ}(0) = 0. \quad (4.52)$$

The latter condition must hold for the consistency of eqs. (4.45) since its left hand side vanishes at $\mu^I = 0$. In the expression (4.51) it guarantees that the integral over t converges. In the case under consideration the condition (4.52) is met as follows from definition (4.46).

Taking Lemma 3.2 and the explicit form (4.46) of R_{IJ} into account one concludes that eq. (4.51) uniquely expresses the \varkappa -trace of special polynomials in terms of the \varkappa -traces of polynomials of lower degrees or in terms of the \varkappa -traces of special polynomials of the same degree multiplied by elements of $W(\mathcal{R})$ with a smaller value of E provided that the μ -independent term $\Phi_g(0)$ is an arbitrary solution of the Ground Level Conditions. This completes the proof of Theorem 4.1. \square

5. Non-deformed skew product $H_{W(\mathcal{R})}(0)$ of the Weyl superalgebra and a finite group generated by reflections

Consider $H_{W(\mathcal{R})}(0)$. It has the same number of traces and supertraces as $H_{W(\mathcal{R})}(\eta)$ for an arbitrary η and whose generation functions of these traces and supertraces can be written down explicitly. This algebra is the skew product of the Weyl superalgebra and the group algebra of the finite group $W(\mathcal{R})$ generated by a root system $\mathcal{R} \subset V = \mathbb{R}^N$. Algebras of this type, and their generalizations, were considered in [13].

The superalgebra $H_{W(\mathcal{R})}(0)$ is an associative superalgebra of polynomials in $a_{\alpha i}$, where $\alpha = 0, 1$ and $i = 1, \dots, N$, with coefficients in the group algebra $\mathbb{C}[W(\mathcal{R})]$ subject to relations

$$ga_{\alpha i} = \sum_{k=1}^N g_i^k a_{\alpha k} g \text{ for any } g \in W(\mathcal{R}) \text{ and } a_{\alpha i}, \quad (5.1)$$

$$[a_{\alpha i}, a_{\beta j}] = \varepsilon_{\alpha\beta} \delta_{ij}, \quad (5.2)$$

where $\varepsilon_{\alpha\beta}$ is the antisymmetric tensor, $\varepsilon_{01} = 1$, and (g_i^k) is a matrix realizing the representation of $g \in W(\mathcal{R})$ in $End(V)$. The commutation relations (5.1)–(5.2) suggest to define the *parity* π by

setting:

$$\pi(a_{\alpha i}) = 1 \text{ and } \pi(g) = 0 \text{ for any } g \in W(\mathcal{R}). \quad (5.3)$$

Unifying indices i and α in one index I one can rewrite eq. (5.2) as

$$[a_I, a_J] = \omega_{IJ}, \text{ where } \omega_{IJ} \text{ is a symplectic form.} \quad (5.4)$$

It is easy to find the general solution of eqs. (4.31) and (4.32) for the generating function of \varkappa -traces: (1) If $g \in W(\mathcal{R})$ and $E(g) \neq 0$, then $sp(P(a_I)g) = 0$ for any polynomial P .

(2) If $g \in W(\mathcal{R})$ and $E(g) = 0$, then $sp(g)$ is an arbitrary central function on $W(\mathcal{R})$.

(3) Let $E(g) = 0$. There exists a complete set $b_{\alpha,k}$ of eigenvectors of g for each α , such that $gb_K = \Lambda_K b_{K,g}$ and $\mathcal{C}_{KL} = [b_K, b_L]$ is nondegenerate skewsymmetric form such that $\mathcal{C}_{KL} \neq 0$ only if $\lambda_K \lambda_L = 1$. In this notation, let

$$S(\mu, b) = \sum_K \mu^K b_K, \\ Q(\mu) = \frac{1}{4} \sum_{KL} \mu^K \mu^L \tilde{\mathcal{C}}_{KL},$$

where

$$\tilde{\mathcal{C}}_{KL} = -\frac{1 + \varkappa \lambda_K}{1 - \varkappa \lambda_K} \mathcal{C}_{KL} = \tilde{\mathcal{C}}_{LK}.$$

Then

$$sp\left(e^{S(\mu, b)} g\right) = e^{Q(\mu)} sp(g). \quad (5.5)$$

The solution of eq. (5.5) can be also obtained in the initial basis. Let $S = \sum_{\alpha i} \mu^{\alpha i} a_{\alpha i}$, $\Psi(g, \mu, t) = sp(e^{tS} g)$, $\Psi(g, \mu) = sp(e^S g) = \Psi(g, \mu, 1)$. Then

$$sp([a_{\alpha i}, e^{tS} g]_{\varkappa}) = sp\left(t \varepsilon_{\alpha\beta} \delta_{ij} \mu^{\beta j} e^{tS} g + e^{tS} a_{\alpha j} g p_i^j\right), \text{ where } p_i^j = (1 - \varkappa g)_i^j. \quad (5.6)$$

Since $E(g) = 0$, the matrix (p_i^j) is invertible, so eq. (5.6) gives

$$\frac{d}{dt} \Psi(g, \mu, t) = -\mu^{\alpha i} \varepsilon_{\alpha\beta} q_i^k \delta_{kj} \mu^{\beta j} \Psi(g, \mu, t), \text{ where } q_i^k = \left(\frac{1}{1 - \varkappa g}\right)_i^k = \frac{1}{2} \left(\frac{1 + \varkappa g}{1 - \varkappa g}\right)_i^k + \frac{1}{2} \delta_i^k$$

So

$$\frac{d}{dt} \Psi(g, \mu, t) = -\mu^{\alpha i} \varepsilon_{\alpha\beta} \tilde{\omega}_{ij} \mu^{\beta j} \Psi(g, \mu, t), \text{ where } \tilde{\omega}_{ij} = \frac{1}{2} \left(\frac{1 + \varkappa g}{1 - \varkappa g}\right)_i^k \delta_{kj} = -\tilde{\omega}_{ji}$$

and finally

$$\Psi(g, \mu) = \exp\left(-\frac{1}{2} \mu^{\alpha i} \varepsilon_{\alpha\beta} \tilde{\omega}_{ij} \mu^{\beta j}\right) sp(g).$$

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Appendix A. Proof of consistency conditions

A.1. Proof of consistency condition (4.33) for $\lambda \neq \varkappa$.

Let parameters $\mu_1 \stackrel{def}{=} \mu^{K_1}$ and $\mu_2 \stackrel{def}{=} \mu^{K_2}$ be such that $\lambda_1 \neq \varkappa$ and $\lambda_2 \neq \varkappa$, where $\lambda_1 \stackrel{def}{=} \lambda_{K_1}$ and $\lambda_2 \stackrel{def}{=} \lambda_{K_2}$. Let $b^1 \stackrel{def}{=} b_{K_1}$ and $b^2 \stackrel{def}{=} b_{K_2}$. Let us prove by induction that conditions (4.33) hold. To implement induction, we select a part of degree k in μ from eq. (4.31) and observe that this part contains a degree k polynomial in b_M in the left-hand side of eq. (4.31) while the part on the right hand side of the differential version (4.31) of eq. (4.24) which is of the same degree in μ has degree $k - 1$ as polynomial in b_M .

This happens because of the presence of the commutator $[b_L, S]$ which is a zero degree polynomial due to the basic relations (1.7). As a result, the inductive hypothesis allows us to use the properties of \varkappa -trace provided that the above commutator is always handled as the right hand side of eq. (1.7), i.e., we are not allowed to represent it again as a difference of the second-degree polynomials.

Direct differentiation of Eq. (4.31) with the help of eq. (4.29) gives

$$\begin{aligned} & (1 - \varkappa\lambda_2) \frac{\partial}{\partial \mu_2} \int (-\varkappa\lambda_1 t_1 - t_2) sp \left(e^{t_1 S} [b^1, S] e^{t_2 S} g \right) D^1 t - (1 \leftrightarrow 2) = \\ & = \left(\int (1 - \varkappa\lambda_2) (-\varkappa\lambda_1 t_1 - t_2) sp \left(e^{t_1 S} [b^1, b^2] e^{t_2 S} g \right) D^1 t - (1 \leftrightarrow 2) \right) + \\ & + \left(\int (1 - \varkappa\lambda_2) (-\varkappa\lambda_1 (t_1 + t_2) - t_3) sp \left(e^{t_1 S} b^2 e^{t_2 S} [b^1, S] e^{t_3 S} g \right) D^2 t - (1 \leftrightarrow 2) \right) + \\ & + \left(\int (1 - \varkappa\lambda_2) (-\varkappa\lambda_1 t_1 - t_2 - t_3) sp \left(e^{t_1 S} [b^1, S] e^{t_2 S} b^2 e^{t_3 S} g \right) D^2 t - (1 \leftrightarrow 2) \right). \end{aligned} \quad (A.1)$$

We have to show that the right hand side of eq. (A.1) vanishes. Let us first transform the second and the third terms on the right-hand side of eq. (A.1). The idea is to move the operators b^2 through the exponentials towards the commutator $[b^1, S]$ in order to use then the Jacobi identity for the double commutators. This can be done in two different ways inside the \varkappa -trace so that one has to fix appropriate weight factors for each of these processes. The correct weights turn out to be

$$\begin{aligned} & D^2 t (-\varkappa\lambda_1 (t_1 + t_2) - t_3) b^2 \equiv D^2 t (-\varkappa\lambda_1 - t_3 (1 - \varkappa\lambda_1)) b^2 = \\ & = D^2 t \left(\left(\frac{\lambda_1 \lambda_2}{1 - \varkappa\lambda_2} - t_3 (1 - \varkappa\lambda_1) \right) \overrightarrow{b^2} + \frac{-\varkappa\lambda_1}{1 - \varkappa\lambda_2} \overleftarrow{b^2} \right) \end{aligned} \quad (A.2)$$

and

$$\begin{aligned} & D^2 t (-\varkappa\lambda_1 t_1 - t_2 - t_3) b^2 \equiv D^2 t ((-\varkappa\lambda_1 + 1) t_1 - 1) b^2 = \\ & = D^2 t \left(\left(t_1 (1 - \varkappa\lambda_1) - \frac{1}{1 - \varkappa\lambda_2} \right) \overleftarrow{b^2} - \frac{-\varkappa\lambda_2}{1 - \varkappa\lambda_2} \overrightarrow{b^2} \right) \end{aligned} \quad (A.3)$$

for the second and third terms in the right hand side of eq. (A.1), respectively. Here the notation \overrightarrow{A} and \overleftarrow{A} imply that the operator A has to be moved from its position to the right and to the left, respectively. Using eq. (4.30) along with the simple formula

$$\int \phi(t_3, \dots, t_{n+1}) D^n t = \int t_1 \phi(t_2, \dots, t_n) D^{n-1} t \quad (\text{A.4})$$

we find that all terms which involve both $[b^1, S]$ and $[b^2, S]$ pairwise cancel after antisymmetrization $1 \leftrightarrow 2$.

As a result, one is left with some terms involving double commutators which, thanks to the Jacobi identities and antisymmetrization, are all reduced to

$$\int \left(\lambda_1 \lambda_2 t_1 + t_2 - t_1 t_2 (1 - \varkappa \lambda_1) (1 - \varkappa \lambda_2) \right) sp \left(\exp(t_1 S) [S, [b^1, b^2]] \exp(t_2 S) g \right) D^1 t. \quad (\text{A.5})$$

Finally, we observe that this expression can be equivalently rewritten in the form

$$\int \left(\lambda_1 \lambda_2 t_1 + t_2 - t_1 t_2 (1 - \varkappa \lambda_1) (1 - \varkappa \lambda_2) \right) \left(\frac{\partial}{\partial t_1} - \frac{\partial}{\partial t_2} \right) sp \left(\exp(t_1 S) [b^1, b^2] \exp(t_2 S) g \right) D^1 t \quad (\text{A.6})$$

and after integration by parts cancel the first term on the right-hand side of eq. (A.1). Thus, it is shown that eqs. (4.31) are compatible for the case $\lambda_{1,2} \neq \varkappa$.

Analogously, we can show that eqs. (4.31) are compatible with eq. (4.32). Indeed, let $\lambda_1 = \varkappa$, $\lambda_2 \neq \varkappa$. Let us prove that

$$\frac{\partial}{\partial \mu_2} sp \left([b^1, \exp(S)] g \right) = 0 \quad (\text{A.7})$$

provided the \varkappa -trace is well-defined for the lower degree polynomials. The explicit differentiation gives

$$\begin{aligned} \frac{\partial}{\partial \mu_2} sp \left([b^1, \exp(S)] g \right) &= \int sp \left([b^1, \exp(t_1 S) b^2 \exp(t_2 S)] g \right) D^1 t = \\ &= (1 - \varkappa \lambda_2)^{-1} sp \left([b^1, (b^2 \exp(S) - \varkappa \lambda_2 \exp(S) b^2)] g \right) + \dots \end{aligned} \quad (\text{A.8})$$

where dots denote some terms of the form $sp \left([b^1, B] g \right)$ involving more commutators inside B , which therefore amount to some lower degree polynomials and vanish by the inductive hypothesis. As a result, we find that

$$\begin{aligned} \frac{\partial}{\partial \mu_2} sp \left([b^1, \exp(S)] g \right) &= (1 - \varkappa \lambda_2)^{-1} sp \left((b^2 [b^1, \exp(S)] - \varkappa \lambda_2 [b^1, \exp(S)] b^2) g \right) + \\ &+ (1 - \varkappa \lambda_2)^{-1} sp \left(([b^1, b^2] \exp(S) - \varkappa \lambda_2 \exp(S) [b^1, b^2]) g \right). \end{aligned} \quad (\text{A.9})$$

This expression vanishes by the inductive hypothesis, too.

A.2. The proof of consistency conditions (4.50) (the case of special polynomials)

In order to prove eq. (4.50) we use the inductive hypothesis (i). In this appendix we use the convention that any expression with the coinciding upper or lower indices are automatically symmetrized,

e.g., $F^{II} \stackrel{def}{=} \frac{1}{2}(F^{I_1 I_2} + F^{I_2 I_1})$. Let us write the identity

$$0 = \sum_M sp \left(\left[\exp(S') \{b_I, f_{IM}\} \mu^M, b_J b_J \right] g \right) - (I \leftrightarrow J) \tag{A.10}$$

which holds due to Lemma 3.3 for all terms of degree $k - 1$ in μ with $E(g) \leq l + 1$ and for all lower degree polynomials in μ (one can always move f_{IJ} to g in eq. (A.10) combining $f_{IJ}g$ into a combination of elements of $W(\mathcal{R})$ analyzed in Lemma 3.3).

Straightforward calculation of the commutator in the right-hand-side of eq. (A.10) gives $0 = X_1 + X_2 + X_3$, where

$$\begin{aligned} X_1 &= - \sum_{M,L} \int sp \left(\exp(t_1 S') \{b_J, F_{JL}\} \mu^L \exp(t_2 S') \{b_I, f_{IM}\} \mu^M g \right) D^1 t - (I \leftrightarrow J), \\ X_2 &= \sum_M sp \left(\exp(S') \left\{ \{b_J, F_{IJ}\}, f_{IM} \right\} \mu^M g \right) - (I \leftrightarrow J), \\ X_3 &= \sum_M sp \left(\exp(S') \left\{ b_I, \{b_J, [f_{IM}, b_J]\} \right\} \mu^M g \right) - (I \leftrightarrow J). \end{aligned} \tag{A.11}$$

The terms of X_1 bilinear in f cancel due to the antisymmetrization $(I \leftrightarrow J)$ and the inductive hypothesis (i). As a result, one can transform X_1 to the form

$$X_1 = \left(-\frac{1}{2} [L_{JJ}, R_{II}] + 2sp \left(e^{S'} \{b_I, f_{IJ}\} \mu_J g \right) \right) - (I \leftrightarrow J). \tag{A.12}$$

Substituting $F_{IJ} = \mathcal{C}_{IJ} + f_{IJ}$ and $f_{IM} = [b_I, b_M] - \mathcal{C}_{IM}$ one transforms X_2 to the form

$$X_2 = 2\mathcal{C}_{IJ} R_{IJ} - 2 \left(sp \left(e^{S'} \{b_J, f_{IJ}\} \mu_I g \right) - (I \leftrightarrow J) \right) + Y, \tag{A.13}$$

where

$$Y = sp \left(e^{S'} \left\{ \{b_J, f_{IJ}\}, [b_I, S'] \right\} g \right) - (I \leftrightarrow J). \tag{A.14}$$

Using that

$$sp \left(\exp(S') [P f_{IJ} Q, S'] g \right) = 0 \tag{A.15}$$

provided that the inductive hypothesis can be used, one transforms Y to the form

$$Y = sp \left(e^{S'} \left(-[f_{IJ}, (b_I S' b_J + b_J S' b_I)] - b_I [f_{IJ}, S'] b_J - b_J [f_{IJ}, S'] b_I + [f_{IJ}, \{b_I, b_J\}] S' \right) g \right). \tag{A.16}$$

Let us rewrite X_3 in the form $X_3 = X_3^s + X_3^a$, where

$$\begin{aligned} X_3^s &= \frac{1}{2} \sum_M sp \left(e^{S'} \left(\{b_I, \{b_J, [f_{IM}, b_J]\}\} + \{b_J, \{b_I, [f_{IM}, b_J]\}\} \right) \mu^M g \right) - (I \leftrightarrow J), \\ X_3^a &= \frac{1}{2} \sum_M sp \left(e^{S'} \left(\{b_I, \{b_J, [f_{IM}, b_J]\}\} - \{b_J, \{b_I, [f_{IM}, b_J]\}\} \right) \mu^M g \right) - (I \leftrightarrow J). \end{aligned}$$

With the help of the Jacobi identity $[f_{IM}, b_J] - [f_{JM}, b_I] = [f_{IJ}, b_M]$ one expresses X_3^s in the form

$$X_3^s = \frac{1}{2} sp \left(e^{S'} \left(\{b_I, b_J\} [f_{IJ}, S'] + [f_{IJ}, S'] \{b_I, b_J\} + 2b_I [f_{IJ}, S'] b_J + 2b_J [f_{IJ}, S'] b_I \right) g \right).$$

Let us transform this expression for X_3^a to the form

$$X_3^a = \frac{1}{2} \sum_M sp \left(e^{S'} [F_{IJ}, [f_{IM}, b_J]] \mu^M g \right) - (I \leftrightarrow J). \quad (\text{A.17})$$

Substitute $F_{IJ} = \mathcal{C}_{IJ} + f_{IJ}$ and $f_{IM} = [b_I, b_M] - \mathcal{C}_{IM}$ in eq. (A.17). After simple transformations we find that $Y + X_3 = 0$. From eqs. (A.12) and (A.13) it follows that the right hand side of eq. (A.10) is equal to

$$\frac{1}{2} ([L_{II}, R_{JJ}] - [L_{JJ}, R_{II}]) + 2\mathcal{C}_{IJ}R_{IJ}.$$

This completes the proof of the consistency conditions (4.50).

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