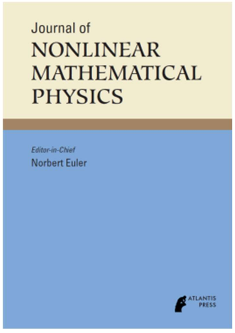

Journal of Nonlinear Mathematical Physics

ISSN (Online): 1776-0852 ISSN (Print): 1402-9251
Journal Home Page: https://www.atlantis-press.com/journals/jnmp

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To cite this article: S.E. Konstein, R. Stekolshchik (2013) Klein operator and the Number of independent Traces and Supertraces on the Superalgebra of Observables of Rational Calogero Model based on the Root System, Journal of Nonlinear Mathematical Physics 20:2, 295–308, DOI: https://doi.org/10.1080/14029251.2013.820410

To link to this article: https://doi.org/10.1080/14029251.2013.820410

Published online: 04 January 2021

Journal of Nonlinear Mathematical Physics, Vol. 20, No. 2 (June 2013), 295-308

Klein operator and the Number of independent Traces and Supertraces on the Superalgebra of Observables of Rational Calogero Model based on the Root System

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Received 2 December 2012
Accepted 1 June 2013

In the Coxeter group $W(\mathscr{R})$ generated by the root system $\mathscr{R}$, let $T(\mathscr{R})$ be the number of conjugacy classes having no eigenvalue $+1$ and let $S(\mathscr{R})$ be the number of conjugacy classes having no eigenvalue -1. The algebra $H_{W(\mathscr{R})}$ of observables of the rational Calogero model based on the root system $\mathscr{R}$ possesses $T(\mathscr{R})$ independent traces; the same algebra, considered as an associative superalgebra with respect to a certain natural parity, possesses $S(\mathscr{R})$ even independent supertraces and no odd trace or supertrace. The numbers $T(\mathscr{R})$ and $S(\mathscr{R})$ are determined for all irreducible root systems (hence for all root systems). It is shown that $T(\mathscr{R}) \leq S(\mathscr{R})$, and $T(\mathscr{R}) = S(\mathscr{R})$ if and only if superalgebra $H_{W(\mathscr{R})}$ contains a Klein operator (or, equivalently, $W(\mathscr{R}) \ni -1$).

Keywords: Trace; supertrace; Cherednik algebra; algebra of observables; Calogero model.

2000 *Mathematics Subject Classification*: 17B80, 16W55

1. Definitions and generalities

1.1. *Traces*

Let $\mathscr{A}$ be an associative superalgebra with parity π. All expressions of linear algebra are given for homogeneous elements only and are supposed to be extended to inhomogeneous elements via linearity.

A linear function str on $\mathscr{A}$ is called a *supertrace* if

$$str(fg) = (-1)^{\pi(f)\pi(g)} str(gf) \quad \text{for all } f,g \in \mathscr{A}.$$

A linear function tr on $\mathscr{A}$ is called a *trace* if

$$tr(fg) = tr(gf) \quad \text{for all } f,g \in \mathscr{A}.$$

A linear function L is *even* if $L(f) = 0$ for any $f \in \mathscr{A}$ such that $\pi(f) = 1$, it is *odd* if $L(f) = 0$ for any $f \in \mathscr{A}$ such that $\pi(f) = 0$.

Co-published by Atlantis Press and Taylor & Francis

Let $\mathscr{A}_1$ and $\mathscr{A}_2$ be associative superalgebras with parities π_1 and π_2, respectively. Define the tensor product $\mathscr{A} = \mathscr{A}_1 \otimes \mathscr{A}_2$ as a superalgebra with the product $(a_1 \otimes a_2)(b_1 \otimes b_2) = (a_1 b_1) \otimes (a_2 b_2)$ (no sign factors in this formula) and the parity π defined by the formula $\pi(a \otimes b) = \pi_1(a) + \pi_2(b)$.

Let T_i be a trace on $\mathscr{A}_i$. Clearly, the function T such that $T(a \otimes b) = T_1(a) T_2(b)$ is a trace on $\mathscr{A}$.

Let S_i be an *even* supertrace on $\mathscr{A}_i$. Clearly, the function S such that $S(a \otimes b) = S_1(a) S_2(b)$ is an even supertrace on $\mathscr{A}$.

1.2. *Klein operator*

Let $\mathscr{A}$ be an associative superalgebra with parity π. Following M.Vasiliev, see, e.g. [21], we say that an element $K \in \mathscr{A}$ is a *Klein operator*[a] if $\pi(K) = 0$, $Kf = (-1)^{\pi(f)} fK$ for any $f \in \mathscr{A}$ and $K^2 = 1$. Every Klein operator belongs to the *anticenter* of the superalgebra $\mathscr{A}$, see [18], p.41.[b]

Any Klein operator, if exists, establishes an isomorphism between the space of even traces and the space of even supertraces on $\mathscr{A}$. Namely, if $f \mapsto T(f)$ is an even trace, then $f \mapsto T(fK)$ is a supertrace, and if $f \mapsto S(f)$ is an even supertrace, then $f \mapsto S(fK)$ is a trace.

1.3. *Group algebra*

Let $V = \mathbb{R}^n$ and $G \subset End(V)$ be a finite group. The *group algebra* $\mathbb{C}[G]$ of G consists of all linear combinations $\sum_{g \in W(\mathscr{R})} \alpha_g \bar{g}$, where $\alpha_g \in \mathbb{C}$. We distinguish g considered as an element of the group $G \subset End(V)$ from the same element $\bar{g} \in \mathbb{C}[G]$ considered as an element of the group algebra. The addition in $\mathbb{C}[G]$ is defined as follows:

$$\sum_{g \in G} \alpha_g \bar{g} + \sum_{g \in G} \beta_g \bar{g} = \sum_{g \in G} (\alpha_g + \beta_g) \bar{g}$$

and the multiplication is defined by setting $\overline{g_1}\,\overline{g_2} = \overline{g_1 g_2}$.

Note that the additions in $\mathbb{C}[G]$ and in $End(V)$ differ. For example, if $I \in G$ is unity and the matrix $J = -I$ from $End(V)$ belongs to G, then $I + J = 0$ in $End(V)$ while $\bar{I} + \bar{J} \neq 0$ in $\mathbb{C}[G]$.

1.4. *Root systems*

Let $V = \mathbb{R}^N$ be endowed with a non-degenerate symmetric bilinear form $(\cdot,\cdot)$ and the vectors $\vec{a}_i$ constitute an orthonormal basis in V, i.e.

$$(\vec{a}_i, \vec{a}_j) = \delta_{ij}.$$

Let x^i be the coordinates of $\vec{x} \in V$, i.e. $\vec{x} = \vec{a}_i x^i$. Then $(\vec{x}, \vec{y}) = \sum_{i=1}^N x^i y^i$ for any $\vec{x}, \vec{y} \in V$. The indices i are raised and lowered by means of the forms δ_{ij} and δ^{ij}.

[a] In honor of Oskar Klein.

[b] Let $\mathscr{A}$ be an associative superalgebra with parity π. Its *anticenter* $AC(\mathscr{A})$ is defined by the formula

$$AC(\mathscr{A}) = \{a \in \mathscr{A} \mid ax - (-1)^{\pi(x)(\pi(a)+1)} xa = 0 \text{ for any } x \in \mathscr{A}\}.$$

For any nonzero $\vec{v} \in V = \mathbb{R}^N$, define the *reflections* $R_{\vec{v}}$ as follows:

$$R_{\vec{v}}(\vec{x}) = \vec{x} - 2\frac{(\vec{x}, \vec{v})}{(\vec{v}, \vec{v})}\vec{v} \qquad \text{for any } \vec{x} \in V. \tag{1.1}$$

The reflections (1.1) have the following properties

$$R_{\vec{v}}(\vec{v}) = -\vec{v}, \qquad R_{\vec{v}}^2 = 1, \qquad (R_{\vec{v}}(\vec{x}), \vec{u}) = (\vec{x}, R_{\vec{v}}(\vec{u})) \quad \text{for any } \vec{v}, \vec{x}, \vec{u} \in V.$$

A finite set of vectors $\mathscr{R} \subset V$ is said to be a *root system* if the following conditions hold:
i) $\mathscr{R}$ is $R_{\vec{v}}$-invariant for any $\vec{v} \in \mathscr{R}$,
ii) if $\vec{v}_1, \vec{v}_2 \in \mathscr{R}$ are collinear, then either $\vec{v}_1 = \vec{v}_2$ or $\vec{v}_1 = -\vec{v}_2$.

Clearly, the group $W(\mathscr{R}) \subset O(N,\mathbb{R}) \subset End(V)$ generated by all reflections $R_{\vec{v}}$ with $\vec{v} \in \mathscr{R}$ is finite.

Let $V = V_1 \oplus V_2$, where $V_1 \neq \{0\}$ and $V_2 \neq \{0\}$ are orthogonal with respect to the form $(\cdot,\cdot)$, and let a root system on V have a decomposition: $\mathscr{R} = \mathscr{R}_1 \bigcup \mathscr{R}_2$, where $\mathscr{R}_i \subset V_i$ for $i = 1,2$. Then each $\mathscr{R}_i \subset V_i$ is a root system. We say in this case that $\mathscr{R}$ is *reducible*, and denote this fact as $\mathscr{R} = \mathscr{R}_1 + \mathscr{R}_2$. Note, that each $\mathscr{R}_i$ can be empty. A root system which is not reducible is called *irreducible*.

If $\mathscr{R} = \mathscr{R}_1 + \mathscr{R}_2$, then $W(\mathscr{R}) = W(\mathscr{R}_1) \times W(\mathscr{R}_2)$.

Any root system has a decomposition $\mathscr{R} = \mathscr{R}_1 + \mathscr{R}_2 + \ldots + \mathscr{R}_n$, where the $\mathscr{R}_j$ are irreducible root systems.

All irreducible root systems are listed in numerous literature (see, e.g., [1], [8], [17], [2], [20], [6], [19]). As it follows from the definition of a root system given above, we consider both crystallographic ($A_n, B_n, C_n, D_n, E_6, E_7, E_8, F_4, G_2$) and non-crystallographic ($H_3, H_4, I_2(n)$) root systems.

We consider also the empty root system, assuming that it generates the trivial group consisting of the unity element only.

The definition of reducible root system implies that the empty root system in $\mathbb{R}^N$ is reducible for any $N > 1$. The *irreducible empty root system* — we denote it A_0 — belongs to $\mathbb{R}$.

1.5. *The superalgebra of observables*

Let $\mathscr{R}$ be a finite root system. Let η be a set of constants $\eta_{\vec{v}}$ with $\vec{v} \in \mathscr{R}$ such that $\eta_{\vec{v}} = \eta_{\vec{w}}$ if $R_{\vec{v}}$ and $R_{\vec{w}}$ belong to one conjugacy class of $W(\mathscr{R})$.

Let $\mathscr{H}^\alpha$, where $\alpha = 0,1$, be two copies of V with orthonormal bases $a_{\alpha i} \in \mathscr{H}^\alpha$, where $i = 1, \ldots, N$.

Definition 1.1. The superalgebra $H_{W(\mathscr{R})}(\eta)$ is an associative superalgebra with unity **1**; it is the superalgebra of polynomials in the $a_{\alpha i}$ with coefficients in the group algebra $\mathbb{C}[W(\mathscr{R})]$ subject to the relations

$$\overline{g}h_\alpha = g(h_\alpha)h_\alpha\overline{g} \text{ for any } g \in W(\mathscr{R}) \text{ and } h_\alpha \in \mathscr{H}^\alpha, \tag{1.2}$$

$$[x_\alpha\overline{I}, y_\beta\overline{I}] = \varepsilon_{\alpha\beta}\left((\vec{x}, \vec{y})\overline{1}\overline{I} + \sum_{\vec{v}\in\mathscr{R}} \eta_{\vec{v}}\frac{(\vec{x}, \vec{v})(\vec{y}, \vec{v})}{(\vec{v}, \vec{v})}\overline{1}\overline{R_{\vec{v}}}\right) \text{ for any } x_\alpha \in \mathscr{H}^\alpha \text{ and } y_\beta \in \mathscr{H}^\beta, \tag{1.3}$$

where $\varepsilon_{\alpha\beta}$ is the antisymmetric tensor, $\varepsilon_{01} = 1$, and $\bar{1}$ is the unity in $\mathbb{C}[a_{\alpha i}]$. The element $\mathbf{1} = \bar{1} \cdot \bar{I}$ is the unity of $H_{W(\mathscr{R})}(\eta)$.[c] The action of any operator $g \in End(V)$ is given by a matrix (g_i^j):

$$g(a_{\alpha i}h^i) = a_{\alpha i}g_i^j h^j, \quad g_1(g_2(h_\alpha)) = (g_1 g_2)(h_\alpha) \text{ for any } h_\alpha = a_{\alpha i}h^i \in \mathscr{H}^\alpha, \tag{1.4}$$

$$g(\bar{1}) = \bar{1}. \tag{1.5}$$

The commutation relations (1.3) suggest to define the *parity* π by setting:

$$\pi(a_{\alpha i}\bar{g}) = 1 \text{ for any } \alpha,\ i \text{ and } g \in \mathbb{C}[W(\mathscr{R})]; \qquad \pi(\bar{1}\bar{g}) = 0 \text{ for any } g \in \mathbb{C}[W(\mathscr{R})].$$

We say that $H_{W(R)}(\eta)$ is a *superalgebra of observables of the Calogero model based on the root system* $\mathscr{R}$.

Clearly, $H_{W(\mathscr{R}_1+\mathscr{R}_2)}(\eta) = H_{W(\mathscr{R}_1)}(\eta) \otimes H_{W(R_2)}(\eta)$.

These algebras $H_{W(R)}(\eta)$ (with parity forgotten) are particular cases of *Symplectic Reflection Algebras* [3] and are also known as *rational Cherednik algebras* (see, e.g., [4]).

It follows from eqs. (1.4) and (1.2) that if $I \in W(\mathscr{R}) \subset End(V)$ is the unity and $J = -I \in End(V)$ belongs to $W(\mathscr{R})$, then $K := \bar{1}\bar{J} \in H_{W(\mathscr{R})}(\eta)$ is a Klein operator in $H_{W(\mathscr{R})}(\eta)$.

2. Traces and supertraces on $H_{W(R)}(\eta)$

The following facts were proved in [14]:

Theorem 2.1. *Let the Coxeter group $W(\mathscr{R}) \subset End(\mathbb{R}^N)$ generated by the finite root system $\mathscr{R} \subset \mathbb{R}^N$ have $T(\mathscr{R})$ conjugacy classes without eigenvalue* 1 *and $S(\mathscr{R})$ conjugacy classes without eigenvalue* -1.

Then the superalgebra $H_{W(\mathscr{R})}(\eta)$ possesses $T(\mathscr{R})$ independent traces and $S(\mathscr{R})$ independent supertraces.

Theorem 2.2. *Each trace and each supertrace on the superalgebta $H_{W(\mathscr{R})}(\eta)$ is even.*

Theorem 2.1 helps to find the numbers $T(\mathscr{R})$ and $S(\mathscr{R})$ for an arbitrary root system $\mathscr{R}$.

Theorem 2.2 implies, clearly, the following statement

Theorem 2.3. *In the terms of Theorem 2.1, the following relations are satisfied:*

$$T(\mathscr{R}_1 + \mathscr{R}_2) = T(\mathscr{R}_1)T(\mathscr{R}_2), \tag{2.1}$$

$$S(\mathscr{R}_1 + \mathscr{R}_2) = S(\mathscr{R}_1)S(\mathscr{R}_2). \tag{2.2}$$

Therefore, the problem of finding $T(\mathscr{R})$ and $S(\mathscr{R})$ is reduced to the problem of finding $T(\mathscr{R})$ and $S(\mathscr{R})$ for irreducible root systems $\mathscr{R}$.

Here, the number $T(\mathscr{R})$ of traces and the number $S(\mathscr{R})$ of supertraces for all irreducible root systems are found and compared. The result is presented in Sections 3 and 4.

It follows from the results presented in Section 3, that if $T(\mathscr{R}) = S(\mathscr{R})$ for some irreducible root system $\mathscr{R}$, then $-I \in W(\mathscr{R})$, and so $H_{W(\mathscr{R})}(\eta)$ has a Klein operator.

The results of this paper were preprinted in [13] and [12].

[c]Clearly, $H_{W(\mathscr{R})}$ contains neither $\bar{1} \in \mathbb{C}$ nor $\bar{I}$.

3. The numbers $T(\mathscr{R})$ of traces and $S(\mathscr{R})$ of supertraces for irreducible root system $\mathscr{R}$ if $T(\mathscr{R}) = S(\mathscr{R})$

$\mathscr{R}$	$T(\mathscr{R}) = S(\mathscr{R})$	presence of $-I$ in $W(\mathscr{R})$ proved in	proof in:
A_1	1	[1], Table I (XI)	Appendix A.2
B_n, C_n	the number of partitions of n into the sum of positive integers	[1], Tables II, III (XI)	Appendix A.3
D_{2n}	the number of partitions of $2n$ into the sum of positive integers with an even number of summands	[1], Table IV (XI)	Appendix A.4
E_7	12	[1], Table VI (XI)	Appendix A.6
E_8	30	[1], Table VII (XI)	Appendix A.7
F_4	9	[1], Table VIII (XI)	Appendix A.8
G_2	3	[1], Table IX (XI)	Appendix A.9
H_3	4	[6], p.160; Appendix A.10	Appendix A.10
H_4	20	[6], Table 3, K_2	Appendix A.11
$I_2(2n)$	n	Appendix A.12	Appendix A.12

4. The numbers $T(\mathscr{R})$ of traces and $S(\mathscr{R})$ of supertraces for irreducible root system $\mathscr{R}$ if $W(R) \not\ni -I$

$\mathscr{R}$	$T(\mathscr{R})$	$S(\mathscr{R})$	proof in:
A_0	0	1	Appendix A.1
$A_{n-1}, n \geq 3$	1	the number of partitions of n into the sum of odd positive integers	Appendix A.2
D_{2n+1}	the number of partitions of $2n+1$ into the sum of positive integers with an even number of summands	the number of partitions of $2n+1$ into the sum of positive integers with an odd number of summands	Appendix A.4
E_6	5	9	Appendix A.5
$I_2(2n+1)$	n	$n+1$	Appendix A.12

The Weyl superalgebra. Let W_n be Weyl superalgebra with n pairs of generating elements: $W_n = \mathbb{C}[a_{\alpha i}]$, where $\alpha = 0, 1$ and $i = 1, ..., n$, subject to relations $[a_{\alpha i}, a_{\beta j}] = \varepsilon_{\alpha\beta}\delta_{ij}$ and with parity defined by $\pi(a_{\alpha i}) = 1$. Clearly, $W_n = (W_1)^{\otimes n}$. Further, superalgebra $H_{W(A_0)}(\eta)$ does not depend on η and since A_0 is irreducible, $H_{W(A_0)}(\eta) = W_1$. So, due to first row of Table 4 and Theorem 2.3, the Weyl superalgebra W_n has 1 supertrace and 0 traces.

5. Inequality Theorem

Theorem 5.1. *Let $H_{W(\mathscr{R})}(\eta)$ has $T(\mathscr{R})$ traces and $S(\mathscr{R})$ supertraces. Then* *i)* $S(\mathscr{R}) > 0$,

ii) $T(\mathscr{R}) \leq S(\mathscr{R})$,

iii) $T(\mathscr{R}) = S(\mathscr{R})$ *if and only if* $W(\mathscr{R})$ *contains* $-I$*. Equivalently,* $T(\mathscr{R}) = S(\mathscr{R})$ *if and only if* $H_{W(\mathscr{R})}(\eta)$ *contains a Klein operator.*

Proof. Since each group contains unity I and spectrum of I does not contain -1, it follows that $S(\mathscr{R}) > 0$.

Let $K \in H_{W(\mathscr{R}}(\eta)$ be a Klein operator. Then K establishes one-to-one correspondence between traces and supertraces:

$$\mathrm{tr}(f) = \mathrm{s}tr(fK), \; \mathrm{s}tr(g) = \mathrm{tr}(gK)$$

Let $T(\mathscr{R}) = S(\mathscr{R})$. Then the decomposition of $\mathscr{R}$ in the sum of irreducible root systems does not contain root systems from Table 4, namely A_0, A_n for $n \geq 2$, D_{2n+1} for $n \geq 1$ and E_6, because $T(\mathscr{R}_i) < S(\mathscr{R}_i)$ for all these root systems. So, this decomposition contains the root systems listed in Table 3 only, each of these groups has the element $-I$ and the direct product of all these $-I$s is $-I$ in $W(\mathscr{R})$.

It remains to prove the inequalities

$$T(A_n) < S(A_n) \qquad \text{for } n = 0 \text{ and } n \geq 2, \tag{5.1}$$

$$T(D_{2n+1}) < S(D_{2n+1}) \;\; \text{for } n \geq 1, \tag{5.2}$$

$$T(E_6) < S(E_6). \tag{5.3}$$

Inequalities (5.1) and (5.3) manifestly follow from Table 4, and inequality (5.2) follows from Table 4 and Lemma 5.1, ii) below. ☐

Lemma 5.1.[d] *Let* $E(n)$ *be the number of partitions of* n *into the sum of positive integers with an even number of summands. Let* $O(n)$ *be the number of partitions of* n *into the sum of positive integers with an odd number of summands. Then*

i) $E(2k) > O(2k)$ *for* $k \geq 2$,

ii) $E(2k-1) < O(2k-1)$ *for* $k \geq 1$,

iii) $E(2) = O(2)$,

iv) $|E(n) - O(n)| = R(n)$*, where* $R(n)$ *is the number of partitions of* n *into the sum of different positive odd integers.*

Proof. Let a_{mn} be the number of partitions of n into the sum of positive integers with m summands, $a_{m0} = \delta_{m0}$. Introduce the generating function

$$F(t,x) = \sum_{m,n=0}^{\infty} a_{mn} t^m x^n.$$

Then

$$\sum_n E(n)x^n = \frac{1}{2}(F(t,x) + F(-t,x))|_{t=1} \text{ and } \sum_n O(n)x^n = \frac{1}{2}(F(t,x) - F(-t,x))|_{t=1}.$$

[d] This Lemma is a simple exercise from Partitions Theory, see, e.g., [7], [16] and references therein.

Hence,

$$\sum_n (O(n) - E(n))x^n = -F(-t,x)|_{t=1}.$$

Further,

$$\begin{aligned} F(t,x) &= (1+tx+(tx)^2+(tx)^3+\cdots)(1+tx^2+(tx^2)^2+(tx^2)^3+\ldots)\ldots \\ &= \frac{1}{(1-tx)(1-tx^2)(1-tx^3)\ldots} \end{aligned}$$

So

$$-F(-t,x)|_{t=1} = -\frac{1}{(1+x)(1+x^2)(1+x^3)\ldots} \tag{5.4}$$

Multiplying both terms of fraction (5.4) by $(1-x)(1-x^2)(1-x^3)\ldots$ we obtain

$$-F(-t,x)|_{t=1} = -\frac{(1-x)(1-x^2)(1-x^3)\ldots}{(1-x^2)(1-x^4)(1-x^6)\ldots} = -(1-x)(1-x^3)(1-x^5)\ldots \tag{5.5}$$

Now, it suffices to notice that eq. (5.5) can be rewritten in the form

$$\sum_n (O(n) - E(n))x^n = -F(-t,x)|_{t=1} = \sum_{n \text{ is odd}} R(n)x^n - \sum_{n \text{ is even}} R(n)x^n .$$

□

Acknowledgments

This work was supported by the Russian Foundation for Basic Research, Grant No. 11-02-00685.

Appendix A. Computing the number of traces and supertraces for all irreducible root systems

A.1. *Root system* A_0

The Weyl algebra $\mathbb{C}[a, a^+]$ generated by elements a and a^+ satisfying the relation $[a,a^+] = 1$ may be considered as the algebra of observables of the Calogero model based on the empty irreducible root system A_0, which generates the trivial group consisting only of the unity 1. This group has 1 conjugacy class without -1 in its spectrum and 0 conjugacy classes without 1 in its spectrum. So, this Weyl algebra has 0 traces and 1 supertrace.

A.2. *Root systems* A_{n-1} *for* $n > 1$

It is well-known that $W(A_{n-1}) = S_n$ and $V = span(e_1 - e_2, e_2 - e_3, \ldots, e_{n-1} - e_n)$. Each element of S_n can be decomposed in the product of cycles of the form

$$\sigma : e_{i_1} \to e_{i_2} \to \ldots \to e_{i_k} \to e_{i_1}.$$

A.2.1. *The number of traces*

If $k<n$, then each cycle $\sigma: e_{i_1}\to e_{i_2}\to\ldots\to e_{i_k}\to e_{i_1}$ has eigenvalue $+1$ with eigenvector

$$e_{i_1}+e_{i_2}+\ldots+e_{i_k}-\frac{k}{n}\sum_{s=1}^{n}e_s.$$

The only conjugacy class without eigenvalue $+1$ is the one containing the cycle of maximal length n:

$$\sigma:\ e_1\to e_2\to\ldots\to e_n\to e_1$$

because its characteristic polynomial has the form

$$f(t)=1+t+...+t^{n-1}.$$

A.2.2. *The number of supertraces (see also [15])*

The cycle $\sigma:\ e_{i_1}\to e_{i_2}\to\ldots\to e_{i_k}\to e_{i_1}$ has eigenvalue -1 if and only if k is even. The corresponding eigenvector has the form

$$e_{i_1}-e_{i_2}+\ldots-e_{i_k}.$$

So, the number of conjugacy classes without eigenvalue -1 is equal to the number of partitions of n into the sum of positive odd integers [15].

A.2.3. *Presence of $-I$ in the group $W(A_{n-1})$, where $n>1$.*

If $A_{n-1}\ni -I$, then $n=2$ (see [1], Table I (XI)). The group $W(A_1)$ consists of two elements: I and $-I$.

A.3. *Root systems B_n and C_n*

The Coxeter group $G=W(B_n)=W(C_n)$ is generated by the permutation group S_n and reflections $R_i:\ e_i\to -e_i,\ e_j\to e_j$ for $i\neq j$, see [1]. Each element $g\in G$ can be represented in the form

$$g=\sigma\prod_{i=1}^{n}R_i^{\alpha_i},\text{ where }\sigma\in S_n\text{ and }\alpha_i\in\{0,1\}.$$

The set $(\sigma,\alpha_1,...\alpha_n)$ unambiguously defines every element of G.

Since each permutation can be decomposed in the product of commuting cycles,

$$\sigma=\prod\hat{\sigma}_k,\quad\text{where }\hat{\sigma}_k:\ e_{i_1}\to e_{i_2}\to\ldots\to e_{i_k}\to e_{i_1},$$

we can introduce what we call *R-cycles* by the formula

$$\tilde{\sigma}_k=\hat{\sigma}_kR_{i_1}^{\alpha_{i_1}}R_{i_2}^{\alpha_{i_2}}\ldots R_{i_k}^{\alpha_{i_k}}$$

So, each element $g\in G$ has the form

$$g=\prod_p\tilde{\sigma}_p.\tag{A.1}$$

We say that the value $\varepsilon_R(\tilde{\sigma})=|\alpha_{i_1}+\alpha_{i_2}\ldots+\alpha_{i_k}|_{\mathrm{mod}\,2}$ is the *R-parity of the R-cycle* $\tilde{\sigma}$. Let $l(\tilde{\sigma})=k$, where k is the length of the cycle $\hat{\sigma}$.

It is easy to prove that an R-cycle $\tilde{\sigma}_1$ is conjugated to an R-cycle $\tilde{\sigma}_2$ if and only if $l(\tilde{\sigma}_1) = l(\tilde{\sigma}_2)$ and $\varepsilon_R(\tilde{\sigma}_1) = \varepsilon_R(\tilde{\sigma}_2)$.

So, a conjugacy class in G is characterized by the numbers $p_1, p_2, \ldots, p_n$ and $q_1, q_2, \ldots, q_n$, where p_i is the number of R-cycles of length i and R-parity 0, and q_i is the number of R-cycles of length i and R-parity 1, in the presentation of g in the form (A.1).

The numbers p_i and q_i satisfy the relation

$$\sum_{i=1}^{n} (i p_i + i q_i) = n. \tag{A.2}$$

A.3.1. *The number of traces*

The characteristic polynomial of the R-cycle $\tilde{\sigma}$ has the form

$$(-1)^{l(\tilde{\sigma})} (t^{l(\tilde{\sigma})} - (-1)^{\varepsilon(\tilde{\sigma})}). \tag{A.3}$$

It has no root $+1$ if $\varepsilon(\tilde{\sigma}) = 1$.

So, if given conjugacy class has no eigenvalue $+1$, then $p_i = 0$ and $\sum_i i q_i = n$.

A.3.2. *The number of supertraces (see also [9])*

The characteristic polynomial of an R-cycle $\tilde{\sigma}$ (A.3) has no root -1 if either $l(\tilde{\sigma})$ is even and $\varepsilon(\tilde{\sigma}) = 1$ or if $l(\tilde{\sigma})$ is odd and $\varepsilon(\tilde{\sigma}) = 0$.

So, if a given conjugacy class has no eigenvalue -1, then $p_{2k} = 0$ and $q_{2k+1} = 0$ and eq. (A.2) gives $p_1 + 2q_2 + 3p_3 + 4q_4 + \ldots = n$.

A.3.3. *Presence of $-I$ in the group $W(B_n)$.*

It is easy to see that $-I = \Pi_{i=1}^{n} R_i$.

A.4. *Root systems D_n*

The Coxeter group $W(D_n)$ is a subgroup of $W(B_n)$, namely, $g = \prod \tilde{\sigma}_s$ belongs to $W(D_n)$ if $(\sum_s \varepsilon_R(\tilde{\sigma}_s))|_{\mathrm{mod}\,2} = 0$ [1].

A.4.1. *The number of traces*

So, g has no eigenvalue $+1$ if $p_i = 0$, $\sum_i i m_i = n$ and $(\sum_i m_i)|_{\mathrm{mod}\,2} = 0$.

This implies that $T(D_n)$ is equal to the number of partition of n into the sum of positive integers with an even number of summands.

A.4.2. *The number of supertraces (see also [9])*

Analogously, $S(D_n)$ is equal to the number of partitions of n into the sum of positive integers with an even number of even integers.

Clearly, if n is even, then $S(D_n)$ is equal to the number of partitions of n into the sum of positive integers with an even number of summands, and if n is odd, then $S(D_n)$ is equal to the number of partitions of n into the sum of positive integers with an odd number of summands.

A.4.3. *Presence of $-I$ in the group $W(D_n)$.*

If n is even, then $-I = \Pi_{k=1}^{n} R_k$.

A.5. *Root system* E_6

The conjugacy classes of the Weyl group E_6 are described in Table 9 of [2][e].

A.5.1. *The number of traces*

The following 5 classes do not have the root $+1$:

$$A_2^3,\ A_5 \times A_1,\ E_6,\ E_6(a_1),\ E_6(a_2).$$

A.5.2. *The number of supertraces*

The following 9 classes do not have the root -1:

$$\phi,\ A_2,\ A_2^2,\ A_4,\ D_4(a_1),\ A_2^3,\ E_6,\ E_6(a_1),\ E_6(a_2).$$

A.6. *Root system* E_7

The conjugacy classes of the Weyl group E_7 are described in Table 10 of [2][e].

A.6.1. *The number of traces*

The last 12 classes in Table 10 of [2] do not have the root $+1$:

$$A_1^7,\ A_3^2 \times A_1,\ A_5 \times A_2,\ A_7,\ D_4 \times A_1^3,\ D_6 \times A_1,\ D_6(a_2) \times A_1,$$
$$E_7,\ E_7(a_1),\ E_7(a_2),\ E_7(a_3),\ E_7(a_4).$$

A.6.2. *The number of supertraces*

The following 12 classes do not have the root -1:

$$\phi,\ A_2,\ A_2^2,\ A_4,\ D_4(a_1),\ A_2^3,\ A_4 \times A_2,\ A_6,\ D_6(a_1),\ E_6,\ E_6(a_1),\ E_6(a_2).$$

A.7. *Root system* E_8

The conjugacy classes of the Weyl group E_8 are described in Table 11 of [2][e].

A.7.1. *The number of traces*

The last 30 classes in Table 11 of [2] do not have the root $+1$:

$$A_1^8,\ A_2^4,\ A_3^2 \times A_1^2,\ A_4^2,\ A_5 \times A_2 \times A_1,\ A_7 \times A_1,\ A_8,\ D_4 \times A_1^4,\ D_4^2,\ D_4(a_1)^2,$$
$$D_5(a_1) \times A_3,\ D_6 \times A_1^2,\ D_8,\ D_8(a_1),\ D_8(a_2),\ D_8(a_3),\ E_6 \times A_2,\ E_6(a_2) \times A_2,$$
$$E_7 \times A_1,\ E_7(a_2) \times A_1,\ E_7(a_4) \times A_1,\ E_8,\ E_8(a_i)\ (i = 1, \ldots 8).$$

[e] In [2], the Carter diagrams are used to describe the conjugacy classes of the finite Weyl groups. Additional results on the Carter diagrams are recently obtained in [20]. In particular, two problems are discussed:
i) two Carter diagrams can correspond to one conjugacy class ([20], Theorem 4.1) and
ii) certain Carter diagrams can correspond to two distinct conjugacy classes ([20], Theorem 6.5).

A.7.2. *The number of supertraces*

The following 30 classes do not have the root -1:

$$\phi,\ A_2,\ A_2^2,\ A_4,\ D_4(a_1),\ A_2^3,\ A_4\times A_2,\ A_6,\ D_4(a_1)\times A_2,\ D_6(a_1),$$
$$E_6,\ E_6(a_1),\ E_6(a_2),\ A_2^4,\ A_4^2,\ A_8,\ D_4(a_1)^2,\ D_8(a_1),\ D_8(a_3),$$
$$E_6\times A_2,\ E_6(a_2)\times A_2,\ E_8,\ E_8(a_i)\ (i=1,\dots 8).$$

A.8. ***Root system*** F_4

The conjugacy classes of the Weyl group F_4 are described in Table 8 of [2][e].

A.8.1. *The number of traces*

The last 9 classes in Table 8 of [2] do not have the root $+1$:

$$A_1^4,\ A_2\times\tilde{A}_2,\ A_3\times\tilde{A}_1,\ C_3\times A_1,\ D_4,\ D_4(a_1),\ B_4,\ F_4,\ F_4(a_1).$$

A.8.2. *The number of supertraces*

The following 9 classes do not have the root -1:

$$\phi,\ A_2,\ \tilde{A}_2,\ B_2,\ A_2\times\tilde{A}_2,\ D_4(a_1),\ B_4,\ F_4,\ F_4(a_1).$$

A.9. ***Root system*** G_2

The conjugacy classes of the Weyl group G_2 are described in Table 7 of [2][e].

A.9.1. *The number of traces*

The last 3 classes in Table 7 of [2] do not have the root $+1$:

$$A_1\times\tilde{A}_1,\ A_2,\ G_2,.$$

A.9.2. *The number of supertraces (see [10])*

The following 3 classes in Table 7 of [2] do not have the root -1:

$$\phi,\ A_2,\ G_2,.$$

The fact that G_2 has 3 conjugacy classes without eigenvalue $+1$ and 3 conjugacy classes without eigenvalue -1 can be derived also from Appendix A.12 because $G_2 = I_2(6)$.

A.10. ***Root system*** H_3

Let $k = \frac{1}{2}(\sqrt{5}+1)$. Then the reflections

$$a = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \qquad b = \frac{1}{2}\begin{pmatrix} 1 & k & k-1 \\ k & 1-k & -1 \\ k-1 & -1 & k \end{pmatrix}, \qquad c = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

corresponding to the roots $\vec{e}_2$, $\frac{1}{2}(-\vec{e}_1 + k\vec{e}_2 + k^{-1}\vec{e}_3)$ and $\vec{e}_1$, respectively, satisfy the relations

$$a^2 = b^2 = c^2 = 1, \ \ (ab)^5 = (bc)^3 = (ac)^2 = 1$$

and generate the Coxeter group H_3.

As $H_3 = S_5^e \times C_2$ (See [1], Ch.VI, Sect. 4, Ex.11 d), p. 284; [6], p.160 and references therein), where S_5^e is the group of even permutations of 5 elements, $C_2 = \{1, -1\}$, it follows that H_3 has 10 conjugacy classes, 5 with a positive determinant and 5 with a negative one. (Observe that $|S_5^e| = 60$, hence $|H_3| = 120$.)

The conjugacy classes with positive determinant are described by their representatives

The representative	The characteristic polynomial
I	$(1-t)^3$
ac	$(1-t)(1+t)^2$
bc	$(1-t)(t^2+t+1)$
ab	$(1-t)[t^2+(1-k)t+1]$
$abab$	$(1-t)(t^2+kt+1)$

(A.4)

A.10.1. *The number of supertraces*

Only four conjugacy classes have no roots -1, their representatives are I, bc, ab and $abab$.

A.10.2. *The number of traces*

Each of the characteristic polynomials (A.4) has the root $+1$. Besides, the conjugacy class with representative $-ac$ has the root $+1$, so only four conjugacy classes with negative determinant have no roots $+1$. Their representatives are $-I$, $-bc$, $-ab$ and $-abab$.

So, the number of conjugacy classes without root $+1$ is equal to 4.

A.10.3. *Presence of* $J = -I$ *in* $W(H_3)$.

The group H_3 contains the element $J = -I = (ababc)^3$ (see [5], p.11).

A.11. *Root system* H_4

According to [6], all 34 conjugacy classes of the Coxeter group H_4 are described by their representatives acting on the space of quaternions

$$g_{lr}: \quad x \mapsto lxr^*, \tag{A.5}$$
$$g^*_p: \quad x \mapsto px^*. \tag{A.6}$$

All 25 pairs of unit quaternions l and r and 9 unit quaternions p are listed in Table 3 of [6].

A.11.1. *The number of traces*

Each operator (A.6) has the root $+1$. Indeed, the equation $px^* = x$ has a nonzero solution $x = 1 + p$ if $p \neq -1$, and x is an arbitrary imaginary quaternion if $p = -1$.

Each operator (A.5) does not have the root $+1$ if and only if

$$l_0 - r_0 \neq 0. \tag{A.7}$$

Indeed, the determinant of the map $x \mapsto lx - xr$ is equal to $4(l_0 - r_0)^2$.

There are 20 pairs (l, r) in Table 3 of [6] satisfying the condition (A.7), namely, K_i with i = 2, 5, 7, 9 to 25.

A.11.2. *The number of supertraces*

Each operator (A.6) has the root -1. Indeed, the equation $px^* = -x$ has nonzero solution $x = 1 - p$ if $p \neq 1$, and x is an arbitrary imaginary quaternion if $p = 1$.

Each operator (A.5) does not have the root -1 if and only if

$$l_0 + r_0 \neq 0. \tag{A.8}$$

Indeed, the determinant of the map $x \mapsto lx - xr$ is equal to $4(l_0 + r_0)^2$.

There are 20 pairs of l, r in Table 3 of [6] satisfying the condition (A.8), namely, K_i with i = 1, 4, 6, 8, 10 to 25.

A.11.3. *Presence of element $J = -I$ in $W(H_4)$.*

The element K_2 in Table 3 of [6] with $l = -r = 1$ is $-I$ in H_4.

A.12. *Root systems* $I_2(n)$

It is convenient to use $\mathbb{C}$ instead of $\mathbb{R}^2$ to describe $W(I_2(n))$. The root system $I_2(n)$ contains $2n$ vectors $v_k = \exp(\pi i k/n)$, where $k = 0, 1, ..., 2n-1$. The corresponding Coxeter group $W(I_2(n))$ has $2n$ elements, n reflections R_k acting on $z,\ z^* \in \mathbb{C}$ as follows

$$R_k z = -z^* v_k^2 R_k,$$
$$R_k z^* = -z v_k^{*2} R_k, \qquad k \in \mathbb{Z}_n$$

and n elements of the form $S_k = R_k R_0$, where S_0 is the unity in $W(I_2(n))$. These elements satisfy the following relations

$$R_k R_l = S_{k-l}, \qquad S_k S_l = S_{k+l}, \qquad R_k S_l = R_{k-l}, \qquad S_k R_l = R_{k+l}.$$

Obviously, the reflections R_{2k} lie in one conjugacy class and R_{2k+1} lie in another one if n is even. If n is odd, then all reflections R_k lie in one conjugacy class. Each reflection has both eigenvalues $+1$ and -1.

The rotation S_k has no eigenvalues -1 if $k \neq n/2$, and has no eigenvalues $+1$ if $k \neq 0$. If n is even, then $S_{n/2} = -I$.

Rotations S_k and S_{-k} form a conjugacy class.

So, the number of conjugacy classes without $+1$ is equal to $\left[\frac{n}{2}\right]$, and the number of conjugacy classes without -1 is equal to $\left[\frac{n+1}{2}\right]$, see [10].

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Co-published by Atlantis Press and Taylor & Francis